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A Characterization of Perfect $\{0, \pm 1\}$-Matrices

by

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Abstract

The concept of perfect $\{0, \pm 1\}$-matrix was introduced recently as a generalization of perfect $\{0, 1\}$-matrices. A $\{0, \pm 1\}$-matrix A is called perfect if the associated generalized set-packing polytope

$$P(A) = \{x \mid \mathbf{0} \le x \le \mathbf{1}, Ax \le \mathbf{1} - \mathrm{n}(A)\}$$

is integral, where $\mathrm{n}(A)$ denote the vector which r-th component is the number of negative components in the r-th row of A. Several characterizations for perfect $\{0, \pm 1\}$-matrices are already known. Here, we associate a bidirected graph for a $\{0, \pm 1\}$-matrix in an obvious way and give a simple characterization of perfect $\{0, \pm 1\}$-matrices in terms of bidirected graphs.

1. Introduction

Let A be a $\{0, \pm 1\}$-matrix of size $m \times n$. The *generalized set-packing polytope* $P(A)$ associated with A is defined as

$$P(A) = \{x \mid x \in \mathbf{R}^n, \mathbf{0} \le x \le \mathbf{1}, Ax \le \mathbf{1} - \mathrm{n}(A)\}, \tag{1.1}$$

where for any scalar $\gamma \in \mathbf{R}$ we denote by $\boldsymbol{\gamma}$ the vector of dimension m which component are all γ and $\mathrm{n}(A)$ denotes the vector which r-th component is the number of negative components in the r-th row of A. A $\{0, \pm 1\}$-matrix is called *perfect* if its associated generalized set-packing polytope $P(A)$ is integral. The concept of perfectness was introduced by Conforti, Cornuèjols and de Francesco [6]. Perfect matrices encompass a large variety of classes of $\{0, \pm 1\}$-matrices related to integral polyhedra such as totally unimodular matrices and balanced $\{0, \pm 1\}$-matrices ([15]; see also [5]).

Recent years, perfect $\{0, \pm 1\}$-matrices have received considerable attention and several authors gave characterizations of perfectness. Conforti, Cornuéjols and de Francesco [6] and Guenin [9] characterized perfectness in terms of associated $\{0,1\}$-matrices. Boros and Čepek [2] gave a characterization is in terms of an associated undirected graph.

Let us recall a characterization of perfect $\{0,1\}$-matrices. For a $\{0,1\}$-matrix with m rows and n columns, we associate a graph G_A which has the vertex set as the set of column indices $V = \{1, \cdots, n\}$ and has an edge $\{v, w\}$ if and only if there is a row $a_r = (\alpha_{r1}, \cdots, \alpha_{rn})$ of A such that a_r covers $\{v, w\}$, i.e., we have $\alpha_{rv} = \alpha_{rw} = 1$. A graph G is called *perfect* if for any induced subgraph G' the clique number $\omega(G')$ of G' (the maximum size of a clique of G') is equal to the chromatic number $\chi(G')$ of G' (the minimum number of pairwise disjoint stable sets of G') (see, e.g., [13]).

Theorem 1.1 (Chvátal [4], Fulkerson [8], Lovász [13]): *A $\{0,1\}$-matrix A is perfect if and only if*

(i) *The associated graph G_A is perfect, and*
(ii) *Each maximal clique of G_A appears as a row of A.*

□

The purpose of this paper is to give a complete analogue of Theorem 1.1, where we associate a *bidirected graph* ([7]; see also [12]) for each $\{0, \pm 1\}$-matrix and characterize the perfectness in terms of the bidirected graph. (Boros and Čepek's characterization [2] has similar flavor as Theorem 1.1. However, they associate with a $\{0, \pm 1\}$-matrix an undirected graph, which construction seems less obvious.)

A bidirected graph $G = (V, A)$ is defined as a pair of a finite set V called *vertex set* and a finite set A called *arc set*, where for arc $a \in A$ either a has two tails v and w, a has two heads v and w, or a has one tail v and one head w. (Later, we will give another pricise definition of bidirected graph.) For any $\{0, \pm 1\}$-matrix A of size $m \times n$, we associate a bidirected graph $G_A = (V, A)$, where $V = \{1, \cdots, n\}$ and the arc set A is defined as follows:

(i) there is an arc $a \in A$ which has two tails v and w if and only if there is a row a_r of A such that $\alpha_{rv} = \alpha_{rw} = 1$.
(ii) there is an arc $a \in A$ which has two heads v and w if and only if there is a row a_r of A such that $\alpha_{rv} = \alpha_{rw} = -1$.
(iii) there is an arc $a \in A$ which has one tail v and one head w if and only if there is row a_r of A such that $\alpha_{rv} = -\alpha_{rw} = 1$.

A crucial step for proving the analogue of Theorem 1.1 is the characterizations of *perfect* bidirected graph obtained recent years ([10], [14]). This corresponds to the polyhedral characterization of perfect graphs of Chvátal [4] in the proof of Theorem 1.1. The rest of the proof of our result is not so immediate as in the case for $\{0,1\}$-matrices. Another important step is the reduction of $\{0, \pm 1\}$-matrices

used by Boros and Čepek [2]. However, for our purpose, the reduction of Boros and Čepek [2] is not enough. We must further reduce $\{0, \pm 1\}$-matrices.

Our characterization has the following features as was the case for Theorem 1.1: (i) The associated bidirected graph is naturally defined; and (ii) Our characterization gives an insight to structures of integral generalized set-packing polytopes. Also, our characterization, if specialized to $\{0, 1\}$-matrices, is easily seen to yield Theorem 1.1.

The remaining sections of this paper are organized as follows. In Section 2 we review the concept of bidirected graph and the characterizations of bidirected graphs. In Section 3 we describe the reduction of $\{0, \pm 1\}$-matrices obtained in [2] (see also [3]). Finally, in Section 4 we describe a further reduction step for $\{0, \pm 1\}$-matices and prove the main theorem characterizing perfect $\{0, \pm 1\}$-matrices.

2. Perfect Bidirected Graphs

In this section we review concepts and results concerned with bidirected graphs. Characterizations of perfect bidirected graphs of Sewell [14] and of Ikebe and Tamura [10] are of particular importance.

2.1. Bidirected graphs

A bidirected graph $G = (V, B; \partial)$ consists of a finite set V called the *vertex set* and a finite set B called the *arc set*, and $\partial\colon B \to \mathbf{Z}^V$, where for each $b \in B$ there exists two vertices v and w such that we have either

(i) $\partial b = \chi_v + \chi_w$ (b has two tails v and w),

(ii) $\partial b = -\chi_v - \chi_w$ (b has two heads v and w), or

(iii) $\partial b = \chi_v - \chi_w$ (b has one tail v and one head w),

where $\chi_w \in \mathbf{R}^V$ is defined as

$$\chi_w(v) = \begin{cases} 1 & \text{if } v = w \\ 0 & \text{otherwise} \end{cases} \qquad (v \in V), \tag{2.1}$$

for any $w \in V$. Throughout this paper, we assume $v \neq w$ for (iii). If $b = \pm v \pm w$, we say b is *incident to* v (and w) and if the coefficient of v is positive (respectively, negative), b is said to be *positively* (respectively, *negatively*) incident to v. Two arcs b_1 and b_2 are called *oppositely* incident to v if b_1 and b_2 are incident to a common vertex v but with opposite signs.

We associate polytope $P(G)$ with a bidirected graph $G = (V, B; \partial)$ defined as

$$P(G) = \{x \mid x \in \mathbf{R}^V, 0 \le x \le 1, \forall b \in B\colon \langle \partial b, x\rangle \le 1 - n(\partial b)\}, \tag{2.2}$$

where $\langle \cdot, \cdot \rangle$ denotes the canonical inner product and for any $z \in \{0, \pm 1\}^V$ we denote by $n(z)$ the number of -1's in z. An integral vector in $P(G)$ is called a *solution* of G. Let $P_I(G)$ be the convex hull of the solutions of G.

For a finite set V let us denote by 3^V the set of ordered pair of disjoint subsets of V, i.e., $3^V = \{(X, Y) \mid X, Y \subseteq V, X \cap Y = \emptyset\}$. Each element (X, Y) of 3^V naturally corresponds to a $\{0, \pm 1\}$-vector $\chi_{(X,Y)}$ defined as

$$\chi_{(X,Y)}(v) = \begin{cases} 1 & \text{if } v \in X \\ -1 & \text{if } v \in Y \\ 0 & \text{otherwise} \end{cases} \qquad (v \in V) \tag{2.3}$$

and conversely. Vector $\chi_{(X,Y)}$ is called the *characteristic vector* of (X, Y). We also call each element in 3^V a *signed subset* of V.

For any two signed subsets $(X_1, Y_1), (X_2, Y_2) \in 3^V$, if $X_1 \subseteq X_2$ and $Y_1 \subseteq Y_2$, we write $(X_1, Y_1) \sqsubseteq (X_2, Y_2)$.

For a $\{0, \pm 1\}$-matrix A of size $m \times n$ we associate a bidirected graph G_A as follows. G_A has the vertex set as the set of columns $V = \{1, \cdots, n\}$ and there is an arc b of G_A if and only if there is a row a_r of A such that a_r covers ∂b, i.e., $\partial b \sqsubseteq a_r$. We call G_A the *quadratic covering (bidirected) graph of* $\{0, \pm 1\}$-matrix A (see [3]).

Let $P_I(A)$ be the integer hull of $P(A)$, i.e., the convex hull of the integral vectors in $P(A)$.

Proposition 2.1 ([3]): *Let A be a $\{0, \pm 1\}$-matrix and G_A its quadratic covering graph. Then we have*

$$P(A) \subseteq P(G_A), \tag{2.4}$$

$$P_I(A) = P_I(G_A). \tag{2.5}$$

□

Note that we may have a strict inclusion in (2.4), e.g., $P(G_A)$ is always nonempty (since $\frac{1}{2} \in P(G_A)$) while $P(A)$ can be empty.

For a bidirected graph G an alternate sequence of arcs and vertices

$$P = (v_0, b_1, v_1, \cdots, b_l, v_l) \tag{2.6}$$

of G is called a *path* of G if

(i) b_1 is incident to v_0 and b_l is incident to v_l.
(ii) b_i and b_{i+1} oppositely incident to v_i for $i = 1, \cdots, l-1$.

A path is called *cycle* if $v_0 = v_l$. A bidirected graph is called *acyclic* if it contains no cycle.

A bidirected graph G is called *transitively closed* if for any two arcs b_1 and b_2 oppositely incident to a vertex v there exists an arc b_3 of G such that $\partial b_3 = \partial b_1 + \partial b_2$.

The *transitive closure* of $G = (V, B; \partial)$ is the minimal transitively closed bidirected graph $G^* = (V, B^*, \partial^*)$ with $B \subseteq B^*$ and ∂ being the restriction of ∂^*. It is obvious that the set of solutions of G coincides with the set of solutions of G^*, and hence,

Proposition 2.2: *For any bidirected graph G we have*

$$P_I(G) = P_I(G^*) \tag{2.7}$$

□

2.2. Perfect bidirected graphs

We call an acyclic and transitively closed bidirected graph *closed* bidirected graph for short.

For a closed bidirected graph G, a signed subset $(X, Y) \in 3^V$ is called a *biclique* ([11]) if for each distinct $v, w \in X \cup Y$ there is an arc b of G such that $\partial b = \pm v \pm w$ and $\langle \partial b, (X, Y) \rangle = 2$. The bicliques $(\{v\}, \emptyset), (\emptyset, \{v\})$ $(v \in V)$ are called *trivial*. A biclique of G is *strong* ([11]) if there is no vertex $v \in V - (X \cup Y)$ of G such that

(i) for each $w \in X$ there exists an arc b of G such that $\partial b = \chi_w \pm v$, and
(ii) for each $w \in Y$ there exists an arc b of G such that $\partial b = -\chi_w \pm v$.

Theorem 2.3 (Johnson and Padberg [11]): *Let $G = (V, B; \partial)$ be a closed bidirected graph. Then we have*

(i) *$P_I(G)$ is full-dimensional.*
(ii) *For each strong biclique (X, Y) of G, inequality $x(X) - x(Y) \leq 1 - |Y|$ induces a facet of $P_I(G)$,*

where $x(Z) = \sum_{v \in Z} x(v)$ for any $Z \subseteq V$. □

A closed bidirected graph G is called *perfect* ([11]) if for each $\{0, \pm 1\}$-vector $c: V \to \{0, \pm 1\}$, two objective function values of the following dual linear programming problems are equal.

$$(P) \left| \begin{array}{ll} \text{Maximize} & \sum_{v \in V} c(v) x(v) \\ \text{s.t.} & x(X) - x(Y) \leq 1 - |Y| \quad ((X, Y) \in \mathcal{S}(G)). \end{array} \right. \tag{2.8}$$

$$(D) \left| \begin{array}{ll} \text{Minimize} & \sum_{(X,Y) \in \mathcal{S}(G)} (1 - |Y|) \lambda_{(X,Y)} \\ \text{s.t.} & \sum_{(X,Y) \in \mathcal{S}(G), v \in X} \lambda_{(X,Y)} - \sum_{(X,Y) \in \mathcal{S}(G), v \in Y} \lambda_{(X,Y)} \geq c(v) \quad (v \in V), \end{array} \right. \tag{2.9}$$

where $S(G)$ denote the set of strong bicliques of G.

The following theorem was conjectured by Johnson and Padberg [11] in 1982 and proved by Sewell [14] only recently. For a bidirected graph $G = (V, B; \partial)$ the *underlying graph* $\underline{G} = (V, E)$ of G is the (undirected) graph which edge set E is defined by

$$E = \{\{v, w\} \mid \exists b \in B\colon \partial b = \pm\chi_v \pm \chi_w\}. \qquad (2.10)$$

That is, $\underline{G}$ is the undirected graph disregarding the incidence relation of every arc of G.

Theorem 2.4 (Sewell [14]): *A closed bidirected graph G is perfect if and only if its underlying graph $\underline{G}$ is perfect.* □

For a closed bidirected graph G define a polytope $Q(G) \subseteq \mathbf{R}^V$ as

$$Q(G) = \{x | x \in \mathrm{R}^V, x(X) - x(Y) \leq 1 - |Y| \text{ ((X,Y): a strong biclique of } G)\}. \qquad (2.11)$$

Ikebe and Tamura [10] characterized bidirected graph G for which $Q(G)$ is integral.

Theorem 2.5 (Ikebe and Tamura [10]): *For a closed bidirected graph G, $Q(G)$ is integral if and only if its underlying graph $\underline{G}$ is perfect.* □

Combining the two theorems above we obtain the following, which is crucial for deriving our main theorem.

Corollary 2.6: *Suppose G is a closed bidirected graph. Then, G is perfect if and only if $Q(G)$ is integral.* □

3. Horn Matrices

In this section we review a sequence of reductions of a $\{0, \pm 1\}$-matrix developed in [2].

3.1. Horn matrices

Let A be a $\{0, \pm 1\}$-matrix of size $m \times n$. Then, we have either

(i) $P(A)$ is empty (A is trivially perfect),
(ii) $P(A)$ is nonempty while $P(A) \cap \{0, 1\}^V$ is empty (A is not perfect), or
(iii) $P(A) \cap \{0, 1\}^V \neq \emptyset$.

(3.1)

Lemma 3.1 ([3]; see also [2, Corollary 2.2]): *There is an* $O(mn)$ *algorithm that checks which case of* (i)~(iii) *in* (3.1) *occurs, and finds a vector* $x^* \in P(A) \cap \{0,1\}^V$ *if Case* (iii) *occurs.* □

A $\{0, \pm 1\}$-matrix A is *disguised Horn* ([3]) if $P(A) \cap \{0,1\}^V \neq \emptyset$. It follows from Lemma 3.1 that we only have to consider disguised Horn matrices. Suppose that A is disguised Horn and $x^* \in P(A) \cap \{0,1\}^V$. (Such a vector x^* can be found in $O(mn)$ time.) Define $S \subseteq V$ by

$$S = \{v \mid v \in V, x^*(v) = 1\}. \tag{3.2}$$

Let A' be a $\{0, \pm 1\}$-matrix obtained from A by multiplying -1 for each column $v \in S$. Then the mapping $P(A) \ni x \mapsto x' \in P(A')$ defined by

$$x'(v) = \begin{cases} 1 - x(v) & \text{if } v \in S \\ x(v) & \text{otherwise} \end{cases} \qquad (v \in V) \tag{3.3}$$

gives an isomorphism between polyhedra $P(A)$ and $P(A')$. In particular, $P(A)$ is integral if and only if $P(A')$ is integral. Therefore, our attention may be restricted to such a matrix A' that we have $\mathbf{0} \in P(A')$.

We call a $\{0, \pm 1\}$-matrix A *Horn* ([2]) if $\mathbf{0} \in P(A)$, or equivalently, $\mathrm{n}(A) \leq 1$.

3.2. Horn matrices with no forced zero

Let A be a Horn matrix. For each row $r = 1, \cdots, m$ of A let us denote by N_r the set of column indices for which r-th row of A has positive entries, i.e.,

$$N_r = \{v \mid v \in V, \alpha_{rv} = 1\}, \tag{3.4}$$

where α_{rv} is the (r, v) component of matrix A. Let us consider the quadratic covering graph G_A of A. Since A is Horn, there is no two-head arcs in G_A. Let the directed graph D_A obtained by deleting all the two-tail arcs of the quadratic covering (bidirected) graph G_A[1]. Suppose that $C_1, \cdots, C_s$ are the strongly connected component of directed graph D_A (or equivalently, strongly connected component of bidirected graph G_A (see [1])).

Lemma 3.2 ([2, Corollary 3.2]): *Let A be a Horn matrix and $x \in P(A)$. Then, for each strong component C_t $(t = 1, \cdots, s)$ of D_A, x is constant on C_t, i.e., we have $x(v) = x(w)$ for $v, w \in C_t$.* □

Horn matrix A is said to have *no forced zero* if there exits a vector $x \in P(A)$ such that $x(v) > 0$ for each $v \in V$.

Let A be a Horn matrix of size $m \times n$. For each row a_r $(r = 1, \cdots, m)$ of A with $n(a_r) = 1$ let us denote by $c(r)$ the unique column index for which a_r has the negative entry.

[1] The definition of D_A is slightly different from that in [2]. However, the difference is inessential.

Lemma 3.3 ([2, Lemmas 3.3, 3.4 and 3.6]): *Let A be a Horn matrix of size $m \times n$. Then A has no forced zero if and only if for each row a_r $(r = 1, \cdots, m)$ of A with $n(a_r) = 1$ we have*

(i) $|N_r| = 1$, *or*

(ii) *For the strong component C_t containing $c(r)$ we have $N_r \cap C_t = \emptyset$.* □

Lemma 3.4 ([2, Corollary 3.7]): *Given an $m \times n$ Horn matrix A, one can obtain in $O(mn)$ time another Horn matrix A' of size $m' \times n'$ ($m' \leq m$ and $n' \leq n$) which has no forced zero. Moreover, $P(A')$ is integral if and only if $P(A)$ is integral.* □

3.3. Non-reconverging Horn matrices

Let us denote by U_v the set of vertices reachable from vertex v in the directed graph D_A, i.e.,

$$U_v = \{w \mid w \in V, \text{ there is a path from } v \text{ to } w\}. \qquad (3.5)$$

A Horn matrix A is said to *reconverge from vertex v* if there exists a row r of A such that $|N_r \cap U_v| \geq 2$. A is called *non-reconverging* if there is no such vertex.

Lemma 3.5 ([2, Lemma 4.1]): *Let A be a Horn matrix with no forced zero and which is reconverging from a vertex v. Then, $P(A)$ is not integral.* □

Boros and Čepek also showed that non-reconverging property of a Horn matrix with no forced zero can be checked in $O(n^2 m)$ time. It follows from Lemma 3.5 that it suffices to consider non-reconverging Horn matrices with no forced zero.

4. Perfect $\{0, \pm 1\}$-Matrices

Let A be a non-reconverging Horn matrix with no forced zero and D_A be the directed graph after deleting the two-tail arcs of the quadratic covering graph G_A of A. Let us note that for each strongly connected component C_t $(t = 1, \cdots, s)$ of D_A and each row r of A we have $|N_r \cap C_t| \leq 1$ by the definition of non-reconverging property. Let us define an (apparently Horn) matrix A' of size $m \times s$ by

$$\alpha'_{rt} = \begin{cases} 1 & \text{if } |N_r \cap C_t| = 1 \text{ and } c(r) \notin C_t \\ -1 & \text{if } N_r \cap C_t = \emptyset \text{ and } c(r) \in C_t \\ 0 & \text{otherwise} \end{cases} \qquad (r = 1, \cdots, m; t = 1, \cdots, s). \quad (4.1)$$

For each strong component C_t $(t = 1, \cdots, s)$ of D_A, choose an arbitrary but fixed vertex $v_t \in C_t$. Given an $x \in \mathbf{R}^V$, we define $x' \in \mathbf{R}^s$ as

$$x'(t) = x(v_t) \quad (t = 1, \cdots, s). \qquad (4.2)$$

Conversely, for any $x' \in \mathbf{R}^s$ we define $x \in \mathbf{R}^V$ as

$$x(v) = x'(t) \quad (v \in C_t, t = 1, \cdots, s). \tag{4.3}$$

Lemma 4.1: *Let A be a non-reconverging Horn matrix with no forced zero and A' be defined in* (4.1). *Then $x \in P(A)$ if and only if $x' \in P(A')$, where $x \in \mathbf{R}^V$ and $x' \in \mathbf{R}^s$ correspond under relations* (4.2) *and* (4.3).
(Proof) We prove for each row index $r = 1, \cdots, m$ we have $a_r x \leq 1 - n(a_r)$ if and only if $a'_r x' \leq 1 - n(a'_r)$. Let $r = 1, \cdots, m$ be any row index.
Case 1: $n(a_r) = 0$. We have $n(a'_r) = 0$ and by Lemma 3.2 that

$$\begin{aligned} a_r x &= x(N_r) \\ &= \sum\{x(N_r \cap C_t) \mid t = 1, \cdots, s\} \\ &= \sum\{x'(t) \mid t = 1, \cdots, s, |N_r \cap C_t| = 1\} \\ &= x'(N'_r) \\ &= a'_r x', \end{aligned} \tag{4.4}$$

where N'_r denotes the set of column indices for which a'_r has positive entries.
Case 2: $n(a_r) = 1$ and for the strong component C_{t^*} containing $c(r)$ we have $N_r \cap C_{t^*} = \emptyset$. We have $n(a'_r) = 1$ and by Lemma 3.2 that

$$\begin{aligned} a_r x &= x(N_r) - x(c(r)) \\ &= \sum\{x(N_r \cap C_t) \mid t = 1, \cdots, s\} - x(c(r)) \\ &= \sum\{x'(t) \mid t = 1, \cdots, s, |N_r \cap C_t| = 1\} - x'(t^*) \\ &= x'(N'_r) - x'(t^*) \\ &= a'_r x'. \end{aligned} \tag{4.5}$$

Case 3: $n(a_r) = 1$ and for the strong component C_{t^*} containing $c(r)$ we have $N_r \cap C_{t^*} \neq \emptyset$. In this case, we have by Lemma 3.3 that $|N_r| = 1$. Letting $\{v^*\} = N_r$ it holds that

$$a_r x = x(v^*) - x(c(r)). \tag{4.6}$$

Since v^* and $c(r)$ belong to the same strong component C_{t^*}, we have $a_r x = 0$ for any $x \in \mathbf{R}^V$ defined by (4.3). On the other hand, we have $a'_r = \mathbf{0}$ by definition of A' and $|N_r| = 1$. Therefore, inequality $a'_r x' \leq 1$ is vacuous. □

Then, we have

Lemma 4.2: *Let A be a non-reconverging Horn matrix A with no forced zero. Then the matrix A' defined in* (4.1) *is again non-reconverging Horn matrix with no forced zero. Furthermore, $P(A)$ is integral if and only if $P(A')$ is.*
(Proof) Let ϕ: $P(A) \to P(A')$ be defined as $\phi(x) = x'$, where x' is defined in (4.3).

It follows form Lemmas 4.1 and 3.2 that $\phi\colon P(A) \to P(A')$ is a bijection. Also, since ϕ can be regarded as a projection of $P(A)$ onto some coordinates, $\phi\colon P(A) \to P(A')$ defines an isomorphism of polyhedra. The assertions of the present lemma now follows. □

By definition, for the matrix A' defined by (4.1) each strong component of $D_{A'}$ is singleton. For a Horn matrix A let us call A *reduced*[2] if each strong component of D_A (or equivalently, of G_A) is singleton. It follows from Lemma 4.2 that we consider only non-reconverging reduced Horn matrix with no forced zero.

We are now ready to prove the following main theorem.

Theorem 4.3: *A non-reconverging reduced Horn matrix A with no forced zero is perfect if and only if the following two conditions hold:*

(i) *The transitive closure G_A^* of the quadratic covering graph G_A is perfect.*

(ii) *The characteristic vector of each nontrivial strong biclique of G_A^* appears as a row of A.*

(Proof) First, note that for a non-reconverging reduced Horn matrix A with no forced zero, the quadratic covering graph G_A is acyclic, and hence, its transitive closure G_A^* is closed bidirected graph.

Suppose $P(A)$ is integral. Then, we have

$$P(A) = P_I(A) = P_I(G_A) = P_I(G_A^*) \subseteq Q(G_A^*). \qquad (4.7)$$

On the other hand, since the the characteristic vector of each strong biclique of G_A^* induces a facet of $P_I(G_A^*) = P(A)$ due to Theorem 2.3, it appears as a row of A or is trivial, i.e., the facet is induced by $0 \le x(v)$ or $x(v) \le 1$ for some $v \in V$. Therefore, we have $Q(G_A^*) \subseteq P(A)$. Then we have $Q(G_A^*) = P(A)$ and it follows from Corollary 2.6 that G_A^* is perfect.

Conversely, suppose that Conditions (i) and (ii) hold. Then, we have

$$P_I(A) = P_I(G_A^*) = Q(G_A^*) \supseteq P(A). \qquad (4.8)$$

Hence, we have $P_I(A) = P(A)$, i.e., A is perfect. □

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[2] In [6] and [2] the term irreducible were used in different meaning. Still, we prefer to use the same term here since it is "more reduced" than that of [6] and [2].

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