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Global Convergence of a Class of Non-Interior-Point Algorithms Using Chen-Harker-Kanzow Functions for Nonlinear Complementarity Problems *

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Abstract

We propose a class of non-interior-point algorithms for solving the complementarity problems(CP): Find a nonnegative pair $(x, y) \in \mathbb{R}^{2n}$ satisfying $y = f(x)$ and $x_i y_i = 0$ for every $i \in \{1, 2, \ldots, n\}$, where f is a continuous mapping from $\mathbb{R}^n$ to $\mathbb{R}^n$. The algorithms are based on the Chen-Harker-Kanzow smooth functions for the CP, and have the following features; (a) it traces a trajectory in $\mathbb{R}^{3n}$ which consists of solutions of a family of systems of equations with a parameter, (b) it can be started from arbitrary (not necessarily positive) point in $\mathbb{R}^{2n}$ in contrast to most of interior-point methods, and (c) its global convergence is ensured for a class of problems including (not strongly) monotone complementarity problems having a feasible-interior-point. To construct the algorithms, we give a homotopy and show the existence of a trajectory leading to a solution under a relatively mild condition, and propose a class of algorithms involving suitable neighborhoods of the trajectory. We also give a sufficient condition on the neighborhoods for global convergence and two examples satisfying it.

1 Introduction

We consider the (standard) complementarity problem(CP):

$$\begin{array}{ll} \text{Find} & (x, y) \in \mathbb{R}^{2n} \\ \text{such that} & y = f(x),\ (x, y) \geq 0,\ x_i y_i = 0\ (i \in N). \end{array}$$

where $N = \{1, 2, \ldots, n\}$ and f is a mapping from $\mathbb{R}^n$ to $\mathbb{R}^n$. If the mapping f is linear, i.e., $f(x) = Mx + q$ for some $n \times n$ matrix M and $q \in \mathbb{R}^n$, then it is called a linear complementarity problem (LCP), and if the mapping f is monotone, i.e., $(x^1 - x^2)^T (f(x^1) - f(x^2)) \geq 0$ for every

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$x^1, x^2 \in \mathbb{R}^n$, then a monotone complementarity problem (monotone CP). It is well-known that many optimization problems can be put into the class of CPs. For instance, we can model any convex programming as a monotone CP and any linear programming problem(LP) as an LCP with a skew-symmetric matrix M.

While there have been many algorithms for solving CP (see [33, 4, 49], etc.), we focus on the following two types of the algorithms:

Interior-Point Method: It generates a sequence $\{(x^k, y^k)\}$ in the positive orthant of $\mathbb{R}^{2n}$ ([1, 12, 13, 14, 15, 9, 10, 23, 22, 21, 20, 25, 28, 31, 30, 29, 32, 37, 41, 38, 39, 40, 43, 44, 45, 46, 48, 50],etc.).

(Equation-based) Non-interior-Point Continuation Method: It does not confine the generated sequence to the positive orthant of $\mathbb{R}^{2n}$ ([2, 3, 6, 17, 19, 16, 18, 26, 35, 36], etc., also see [11, 8, 27, 34] for other non-interior-point methods including merit function algorithms).

Our work is motivated by the following observations:

Many of interior-point methods solve a class of CPs including LPs and QCPs polynomially, and can be regarded as path-following algorithms for tracing a trajectory leading to a solution of the problem (see Kojima[22]). However, they lack flexibility in the choice of the trajectory; the trajectory must be confined in the positive orthant.

The non-interior-point methods can be started from any point in $\mathbb{R}^{2n}$. However, most of them require either of the assumptions below to show the global convergence properties:

Condition 1.1. *The mapping f is strongly monotone, i.e., there exists a $\omega \in (0, \infty)$ such that*

$$(x^1 - x^2)^T (f(x^1) - f(x^2)) \geq \omega \|x^1 - x^2\|^2$$

for every $x^1, x^2 \in \mathbb{R}^n$.

Condition 1.2. *The mapping f is linear, i.e., $f(x) = Mx + q$ and the matrix M belongs to the class $P_0 \cap R_0$. Here $M \in P_0$ iff all the principal minors are nonnegative, and $M \in R_0$ iff $x^T Mx = 0$ implies $x = 0$. It is well-known that the class P_0 can be characterized equivalently as the set of matrices satisfy that for any nonzero vector $x \in \mathbb{R}^n$, there exists an index $i \in N$ such that $x_i[Mx]_i \geq 0$ where $[Mx]_i$ denotes the ith component of the vector Mx (see [4]).*

It should be noted that the mapping f of the CP arising from LP is an LCP with a skew symmetric matrix M, which implies that f is not strongly monotone and that the matrix M does not belong to R_0. Thus the global convergence does not necessarily ensured as long as we directly apply the continuation methods to such CPs.

In this paper, we will propose a non-interior homotopy continuation method for which we can choose any (not necessarily positive) initial point (x^1, y^1) in $\mathbb{R}^{2n}$, whenever the following condition holds.

Condition 1.3.

(i) *The mapping f is monotone, i.e.,*

$$(x^1 - x^2)^T (f(x^1) - f(x^2)) \geq 0.$$

for every $x^1 \in \mathbb{R}^n$ and $x^2 \in \mathbb{R}^n$.

(ii) *There exists a feasible-interior-point (x, y) of the CP, i.e.,*

$$(x, y) > 0 \quad \textit{and} \quad y = f(x).$$

This condition has been used in many interior-point algorithms for solving the CP (see [24, 21, 25, 13, 43], etc.). We should mention some relationships among Conditions 1.1, 1.2 and 1.3. Suppose that Condition 1.1. It is obvious that the requirement (i) of Condition 1.3 is satisfied. Moreover, we can see that

$$\max_{i \in N}(x_i^1 - x_i^2)(f_i(x^1) - f_i(x^2)) \geq (\omega/n)\|x^1 - x^2\|^2$$

for every $x^1, x^2 \in \mathbb{R}^n$, which implies that f is a uniform P-function. The CP with a uniform P-function f has an feasible-interior-point (see Section 3 of [21]). Thus Condition 1.3 holds whenever Condition 1.1. Also, since the LCP with a matrix $M \in P_0 \cap R_0$ has a feasible-interior-point (see Remark 5.9.6 of [4]), the requirement (ii) of Condition 1.3 is satisfied if Condition 1.2 holds. However, the monotonicity of the mapping f does not necessarily guaranteed. To see this, let us consider the following matrix

$$M = \begin{pmatrix} 1 & -1 \\ 4 & 1 \end{pmatrix}.$$

Then $M \in P_0$ since all of the principal minors are nonnegative. In addition, we have

$$x^T M x = (x_1 + x_2)^2 + x_1 x_2$$

for every $x = (x_1, x_2)^T \in \mathbb{R}^2$ which implies that $M \in R_0$, i.e., if $x \geq 0$ $Mx \geq 0$ and $x^T Mx = 0$ then $x = 0$. However, it is obvious that M is not positive semi-definite. We will discuss again this subject in Remark 2.4.

Our approach is based on the use of Chen-Harker-Kanzow smooth function

$$\phi_\mu(a, b) := a + b - \sqrt{(a - b)^2 + 4\mu}$$

with a positive number $\mu > 0$. This function was given by Chen and Harker[2] to construct the first non-interior path-following method for the LCP and then by Kanzow[18]. It can be regarded as a modification of the function

$$\phi(a,b) := a + b - \sqrt{(a-b)^2}$$

introduced by Fischer[5], which is a so–called *complementarity function (CP-function)* since the equation $\phi(a,b) = 0$ is equivalent to the system

$$(a,b) \geq 0, \quad \text{and} \quad ab = 0.$$

Many other CP-functions and their applications can be found in [3, 6, 16, 19, 26, 35, 36, 47], etc.

Kanzow[18] shows that for every $\mu \geq 0$, $\phi_\mu(a,b) := a + b - \sqrt{(a-b)^2 + 4\mu} = 0$ if and only if $(a,b) \geq 0$ and $ab = \mu \geq 0$. It follows that if $(x,y) \in \mathbb{R}^{2n}$ satisfies $\phi_\mu(x_i, y_i) = 0$ $(i \in N)$ and $y = f(x)$ for some $\mu > 0$ then the point (x,y) is an analytical center of the CP, i.e.,$(x,y) > 0$, $x_i y_i = \mu$ $(i \in N)$, $y = f(x)$. Moreover, we can easily obtain the following lemma:

Lemma 1.4. *For every nonnegative number $\mu \geq 0$, a triplet $(a,b,c) \in \mathbb{R}^3$ satisfies $\phi_\mu(a,b) := a+b-\sqrt{(a-b)^2+4\mu} = c$ if and only if $((a-c/2),(b-c/2)) \geq 0$ and $(a-c/2)(b-c/2) = \mu \geq 0$.*

Therefore, if $(\bar{x}, \bar{y}) \in \mathbb{R}^{2n}$ satisfies

$$\phi_\mu(\bar{x}_i, \bar{y}_i) = \bar{v}_i \ (i \in N) \text{ and } \ \bar{y} - f(\bar{x}) = \bar{r}$$

for some $\mu > 0$, $\bar{v} \in \mathbb{R}^n$ and $\bar{r} \in \mathbb{R}^n$, then

$$((\bar{x}_i - \bar{v}_i/2), (\bar{y}_i - \bar{v}_i/2)) > 0, \ (\bar{x}_i - \bar{v}_i/2)(\bar{y}_i - \bar{v}_i/2) = \mu > 0, \ \bar{y} = f(\bar{x}) + r,$$

which implies that the point $(\bar{x} - \bar{v}/2, \bar{y} - \bar{v}/2) \in \mathbb{R}^{2n}$ is an analytical center of the perturbed problem CP$(\bar{v}, \bar{r})$ given by

$$\begin{array}{ll} \text{Find} & (x', y') \in \mathbb{R}^{2n} \\ \text{such that} & y' = f(x'), \ (x', y') \geq 0, \ x'_i y'_i = 0 \ (i \in N) \end{array}$$

where $f'(x') = f(x' + \bar{v}/2) + \bar{r} - \bar{v}/2$. Figure 1 illustrates a perturbed problem for the CP with $n = 1$ and $f(x) = 1$.

Base on this idea, we will develop a new homotopy continuation method for solving CP's. The rest of this paper is organized as follows. In Section 2, a new homotopy map will be presented and several properties of this map will be stated. In Section 3, we will prove the existence of the trajectory leading to a solution of the CP under a relatively mild condition, Condition 2.2. We will also show that the trajectory can be started from any point (x,y) in $\mathbb{R}^{2n}$ as long as Condition 1.3 holds. In Section 4, a class of path-following algorithms will be described to trace the trajectory involving its suitable neighborhoods. The requirements for the neighborhoods will be collected in Condition 4.4. After establishing the global and monotone convergence of the algorithm in Section 5, two examples of the neighborhoods having the desired properties will be presented in Section 6.

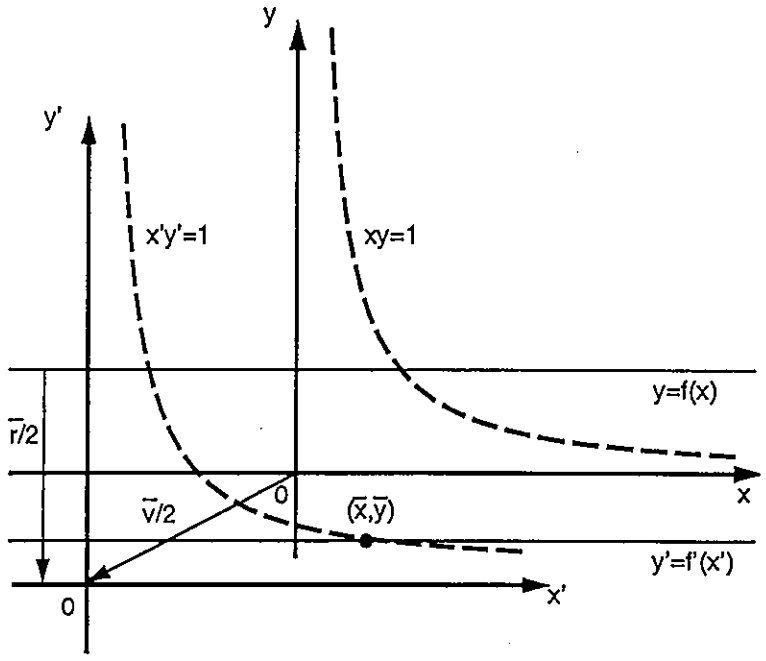

Figure 1: A perturbed problem for the CP with $n = 1$ and $f(x) = 1$.

Recently, Xu and Burke[47] proposed an interior-point method based on Chen-Harker-Kanzow techniques and showed its polynomial complexity. Their result suggests a possibility of deriving a similar result for our non-interior continuation method.

Throughout this paper, we use the symbols $\mathbb{R}^n_+$, $\mathbb{R}^n_{++}$, $\mathbb{R}^n_-$ and $\mathbb{R}^n_{--}$ to denote the nonnegative orthant, the positive orthant, the nonpositive orthant and the negative orthant of $\mathbb{R}^n$, respectively. The triplet (u, x, y) (the pair (x, y)) denotes the column vector

$$(u, x, y) := (u^T, x^T, y^T)^T \quad ((x, y) := (x^T, y^T)^T).$$

for given column vectors u,x and y. Also the symbol e denotes the vector with all components equal to one. For each mapping h with the domain X and each subset $D \subset X$, we define

$$h(D) := \{\bar{y} : y(x) = \bar{y} \text{ for some } x \in D\}.$$

For ease of notation, we often use the symbols z and w to denote the triplets (u, x, y) and (u, v, r), respectively.

2 A homotopy map for the CP

To construct a continuation method, we introduce the following mappings based on the function ϕ_μ:

$$v \ : \ \mathbb{R}^n_+ \times \mathbb{R}^{2n} \to \mathbb{R}^n,$$

$$
\begin{aligned}
& v = (v_1, v_2, \ldots, v_n)^T, \\
& v_i(u,x,y) := (x_i + y_i) - \sqrt{(x_i - y_i) + 4u_i} \ (i \in N), \\
\tau \ : \ & \mathbb{R}^n_+ \times \mathbb{R}^{2n} \to \mathbb{R}^n, \\
& \tau(u,x,y) := y - f(x), \\
V \ : \ & \mathbb{R}^n_+ \times \mathbb{R}^{2n} \to \mathbb{R}^{2n}, \\
& V(u,x,y) := (v(u,x,y), \tau(u,x,y)), \\
U \ : \ & \mathbb{R}^n_+ \times \mathbb{R}^{2n} \to \mathbb{R}^n_+ \times \mathbb{R}^{2n}, \\
& U(u,x,y) := (u, v(u,x,y), \tau(u,x,y)),
\end{aligned}
$$

Our intention is to find a $(\bar{u}, \bar{v}, \bar{\tau}) \in \mathbb{R}^n_{++} \times \mathbb{R}^{2n}$ for which

$$\bar{W} := \{\theta(\bar{u}, \bar{v}, \bar{\tau}) \in \mathbb{R}^n_{++} \times \mathbb{R}^{2n} : \ \theta \in (0,1]\} \subset U(\mathbb{R}^n_{++} \times \mathbb{R}^{2n})$$

and the set

$$U^{-1}(\bar{W}) := \{(u,x,y) \in \mathbb{R}^n_{++} \times \mathbb{R}^{2n} : \ U(z) = \theta(\bar{u}, \bar{v}, \bar{\tau}) \ \text{ for some } \theta \in (0,1]\}$$

forms a one-dimensional curve (subtrajectory) leading to a solution of the CP.

The following results are useful to find such a point $(\bar{u}, \bar{v}, \bar{\tau})$:

Lemma 2.1.

(i) $V(\mathbb{R}^n_{++} \times \mathbb{R}^{2n})$ *is an open subset of* $\mathbb{R}^{2n}$.

(ii) *If* $(\bar{v}, \bar{\tau}) \in V(\mathbb{R}^n_{++} \times \mathbb{R}^{2n})$ *then*

$$(\bar{v} + \mathbb{R}^n_-) \times (\bar{\tau} + \mathbb{R}^n_+) \subset V(\mathbb{R}^n_{++} \times \mathbb{R}^{2n})$$

(iii) *Specially, if* $(0,0) \in V(\mathbb{R}^n_{++} \times \mathbb{R}^{2n})$, *which is equivalent to that the CP has a feasible-interior-point, then*

$$\mathbb{R}^n_- \times \mathbb{R}^n_+ \subset V(\mathbb{R}^n_{++} \times \mathbb{R}^{2n}).$$

Proof: Suppose that $(\bar{v}, \bar{\tau}) \in V(\mathbb{R}^n_{++} \times \mathbb{R}^{2n})$. Then, by the definition of the set $V(\mathbb{R}^n_{++} \times \mathbb{R}^{2n})$ and by Lemma 1.4, there exists a point $(\bar{u}, \bar{x}, \bar{y}) \in \mathbb{R}^n_{++} \times \mathbb{R}^{2n}$ such that

$$((\bar{x} - \bar{v}/2), (\bar{y} - \bar{v}/2)) > 0, \ \ (\bar{x}_i - \bar{v}_i/2)(\bar{y}_i - \bar{v}_i/2) = \bar{u}_i > 0 \ (i \in N) \ \text{ and } \ \bar{y} = f(\bar{x}) + \bar{\tau}.$$

Let us define

$$\epsilon := \min\{(\bar{x}_i - \bar{v}_i/2), \ (\bar{y}_i - \bar{v}_i/2) \ (i \in N)\} > 0.$$

Then for every $d = (dv, dr) \in \mathbb{R}^{2n}$ such that $\|d\| < \epsilon/2$, we obtain that

$$\begin{aligned} &((\bar{x}_i - (\bar{v}_i + dv_i)/2), ((\bar{y}_i + dr_i) - (\bar{v}_i + dv_i)/2)) > 0 \ (i \in N), \\ &\bar{y} + dr = f(\bar{x}) + (\bar{r} + dr). \end{aligned} \tag{1}$$

Thus $(\bar{v} + dv, \bar{r} + dr) \in V(\mathbb{R}^n_{++} \times \mathbb{R}^{2n})$ and the assertion (i) follows. Since the relation (1) still holds if $dv \leq 0$ and $dr \geq 0$, we also obtain (ii). The assertion (iii) directly follows from (ii). ∎

Here we provide a condition which is relatively mild compared with Condition 1.3.

Condition 2.2.

(i) *The mapping f is a P_0-function, i.e., for every $x^1, x^2 \in \mathbb{R}^n$ with $x^1 \neq x^2$ there exists an index i such that*

$$x^1_i \neq x^2_i \quad \textit{and} \quad (x^1_i - x^2_i)(f_i(x^1) - f_i(x^2)) \geq 0.$$

(ii) *There exists a feasible-interior-point (x, y) of the CP, i.e.,*

$$(x, y) > 0 \quad \textit{and} \quad y = f(x).$$

(iii) $$U^{-1}(D) := \{(u, x, y) \in \mathbb{R}^n_+ \times \mathbb{R}^{2n} : \ U(u, x, y) \in D\}$$
is bounded for every compact subset D of $\mathbb{R}^n_+ \times V(\mathbb{R}^n_{++} \times \mathbb{R}^{2n})$.

Lemma 2.3. *If Condition 1.3 holds so does the Condition 2.2.*

Proof: It follows immediately that the requirement (i) and (ii) of Condition 2.2 hold. To show (iii) of Condition 2.2, assume on the contrary that the set

$$U^{-1}(D) := \{(u, x, y) \in \mathbb{R}^n_+ \times \mathbb{R}^{2n} : \ U(u, x, y) \in D\}$$

is unbounded for some compact subset $D \subset \mathbb{R}^n_+ \times V(\mathbb{R}^n_{++} \times \mathbb{R}^{2n})$. Then, since $\{u^k\}$ is bounded by the definition of the map u and by the assumption, we can take a sequence $\{(u^k, x^k, y^k) \in U^{-1}(D) \ (k = 1, 2, \ldots)\}$ such that $\|(x^k, y^k)\| \to \infty$ and

$$\lim_{k \to \infty} v(u^k, x^k, y^k) = \bar{v} \quad \text{and} \quad \lim_{k \to \infty} r(u^k, x^k, y^k) = \bar{r}$$

for some $(\bar{v}, \bar{r}) \in V(\mathbb{R}^n_{++} \times \mathbb{R}^{2n})$. Since $V(\mathbb{R}^n_{++} \times \mathbb{R}^{2n})$ is an open subset of $\mathbb{R}^{2n}$ (see Lemma 2.1), we can find a $(\tilde{v}, \tilde{r}) \in V(\mathbb{R}^n_{++} \times \mathbb{R}^{2n})$ such that

$$v(u^k, x^k, y^k) \leq \tilde{v} \quad \text{and} \quad r(u^k, x^k, y^k) \geq \tilde{r}$$

for every sufficiently large k. Since $(\tilde{v}, \tilde{r}) \in V(\mathbb{R}^n_{++} \times \mathbb{R}^{2n})$, by Lemma 1.4, there exists a point $(\tilde{u}, \tilde{x}, \tilde{y}) \in \mathbb{R}^n_{++} \times \mathbb{R}^{2n}$ such that

$$((\tilde{x}-\tilde{v}/2),(\tilde{y}-\tilde{v}/2))>0,\quad (\tilde{x}_i-\tilde{v}_i/2)(\tilde{y}_i-\tilde{v}_i/2)=\tilde{u}_i>0\ (i\in N)\ \text{ and }\ \tilde{y}=f(\tilde{x})+\tilde{r}.$$

Then, by the monotonicity of the mapping f, we obtain that

$$\begin{aligned}
0 &\le (x^k-\tilde{x})^T(f(x^k)-f(\tilde{x}))\\
&= (x^k-\tilde{x})^T\{(y^k-r^k)-(\tilde{y}-\tilde{r})\}\\
&= (x^k-\tilde{x})^T\{(y^k-\tilde{y})-(r^k-\tilde{r})\}\\
&= \{(x^k-v^k/2)-(\tilde{x}-\tilde{v}/2)+(v^k-\tilde{v})/2)^T\\
&\quad \{(y^k-v^k/2)-(\tilde{y}-\tilde{v}/2)+(v^k-\tilde{v})/2-(r^k-\tilde{r})\}\\
&= e^Tu^k\\
&\quad -\{(\tilde{y}-\tilde{v}/2)+(\tilde{v}-v^k)/2+(r^k-\tilde{r})\}^T(x^k-v^k/2)\\
&\quad -\{(\tilde{x}-\tilde{v}/2)+(\tilde{v}-v^k)/2\}^T(y^k-v^k/2)\\
&\quad +\{(\tilde{y}-\tilde{v}/2)+(\tilde{v}-v^k)/2+(r^k-\tilde{r})\}^T\{(\tilde{x}-\tilde{v}/2)+(\tilde{v}-v^k)/2\}.
\end{aligned}$$

Since (v^k,r^k) lies in the bounded set D for every k, we can find a positive number α such that

$$e^Tu^k+\{(\tilde{y}-\tilde{v}/2)+(\tilde{v}-v^k)/2+(r^k-\tilde{r})\}^T\{(\tilde{x}-\tilde{v}/2)+(\tilde{v}-v^k)/2\}\le\alpha.$$

Also, since $(\tilde{x}-\tilde{v}/2,\tilde{y}-\tilde{v}/2)>0$, $(x^k-v^k/2,y^k-v^k/2)>0$, $\tilde{v}-v^k\ge 0$ and $r^k-\tilde{r}\ge 0$, we have

$$\begin{aligned}
(\tilde{y}-\tilde{v}/2)^T(x^k-v^k/2) &\le \{(\tilde{y}-\tilde{v}/2)+(\tilde{v}-v^k)/2+(r^k-\tilde{r})\}^T(x^k-v^k/2),\\
(\tilde{x}-\tilde{v}/2)^T(y^k-v^k/2) &\le \{(\tilde{x}-\tilde{v}/2)+(\tilde{v}-v^k)/2\}^T(y^k-v^k/2)
\end{aligned}$$

and

$$(\tilde{y}-\tilde{v}/2)^T(x^k-v^k/2)+(\tilde{x}-\tilde{v}/2)^T(y^k-v^k/2)\le\alpha.$$

Moreover, the boundedness of D also ensures that there exists positive numbers β and γ such that

$$(\tilde{y}-\tilde{v}/2)^Tv^k/2+(\tilde{x}-v^k/2)^Tv^k/2\le\beta$$

for every k and

$$x_i^k>v_i^k/2\ge\gamma,\quad y_i^k>v_i^k/2\ge\gamma$$

for every $i\in N$ and k. Thus the bounded set

$$\{(x,y)\in\mathbb{R}^{2n}:\ x\ge\gamma e,\ y\ge\gamma e,\ (\tilde{y}-\tilde{v}/2)^Tx+(\tilde{x}-\tilde{v}/2)^Ty\le\alpha+\beta\}$$

contains $\{(x^k,y^k)\}$ for every sufficiently large k, which contradicts $\|(x^k,y^k)\|\to\infty$. ∎

Remark 2.4. Lemma 2.3 ensures that the CP arising from LP satisfies Condition 2.2. Here we observe some conditions which have been imposed so far to analyze the interior-point algorithms for the CP, and compare them with the condition above. Let us define

$$\begin{aligned} u' &: \mathbb{R}^{2n}_{+} \to \mathbb{R}^{n}_{+}, \\ &\quad u'(x,y) = (x_1y_1, x_2y_2, \ldots, x_ny_n)^T, \\ U' &: \mathbb{R}^{2n}_{+} \to \mathbb{R}^{n}_{+} \times \mathbb{R}^{n}, \\ &\quad U'(x,y) := (u'(x,y), r(u,x,y)). \end{aligned}$$

In the paper [21], the authors used the following condition and showed that the condition holds if Condition 1.3 does:

Condition 2.5.

(i) *The mapping f is a P_0-function, i.e., for every $x^1, x^2 \in \mathbb{R}^n$ with $x^1 \neq x^2$ there exists an index i such that*

$$x^1_i \neq x^2_i \quad \textit{and} \quad (x^1_i - x^2_i)(f_i(x^1) - f_i(x^2)) \geq 0.$$

(ii) *There exists a feasible-interior-point (x,y) of the CP, i.e.,*

$$(x,y) > 0 \quad \textit{and} \quad y = f(x).$$

(iii) $$U'^{-1}(D) := \{(x,y) \in \mathbb{R}^{2n}_{+} : \ U'(x,y) \in D'\}$$
is bounded for every compact subset D' of $\mathbb{R}^n_+ \times r(\mathbb{R}^{2n}_{++})$.

In view of Lemma 1.4, we can see that

$$(x,y) \in \mathbb{R}^{2n}_{+}, \quad U'(x,y) = (\bar{u}', \bar{r}')$$

if and only if

$$U(\bar{u}', x, y) = (\bar{u}', 0, \bar{r}').$$

By using this relation, it is easy to see that Condition 2.5 holds whenever Condition 2.2 does. However, the converse is not obvious. In linear cases, i.e., the mapping f is given by $f(x) = Mx + q$ with an $n \times n$ matrix M and an n-dimensional vector q, the next condition has been proposed in [22]:

Condition 2.6.

(i) *The matrix M is a P_0-matrix, i.e., for every $x^1, x^2 \in \mathbb{R}^n$ with $x^1 \neq x^2$ there exists an index i such that*

$$x^1_i \neq x^2_i \quad \textit{and} \quad (x^1_i - x^2_i)M(x^1_i - x^2_i) \geq 0.$$

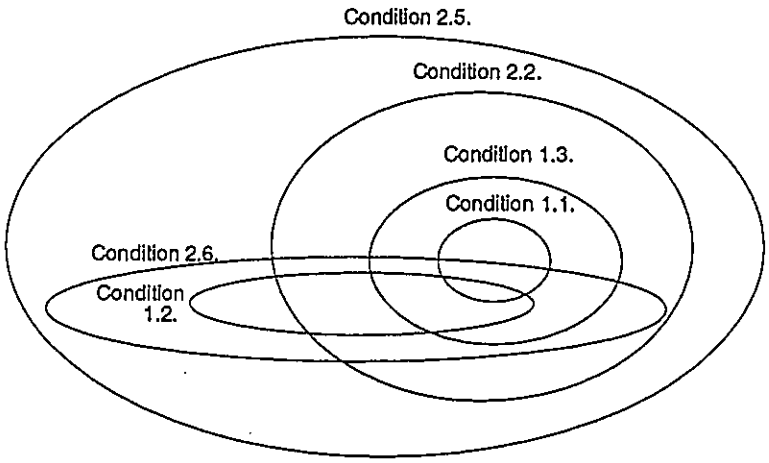

Figure 2: Some relationships among the conditions.

(ii) *There exists a feasible-interior-point* (x, y) *of the CP, i.e.,*

$$(x, y) > 0 \quad \textit{and} \quad y = f(x).$$

(iii) $$S_+(t) := \{(x, y) \in \mathbb{R}^{2n}_+ : y = Mx + q, \quad x^T y \leq t\}$$
is bounded for every $t \geq 0$.

It is also easy to see that Condition 2.2 holds if f is linear and Condition 2.6 holds. Kojima et al. [22] showed that the monotone and linear CP, i.e., the matrix M of f is positive semi-definite, satisfies this condition. In addition, Kanzow[18] derived an interesting result concerning the relationship between Condition 1.2 and Condition 2.6: If we enforce a more strict requirement such that the set $S_+(t)$ is bounded for every $q \in \mathbb{R}^n$ and for every $t \geq 0$ on Condition 2.6, then the enforced condition is equivalent to Condition 1.2. See Figure 2 in which the discussion above is summarized.

Note that by (iii) of Lemma 2.1, we can easily obtain the following lemma which will be often used in the further discussions:

Lemma 2.7. *If Condition 2.2 holds then*

$$U^{-1}(D) := \{(u, x, y) \in \mathbb{R}^n_+ \times \mathbb{R}^{2n} : \ U(u, x, y) \in D\}$$

is bounded for every bounded subset D *of* $\mathbb{R}^n_+ \times \mathbb{R}^n_- \times \mathbb{R}^n_+$.

In the remainder of this section we show that the mapping U gives a homeomorphism.

Lemma 2.8. *Assume that (i) of Condition 2.2 holds. Then the mapping U is one-to-one on $\mathbb{R}^n_{++} \times \mathbb{R}^{2n}$.*

Proof: Assume on the contrary that

$$U(u^1, x^1, y^1) = U(u^2, x^2, y^2)$$

for some distinct (u^1, x^1, y^1) and (u^2, x^2, y^2). By the definition of the mapping U, we can assume that $u^1 = u^2 = \bar{u}$ and $x^1 \neq x^2$. Let us define

$$(\bar{u}, \bar{v}, \bar{r}) := U(\bar{u}, x^1, y^1) = U(\bar{u}, x^2, y^2).$$

Then we obtain that

$$\begin{aligned} & f(x^1) - f(x^2) = y^1 - y^2, \\ & (x^1_i - \bar{v}_i/2)(y^1_i - \bar{v}_i/2) = (x^2_i - \bar{v}_i/2)(y^2_i - \bar{v}_i/2) = \bar{u}_i > 0. \end{aligned} \tag{2}$$

By the assumption on the mapping f, there exists an index k such that $x^1_k \neq x^2_k$ and

$$\begin{aligned} 0 \;&\leq\; (x^1_k - x^2_k)(f_k(x^1) - f_k(x^2)) \\ &=\; \{(x^1_k - \bar{v}_k/2) - (x^2_k - \bar{v}_k/2)\}\{(y^1_k - \bar{v}_k/2) - (y^2_k - \bar{v}_k/2)\}. \end{aligned}$$

Here we may assume without loss of generality that

$$x^1_k - \bar{v}_k/2 > x^2_k - \bar{v}_k/2 > 0,$$

and it follows that

$$y^1_k - \bar{v}_k/2 \geq y^2_k - \bar{v}_k/2 > 0.$$

This contradicts the equality (2). ∎

Lemma 2.9. *Assume that (i) and (iii) of Condition 2.2.*

(i) *For every $(\bar{u}, \bar{v}, \bar{r}) \in \mathbb{R}^n_+ \times V(\mathbb{R}^n_{++} \times \mathbb{R}^{2n})$, the system*

$$U(u, x, y) = (\bar{u}, \bar{v}, \bar{r})$$

has a solution.

(ii) $$U(\mathbb{R}^n_{++} \times \mathbb{R}^{2n}) = \mathbb{R}^n_{++} \times V(\mathbb{R}^n_{++} \times \mathbb{R}^{2n})$$

Proof: Let $(\bar{u}, \bar{v}, \bar{r}) \in \mathbb{R}^n_+ \times V(\mathbb{R}^n_{++} \times \mathbb{R}^{2n})$. Since $(\bar{v}, \bar{r}) \in V(\mathbb{R}^n_{++} \times \mathbb{R}^{2n})$, there exists a point $(\hat{u}, \hat{x}, \hat{y}) \in \mathbb{R}^n_{++} \times \mathbb{R}^{2n}$ such that

$$\begin{aligned} & (\hat{x}_i - \bar{v}_i/2)(\hat{y}_i - \bar{v}_i/2) = \hat{u}_i \quad (i \in N), \\ & \hat{y} = f(\hat{x}) + \bar{r}. \end{aligned}$$

Consider the family of equations with the parameter $\theta \in [0, 1]$

$$U((1-\theta)\hat{u}+\theta\bar{u},x,y)=((1-\theta)\hat{u}+\theta\bar{u},\bar{v},\bar{r}) \quad \text{and} \quad (x,y)\in\mathbb{R}^{2n}. \tag{3}$$

Let $\bar{\theta}\leq 1$ be the supremum of the $\hat{\theta}$'s such that (3) has a solution for every $\theta\in[0,\hat{\theta}]$. Then there exists a sequence $\{(x^k,y^k,\theta^k)\}$ of the system (3) such that $\lim_{k\to\infty}\theta^k=\bar{\theta}$. Since $((1-\theta)\hat{u}+\theta\bar{u},\bar{v},\bar{r})$ lies in the compact subset

$$D=\{(1-\theta)\hat{u}+\theta\bar{u},\bar{v},\bar{r}):\ \theta\in[0,1]\}$$

of $\mathbb{R}^n_+\times V(\mathbb{R}^n_{++}\times\mathbb{R}^{2n})$, (iii) of Condition 2.2 ensures that $\{(x^k,y^k)\}$ is bounded. Thus we may assume that there exists a point $(\bar{x},\bar{y})$ such that $(x^k,y^k)\to(\bar{x},\bar{y})$. Note that $(\bar{x},\bar{y},\bar{\theta})$ satisfies (3) by the continuity of the mapping U. Hence if $\bar{\theta}=1$ then the desired result follows. Assume on the contrary that $\bar{\theta}<1$. Then we have

$$\begin{aligned}&(\bar{x}_i-\bar{v}_i/2)(\bar{y}_i-\bar{v}_i/2)=(1-\bar{\theta})\hat{u}+\bar{\theta}\bar{u}>0,\\&\bar{y}=f(\bar{x})+\bar{r},\end{aligned}$$

which implies that

$$((1-\bar{\theta})\hat{u}+\theta\bar{u},\bar{v},\bar{r})\in U(\mathbb{R}^n_{++}\times\mathbb{R}^{2n}).$$

It follows from Lemma 2.8 that the mapping U is local homeomorphism at $((1-\bar{\theta})\hat{u}+\bar{\theta}\bar{u},\bar{x},\bar{y})$ (See the domain invariance theorem [42]). This implies that the system (3) has a solution for every θ sufficiently close to $\bar{\theta}$, which contradicts the definition of $\bar{\theta}$. Thus the assertion (i) is obtained. Note that the assertion (i) implies the relation of inclusion

$$U(\mathbb{R}^n_{++}\times\mathbb{R}^{2n})\subset\mathbb{R}^n_{++}\times V(\mathbb{R}^n_{++}\times\mathbb{R}^{2n}).$$

Since we immediately see that

$$U(\mathbb{R}^n_{++}\times\mathbb{R}^{2n})\supset\mathbb{R}^n_{++}\times V(\mathbb{R}^n_{++}\times\mathbb{R}^{2n}),$$

the assertion (ii) is also obtained. ∎

Thus, the mapping U is one-to-one on the open subset $\mathbb{R}^n_{++}\times\mathbb{R}^{2n}$ of R^{3n} (Lemma 2.8) and the image $U(\mathbb{R}^n_{++}\times\mathbb{R}^{2n})$ is given by $\mathbb{R}^n_{++}\times V(\mathbb{R}^n_{++}\times\mathbb{R}^{2n})$ ((ii) of Lemma 2.9) if (i) and (iii) of Condition 2.2 hold. The following theorem follows from the domain invariance theorem [42]).

Theorem 2.10. *Assume that (i) and (iii) of Condition 2.2 hold. Then the mapping U maps $\mathbb{R}^n_{++}\times\mathbb{R}^{2n}$ onto $\mathbb{R}^n_{++}\times V(\mathbb{R}^n_{++}\times\mathbb{R}^{2n})$ homeomorphically.*

3 The existence of the trajectory

Let $(\bar{u}, \bar{v}, \bar{r}) \in U(\mathbb{R}^n_{++} \times \mathbb{R}^{2n}) = \mathbb{R}^n_{++} \times V(\mathbb{R}^n_{++} \times \mathbb{R}^{2n})$ and define

$$\bar{W} := \{\theta(\bar{u}, \bar{v}, \bar{r}) \in \mathbb{R}^n_{++} \times \mathbb{R}^{2n} : \theta \in (0, 1]\}.$$

We have already seen that $\mathbb{R}^n_{++} \times \mathbb{R}^n_{-} \times \mathbb{R}^n_{+} \subset U(\mathbb{R}^n_{++} \times \mathbb{R}^{2n}) = \mathbb{R}^n_{++} \times V(\mathbb{R}^n_{++} \times \mathbb{R}^{2n})$ if (i) of Condition 2.2 holds (Lemmas 2.1 and 2.9) and U maps $\mathbb{R}^n_{++} \times \mathbb{R}^{2n}$ onto $\mathbb{R}^n_{++} \times V(\mathbb{R}^n_{++} \times \mathbb{R}^{2n})$ homeomorphically (Theorem 2.10). Thus, if Condition 2.2 holds then for every $(\bar{u}, \bar{v}, \bar{r}) \in \mathbb{R}^n_{++} \times \mathbb{R}^n_{-} \times \mathbb{R}^n_{+}$, the subtrajectory

$$U^{-1}(\bar{W}) \in \mathbb{R}^n_{++} \times \mathbb{R}^{2n}$$

exists.

Moreover, if the more strict condition, Condition 1.3 (the monotone CP with a feasible-interior-point), holds then we can see that the trajectory exists for every $(\bar{u}, \bar{v}, \bar{r}) \in U(\mathbb{R}^n_{++} \times \mathbb{R}^{2n}) = \mathbb{R}^n_{++} \times V(\mathbb{R}^n_{++} \times \mathbb{R}^{2n})$. We first show the following lemma.

Lemma 3.1. *Assume that (i) of Condition 1.3 holds, i.e., the problem is a monotone CP. Let* $(\bar{u}^1, \bar{v}^1, \bar{r}^1), (\bar{u}^2, \bar{v}^2, \bar{r}^2) \in U(\mathbb{R}^n_{++} \times \mathbb{R}^{2n}) = \mathbb{R}^n_{++} \times V(\mathbb{R}^n_{++} \times \mathbb{R}^{2n})$ *and define*

$$(u(\theta), v(\theta), r(\theta)) := (1-\theta)(\bar{u}^1, \bar{v}^1, \bar{r}^1) + \theta(\bar{u}^2, \bar{v}^2, \bar{r}^2)$$

for every $\theta \in [0, 1]$. *Consider the set*

$$\begin{aligned}&\mathcal{P}((\bar{u}^1, \bar{v}^1, \bar{r}^1), (\bar{u}^2, \bar{v}^2, \bar{r}^2)) \\ &\quad := \{(u, x, y) \in \mathbb{R}^n_{++} \times \mathbb{R}^{2n} : \ U(u, x, y) = (u(\theta), v(\theta), r(\theta)) \ \text{ for some } \theta \in [0, 1]\}\end{aligned}$$

Then there exists a bounded set $\Lambda((\bar{u}^1, \bar{v}^1, \bar{r}^1), (\bar{u}^2, \bar{v}^2, \bar{r}^2))$ *such that*

$$\Lambda((\bar{u}^1, \bar{v}^1, \bar{r}^1), (\bar{u}^2, \bar{v}^2, \bar{r}^2)) \subset \mathcal{P}((\bar{u}^1, \bar{v}^1, \bar{r}^1), (\bar{u}^2, \bar{v}^2, \bar{r}^2))$$

for every $(\bar{u}^1, \bar{v}^1, \bar{r}^1), (\bar{u}^2, \bar{v}^2, \bar{r}^2) \in U(\mathbb{R}^n_{++} \times \mathbb{R}^{2n}) = \mathbb{R}^n_{++} \times V(\mathbb{R}^n_{++} \times \mathbb{R}^{2n})$.

Proof: Let $(\bar{u}^1, \bar{v}^1, \bar{r}^1), (\bar{u}^2, \bar{v}^2, \bar{r}^2) \in U(\mathbb{R}^n_{++} \times \mathbb{R}^{2n})$ and consider the line segment

$$\tilde{W} := \{(u(\theta), v(\theta), r(\theta)) : \ (u(\theta), v(\theta), r(\theta)) = (1-\theta)(\bar{u}^1, \bar{v}^1, \bar{r}^1) + \theta(\bar{u}^2, \bar{v}^2, \bar{r}^2), \quad \theta \in [0, 1]\}.$$

Suppose that $(u(\theta), v(\theta), r(\theta)) \in U(\mathbb{R}^n_{++} \times \mathbb{R}^{2n})$ for some $\theta \in [0, 1]$. Then, by the definition of the mapping U, there exist a point $(u(\theta), x(\theta), y(\theta))$ such that

$$\begin{aligned}&((x(\theta)_i - v(\theta)_i/2), (y(\theta)_i - v(\theta)_i/2)) > 0, \\ &(x(\theta)_i - v(\theta)_i/2)(y(\theta)_i - v(\theta)_i/2)) = u(\theta)_i > 0, \\ &y(\theta) = f(x(\theta)) + r(\theta).\end{aligned} \tag{4}$$

We denote the point $(u(\theta), x(\theta), y(\theta))$ with $\theta = 0$ by $(\bar{u}^1, \bar{x}^1, \bar{y}^1)$ and the one with $\theta = 1$ by $(\bar{u}^2, \bar{x}^2, \bar{y}^2)$, respectively.

Let us assume that we have two points $(u(\theta^1), v(\theta^1), r(\theta^1))$, $(u(\theta^2), v(\theta^2), r(\theta^2))$ for some $\theta^1, \theta^2 \in [0, 1]$ both of which belong to the set $U(\mathbb{R}^n_{++} \times \mathbb{R}^{2n})$. Define $(u^1, x^1, y^1) := (u(\theta^1), x(\theta^1), y(\theta^1))$ and $(u^2, x^2, y^2) := (u(\theta^2), x(\theta^2), y(\theta^2))$. The monotonicity of the mapping f implies that

$$\begin{aligned} 0 &\leq (x^1 - x^2)^T \{f(x^1) - f(x^2)\} \\ &= (x^1 - x^2)^T \{(y^1 - r^1) - (y^2 - r^2)\} \\ &= (x^1 - x^2)^T \{(y^1 - y^2) - (r^1 - r^2)\} \\ &= (x^1 - x^2)^T (y^1 - y^2) - (x^1 - x^2)^T (r^1 - r^2). \end{aligned} \tag{5}$$

It follows from (4) that

$$\begin{aligned} &(x^1 - x^2)^T (y^1 - y^2) \\ &= \{(x^1 - v^1/2) - (x^2 - v^2/2) + (v^1 - v^2)/2\}^T \{(y^1 - v^1/2) - (y^2 - v^2/2) + (v^1 - v^2)/2\} \\ &= e^T u^1 + e^T u^2 \\ &\quad -(x^1 - v^1/2)^T (y^2 - v^2/2) - (x^2 - v^2/2)^T (y^1 - v^1/2) \\ &\quad +(v^1 - v^2)^T \{(x^1 - v^1/2) - (x^2 - v^2/2) + (y^1 - v^1/2) - (y^2 - v^2/2)\}/2 \\ &\quad +\|v^1 - v^2\|^2/4 \\ &= e^T u^1 + e^T u^2 \\ &\quad -(x^1 - v^1/2)^T (y^2 - v^2/2) - (x^2 - v^2/2)^T (y^1 - v^1/2) \\ &\quad +(v^1 - v^2)^T \{(x^1 - x^2) + (y^1 - y^2) - (v^1 - v^2)\}/2 \\ &\quad +\|v^1 - v^2\|^2/4 \\ &= e^T u^1 + e^T u^2 \\ &\quad -(x^1 - v^1/2)^T (y^2 - v^2/2) - (x^2 - v^2/2)^T (y^1 - v^1/2) \\ &\quad +(v^1 - v^2)^T \{(x^1 - x^2) + (y^1 - y^2)\}/2 \\ &\quad -\|v^1 - v^2\|^2/4. \end{aligned} \tag{6}$$

Combining (5) and (6), we have

$$\begin{aligned} &(x^1 - v^1/2)^T (y^2 - v^2/2) + (x^2 - v^2/2)^T (y^1 - v^1/2) \\ &\leq e^T u^1 + e^T u^2 - \|v^1 - v^2\|^2/4 \\ &\quad +(v^1 - v^2)^T \{(x^1 - x^2) + (y^1 - y^2)\}/2 - (r^1 - r^2)^T (x^1 - x^2). \end{aligned}$$

Now we consider the following two special cases. First, let $\theta^1 = 0$ and $\theta^2 = \theta$. Then by (4) and by the definition $(u(\theta), v(\theta), r(\theta))$, we see that

$$\begin{aligned} &(\bar{y}^1 - \bar{v}^1/2)^T [x(\theta) - \{(1-\theta)\bar{v}^1 + \theta\bar{v}^2\}/2] + (\bar{x}^1 - \bar{v}^1/2)^T [y(\theta) - \{(1-\theta)\bar{v}^1 + \theta\bar{v}^2\}/2] \\ &\leq e^T \bar{u}^1 + e^T \{(1-\theta)\bar{u}^1 + \theta\bar{u}^2\} - \theta^2 \|\bar{v}^1 - \bar{v}^2\|^2/4 \\ &\quad + \theta(\bar{v}^1 - \bar{v}^2)^T \{(\bar{x}^1 - x(\theta)) + (\bar{y}^1 - y(\theta))\}/2 - \theta(\bar{r}^1 - \bar{r}^2)^T (\bar{x}^1 - x(\theta)). \end{aligned} \tag{7}$$

Second, let $\theta^1 = 1$ and $\theta^2 = \theta$. Then,

$$\begin{aligned} &(\bar{y}^2 - \bar{v}^2/2)^T [x(\theta) - \{(1-\theta)\bar{v}^1 + \theta\bar{v}^2\}/2] + (\bar{x}^2 - \bar{v}^2/2)^T [y(\theta) - \{(1-\theta)\bar{v}^1 + \theta\bar{v}^2\}/2] \\ &\leq e^T \bar{u}^2 + e^T \{(1-\theta)\bar{u}^1 + \theta\bar{u}^2\} - (1-\theta)^2 \|\bar{v}^1 - \bar{v}^2\|^2/4 \\ &\quad - (1-\theta)\theta(\bar{v}^1 - \bar{v}^2)^T \{(\bar{x}^2 - x(\theta)) + (\bar{y}^2 - y(\theta))\}/2 + (1-\theta)(\bar{r}^1 - \bar{r}^2)^T (\bar{x}^2 - x(\theta) \end{aligned} \tag{8}$$

Multiplying (7) by $(1-\theta) \geq 0$ and (8) by $\theta \geq 0$, and adding two inequalities, we obtain that

$$\begin{aligned}
&\tilde{y}^T[x - \{(1-\theta)\bar{v}^1 + \theta\bar{v}^2\}/2] + \tilde{x}^T[y - \{(1-\theta)\bar{v}^1 + \theta\bar{v}^2\}/2] \\
&\quad \leq \; 2e^T\{(1-\theta)\bar{u}^1 + \theta\bar{u}^2\} - \theta(1-\theta)\|\bar{v}^1 - \bar{v}^2\|^2/4 \\
&\qquad +\theta(1-\theta)(\bar{v}^1 - \bar{v}^2)^T\{(\bar{x}^1 - \bar{x}^2) + (\bar{y}^1 - \bar{y}^2)\}/2 - \theta(1-\theta)(\bar{r}^1 - \bar{r}^2)^T(\bar{x}^1 - \bar{x}^2)
\end{aligned}$$

where

$$\begin{aligned}
\tilde{y} &:= (1-\theta)(\bar{y}^1 - \bar{v}^1/2) + \theta(\bar{y}^2 - \bar{v}^2/2) > 0, \\
\tilde{x} &:= (1-\theta)(\bar{x}^1 - \bar{v}^1/2) + \theta(\bar{x}^2 - \bar{v}^2/2) > 0,
\end{aligned}$$

Let us define

$$\begin{aligned}
\gamma_1 &:= \max\{\bar{x}_i^1 - \bar{v}_i^1/2,\ \bar{y}_i^1 - \bar{v}_i^1/2,\ \bar{x}_i^2 - \bar{v}_i^2/2,\ \bar{y}_i^2 - \bar{v}_i^2/2 \ (i \in N)\} > 0, \\
\gamma_2 &:= \max\{\|\bar{x}^1 - \bar{v}^1/2\|,\ \|\bar{y}^1 - \bar{v}^1/2\|,\ \|\bar{x}^2 - \bar{v}^2/2\|,\ \|\bar{y}^2 - \bar{v}^2/2\|\} > 0, \\
\gamma_3 &:= \max\{\|\bar{v}^1\|,\ \|\bar{v}^2\|\} \geq 0, \\
\gamma_4 &:= \max\{e^T\bar{u}^1, e^T\bar{u}^2\}.
\end{aligned}$$

Then we can see that

$$(\gamma_1 e)^T x(\theta) + (\gamma_1 e)^T y(\theta) \leq \gamma_5 \tag{9}$$

where γ_5 is a positive constant given by

$$\begin{aligned}
\gamma_5 &:= \gamma_2\gamma_3/2 + 2\gamma_4 \\
&\quad +\|\bar{v}^1 - \bar{v}^2\|^2/4 + \|\bar{v}^1 - \bar{v}^2\|(\|\bar{x}^1 - \bar{x}^2\| + \|\bar{y}^1 - \bar{y}^2\|)/2 + \|\bar{r}^1 - \bar{r}^2\|\|\bar{x}^1 - \bar{x}^2\|.
\end{aligned}$$

Moreover, letting

$$\gamma_6 := \min\{\bar{v}_i^1/2,\ \bar{v}_i^2/2 \ (i \in N)\}$$

we have

$$\begin{aligned}
x_i &> (1-\theta)\bar{v}_i^1/2 + \theta\bar{v}_i^2/2 \geq \gamma_6, \\
y_i &> (1-\theta)\bar{v}_i^1/2 + \theta\bar{v}_i^2/2 \geq \gamma_6
\end{aligned}$$

for every $i \in N$. Thus, the point $(u(\theta), x(\theta), y(\theta))$ lies in the bounded set

$$\begin{aligned}
\Lambda((\bar{u}^1, \bar{v}^1, \bar{r}^1), (\bar{u}^2, \bar{v}^2, \bar{r}^2)) \;:=\; &\{(u,x,y) \in \mathbb{R}^n_{++} \times \mathbb{R}^{2n} :\ (\gamma_1 e)^T x + (\gamma_1 e)^T y \leq \gamma_5, \\
&(x,y) \geq \gamma_6(e,e),\ u = (1-\theta)\bar{u}^1 + \theta\bar{u}^2,\ \theta \in [0,1]\}.
\end{aligned}$$

∎

Theorem 3.2. *Assume that (i) of Condition 1.3 holds, i.e., the problem is a monotone CP. Then the image $U(\mathbb{R}^n_{++} \times \mathbb{R}^{2n}) = \mathbb{R}^n_{++} \times V(\mathbb{R}^n_{++} \times \mathbb{R}^{2n})$ of U on $\mathbb{R}^n_{++} \times \mathbb{R}^{2n}$ is an open convex subset of $\mathbb{R}^n_{++} \times \mathbb{R}^{2n}$.*

Proof: Let $(\bar{u}^1, \bar{v}^1, \bar{r}^1), (\bar{u}^2, \bar{v}^2, \bar{r}^2) \in U(\mathbb{R}^n_{++} \times \mathbb{R}^{2n})$ and consider the line segment

$$\tilde{W} := \{(u(\theta), v(\theta), r(\theta)) : \ (u(\theta), v(\theta), r(\theta)) = (1-\theta)(\bar{u}^1, \bar{v}^1, \bar{r}^1) + \theta(\bar{u}^2, \bar{v}^2, \bar{r}^2), \quad \theta \in [0,1]\}.$$

Let

$$\Theta := \{\theta \in [0,1] : \ (u(\theta), v(\theta), r(\theta)) \in U(\mathbb{R}^n_{++} \times \mathbb{R}^{2n})\}$$

and define

$$\theta^* := \inf\{\theta \in \Theta : \ [\theta, 1] \in \Theta\}.$$

Since $1 \in \Theta$, by the openness of the set $U(\mathbb{R}^n_{++} \times \mathbb{R}^{2n}) = \mathbb{R}^n_{++} \times V(\mathbb{R}^n_{++} \times \mathbb{R}^{2n})$ (see Lemmas 2.1 and 2.9), we know that $\theta^* < 1$ and $\theta^* \notin \Theta$. If $\theta < 0$ then $(u(\theta), v(\theta), r(\theta)) \in U(\mathbb{R}^n_{++} \times \mathbb{R}^{2n})$ for every $\theta \in [0,1]$ and the convexity follows. Assume on the contrary that $\theta^* \geq 0$. Let $\{\theta^k \in (\theta^*, 1]\}$ be a sequence converging to θ^*. Then for every $k = 1, 2, \ldots$, there exists a point (u^k, x^k, y^k) such that $U(u^k, v^k, r^k) = (u(\theta^k), v(\theta^k), r(\theta^k))$. By Lemma 3.1, (u^k, x^k, y^k) lies in a bounded set $\Lambda((\bar{u}^1, \bar{v}^1, \bar{r}^1), (\bar{u}^2, \bar{v}^2, \bar{r}^2))$ for every k. Hence we may assume without loss of generality that $\{(u^k, x^k, y^k)\}$ converges to some $(\bar{u}, \bar{x}, \bar{y})$. By the continuity of the mapping U, we have

$$U(\bar{u}, \bar{x}, \bar{y}) = (u(\theta^*), v(\theta^*), r(\theta^*))$$

which implies that $(u(\theta^*), v(\theta^*), r(\theta^*)) \in U(\mathbb{R}^n_{++} \times \mathbb{R}^{2n})$ and $\theta^* \in \Theta$ which contradicts $\theta^* \notin \Theta$. Thus we have shown that $U(\mathbb{R}^n_{++} \times \mathbb{R}^{2n})$ is convex. ∎

Thus, we are ready to show the following main theorem of this section.

Theorem 3.3. (I) *Assume that Condition 2.2 holds. Let* $(\bar{u}, \bar{v}, \bar{r}) = (\bar{u}, \bar{v}, \bar{r}) \in \mathbb{R}^n_{++} \times \mathbb{R}^n_{-} \times \mathbb{R}^n_{+}$ *and* $\bar{W} = \{\theta(\bar{u}, \bar{v}, \bar{r}) \in \mathbb{R}^n_{++} \times \mathbb{R}^n_{-} \times \mathbb{R}^n_{+} : \ \theta \in (0,1]\}$.

- (i) *For every* $\theta \in (0,1]$, *the system* $U(u,x,y) = \theta(\bar{u}, \bar{v}, \bar{r})$ *has a unique solution* $(u(\theta), x(\theta), y(\theta))$ *which is continuous in* θ. *Hence* $U^{-1}(\bar{W})$ *forms a subtrajectory.*
- (ii) *The subtrajectory* $\{(u(\theta), x(\theta), y(\theta)) : \ \theta \in (0,1]\}$ *is bounded; hence there is at least one limiting point of* $(u(\theta), x(\theta), y(\theta))$ *as* $\theta \to 0$.
- (iii) *Every limiting point of* $(x(\theta), y(\theta))$ *is a solution of the CP.*
- (iv) *If* f *is a linear mapping of the form* $f(x) = Mx + q$ *then* $(x(\theta), y(\theta))$ *converges to a solution of the CP as* $\theta \to 0$.

(II) *Assume that the Condition 1.3 holds. Then the above assertions (i) – (iv) in (I) hold even if we replace the set* $\mathbb{R}^n_{++} \times \mathbb{R}^n_{-} \times \mathbb{R}^n_{+}$ *by the set* $U(\mathbb{R}^n_{++} \times \mathbb{R}^{2n}) = \mathbb{R}^n_{++} \times V(\mathbb{R}^n_{++} \times \mathbb{R}^{2n}) \supset \mathbb{R}^n_{++} \times \mathbb{R}^n_{-} \times \mathbb{R}^n_{+}$.

Proof: (I): By (ii) of Lemma 2.1, we observe that

$$\theta(\bar{u},\bar{v},\bar{r}) \in \bar{W} \subset \mathbb{R}^n_{++} \times \mathbb{R}^n_{-} \times \mathbb{R}^n_{+} \subset \mathbb{R}^n_{++} \times V(\mathbb{R}^n_{++} \times \mathbb{R}^{2n}) = U(\mathbb{R}^n_{++} \times \mathbb{R}^{2n})$$

for every $\theta > 0$. Hence the assertion (i) follows from Lemma 2.9. If we take $D = \{\theta(\bar{u},\bar{v},\bar{r}) : 0 < \theta \leq 1\}$ then by Lemma 2.7, the set

$$U^{-1}(D) = \{(u,x,y) \in \mathbb{R}^n_{++} \times \mathbb{R}^{2n} : \ U(u,x,y) \in D\} = \{(u(\theta),x(\theta),y(\theta)) : \ 0 < \theta \leq 1\}$$

is bounded. Thus we obtain (ii). Let us show the assertion (iii). By the continuity of the mapping U if (u,x,y) is a limiting point of $(u(\theta),x(\theta),y(\theta))$ then we have $U(u,x,y) = (0,0,0)$. By the definition of the mappings u,v,r, we have

$$(x,y) \geq 0, \quad x_i y_i = 0 \ (i \in N), \quad y = f(x)$$

which implies that (x,y) is a complementarity solution. The assertion (iv) follows from a similar discussion which can be seen in the proof of Theorem 4.4 of [21].

(II): By Theorem 3.2, $U(\mathbb{R}^n_{++} \times \mathbb{R}^{2n})$ is a convex set and $0 \in U(\mathbb{R}^n_{++} \times \mathbb{R}^{2n})$ under Condition 2.2. We can prove (II) similarly as in the proofs of (i) – (iv) of (I). ∎

4 A class of methods for tracing the trajectory

In the remainder of this paper, we use the symbol z and w to denote the triplets $(u,x,y) \in \mathbb{R}^{3n}$ and $(u,v,r) \in \mathbb{R}^{3n}$, respectively.

Suppose that Condition 2.2 holds. Let us choose a point $\bar{w} = (\bar{u},\bar{v},\bar{r})$ from the set $\mathbb{R}^n_{++} \times V(\mathbb{R}^n_{++} \times \mathbb{R}^{2n})$ if Condition 1.3 holds, and otherwise from the set $\mathbb{R}^n_{++} \times \mathbb{R}^n_{-} \times \mathbb{R}^n_{+}$ which is a subset of $\mathbb{R}^n_{++} \times V(\mathbb{R}^n_{++} \times \mathbb{R}^{2n})$. Define $\bar{W} := \{\theta\bar{w} : \ \theta \in (0,1]\}$. Then $U^{-1}(\bar{W}) \subset \mathbb{R}^n_{++} \times \mathbb{R}^{2n}$ forms a subtrajectory leading to a solution of the CP (cf. Theorem 3.3). Based on this fact, we propose a class of iterative methods for tracing the subtrajectory $U^{-1}(\bar{W})$ that involves

- the merit function

$$\psi(z) := \bar{w}^T U(z)/\|\bar{w}\|^2, \tag{10}$$

- a suitable neighborhood C of the trajectory which confines the generated sequence in a bounded set,

- the Newton direction to the system of equations

$$U(z') = \beta\psi(z)\bar{w} \tag{11}$$

  with a constant $\beta \in (0,1)$ at a point z,

- an inexact line search procedure.

First, we impose the following condition on the mapping f:

Condition 4.1. *The mapping f is continuously differentiable on $\mathbb{R}^n$.*

Then we obtain the following results on the Jacobian matrix $\mathcal{D}U(u,x,y)$ of the mapping U.

Lemma 4.2. *Assume that (i) of Condition 2.2 and Condition 4.1 hold, i.e., f is a continuously differentiable P_0-function.*

(i) *The Jacobian matrix $\mathcal{D}f(x)$ is a P_0-matrix at every $x \in \mathbb{R}^n$.*

(ii) *The Jacobian matrix $\mathcal{D}U(u,x,y)$ is given by*

$$\mathcal{D}U(u,x,y) = \begin{pmatrix} I & O & O \\ -2D & I-(X-Y)D & I+(X-Y)D \\ O & -\mathcal{D}f(x) & I \end{pmatrix}$$

where

$$X = \mathrm{diag}\,\{x_i(i \in N)\},\ Y = \mathrm{diag}\,\{y_i(i \in N)\},\ D = \mathrm{diag}\,\{d_i(i \in N)\}$$

and

$$d_i = 1/\sqrt{(x_i - y_i)^2 + 4u_i}\quad (i \in N).$$

for every $(u,x,y) \in \mathbb{R}^n_{++} \times \mathbb{R}^{2n}$.

(iii) $0 < 1-(x_i-y_i)d_i < 2,\ \ 0 < 1+(x_i-y_i)d_i < 2$

and $I-(X-Y)D$ and $I+(X-Y)D$ are positive diagonal matrices for every $(z) \in \mathbb{R}^n_{++} \times \mathbb{R}^{2n}$.

(iv) *$\mathcal{D}U(u,x,y)$ is a $3n \times 3n$ nonsingular matrix for every $(u,x,y) \in \mathbb{R}^n_{++} \times \mathbb{R}^{2n}$.*

Proof: (i): The proof has been given in Lemma 5.4 of [21].

(ii): Recall that the function v_i is given by

$$v_i(z) = (x_i + y_i) - \sqrt{(x_i - y_i)^2 + 4u_i}$$

for every $i \in N$. We can easily see that

$$\frac{\partial v_i}{\partial u_j} = \begin{cases} 0 & (i \neq j), \\ -2/\sqrt{(x_i-y_i)^2+4u_i} & (i=j), \end{cases}$$

$$\frac{\partial v_i}{\partial x_j} = \begin{cases} 0 & (i \neq j), \\ 1-(x_i-y_i)/\sqrt{(x_i-y_i)^2+4u_i} & (i=j), \end{cases}$$

$$\frac{\partial v_i}{\partial y_j} = \begin{cases} 0 & (i \neq j), \\ 1+(x_i-y_i)/\sqrt{(x_i-y_i)^2+4u_i} & (i=j), \end{cases}$$

for every $(u,x,y) \in \mathbb{R}^n_{++} \times \mathbb{R}^{2n}$. By the definition of d, we obtain the assertion (ii). (iii): It follows from a direct calculation. (iv): It is known that the $2n \times 2n$ matrix

$$\begin{pmatrix} D_x & D_y \\ -M & I \end{pmatrix}$$

is nonsingular for every positive diagonal matrices D_x, D_y if and only if the matrix M is a P_0-matrix (see, e.g., Lemma 4.1 of Kojima, Megiddo, Noma and Yoshise). Thus by the assertion (i) the matrix

$$\begin{pmatrix} I-(X-Y)D & I+(X-Y)D \\ -\mathcal{D}f(x) & I \end{pmatrix}$$

is nonsingular for every $(z) \in \mathbb{R}^n_{++} \times \mathbb{R}^{2n}$ and so is the Jacobian matrix $\mathcal{D}(z)$. ∎

Remark 4.3. It should be noted that Theorem 2.10 and (iv) of Lemma 4.2 ensure that U maps $\mathbb{R}^n_{++} \times \mathbb{R}^{2n}$ onto $U(\mathbb{R}^n_{++} \times \mathbb{R}^{2n}) = \mathbb{R}^n_{++} \times V(\mathbb{R}^n_{++} \times \mathbb{R}^{2n})$ diffeomorphically under Conditions 2.2 and 4.1.

Thus the Newton direction for the system (11) can be defined at every point $z \in \mathbb{R}^n_{++} \times \mathbb{R}^{2n}$ and given by

$$\mathcal{D}U(z)\Delta z = -U(z) + \beta\psi(z)\bar{w} \tag{12}$$

with a $\beta \in (0,1)$. Define

$$H_w := \{w \in \mathbb{R}^{3n} : \bar{w}^T w \le \|\bar{w}\|^2\}. \tag{13}$$

$$H_+ := \{w \in \mathbb{R}^{3n} : \bar{w}^T w \ge 0\}. \tag{14}$$

Here we introduce a condition on the neighborhood C and then describe our algorithm. We will give a proof of its global convergence and some examples of the neighborhoods satisfying the condition in the succeeding sections.

Condition 4.4.

(i) $\bar{W} \subset \mathrm{int}C \subset C \subset H_+$.

(ii) Let us define

$$\Omega := C \cap H_w. \tag{15}$$

Then Ω is a compact subset of

$$U(\mathbb{R}^n_{++} \times \mathbb{R}^{2n}) \cup \{0\} = (\mathbb{R}^n_{++} \times V(\mathbb{R}^n_{++} \times \mathbb{R}^{2n})) \cup \{0\}.$$

(iii) For every sequence $\{z^k : k = 1, 2, \ldots\} \subset \Omega$, $\psi(z^k) \to 0$ implies $U(z^k) \to 0$ (see (10) for the definition of the function ψ).

(iv) Let $\beta \in (0,1)$ be given and let z be a point such that $U(z) \in \mathrm{int}C \cap H_w$. Then there exists a positive number $\bar{\alpha}$ and a continuous and decreasing function $\rho : [0, \bar{\alpha}] \to [0,1]$ such that $\rho(0) = 1$ and

$$U(z + \alpha \Delta z) \in \mathrm{int}\Omega, \tag{16}$$
$$\psi(z + \alpha \Delta z) \leq \rho(\alpha)\psi(z) \tag{17}$$

for every $\alpha \in (0, \bar{\alpha})$. Here Δz is the Newton direction at z satisfying (11).

Now we are ready to propose our algorithm. In the algorithm below, the step length $\bar{\alpha}$ is determined by an inexact line search procedure which finds the smallest nonnegative integer l such that

$$U(z + \delta^l \Delta z) \in \mathrm{int}\Omega, \tag{18}$$
$$\psi(z + \delta^l \Delta z) \leq \rho(\delta^l)\psi(z). \tag{19}$$

Here $\delta \in (0,1)$ is a constant and δ^l is the lth power of δ. The finite termination of the line search procedure in Step 3 is ensured by (iv) of Condition 4.4.

Algorithm.

Step 0. Let $z^1 := U^{-1}(\bar{w})$, $\psi^1 := \psi(z^1)$, $\beta \in (0,1)$ and $k := 1$.

Step 1. If $U(z^k) = 0$ then stop. Otherwise, let $z := z^k$ and $\psi := \psi^k$.

Step 2. Compute the Newton direction Δz which is the unique solution of (11).

Step 3. Let l be the smallest nonnegative integer satisfying (18) and (19) and define

$$\alpha^k := \delta^l,$$
$$z^k := z + \alpha^k \Delta z,$$
$$\psi^k := \psi(z^k).$$

Step 4. Replace k by $k+1$ and go to Step 1.

5 Global and monotone convergence of the algorithm

In this section, we show that the sequence generated by our algorithm globally and monotonically converges to a solution of the CP if Conditions 2.2, 4.1 and 4.4 hold.

Theorem 5.1. *Suppose that Conditions 2.2, 4.1 and 4.4 hold. Let $\{(z^k, \psi^k)\} \subset \Omega \times [0,1]$ be a sequence generated by the algorithm described in Section 4.*

(i) *The sequence $\{\psi^k\}$ is monotonically decreasing and converges to 0 as $k \to \infty$.*

(ii) *The sequence $\{(z^k, \psi^k) = (u^k, x^k, y^k, \psi^k)\}$ is bounded and every limiting point of $\{(x^k, y^k)\}$ is a solution of the CP.*

Proof: (i): We can easily see that $\psi^k > \psi^{k+1} (k = 1, 2, \ldots)$ by construction of the algorithm and by (iv) of Condition 4.4. Hence the sequence $\{\psi^k\}$ is monotonically decreasing. Since $\psi^k \geq 0 (k = 1, 2, \ldots)$ (see (i) of Condition 4.4 and the definition (10) of ψ), there exists a $\tilde{\psi} \geq 0$ such that $\psi^k \to \tilde{\psi}$. If $\tilde{\psi} = 0$ then we obtain the desired result. Suppose that $\tilde{\psi} > 0$. Define a compact set $\tilde{\Omega}$

$$\tilde{\Omega} := \{w \in \Omega : \ \tilde{\psi}\|\bar{w}\|^2 \leq \bar{w}^T w \leq \psi^1 \|\bar{w}\|^2\}.$$

Note that $\tilde{\Omega} \subset U(\mathbb{R}^n_{++} \times \mathbb{R}^{2n})$ since $0 \notin \tilde{\Omega}$ (see (iii) of condition 4.4). Thus, by Theorem 2.10, the set $U^{-1}(\tilde{\Omega})$ containing the sequence $\{z^k\}$ is a compact subset of $\mathbb{R}^n_{++} \times V(\mathbb{R}^n_{++} \times \mathbb{R}^{2n})$. Taking a subsequence if necessary, we may assume that $\{z^k\}$ converges to some $\tilde{z} \in U(\tilde{\Omega})$. It is easy to see that $\tilde{\psi} = \bar{w}^T U(\tilde{z})/\|\bar{w}\|^2$. Moreover, by (iv) of Condition 4.4, there exists a positive number $\tilde{\alpha}$ and a continuous and decreasing function ρ such that for every $\alpha \in (0, \tilde{\alpha})$,

$$\begin{aligned} &U(\tilde{z} + \alpha\tilde{\Delta z}) \in \mathrm{int}\Omega, \\ &\psi(\tilde{z} + \alpha\tilde{\Delta z}) < \rho(\alpha)\psi(z). \end{aligned}$$

Here (ii) of Lemma 4.2 ensures that the Jacobian matrix $\mathcal{D}U(z)$ is nonsingular and continuous at $z = \tilde{z}$. This implies that the Newton direction Δz^k generated at the kth iteration converges to $\tilde{\Delta z}$. Therefore, for a nonnegative integer l such that $\delta^l \in (0, \tilde{\alpha})$, we have

$$\begin{aligned} &U(z^k + \delta^l \Delta z^k) \in \mathrm{int}\Omega, \\ &\psi(z^k + \delta^l \Delta z^k) < \rho(\delta^l)\psi(z^k) \end{aligned}$$

for every sufficiently large k. Let l^k be the nonnegative integer determined at Step 3 of the kth iteration in the algorithm. Then, for every sufficiently large k, we see that $l^k \leq l$ and hence $\delta^{l^k} \geq \delta^l$. Since ρ is a decreasing function, we obtain the following relation

$$\begin{aligned} \psi^{k+1} &\leq \rho(\delta^{l^k})\psi^k \\ &\leq \rho(\delta^l)\psi^k, \end{aligned}$$

which contradicts the fact that the sequence $\{\psi^k\}$ converges to $\tilde{\psi} > 0$.

(ii): By the assertion (i) above and (iii) of Condition 4.4, we have

$$\lim_{k \to \infty} U(z^k) = 0.$$

Since the sequence $\{U(z^k)\}$ is bounded, so is the sequence $\{z^k\}$ by Lemma 2.7. Therefore, by the continuity of the mapping U, we see that $U(\tilde{z}) = 0$ for any limiting point $\tilde{z}$ of the sequence $\{z^k\}$. ∎

6 Some examples of the neighborhood C

Suppose that Condition 2.2 holds. In view of Lemmas 2.1 and 2.9, the set $\mathbb{R}^n_{++} \times \mathbb{R}^n_{-} \times \mathbb{R}^n_{+}$ is a subset of $U(\mathbb{R}^n_{++} \times \mathbb{R}^{2n})$. By using this fact, we propose two examples for the neighborhood C to satisfy Condition 4.4

Let $\bar{z} = (\bar{u}, \bar{x}, \bar{y})$ be a point satisfying $U(\bar{z}) \in \mathbb{R}^n_{++} \times \mathbb{R}^n_{--} \times \mathbb{R}^n_{++}$. In fact, it is not difficult to find such a point: Let $\tilde{z} = (\tilde{u}, \tilde{x}, \tilde{y})$ be an arbitrary point of $\mathbb{R}^n_{++} \times \mathbb{R}^{2n}$. Even if $U(\tilde{z}) \notin \mathbb{R}^n_{++} \times \mathbb{R}^n_{--} \times \mathbb{R}^n_{++}$, we may choose a $(dv, dr) \in \mathbb{R}^{2n}$ so that

$$
\begin{aligned}
&((\tilde{x}_i - (\tilde{v}_i + dv_i)/2), ((\tilde{y}_i + dr_i) - (\tilde{v}_i + dv_i)/2)) > 0 \ (i \in N),\\
&\tilde{y} + dr = f(\tilde{x}) + (\tilde{r} + dr),\\
&\tilde{v} + dv < 0, \quad \tilde{r} + dr > 0.
\end{aligned}
$$

By setting

$$
\begin{aligned}
\bar{u}_i &= ((\tilde{x}_i - (\tilde{v}_i + dv_i)/2)((\tilde{y}_i + dr_i) - (\tilde{v}_i + dv_i)/2)) > 0 \ (i \in N),\\
\bar{x} &= \tilde{x},\\
\bar{y} &= \tilde{y} + dr
\end{aligned}
$$

we obtain a point $\bar{z}$ which satisfies $U(\bar{z}) \in \mathbb{R}^n_{++} \times \mathbb{R}^n_{--} \times \mathbb{R}^n_{++}$.

We consider the following two types of neighborhoods:

$$
C_1(\tau) := \{w \in \mathbb{R}^{3n} : \ \|w - (\bar{w}^T w/\|\bar{w}\|^2)\bar{w}\| \le \tau(\bar{w}^T w/\|\bar{w}\|^2)\} \tag{20}
$$

$$
\begin{aligned}
C_2(\tau_u, \tau_v, \tau_r) := \{&w = (u, v, r) \in \mathbb{R}^n_+ \times \mathbb{R}^n_- \times \mathbb{R}^n_+ :\\
&u \ge \tau_u(e^T u/n)e, \quad v \le \tau_v(e^T v/n)e, \quad r \ge \tau_r(e^T r/n)e\}
\end{aligned} \tag{21}
$$

with parameters $\tau, \tau_u, \tau_v, \tau_r \in (0, 1)$. It is easy to see that

$$
\bar{W} \subset \mathrm{int} C_1(\tau) \subset C_1(\tau) \subset H_+
$$

for every $\tau \in (0, 1)$ (see (14) for the definition of H_+), and for every $\bar{w} \in \mathbb{R}^n_{++} \times \mathbb{R}^n_{--} \times \mathbb{R}^n_{++}$,

$$
C_2(\tau_u, \tau_v, \tau_r) \subset (\mathbb{R}^n_{++} \times \mathbb{R}^n_{--} \times \mathbb{R}^n_{++}) \cup \{0\}
$$

and hence

$$
C_2(\tau_u, \tau_v, \tau_r) \subset H_+.
$$

Moreover, if

$$
\begin{aligned}
&0 < \tau < \min\{|\bar{w}_i| : i = 1, 2, \ldots, 3n\},\\
&0 < \tau_u < n \min\{\bar{u}_i : i \in N\}/(e^T \bar{u}),\\
&0 < \tau_v < n \max\{\bar{v}_i : i \in N\}/(e^T \bar{v}),\\
&0 < \tau_r < n \min\{\bar{r}_i : i \in N\}/(e^T \bar{r})
\end{aligned}
$$

then we can observe that the following relations hold:

$$
C_1(\tau) \subset (\mathbb{R}^n_{++} \times \mathbb{R}^n_{--} \times \mathbb{R}^n_{++}) \cap \{0\}, \quad \bar{W} \subset \mathrm{int} C_2(\tau_u, \tau_v, \tau_r).
$$

Let us consider the set Ω given by (15) in two cases; Case:$C = C_1(\tau)$ and Case:$C = C_2(\tau_u, \tau_v, \tau_r)$. In each case, the definition of the neighborhood and the relation $C_2(\tau_u, \tau_v, \tau_r) \subset \mathbb{R}^n_{++} \times \mathbb{R}^n_{--} \times \mathbb{R}^n_{++} \cup \{0\}$ ensures that

- the set Ω is a compact subset of $\mathbb{R}^n_{++} \times \mathbb{R}^n_{--} \times \mathbb{R}^n_{++} \cup \{0\}$ whenever the parameters satisfy the above relations,
- and hence $\psi(z^k) \to 0$ implies $U(z^k) \to 0$ for every sequence $\{z^k : k = 1, 2, \ldots\} \subset \Omega$ (see (10) for the definition of the function ψ).

Thus we obtain the lemma below:

Lemma 6.1. *Suppose that Condition 2.2 holds. Let* $\bar{w} = (\bar{u}, \bar{v}, \bar{r}) \in \mathbb{R}^n_{++} \times \mathbb{R}^n_{--} \times \mathbb{R}^n_{++}$ *and define*

$$\begin{aligned} \bar{\tau} &:= \min\{|\bar{w}_i| : i = 1, 2, \ldots, 3n\}, \\ \bar{\tau}_u &:= n \min\{\bar{u}_i : i \in N\}/(e^T \bar{u}), \\ \bar{\tau}_v &:= n \max\{\bar{v}_i : i \in N\}/(e^T \bar{v}), \\ \bar{\tau}_r &:= n \min\{\bar{r}_i : i \in N\}/(e^T \bar{r}). \end{aligned} \tag{22}$$

Then the neighborhoods $C_1(\tau)$ *and* $C_2(\tau_u, \tau_v, \tau_r)$ *given by (20) and (21) satisfy the requirements (i) - (iii) of Condition 4.4 whenever* $\tau \in (0, \bar{\tau})$, $\tau_u \in (0, \bar{\tau}_u)$, $\tau_v \in (0, \bar{\tau}_v)$ *and* $\tau_r \in (0, \bar{\tau}_r)$.

To show that these neighborhoods also satisfy the requirement (iv) of Condition 4.4, we use the lemma below.

Lemma 6.2. *Assume that Conditions 2.2 and 4.1 hold. Let* $z \in \mathbb{R}^n_{++} \times \mathbb{R}^{2n}$ *and* Δz *be the Newton direction satisfying (11) with a* $\beta \in (0, 1)$. *Define*

$$\alpha^* := \max\{\alpha \in [0, 1] : \ z + \alpha \Delta z \in \mathbb{R}^n_+ \times \mathbb{R}^{2n}\}, \tag{23}$$

$$\begin{aligned} g(\alpha) &:= (g_u(\alpha), g_v(\alpha), g_r(\alpha)) \\ &:= U(z + \alpha \Delta z) - U(z) - \alpha \mathcal{D}U(z) \Delta z \end{aligned} \tag{24}$$

for every $\alpha \in [0, \alpha^*]$. *Then*

(i) $\lim_{\alpha \to 0} \|g(\alpha)\|/\alpha = 0$,

(ii) $U(z + \alpha \Delta z) = (1 - \alpha) U(z) + \alpha(\beta \psi(z) \bar{w} + g(\alpha)/\alpha)$

for every $\alpha \in (0, \alpha^*]$,

(iii)

$$\psi(z + \alpha \Delta z) \ \leq \ (1 - \alpha)\psi(z) + \alpha \left\{ \beta \psi(z) + \frac{\|g(\alpha)\|}{\alpha \|\bar{w}\|} \right\}$$

for every $\alpha \in (0, \alpha^*]$.

(iv) *Define*

$$\begin{aligned} \alpha_\psi := \sup\{\alpha' \in [0, \alpha^*) : \ & \|g(\alpha)\|/\alpha < (1 - \beta)\psi(z)\|\bar{w}\|/2 \\ & \textit{for every} \ \ \alpha \in (0, \alpha']\}. \end{aligned}$$

Then

$$0 < \alpha_\psi \leq \alpha^* \leq 1 \tag{25}$$

and

$$\psi(z + \alpha\Delta z) < (1 - \alpha(1-\beta)/2)\psi(z)$$

for every $\alpha \in (0, \alpha_\psi)$.

(v) *Let* $\tau \in (0,1)$ *be a constant and let* z *be a point such that* $U(z) \in \mathrm{int}C_1(\tau)$. *Define*

$$\begin{aligned}\alpha_1 \;:=\; \sup\{\alpha' \in [0,\alpha^*) :\;& (2 + \tau/\|\bar{w}\|)\|g(\alpha)\|/\alpha < \tau\beta\psi(z) \\ & \textit{for every}\;\; \alpha \in (0,\alpha']\}.\end{aligned}$$

Then

$$0 < \alpha_1 \leq \alpha^* \leq 1 \tag{26}$$

and

$$U(z + \alpha\Delta z) \in \mathrm{int}C_1(\tau)$$

for every $\alpha \in (0, \alpha_1)$.

(vi) *Let* $\tau_u, \tau_v, \tau_r \in (0,1)$ *be constants and let* z *be a point such that* $U(z) \in \mathrm{int}C_2(\tau_u, \tau_v, \tau_r)$. *Define*

$$\begin{aligned}\alpha_2 \;:=\; \sup\{\alpha' \in [0,\alpha^*) :\;& \\ & (1+\tau_u)\|g(\alpha)\|/\alpha < \min_{i\in\mathcal{N}}\{\beta\psi(z)(\bar{u}_i - \tau_u(e^{iT}\bar{u}/n)\} \\ & (1+\tau_v)\|g(\alpha)\|/\alpha < \min_{i\in\mathcal{N}}\{-\beta\psi(z)(\bar{v}_i - \tau_v(e^{iT}\bar{v}/n))\} \\ & (1+\tau_r)\|g(\alpha)\|/\alpha < \min_{i\in\mathcal{N}}\{\beta\psi(z)(\bar{r}_i - \tau_r(e^{iT}\bar{r}/n))\} \\ & \textit{for every}\;\; \alpha \in (0,\alpha']\}.\end{aligned}$$

Then

$$0 < \alpha_2 \leq \alpha^* \leq 1 \tag{27}$$

and

$$U(z + \alpha\Delta z) \in \mathrm{int}C_2(\tau_u, \tau_v, \tau_r)$$

for every $\alpha \in (0, \alpha_2)$.

Here the neighborhoods $C_1(\tau)$ *and* $C_2(\tau_u, \tau_v, \tau_r)$ *are given by (20) and (21), respectively.*

Proof: It should be noted that $\alpha^* > 0$. The assertion (i) follows from the continuous differentiability of the mapping U. By the definition (24) of g and the Newton equation (11), we can see that

$$U(z+\alpha\Delta z) = U(z) + \alpha\mathcal{D}U(z)\Delta z + g(\alpha),$$
$$U(z) + \alpha\mathcal{D}U(z)\Delta z = (1-\alpha)U(z) + \alpha\beta\psi(z)\bar{w}$$

for every $\alpha \in [0, \alpha^*]$. Thus, we obtain (ii) and, by the definition (10) of ψ, that

$$\begin{aligned}\psi(z+\alpha\Delta z) &= (1-\alpha)\psi(z) + \alpha\{\beta\psi(z) + \bar{w}^T g(\alpha)/(\alpha\|\bar{w}\|^2)\} \\ &\leq (1-\alpha)\theta\|\bar{w}\|^2 + \alpha\{\beta\theta\|\bar{w}\|^2 + \|g(\alpha)\|/(\alpha\|\bar{w}\})\}\end{aligned}$$

for every $\alpha \in [0, \alpha^*]$.

The inequalities (25), (26) and (27) follows from (i) and the assumption on z, i.e., $U(z) \in \mathrm{int}C_1\tau$ and/or $U(z) \in \mathrm{int}C_2(\tau_u, \tau_v, \tau_r)$.

The definition of α_ψ guarantees that

$$\frac{\|g(\alpha)\|}{\alpha\|\bar{w}\|} < (1-\beta)\psi(z)/2$$

for every $\alpha \in (0, \alpha_\psi)$. Thus, (iv) is obtained by the relation (iii).

Also, it follows from the definition of α_ψ that

$$\begin{aligned}&\left\|(\beta\psi(z)\bar{w} + g(\alpha)/\alpha) - \frac{\bar{w}^T(\beta\psi(z)\bar{w} + g(\alpha)/\alpha)}{\|\bar{w}^2\|}\bar{w}\right\| - \tau\frac{\bar{w}^T(\beta\psi(z)\bar{w} + g(\alpha)/\alpha)}{\|\bar{w}^2\|} \\ &= \left\|g(\alpha)/\alpha - \frac{\bar{w}^T g(\alpha)/\alpha}{\|\bar{w}^2\|}\bar{w}\right\| - \tau\left(\beta\psi(z) + \frac{\bar{w}^T g(\alpha)/\alpha}{\|\bar{w}^2\|}\right) \\ &\leq \|g(\alpha)\|/\alpha + \|g(\alpha)\|/\alpha - \tau\beta\psi(z) + \tau\frac{\|g(\alpha)\|/\alpha}{\|\bar{w}\|} \\ &\leq (2 + \tau/\|\bar{w}\|)\|g(\alpha)\|/\alpha - \tau\beta\psi(z) \\ &< 0\end{aligned}$$

for every $\alpha \in (0, \alpha_1)$. Hence we see that

$$\beta\psi(z)\bar{w} + g(\alpha)/\alpha \in \mathrm{int}C_1(\tau).$$

Since $U(z) \in \mathrm{int}C_1(\tau)$, $1-\alpha \geq 0$ and $\alpha > 0$, (v) follows from the relation (ii).

To see (vi), observe that

$$\begin{aligned}&\beta\psi(z)\bar{u} + g_u(\alpha)/\alpha - \tau_u\frac{e^T\beta\psi(z)\bar{u} + g_u(\alpha)/\alpha}{n}e \\ &= \beta\psi(z)(\bar{u} - \tau_u(e^T\bar{u}/n)e) + \left(g_u(\alpha)/\alpha - \tau_u\frac{e^T g_u(\alpha)/\alpha}{n}e\right)\end{aligned}$$

Since $\bar{w} \in \mathrm{int}C_2(\tau_u, \tau_v, \tau_r)$, we have

$$\beta\psi(z)(\bar{u} - \tau(e^T\bar{u}/n)e) > 0.$$

The definition of α_2 gives us the relation

$$\begin{aligned}
&\beta\psi(z)(\bar{u} - \tau_u(e^T\bar{u}/n)e) + g_u(\alpha)/\alpha - \tau_u\frac{e^T g_u(\alpha)/\alpha}{n}e \\
&\geq \beta\psi(z)(\bar{u} - \tau_u(e^T\bar{u}/n)e) - \|g_u(\alpha)\|/\alpha - \tau_u(\|g_u(\alpha)\|/\alpha)e \\
&\geq (\min_{i\in N}\{\beta\psi(z)(\bar{u}_i - \tau_u(e^T\bar{u}/n))\} - (1+\tau)\|g_u(\alpha)\|/\alpha)e \\
&> 0
\end{aligned}$$

for every $\alpha \in (0, \alpha_2)$. In a similar way, we obtain that

$$\beta\psi(z)\bar{v} + g_v(\alpha)/\alpha - \tau_v\frac{e^T\beta\psi(z)\bar{v} + g_v(\alpha)/\alpha}{n}e < 0,$$

$$\beta\psi(z)\bar{r} + g_r(\alpha)/\alpha - \tau_r\frac{e^T\beta\psi(z)\bar{r} + g_r(\alpha)/\alpha}{n}e < 0$$

for every $\alpha \in (0, \alpha_2)$. Thus (vi) follows from the definitions of α_2 and the neighborhood $C_2(\tau_u, \tau_v, \tau_r)$. ∎

The next theorem is the desired result of this section, which can be derived from Lemmas 6.1 and 6.2.

Theorem 6.3. *Suppose that Condition 2.2 holds. Let $\bar{w} = (\bar{u}, \bar{v}, \bar{r}) \in \mathbb{R}^n_{++} \times \mathbb{R}^n_{--} \times \mathbb{R}^n_{++}$ and let $\beta \in (0,1)$ be a constant. Choose the parameters τ, τ_u, τ_v, and τ_r as in Lemma 6.1. Then both of the neighborhoods $C_1(\tau)$ and $C_2(\tau_u, \tau_v, \tau_r)$ satisfy Condition 4.4.*

The neighborhood $C_1(\tau)$ satisfies (iv) of the condition with $\tilde{\alpha} = \min\{\alpha_\psi, \alpha_1\}$ and with $\rho(\alpha) = 1 - \alpha(1-\beta)/2$, and the neighborhood $C_2(\tau_u, \tau_v, \tau_r)$ does with $\tilde{\alpha} = \min\{\alpha_\psi, \alpha_2\}$ and with $\rho(\alpha) = 1 - \alpha(1-\beta)/2$, respectively.

In the above discussion, we only impose a mild condition, Condition 2.2, on the problem. Suppose that another relatively strict condition, Condition 1.3, holds. In view of Theorem 3.2, the image $U(\mathbb{R}^n_{++} \times \mathbb{R}^{2n})$ of U is a open convex subset of $\mathbb{R}^{3n}$. Hence, for arbitrary point $\bar{w} \in U(\mathbb{R}^n_{++} \times \mathbb{R}^{2n})$, the line segment $\bar{W} := \{\theta\bar{w} \in \mathbb{R}^n_{++} \times \mathbb{R}^{2n} : \ \theta \in (0,1]\}$ satisfies $\bar{W} \subset U(\mathbb{R}^n_{++} \times \mathbb{R}^{2n})$ and there exists an open ball

$$B(\bar{w}, \epsilon) := \{w \in \mathbb{R}^n_{++} \times \mathbb{R}^{2n} : \ \|w - \bar{w}\| < \epsilon\}$$

contained in the set $U(\mathbb{R}^n_{++} \times \mathbb{R}^{2n})$. Therefore, if we choose a sufficiently small $\tau \in (0, \epsilon)$ then the cone $C_1(\tau)$ given by (20) satisfies $\bar{W} \subset \mathrm{int}C_1(\tau)$ and $\omega := C_1(\tau) \cap H_w \subset U(\mathbb{R}^n_{++} \times \mathbb{R}^{2n})$. In fact, by a similar discussion, we can see that the cone $C_1(\tau)$ satisfies Condition 4.4. Thus, we can start the algorithm from any point $\bar{z} = (\bar{u}, \bar{x}, \bar{y}) \in \mathbb{R}^n_{++} \times \mathbb{R}^{2n}$ by setting $\bar{w} = U(\bar{z})$, and its global convergence is theoretically guaranteed. In general, it may be difficult to know such an open ball $B(\bar{w}, \epsilon)$. However, since the Newton direction can be computed at every $z = (u, x, y) \in \mathbb{R}^n_{++} \times \mathbb{R}^{2n}$, if we obtain a point $\bar{z}$ for which $\bar{w} = U(\bar{z})$ is sufficiently close to the origin, then it might be possible to obtain a solution of the CP by one step Newton iteration, which implies that the method has a potential for post-optimal analysis and/or solving parametric problems.

7 Concluding remarks

We have proposed a continuation method for the CP using Chen-Harker-Kanzow smooth function. The method does not confine the generated sequence in the positive orthant and, for the monotone CP, it allows us to start arbitrary point $(x, y) \in \mathbb{R}^{2n}$ theoretically. We have also shown a sufficient condition for the neighborhood to achieve the global convergence and two examples satisfying it.

Another approach to construct a non-interior homotopy continuation method would be to analyze the following mappings:

$$\begin{aligned}
\tilde{v}(v, x, y) &= v, \\
\tilde{u}(v, x, y) &= (\tilde{u}_1(v, x, y), \tilde{u}_2(v, x, y), \ldots, \tilde{u}_n(v, x, y))^T, \\
\tilde{u}_i(v, x, y) &= (x_i - v_i)(y_i - v_i) \quad (i \in N), \\
\tilde{r}(v, x, y) &= y - f(x), \\
\tilde{U}(v, x, y) &= (\tilde{v}(v, x, y), \tilde{u}(v, x, y), \ldots, \tilde{r}(v, x, y))
\end{aligned}$$

A similar approach has been studied by R. M. Freund in [7] for the linear programming. It should be noted that the mapping $\tilde{U}$ is not necessarily one-to-one mapping on $\mathbb{R}^{3n}$ even if Condition 1.3 holds (the monotone CP with a feasible-interior-point). For example, let

$$n = 1, \quad f(x) = x + 1, \quad v = -1, \quad u = 2, \quad r = 0.$$

By solving the system

$$\tilde{u}(v, x, y) = (x + 1)(y + 1) = 2, \quad \tilde{r}(v, x, y) = y - x - 1 = 0,$$

we obtain the two solutions, $(0, 1)$ and $(-3, -2)$. In this case, we may have to choose the domain of $\tilde{U}$ as follows:

$$\{(v, x, y) \in \mathbb{R}^{3n} : \ (x - v, y - v) \geq 0\}$$

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