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A NOTE:
SPATIAL DISTRIBUTIONS
MAXIMIZING OR MINIMIZING
GEARY'S SPATIAL CONTIGUITY RATIO

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(1) INTRODUCTION

To measure the magnitude of spatial contiguity, the literature often employs Geary's ratio (1949) defined by

$$G = \frac{n-1}{2\sum_{i=1}^{n} L_i} \frac{\sum_{i=1}^{n}\sum_{j=1}^{n} \delta_{ij}(x_i - x_j)^2}{\sum_{i=1}^{n} x_i^2}, \tag{1}$$

where n is the number of counties covering a region, L_i is that of counties being joined to county i, x_i is an attribute value of county i satisfying

$$\sum_{i=1}^{n} x_i = 0, \tag{2}$$

and

$$\delta_{ij} = \begin{cases} 1 & \text{if county } i \text{ is joined to county } j, \ (i \neq j), \\ 0 & \text{otherwise or } i = j. \end{cases} \tag{3}$$

Concerning the value of this ratio, it can be shown that Geary's ratio always takes positive value. For the numerator of equation (1) is non-positive, and if the ratio takes zero, $x_i = x_j$ for $\delta_{ij} = 1$ which implies $x_i = 0$ for all i, (since all counties are topologically connected), that makes the ratio meaningless. It is also shown that if attribute values are randomly distributed over the region, the expected value of Geary's ratio takes one. (See Cliff and Ord (1973)). It is not explicitly shown, however, what the range of Geary's ratio is.

Furthermore few literature examines what spatial patterns will be observed when Geary's ratio takes the maximum or minimum value. The purpose of this note is to answer these questions explicitly.

(2) SPATIAL DISTRIBUTION MAXIMIZING OR MINIMIZING GEARY'S RATIO

The above question will be examined by considering the maximization (or minimization) problem given by:

$$\text{Max (Min)} \quad G \quad \text{(given by equation (1))}, \tag{4}$$

$$\text{subject to} \quad \sum_{i=1}^{n} x_i = 0, \tag{2}$$

$$\sum_{i=1}^{n} x_i^2 = 1. \tag{5}$$

The last constraint is added because Geary's ratio is unique up to a scale transformation. Alternatively the above problem is written as:

$$\text{Max (Min)} \quad G = \alpha \frac{\mathbf{x}' A \mathbf{x}}{\mathbf{x}'}, \tag{4$'$}$$

subject to

$$\mathbf{x}' \mathbf{e} = 0, \tag{2$'$}$$

$$\mathbf{x}' \mathbf{x} = 1, \tag{5$'$}$$

where $\alpha = (n-1) / \sum_{i=1}^{n} L_i$, $\mathbf{x}' = (x_1, x_2, \ldots, x_n)$, $\mathbf{e}' = (1, 1, \ldots, 1)$

and

$$A = \begin{pmatrix} \sum_{i=1}^{n} \delta_{i1} + \sum_{j=1}^{n} \delta_{ij} - \delta_{12} & \cdots & -\delta_{1n} \\ -\delta_{12} & \ddots & \vdots \\ \vdots & & \\ -\delta_{1n} & \cdots & \sum_{i=1}^{n} \delta_{in} + \sum_{j=1}^{n} \delta_{nj} \end{pmatrix} \tag{6}$$

This problem may be solved by the Lagrange method but the derivation appears to be not straightforward. We shall hence employ the following method.

First the constraint equation (2)' is taken into equaiton (4)' by $x = B\,\bar{x}$, where $\bar{x}' = (x_1, x_2, \ldots, x_{n-1})$ and

$$B = \begin{pmatrix} 1 & \cdot & 0 \\ & \cdot & \\ 0 & \cdot & 1 \\ -1 & \ldots & -1 \end{pmatrix}, \quad \text{a } n \times (n-1) \text{ matrix.} \tag{7}$$

The optimization problem is then written as:

$$\text{Max (Min)} \quad G = \alpha \frac{\bar{x}'\, B'\, A\, B\, \bar{x}}{\bar{x}'\, B'\, B\, \bar{x}}, \tag{4)''$$

subject to

$$\bar{x}'\, B'\, B\, \bar{x} = 1. \tag{5)''$$

Second let us consider whether or not $\bar{x}'\, B'\, B\, \bar{x}$ can be transformed into $y'\, y$. The eigen values $\lambda_1, \lambda_2, \ldots, \lambda_{n-1}$ of matrix $B'B$ is obtained from

$$|B'B - \lambda I| = (1-\lambda)^{n-2}(n-\lambda) = 0, \tag{8}$$

and hence

$$\lambda_1 = \lambda_2 = \ldots = \lambda_{n-2} = 1, \qquad \lambda_{n-1} = n. \tag{9}$$

Since all eigen values are positive and matirx $B'B$ is symmetric, there exists transformation that transforms $\bar{x}' B' B \bar{x}$ into $y' y$. To state explicitly, let c_i be the eigen vector of λ_i that satisfies

$$c'_i c_j = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{otherwise}. \end{cases} \tag{10}$$

Note that such eigen vectors exist and non-unique. For example

$$c'_i = \left(\frac{1}{\sqrt{i(i+1)}}, \ldots, \frac{1}{\sqrt{i(i+1)}}, -\frac{i}{\sqrt{i(i+1)}}, 0 \ldots 0 \right), \quad i = 1, 2, \ldots, n-2, \tag{11}$$

$$c'_{n-1} = \left(\frac{1}{\sqrt{n-1}}, \ldots\ldots, \frac{1}{\sqrt{n-1}} \right). \tag{12}$$

Let

$$C' = (c_1, c_2, \ldots, c_{n-1}), \tag{13}$$

$$D = \begin{pmatrix} 1 & & & 0 \\ & \ddots & & \\ & & 1 & \\ 0 & & & \frac{1}{\sqrt{n}} \end{pmatrix}. \tag{14}$$

Then the transformation $\bar{x}' = C' B y$ does tranform $\bar{x}' B' B \bar{x}$ into $y' y$. The optimization problem is therefore written as:

$$\text{Max (Min)} \quad G = \alpha \frac{y' \, D \, C \, B' \, A \, B \, C' \, D \, y}{y' \, y} \tag{4)'''$$

subject to

$$y' \, y = 1. \tag{5)'''$$

The solution of this optimization problem is well-known: let β_{max} and β_{min} respectively be the maximum and minimum eigen values of matrix $D \, C \, B' \, A \, B \, C' \, D$, (note that the eigen values are real since the matrix is symmetric); then the maximum G_{max} and minimum G_{min} values of Geary's ratio are respectively given by

$$G_{max} = \alpha \, \beta_{max}, \qquad G_{min} = \alpha \, \beta_{min}. \tag{15}$$

The spatial distributions $\mathbf{x}_{max}$ ($\mathbf{x}_{min}$) that maximize (minimize) Geary's ratio is therefore obtained from

$$\mathbf{x}_{max} = B \, C' \, D \, \mathbf{y}_{max}, \qquad \mathbf{x}_{min} = B \, C' \, D \, \mathbf{y}_{min} \tag{16}$$

where $\mathbf{y}_{max}$ and $\mathbf{y}_{min}$ are respectively the eigen vectors of β_{max} and β_{min} that satisfy $\mathbf{y}'_{max} \, \mathbf{y}_{max} = 1$ and $\mathbf{y}'_{min} \, \mathbf{y}_{min} = 1$.
Last it should be noted that these distributions are non-unique for the matrix C is not uniquely determined.

(3) NUMERICAL EXAMPLES

To see the above result in an explicit form, let us consider three numerical examples: i) 3 × 3 county, ii) 5 × 5 county, and

7×7 county cases.

The maximum and minimum values of Geary's ratio are tabulated in table 1. From this table it is first noticed that the minimum value of

the number of counties	G_{max}	G_{min}
3×3	1.57	.54
5×5	1.67	.45
7×7	1.65	.43

Table 1

Geary's ratio decreases as the number of counties increases. Second the maximum value of Geary's ratio appears to be almost constant around 1.6 regardless of the number of counties.

The spatial distributions maximizing and minimizing Geary's ratio are tabulated in tables 2, 3 and 4. It is read from this table that the spatial distribution maximizing Geary's ratio forms like a checker board, while the spatial distribution minimizing Geary's ratio shows the region being splited into two subregions showing positive and negative values.

*** XMAX ***

	1	2	3
1	-1.3508E-01	3.9711E-01	-1.3508E-01
2	3.9711E-01	-1.0481E 00	3.9711E-01
3	-1.3508E-01	3.9711E-01	-1.3508E-01

*** XMIN ***

	1	2	3
1	1.4510E 00	4.5741E-01	-2.0137E-01
2	6.0483E-01	1.3053E-07	-6.0483E-01
3	2.0137E-01	-4.5741E-01	-1.4510E 00

*** XMAX ***

	1	2	3	4	5
1	-3.8513E-02	1.3909E-01	-1.7708E-01	1.3909E-01	-3.8513E-02
2	1.3909E-01	-4.9313E-01	6.3289E-01	-4.9313E-01	1.3909E-01
3	-1.7708E-01	6.3289E-01	-8.0936E-01	6.3289E-01	-1.7708E-01
4	1.3909E-01	-4.9313E-01	6.3289E-01	-4.9313E-01	1.3909E-01
5	-3.8513E-02	1.3909E-01	-1.7708E-01	1.3909E-01	-3.8513E-02

*** XMIN ***

	1	2	3	4	5
1	-1.3873E-01	-9.4948E-02	-1.1154E-01	-1.9078E-01	-3.7008E-01
2	-4.0421E-02	-3.2269E-02	-4.6203E-02	-8.6081E-02	-1.7033E-01
3	5.0717E-02	2.1008E-02	-2.5465E-08	-2.1008E-02	-5.0717E-02
4	1.7034E-01	8.6082E-02	4.6203E-02	3.2269E-02	4.0420E-02
5	3.7008E-01	1.9078E-01	1.1154E-01	9.4947E-02	1.3873E-01

*** XMAX ***

	1	2	3	4	5	6	7
1	1.6438E-02	-6.2544E-02	9.4672E-02	-1.0646E-01	9.4672E-02	-6.2544E-02	1.6438E-02
2	-6.2544E-02	2.3608E-01	-3.5845E-01	4.0216E-01	-3.5845E-01	2.3608E-01	-6.2544E-02
3	9.4673E-02	-3.5845E-01	5.4359E-01	-6.1042E-01	5.4359E-01	-3.5845E-01	9.4672E-02
4	-1.0646E-01	4.0216E-01	-6.1042E-01	6.8502E-01	-6.1042E-01	4.0216E-01	-1.0646E-01
5	9.4672E-02	-3.5845E-01	5.4359E-01	-6.1042E-01	5.4359E-01	-3.5845E-01	9.4672E-02
6	-6.2544E-02	2.3608E-01	-3.5845E-01	4.0216E-01	-3.5845E-01	2.3608E-01	-6.2544E-02
7	1.6438E-02	-6.2544E-02	9.4672E-02	-1.0646E-01	9.4672E-02	-6.2544E-02	1.6438E-02

*** XMIN ***

	1	2	3	4	5	6	7
1	-5.4240E-02	-1.9490E-02	9.2963E-03	4.4539E-02	1.0279E-01	2.1248E-01	4.2784E-01
2	-3.4629E-02	-1.3483E-02	2.8192E-03	2.1432E-02	5.1116E-02	1.0635E-01	2.1440E-01
3	-3.6088E-02	-1.6083E-02	-3.1560E-03	8.6377E-03	2.4893E-02	5.3512E-02	1.0854E-01
4	-5.7472E-02	-2.7655E-02	-1.1146E-02	-1.0944E-08	1.1146E-02	2.7655E-02	5.7472E-02
5	-1.0854E-01	-5.3512E-02	-2.4893E-02	-8.6377E-03	3.1561E-03	1.6084E-02	3.6089E-02
6	-2.1441E-01	-1.0635E-01	-5.1116E-02	-2.1432E-02	-2.8187E-03	1.3484E-02	3.4632E-02
7	-4.2784E-01	-2.1249E-01	-1.0279E-01	-4.4539E-02	-9.2953E-03	1.9492E-02	5.4244E-02