No. 713

# A New View on Statistical Inference.

(Part I)

--- Case of the Uniform Distribution $U[\theta, \theta+1)$ ---

by

Yoshiko Nogami

February 1997

A New View on Statistical Inference.
(Part I)
---Case of the Uniform Distribution $U[\theta,\theta+1)$---

By Yoshiko Nogami

January, 1997

Abstract.

In this paper we again use the uniform distribution with density $f(x|\theta)=1$ for $\theta \leq x < \theta+1$; $=0$, otherwise for real $\theta$. The purpose of this paper is to consider the problem of testing the null hypothesis $H_0$: $\theta=\theta_0$ versus the alternative hypothesis $H_1$: $\theta \neq \theta_0$, and construct the two-sided unbiased test $\phi*$ of size $\alpha$ which is uniformly most powerful(UMP). From direct computation we also see that the power function of $\phi*$ is convex (from below).

## § 1. Introduction.

Let $I_A(x)$ be an indicator function such that $I_A(x)=1$ for $x\in A$; $=0$ for $x\in A$. Throughout this paper the underlined distribution is the uniform with density $f(x|\theta)=I_{(\theta,\ \theta+1)}(x)$ for real $\theta$. We also let $X_{(i)}$ $(i=1,2,\ \dots\ ,n)$ denote the i-th smallest observation of $X_1,\ \dots\ ,X_n$, taken randomly from the population with density $f(x|\theta)$. If we need to test the null hypothesis $H_0:\theta=\theta_0$ against an alternative $H_1:\theta\neq\theta_0$, then the likelihood-ratio tests based on the minimal sufficient statistics are useless.

So, in this paper we use an unbiased estimator $Y\doteq(X_{(1)}+X_{(n)}-1)/2$ to construct the two-sided size-$\alpha$ test $\phi^*$ which is unbiased and see this $\phi*$ is uniformly most powerful(UMP) (in Section 2). In Section 3 the power function based on $\phi^*$ is computed, exactly. Eventually,we can see that for finite n, this power function is convex(from below).

For notational convensions we denote the defining property by $\doteq$ and also let $h'(x)\doteq dh(x)/dx$.

## §2. Optimal Two-Sided Tests.

Let $\alpha$ be a real number such that $0<\alpha<1$. In this section we consi~~der the~~ problem of testing $H_0:\theta=\theta_0$ versus $H_1:\theta\neq\theta_0$ for real $\theta_0$ and construct the optimal two-sided size-$\alpha$ test $\phi^*$. Let $Y\doteq(X_{(1)}+X_{(2)}-1)/2$.

Let $\phi$ be a two-sided test of the following form:

$$\phi(y)=\begin{cases} 1, & ,\text{if } \ y<y_1 \ \text{ or } \ y>y_2, \\ \gamma_i, & ,\text{if } \ y=y_i \ ,\ i=1,2, \\ 0, & ,\text{if } \ y_1<y<y_2, \end{cases} \tag{1}$$

where $y_1$, $y_2$, $\gamma_1$ and $\gamma_2$ are chosen so that $E_{\theta_0}(\phi(Y))=\alpha$.

We let C be the critical region given by

$$C=\{y:\ y\le y_1 \quad \text{or } y\ge y_2\}.$$

To find optimal $y_1$ and $y_2$ we find the p.d.f. of Y.

Applying a variable-transformation to the joint density of $(X_{(1)},$ $X_{(n)})$ and taking marginal probability density function (p.d.f.) we obtain the p. d. f. of Y as follows:

$$(2) \qquad g(y|\theta) = \begin{cases} n(1-2|y-\theta|)^{n-1} & \text{for } -1/2< y-\theta <1/2, \\ 0, & \text{elsewhere.} \end{cases}$$

Let $\pi_C(\theta) \doteq E_\theta(\phi(Y))$. To get size-$\alpha$ unbiased test, we determine y1, y2 which satisfy $\pi_C(\theta_0)=\alpha$ and minimise $\pi_C(\theta)$ at $\theta=\theta_0$. Then, we get optimal region

$$(3) \qquad C^* =\{y: y\le\theta_0-r, \quad y\ge\theta_0+r\}$$

where

$$(4) \qquad r=(1-\alpha^{1/n})/2.$$

Let $\phi^*$ be the two-sided test based on above $C^*$;

$$(5) \qquad \phi^*(y)= \begin{cases} 1, & \text{if } y\le\theta_0-r, \quad y\ge\theta_0+r \\ 0, & \text{if } \theta_0-r<y<\theta_0+r. \end{cases}$$

From the generalized Fundamental Lemma, we can show that UMP unbiased size-$\alpha$ test of testing $H_0:\theta=\theta_0$ versus $H_1:\theta\neq\theta_0$ is of form (1). Hence, $\phi^*$ is certainly UMP-unbiased.

## § 3. The convex power function.

In this section we shall exhibit the power function $\pi_{C^*}(\theta)$ and show that it is convex.

$$
(5) \qquad \pi_C{}^*(\theta) = \begin{cases}
1, & (\theta < \theta_0 - r - 1/2), \\
1 - \{1 - 2(\theta_0 - r - \theta)\}^n/2, & (\theta_0 - r - 1/2 \leqq \theta < \theta_0 + r - 1/2) \\
1 - [\{1 - 2(\theta_0 - r - \theta)\}^n - \{1 - 2(\theta_0 + r - \theta)\}^n]/2, & (\theta_0 + r - 1/2 \leqq \theta < \theta_0 - r) \\
[\{1 - 2(\theta - \theta_0 + r)\}^n + \{1 - 2(\theta_0 + r - \theta)\}^n]/2, & (\theta_0 - r \leqq \theta < \theta_0 + r), \\
1 - [\{1 + 2(\theta_0 + r - \theta)\}^n - \{1 - 2(\theta - \theta_0 + r)\}^n]/2, & (\theta_0 + r \leqq \theta < \theta_0 - r + 1/2) \\
1 - \{1 + 2(\theta_0 + r - \theta)\}^n/2, & (\theta_0 - r + 1/2 \leqq \theta < \theta_0 + r + 1/2), \\
1, & (\theta_0 + r + 1/2 \leqq \theta).
\end{cases}
$$

(Here, we remark that for ($\alpha \geqq 0.001$; $n \geqq 10$), ($\alpha \geqq 0.01$; $n \geqq 7$), ($\alpha \geqq 0.05$; $n \geqq 5$), and ($\alpha \geqq 0.10$; $n \geqq 4$), we have that $\alpha^{1/n} \geqq 1/2$. (See Table, below.) So, calculations led to (5) depend on $\alpha$ and n such that $\alpha^{1/n} \geqq 2^{-1}$.)

*Table.* The Values of $\alpha^{1/n}$.

| $\alpha$ | .10 | .05 | .01 | .001 |
|---|---|---|---|---|
| n=4 | .56 | .47 | .32 | .18 |
| 5 | .63 | .55 | .40 | .25 |
| 6 | .68 | .61 | .46 | .32 |
| 7 | .72 | .65 | .52 | .37 |
| 10 | .79 | .74 | .63 | .50 |
| 50 | .95 | .94 | .91 | .87 |
| 100 | .98 | .97 | .96 | .93 |

Since, $\pi'_C{}^*(\theta) < 0$ for $\theta < \theta_0$, $\pi'_C{}^*(\theta) > 0$ for $\theta > \theta_0$ and $\pi'_C{}^*(\theta_0) = 0$, and since $\pi_C{}^*(\theta_0) = \alpha$ and $\pi_C{}^*(+\infty) = \pi_C{}^*(-\infty) = 1$, $\pi_C{}^*(\theta)$ is a convex (from below) function of $\theta$ and the test based on $C^*$ is an unbiased size-$\alpha$ test. We also see that for all $\theta$ with $\theta \neq \theta_0$, $\pi_C{}^*(\theta) \uparrow 1$ as $n \uparrow \infty$.