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Securitization of Assets into Multiple Securities: An Option-Theoretic Valuation

by

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SECURITIZATION OF ASSETS INTO MULTIPLE SECURITIES: AN OPTION-THEORETIC VALUATION

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SECURITIZATION OF ASSETS
INTO MULTIPLE SECURITIES:
AN OPTION-THEORETIC VALUATION

Abstract

This paper presents a valuation framework for multiple classes of securities which are collateralized by a pool of assets with embedded options. We call these securities *multi-class asset backed securities* (MCABS's). MCABS's require special attention, because cash flows to MCABS's depend on both current and past realizations of a state variable (or state variables). As a result, MCABS's cannot be valued by standard numerical methods for American options as in Brennan and Schwartz (1977) and Cox, Ross and Rubinstein (1979). We value MCABS's by a state augmentation method, and examine the characteristics of MCABS's by simulation.

SECURITIZATION OF ASSETS INTO MULTIPLE SECURITIES:
AN OPTION-THEORETIC VALUATION

1. Introduction

Securitization of assets is a pooling and repackaging of assets into new securities. It began with residential mortgage loans in 1970, and spread to other types of assets, including automobile loans, credit card receivables and commercial bank loans. Initially, as in GNMA pass-through securities, a single class of securities was issued against an underlying pool of assets where cash flows from the underlying assets were "passed through" to the holders of the securities on a *pro rata* basis. Since the introduction of collateralized mortgage obligations in 1983, there have been a major innovation in securitization where multiple classes of securities are issued against a pool of assets, and cash flows from the underlying assets are allocated differently among the securities. The purpose of this paper is to present a simple framework for valuing multiple classes of securities which are collateralized by a pool of assets with embedded options of American type.[1] We call these securities *multi-class asset backed securities* (MCABS's).

There are many examples of MCABS's. A collateralized mortgage obligation (CMO), for example, is a set of MCABS's where the pool consists of residential mortgage loans. Non-mortgage loans, such as automobile loans, credit card receivables, leases and commercial bank loans, have been securitized under the name of asset backed securities (ABS's), some of which have multi-class structures.[2] In addition, some pass-through securities are structured as MCABS's. In senior/subordinated structure, for example, cash flows from the pool are divided differently between two classes of securities, senior and subordinated pass-through securities (SSPT's), so that the latter gives credit support to the former. It should be pointed out that MCABS's are found not only in great variety but also in great quantity. According to *The Mortgage Market Statistical Annual for 1996*, for instance, \$1,368 billion of CMO's were issued from its introduction in 1983 till the end of 1995, and \$277 billion of SSPT's were issued from 1987 till the end of 1995.

The assets that have been securitized into MCABS's thus far are loans, leases and bonds with embedded options of American type. A mortgage loan, for example, has a default option as well as a prepayment option, which gives the mortgagor the right to repay a part or all of the loan any time before the stated maturity.[3] Notice that MCABS's are backed by many of these assets, and that if exercises of embedded options occur at different values of state variables, MCABS's will become path dependent.[4] In reality, this phenomenon of heterogeneous option exercises has been observed on MCABS's to a great extent. Therefore, we address the path dependence of MCABS's as a central issue of this paper.

It should be pointed out that standard numerical methods for American options, as in Brennan and Schwartz (1977) and Cox, Ross and Rubinstein (1979), are not adequate to account for the path dependence of MCABS's, because their nodes do not provide information on past realizations of a state variable (or state variables). Therefore, earlier studies of MCABS's, such as Schwartz and Torous (1989b) and McConnell and Singh (1993), where path dependence occurs from prepayment options of mortgage loans in collaterals, have taken the following approach to account for the path dependence of MCABS's.[5] Specifically, they empirically estimated a prepayment function, which determines the prepayment rate of the entire collateral conditional on the prevailing interest rate and other relevant variables. They also estimated the stochastic process of interest rates empirically. Then they valued MCABS's by a Monte Carlo method. That is, they first generated sample paths for interest rates according to the risk-adjusted version of the estimated interest rates process. Next, using the prepayment function, they determined the cash flow stream of a MCABS along each path, and computed its present value. Finally, they computed the average of the present value over the generated sample paths, and took the average as an approximation to the model price of the MCABS.

In contrast, we take a different approach to valuing MCABS's. First, following Davidson, Herskovitz and Van Drunen (1988) (DHV) and Johnston and Van Drunen (1988) (JV), we model a collateral as a portfolio of assets with embedded options, where the options are assumed to differ in some respects that affect the boundary conditions of option exercises.[6] Therefore, the overall exercises of the embedded options in the collateral are governed by the distribution of the types of assets, where the types are determined by the characteristics of the assets that affect exercises of the embedded options. According to the study of mortgage-backed securities by Stanton (1995), this way of modeling collaterals "captures many of the empirical features of mortgage prepayments."

Notice, however, that applying this model of collateral to MCABS's accompanies the difficulty that DHV and JV did not face. DHV and JV valued asset backed securities with single class structures, where the security backed by a collateral has claim to the entire collateral. Therefore, the security could be valued by summing the values of underlying assets, each of which could be, in turn, valued by the standard numerical methods for American options. By contrast, a MCABS does not have claim to a fixed portion of the collateral. Rather, the portion of the collateral that it has claim to is path dependent, and as a result, cannot be identified at the time of valuation. We tackle this path dependence problem by a form of state augmentation method presented by Kishimoto (1989) instead of a Monte Carlo method.[7] Specifically, assuming that an underlying state variable is a Markov chain, we find auxiliary state variables that summarize the path of the state variable for a MCABS in question. We then value the MCABS by a risk-neutral valuation relationship with respect to the underlying and auxiliary state variables. The advantage of the state augmentation method over Monte Carlo methods is that the state augmentation method yields a model price free of a statistical error which is inherent in Monte Carlo methods.

The remainder of this paper is organized as follows. First, in the next section, we describe CMO's to give an introduction to the institutional aspects of MCABS's. Section 3 presents a general model and a valuation procedure for MCABS's. In Section 4, we apply the general model to hypothetical MCABS's where loans in the collateral differ only in contract interest rates. Although such MCABS's are not found in reality, they provide a simple and clean example of MCABS's to help us understand how the proposed method works on MCABS's. Section 5 examines the characteristics of the MCABS model of Section 4 by simulation. Section 6 concludes the paper with a summary.

2. A Description of MCABS's

All types of MCABS's, including CMO's, ABS's with multi-class structures and SSPT's, share basic common features. Furthermore, CMO's are the largest of all types of MCABS's in terms of principals outstanding. Therefore, we concentrate on CMO's to give an introduction to the institutional aspects of MCABS's.

First, let us describe collaterals of CMO's. A CMO collateral consists of many mortgage pass-through securities, each of which, in turn, represents a *pro rata* share of a pool of residential mortgage loans. The number of mortgage loans in a CMO collateral is usually in the order of thousands. A mortgage loan requires the mortgagor to make monthly payments until the loan is fully repaid. Each monthly payment consists of interest and principal portions, where the interest portion usually decreases with an increase in the principal portion over the life of the loan. In addition, a mortgage loan allows the debtor to pay more than the loan requires before the stated maturity. This payment of principal in excess of the required amount is called a prepayment, and occurs for several reasons. First, if market interest rates are sufficiently below the contract interest rates of outstanding loans, mortgagors will prepay to refinance at lower interest rates. Secondly, if mortgagors sell real estates, and if buyers either cannot or do not assume the loans, the loans must be repaid in full. Thirdly, if mortgagors default, the mortgages will be foreclosed. Fourthly, if real estates are destroyed by natural disasters, insurance proceeds are used to pay off the loans. In any event, a prepayment alters the cash flow pattern of the loan substantially. Hence, prepayments are of critical importance in valuing securities backed by mortgage loans.

Next, we describe CMO's briefly. A CMO is a set of several classes of bonds collateralized by a pool of mortgage pass-through securities, where the classes of bonds differ from one another in principal amount, stated maturity and interest and principal payments. Morgan Stanley Mortgage Trust R, for instance, consists of three classes of bonds, R-1, R-2 and R-3, each having different terms as in Table 1. In general, principals and coupon rates of bonds are set so that cash flows from the collateral are sufficient to meet cash flow requirements of the bonds even under adverse conditions. Therefore, in most cases, there is a positive difference between the total cash inflow from the collateral and the total cash outflow to the bonds. The claim to this difference is called a residual and is either held by the CMO issuer or sold to a third party.

CMO's vary greatly in the number of classes, payment rules and other aspects. For instance, some CMO's have four or five classes of bonds at issuance, while others have more than ten classes. Many CMO's make payments either quarterly or semiannually, while a few others make payments monthly. Some CMO's consist of only fixed-rate bonds, while others consist of both fixed-rate and floating-rate bonds. Despite this wide variation, all CMO's share basic common features. First, every CMO has more than one class of bond. Secondly, each class of bond, except for IO's and PO's, receives cash flows of two types: interest and principal payments. The amount of an interest payment to a class of bond is equal to the interest rate times the principal balance of the bond. The amount of a principal payment to the bond is determined based on the rules set forth in the CMO prospectus. These rules vary greatly across CMO's. In a *sequential-pay* CMO, for instance, principal payments made by mortgagors in the collateral are allocated to each class of bonds in the order of stated maturities of the bonds, so that no payment of principal will be made on any bond until all bonds with earlier stated maturities have been paid in full.[8] In Morgan Stanley Mortgage Trust R, for instance, class R-1 first receives all principal payments, which serve to reduce the principal balance of class R-1. When class R-1 is fully retired, class R-2 starts receiving principal payments, which reduce its principal balance. When class R-2 is fully retired, class R-3 starts receiving all principal payments. After all classes of bonds are fully retired, the residual receives all principal and interest payments.

Figure 1 shows the cash flow to each class of Morgan Stanley Mortgage Trust R under two scenarios of prepayments. Under the first scenario of "50% PSA," prepayments occur at half the speed of the Public Securities Association(PSA) prepayment model. Under the second scenario of "200% PSA," prepayments occur at twice the speed of the PSA prepayment model. The cash flow pattern and the life of each class change drastically depending on the speed of prepayments. Therefore, we can safely conclude that prepayments are critical in valuing CMO's as well as other types of MCABS's backed by mortgage loans.[9]

3. General Model for MCABS's

A. Model for Uncertainty

MCABS's are collateralized by loans, leases or bonds, all of which have interest rates as the major source of uncertainty. Therefore, we present an interest rate model by Ho and Lee (1986) (HL) as our model of uncertainty.[10] The basic assumptions of the model are as follows.

(A1) The market opens at $N+1$ discrete points in time, which are indexed by $n = 0, 1, \cdots, N$, and separated by intervals of equal length.

The discrete point indexed by n is called *time* n and the interval from time $n-1$ to time n is referred to as *period* n.

(A2) The market is frictionless unless otherwise stated.

(A3) At time $n = 1, \cdots, N$, one of two possible events, an up event or a down event, occurs. The current event is independent of past events.

Thus, the resolution of uncertainty through time n is represented by a sequence of n up and down events, which is referred to as a *path* through time n. Let a *statistic* be a mapping which assigns any object whatsoever to each path. For example, the number of up events which occur from time 1 to time n, denoted by $i_n = 0, \cdots, n$, is a statistic. In this paper, a security is said to be *path independent* if its cash flow at time n is a function of at most i_n, and *path dependent* if the determination of its cash flow at time n requires statistics other than i_n.

Next, we define the pure discount bond with T periods to maturity as a security that pays \$1 in T periods with no cash flow at other time.

(A4) At time 0, there exists a pure discount bond with T periods to maturity for every $T = 1, 2, \cdots, N$. At time n, the prices of pure discount bonds are functions of only $i_n = 0, 1, \cdots, n$.

We denote the price of a T-period pure discount bond when $i_n = i$ by $P_i^{(n)}(T)$. Given n and i, $P_i^{(n)}(T)$ represents the price of a pure discount bond as a function of time to maturity, and is referred to as the *discount function* at $i_n = i$. Under assumptions (A1)-(A4) and an additional assumption on the way the discount function shifts, HL derived a unique stochastic process for the discount function.

B. General Model for Collaterals

The MCABS's in existence currently are collateralized by loans, leases or bonds, all of which are termed loans hereafter. The loans in a collateral differ from one another in some respects.[11] They are classified into $\bar{k}$ groups such that loans in the same group have the identical cash flow, per \$1 principal, in every state of the world. Therefore, loans in the same group can be viewed as a single entity, and are indexed by $k = 1, \cdots, \bar{k}$. We assume that each type of loans can be valued by standard numerical methods for American options. D_k denotes the total principal at time 0 of type k loans. Then, $\{D_k | k = 1, \cdots, \bar{k}\}$ represents the time-0 distribution of loans in the collateral by type.

C. General Model for Multi-Class Asset Backed Securities

Multiple classes of securities are issued against a collateral, where cash flows from underlying assets in the collateral are allocated among the securities by prespecified rules. These rules do not allocate cash flows on a *pro rata* basis, and as a result, the securities issued exhibit complex path dependence. We deal with this path dependence by auxiliary state variables, which are referred to as statistics in this paper.

As defined earlier, a statistic associates an object with each path. These objects are called *values* of the statistic and may be numerical values, events or actions. Call the set of all values the statistic may take on at time n the *state space* at time n of the statistic. For example, the state space at time n of i_n is $\{0, 1, \cdots, n\}$. We assume that i_n and another statistic are sufficient to determine the cash flow to a MCABS in question, and that the state space of the additional statistic is finite for all $n = 1, \cdots, N$.[12] Then, the state space at time n of the additional statistic can be represented by a set of nonnegative integers $\{0, 1, \cdots, \bar{h}_n\}$, and the value of the statistic by $h_n \in \{0, 1, \cdots, \bar{h}_n\}$.[13] Given these notations, we make an additional assumption.

(A5) For a MCABS with finite life through time N, there exist statistics (i_n, h_n) such that:

(i) The state space of h_n is finite for all $n = 1, \cdots, N$.

(ii) The cash flow to the MCABS is uniquely determined for all values of (i_n, h_n), $n = 1, \cdots, N$.

(iii) Given (i_n, h_n), h_{n+1} is a function of at most i_{n+1}.

(i) is necessary for the proposed valuation procedure to work. (ii) requires that (i_n, h_n) is sufficient to determine the cash flow to the MCABS. (iii) requires that the value of h_{n+1} is uniquely determined for each of up and down events at time $n + 1$.[14]

D. Valuation Procedure

We now turn to the valuation of MCABS's. Assumption (iii) of (A5) requires that given $(i_n, h_n) = (i, h)$, the value of h_{n+1} is uniquely determined for each of $i_{n+1} = i + 1$ and $i_{n+1} = i$. Let $h^{(u)}$ and $h^{(d)}$ denote the value of h_{n+1} when $i_{n+1} = i+1$ and $i_{n+1} = i$, respectively. In addition, let $X_i^{(n)}(h)$ denote the cash flow to a MCABS in question when $(i_n, h_n) = (i, h)$ and $C_i^{(n)}(h)$ the price of the MCABS when $(i_n, h_n) = (i, h)$. Given these notations, the usual no-arbitrage argument yields the following risk-neutral valuation relationship (RNVR).

$$C_i^{(n)}(h) = P_i^{(n)}(1)[\pi C_{i+1}^{(n+1)}(h^{(u)}) + (1 - \pi)C_i^{(n+1)}(h^{(d)})] + X_i^{(n)}(h), \tag{1}$$

where π is the risk-neutral probability of an up event of HL. Equation (1) implies that the current price of the MCABS is equal to the discounted value of the expectation of its next-period price plus the current cash flow, where the expectation is computed with respect to the risk-neutral probabilities.

Notice that if the security were path independent, i_n with state space $\{0, \cdots, n\}$ would trivially satisfy Assumption (A5), and Equation (1) would reduce to the following RNVR for path independent securities.

$$C_i^{(n)} = P_i^{(n)}(1)[\pi C_{i+1}^{(n+1)} + (1-\pi)C_i^{(n+1)}] + X_i^{(n)}. \tag{2}$$

Clearly, the only difference of Equation (1) from Equation (2) is that the prices are conditioned on the additional statistic h_n. This is what we would expect because Assumption (A5) requires the additional statistic to absorb the path dependence of the MCABS completely.

Given Equation (1), we value the MCABS by applying the one-period RNVR recursively with the boundary conditions of the MCABS. This valuation procedure for MCABS's differs from the one for path independent securities in that nodes are set up not only in time and i_n, but also in the additional statistic h_n. Therefore, it is natural to imagine the nodes in three dimension. Following Kishimoto (1989), we call the collection of the nodes arranged in three dimension the *extended tree* (ET), and the pricing procedure outlined here the *ET method.*

E. Computational Complexity of the ET Method

This section considers the computational complexity of the ET method when it is applied to MCABS's. Formally, let a *node* of the ET be each set of values (n, i_n, h_n) may take on. Also, let $\bar{h}$ denote the maximum of the number of values the statistic h_n may take on at a time over the entire period.

$$\bar{h} = \max_{1 \leq n \leq N} \bar{h}_n. \tag{3}$$

Then, it is easy to show that the total number of nodes of the ET is bounded above by

$$\frac{1}{2}(N+1)(N+2)\bar{h}. \tag{4}$$

Now we apply Equation (4) to an example of MCABS's. The example we consider is a CMO backed by 30-year mortgage loans. If a month is chosen as a period ($N = 360$), and if $\bar{h} = 87$, then the total number of nodes of the ET is *at most* $(\frac{1}{2})(361)(362)(87)$ nodes, which is roughly 5.7×10^6.[15] Thus, even when N is large, the ET method yields model prices with manageable amount of computation.[16]

4. MCABS Model Where Loans Differ Only in Contract Interest Rate

This section presents a simple MCABS model where loans in the collateral differ only in contract interest rate.

A. Model for Collaterals

The collateral consists of many default-free loans, each of which gives the debtor the right to prepay any time before the stated maturity. All loans in the collateral are identical except for contract interest rates. The notations of the loans are as follows.

$N =$ number of periods from time 0 to the stated maturity.

$c_k =$ contract interest rate of loans of type k where $k = 1, \cdots, \bar{k}$ and $c_1 > \cdots > c_{\bar{k}}$.

$F_k^{(n)} =$ principal balance in the n^{th} period of the type k loan, per \$1 initial principal, given that the loan has not been prepaid yet, where $1 = F_k^{(1)} \geq \cdots \geq F_k^{(N)} > 0$.

$D_k =$ total principal balance at time 0 of the type k loans.

Clearly, the initial principal balance B of the collateral is given by

$$B = \sum_{k=1}^{\bar{k}} D_k. \tag{5}$$

The heterogeneity of loans in contract interest rate generates the heterogeneity in prepayment for the following reason. If the market interest rate falls below the contract interest rate, the debtor will prepay to make an arbitrage profit. Notice that the lower the contract interest rate of the loan is, the greater a drop in the market interest rate has to be to induce the prepayment of the loan. Therefore, loans with higher contract interest rates will be prepaid sooner than loans with lower contract interest rates.

It is needless to say that each loan can be valued by a standard numerical method for American options. Specifically, in the framework of this paper, each loan can be valued by using Equation (2) recursively with the boundary condition $V_{i,k}^{(n)} \geq (1 + c_k)F_k^{(n)}$ where $V_{i,k}^{(n)}$ denotes the price of the type k loan, per \$1 initial principal, when $i_n = i$.

B. Model for Multi-Class Asset Backed Securities

Despite the wide variation in the rule for allocating cash flows among individual MCABS's, most rules share the common feature that the allocation is based on the principal balance of the collateral. Therefore, we specialize the general model of Section 3 to the one where the allocation is based on only the principal balance of the collateral. Furthermore, for the convenience of exposition, we consider MCABS's with a very simple structure. Specifically, the MCABS's consist of two classes of securities, Bond A and the residual.[17] B_A is the principal balance at time 0 of Bond A. At the end of every period, if Bond A is not fully retired, it receives all principal payments made by debtors in the collateral, which serve to reduce the principal balance of Bond A.[18] In addition, Bond A receives an interest equal to its interest rate c_A times its principal balance at the time of receipt. The residual receives the difference between the interest payments made by the debtors in the collateral and the interest payment to Bond A. In addition, if Bond A is fully retired, the residual receives all principal payments made by the debtors in the collateral.

Next, we define a statistic to deal with the path dependence of MCABS's. The statistic is the number of types of loans that have been prepaid prior to time n, and is denoted by $h_n = 0, 1, \cdots, \bar{k}$. Notice that if it is optimal to prepay a loan of type h, it must be optimal to prepay loans that have contract interest rates higher than c_h. Thus, if $h_n = h$, loans of type 1 to type h must have been prepaid previously, while loans of type $h+1$ to type $\bar{k}$ must be outstanding immediately before time n. Hence, the amount $Q_n(h)$ of cumulative principal payments made before time n is given by the difference between the initial collateral principal B and the sum of the principal balances of type $h+1$ to type $\bar{k}$ loans immediately before time n.

$$Q_n(h) = B - \sum_{k=h+1}^{\bar{k}} F_k^{(n)} D_k. \tag{6}$$

Next, we will describe how the statistic h_n changes its value over time. Let $\eta_i^{(n)}(h)$ denote the number of types of loans which are prepaid through time n given $(i_n, h_n) = (i, h)$. Also, let $\mathrm{I}[\cdot]$ denote the indicator function which takes on one if the condition in the bracket is satisfied and zero if it is not. Notice that $\eta_i^{(n)}(h)$ is given by adding to h the number of types of loans that are prepaid at time n. Notice also that loans of type k will be prepaid at time n if the boundary condition $V_{i,k}^{(n)} \geq (1+c_k)F_k^{(n)}$ is met.

Therefore, we have

$$\eta_i^{(n)}(h) = h + \sum_{k=h+1}^{\bar{k}} \mathrm{I}[V_{i,k}^{(n)} \geq (1+c_k)F_k^{(n)}]. \tag{7}$$

Given the value of $\eta_i^{(n)}(h)$, we can compute the amount $\bar{Q}_i^{(n)}(h)$ of cumulative principal payments made before and at time n as the difference between the initial collateral principal B and the sum of principal balances immediately after time n of type $\eta+1$ to type $\bar{k}$ loans where η is the value of $\eta_i^{(n)}(h)$.

$$\bar{Q}_i^{(n)}(h) = B - \sum_{k=\eta+1}^{\bar{k}} F_k^{(n+1)} D_k. \tag{8}$$

Now we will show that statistics (i_n, h_n) with state spaces $\{0, \cdots, n\} \times \{0, \cdots, \bar{k}\}$ satisfy Assumptions (ii) and (iii) of (A5) with respect to Bond A. First, from Equations (6)-(8), the values of $Q_n(h)$ and $\bar{Q}_i^{(n)}(h)$ are uniquely determined by $(i_n, h_n) = (i, h)$. If (i, h) is such that the cumulative principal payments made prior to time n exceed the initial principal of Bond A ($Q_n(h) > B_A$), Bond A must have been fully retired before time n. Hence, the current cash flow $X_i^{(n)}(h)$ to Bond A is trivially \$0.

$$X_i^{(n)}(h) = 0. \tag{9}$$

If the cumulative principal payments do not exceed B_A before time n but exceed it immediately after time n ($Q_n(h) \leq B_A < \bar{Q}_i^{(n)}(h)$), Bond A is fully retired at time n, and its current cash flow is equal to its remaining principal plus the interest on it.

$$X_i^{(n)}(h) = (1+c_A)(B_A - Q_n(h)). \tag{10}$$

Finally, if Bond A is outstanding immediately after time n ($B_A > \bar{Q}_i^{(n)}(h)$), its cash flow is equal to the interest on the Bond A principal remaining in the n^{th} period plus all current principal payments made by the debtors in the collateral.

$$X_i^{(n)}(h) = c_A(B_A - Q_n(h)) + (\bar{Q}_i^{(n)}(h) - Q_n(h)). \tag{11}$$

In summary, Equations (9)-(11) uniquely assign a proper cash flow to Bond A conditional on the statistics (i_n, h_n). Thus, Bond A satisfies Assumption (ii) of (A5). In addition, Equation (7) implies that the value of h_{n+1} is uniquely determined by (i_n, h_n).[19] Thus, Bond A satisfies Assumption (iii) of (A5).

C. Valuation of Multi-Class Asset Backed Securities

Next, we turn to the valuation of Bond A by the ET method. Let $T_i^{(n)}(h)$ denote the price at time n of Bond A conditional on $(i_n, h_n) = (i, h)$. Substituting Equations (9)-(11) into Equation (1), we have the Bond A price.

$$T_i^{(n)}(h) = \begin{cases} 0, & \text{if } Q_n(h) > B_A; \\ (1 + c_A)\mathcal{A}_n(h), & \text{if } Q_n(h) \leq B_A \leq \bar{Q}_i^{(n)}(h); \\ c_A \mathcal{A}_n(h) + \mathcal{B}_i^{(n)}(h) + T_i^{(n)}(h)^*, & \text{if } B_A > \bar{Q}_i^{(n)}(h). \end{cases} \tag{12}$$

where

$$\mathcal{A}_n(h) = B_A - Q_n(h), \tag{13a}$$

$$\mathcal{B}_i^{(n)}(h) = \bar{Q}_i^{(n)}(h) - Q_n(h), \tag{13b}$$

$$T_i^{(n)}(h)^* = P_i^{(n)}(1)(\pi T_{i+1}^{(n+1)}(\eta) + (1 - \pi) T_i^{(n+1)}(\eta)), \tag{13c}$$

$$\eta = \eta_i^{(n)}(h). \tag{13d}$$

In other words, if $Q_n(h) > B_A$, Bond A is worthless. If $Q_n(h) \leq B_A \leq \bar{Q}_i^{(n)}(h)$, Bond A is fully retired at time n, and its price is equal to the cash flow at time n. If $B_A > \bar{Q}_i^{(n)}(h)$, Bond A is outstanding after time n, and the time-n price of Bond A is equal to the current cash flow plus the present value of its price at time $n+1$. In short, Equation (12) provides the one-period RNVR for Bond A and the recursive valuation procedure outlined in Section 3.D. can be used to obtain the initial price $T_0^{(0)}(0)$ of Bond A.

To complete the valuation of the MCABS's in question, we consider the valuation of the residual.[20] First, notice that the initial price $V_{0,k}^{(0)}$ of the type k loan per \$1 initial principal can be obtained by standard numerical methods for American options. Thus, the initial price V_c of the collateral is given by summing the initial prices of all loans in the collateral.

$$V_c = \sum_{k=1}^{\bar{k}} V_{0,k}^{(0)} D_k. \tag{14}$$

Next, considering that cash flows from the collateral are divided between Bond A and the residual, the initial price V_r of the residual must be equal to the initial price V_c of the collateral minus the initial price of Bond A.

$$V_r = V_c - T_0^{(0)}(0). \tag{15}$$

5. Effects of Path Dependence

This section examines the MCABS model of Section 4 by simulation. Define an *elementary tranche* (ET) j to be a portfolio of a long position in Bond A of Section 4 with principal equal to j tenths of the principal of the collateral and a short position in Bond A of Section 4 with principal equal to $j-1$ tenths of the principal of the collateral.[21] It follows that the initial price T_j of the j^{th} ET must be equal to the difference between the initial prices of the two hypothetical bonds.

$$T_j = T_0^0(0)|_{B_A=(j/10)B} \;-\; T_0^0(0)|_{B_A=(j-1/10)B}, \qquad j = 1, 2, \cdots, 10, \tag{16}$$

where B and B_A are the initial principals of the collateral and Bond A, respectively. If the ET's were defined on a sufficiently fine partition of the collateral, the price of any MCABS with a sequential-pay structure is obtained by summing the prices of corresponding ET's. For example, assume that Bonds A, B, C are issued against a collateral with a sequential-pay structure where their initial principals are equal to four, two and three tenths of the collateral, respectively. Then, the price of Bond B must be equal to the sum of the prices of ET's 5 and 6. Therefore, the two-class MCABS model of Section 4 suffices to value sequential-pay MCABS's with more than two classes. For this reason, the ET's provide a convenient way to analyze various classes of MCABS's. Therefore, we focus our analysis on the ET's, hereafter.

The prices of the ET's are simulated for the following specification of the MCABS model of Section 4. The length of a period is a year and $N = 10$. The initial yield curve is flat and shifted in parallel from 10.4% to 13% with 0.2% increments. The values of π and δ of HL are 0.8 and 0.98, respectively. These values would generate 15% one-year yield volatility, if the true probability of an upstate were 0.5. There are ten types of loans that are amortized at flat rates over a ten year period with $c_1 = 11\%$, $c_2 = 10.9\%$, $\cdots$, $c_9 = 10.2\%$, $c_{10} = 10.0\%$ and $D_k = \$100$ for all $k = 1, 2, \cdots, 10$. The contract interest rate is 10% for all ET's.

Figure 2 plots the prices of the ET's per \$100 principal on the vertical axis and the initial market interest rate and j on the horizontal axes. The figure shows that the price falls with an increase in interest rate. This phenomenon is easy to see if we notice that an ET holder receives a \$10 coupon every year until she starts receiving principal payments, the timing of which depends on prepayment decisions of debtors in the collateral. Roughly, an ET is a coupon bond with stochastic maturity.[22] Thus, if the market interest rate rises, prepayments of the underlying loans become less likely, and the effective maturities of the ET's become longer. Therefore, the ET's become less valuable due to both longer effective maturities and a higher discount rate.[23] Notice also that this effect is stronger for ET's with greater j's. Thus, the greater the j of the ET, the higher the interest rate sensitivity.

Next let $T_j(r)$ be the price of the j^{th} ET when the market interest rate is r, and Δ the size of an increment in yield curve shift. Also, define the duration $D_j(r)$ of the j^{th} ET when the interest rate is r by

$$D_j(r) = -\frac{1}{T_j(r)} \frac{T_j(r+\Delta) - T_j(r-\Delta)}{2\Delta}. \tag{17}$$

In our simulation, Δ is 0.2%, and the initial yield curve is flat. Thus, $D_j(r)$ yields an approximation to Macaulay's duration measure. Figure 3 plots the duration $T_j(r)$ for all $j = 1, 2, \cdots, 10$ over interest rates 10.4%-12.8%. Greater values of j yield longer durations, which was observed in Figure 2. Also, an interest rate rise tends to lengthen the duration. This is because an interest rate rise delays the redemption of the ET's, and because a longer-term bond has a longer Macaulay's duration than a shorter-term bond.

Next, we define the option value $\Phi_j(r)$ of the j^{th} ET at interest rate r by

$$\Phi_j(r) = T_j(r)|_{\sigma=0\%} \; - \; T_j(r)|_{\sigma=15\%}. \tag{18}$$

$\Phi_j(r)$ represents the difference between the price of the j^{th} ET when the one-year yield volatility σ is zero and that when σ is 15%. Figure 4 shows that an increase in interest rate tends to lower the option values of the ET's. This phenomenon reflects the fact that the higher the interest rate is, the further out of the money the prepayment options of the underlying loans are. In addition, notice that when the interest rate is below 11%, a low- or medium-numbered ET has the highest option value, and when the interest rate is 11% or above, the greater the j of the ET is, the higher the option value is. This pattern can be understood in the following way. When the interest rate is below 11%, there is a high probability that some of the loans are prepaid soon. Thus, low-numbered ET's may be retired very soon and medium-numbered ET's may be retired in the near future. Hence, they have high option values. When the interest rate is 11% or above, there is a low probability that any loans are prepaid soon, but there is still a nontrivial probability that some of them are prepaid in the distant future. Thus, it is not likely that low-numbered ET's are prepaid, but it is still likely that high-numbered ET's are prepaid. Therefore, low-numbered ET's have little option values, but high-numbered ET's have relatively high option values.

6. Summary and Conclusions

The purpose of this paper is to present a framework for valuing multiple classes of securities which are collateralized by a pool of assets with embedded options of American type. We call these securities *multi-class asset backed securities* (MCABS's). MCABS's require special attention because cash flows to MCABS's depend on past realizations of a state variable (or state variables), and as a result, MCABS's cannot be valued by standard numerical methods for American options as in Brennan and Schwartz (1977) and Cox, Ross and Rubinstein (1979).

The proposed framework for valuing MCABS's has two major features. The first feature is that the collateral is modeled as a pool of assets that differ in some respects that affect the boundary conditions of option exercises. In the model of Section 4, for example, the underlying assets differ in contract interest rate. In the earlier version of this paper as well as McConnell and Singh (1994) and Singh and McConnell (1996), the underlying assets differ in transaction costs. According to the study of mortgage-backed securities by Stanton (1995), this way of modeling collaterals "... captures many of the empirical features of mortgage prepayments." The second feature of the proposed framework is that the path dependence of MCABS's is dealt with by a state augmentation method where the path of a state variable is summarized by auxiliary state variables. This method has an advantage over Monte Carlo methods used by prior studies on MCABS's in that it yields a model price free of a statistical error, which is inherent in Monte Carol methods.

In Section 5, we have examined the MCABS model of Section 4 by simulation, and found unique properties of MCABS's due to path dependence. For example, unlike straight bonds, MCABS's tend to have longer duration as the market interest rate gets higher. Furthermore, this phenomenon is found stronger for longer-term MCABS's. Section 5 also examined how the option values of underlying assets are distributed among MCABS's. It is found that the total amount and the distribution of the option values are very sensitive to the level of the market interest rate.

Finally, we conclude this paper with additional comments. First, most MCABS's have allocation rules where the remaining principal of the collateral summarizes the path of a state variable for MCABS's. Therefore, the MCABS model of Section 4 can be easily extended to accommodate other allocation rules of MCABS's. Second, the proposed framework is not limited to a particular model of uncertainty. Rather, it can be implemented in any discrete time model of uncertainty, as long as the uncertainty model permits a one-period RNVR.

FOOTNOTES

1. Most assets that have been securitized so far have embedded options of American type. Therefore, it is crucial to develop a valuation framework which accounts for embedded options explicitly.

2. Examples of ABS's with multi-class structures are found in Appendix B of Rosenthal and Ocampo (1988). We use the term asset backed securities more broadly than this definition elsewhere in this paper.

3. Mortgage loans in the collaterals of MCABS's are typically collateralized through agency pass-through securities. As a result, they are free of default risk from the point of view of investors. For this reason, we do not consider default options in MCABS's backed by mortgage loans.

4. In general, a security whose cash flow depends on not only current but also past realizations of a state variable (or state variables) is referred to as a path dependent security.

5. The valuation of asset backed securities with single class structures was addressed by Dunn and McConnell (1981), Brennan and Schwartz (1985), Hall (1985), Schwartz and Torous (1989a) and many others.

6. The author presented this model of MCABS's in European Finance Association Meetings in both 1989 and 1990 - several years earlier than McConnell and Singh (1994) and Singh and McConnell (1996), who valued CMO's by a Monte Carlo method assuming that a collateral is a portfolio of assets with embedded options.

7. See Cox and Miller (1968) for the exposition of the state augmentation method in the context of stochastic processes and Bertzekas (1987) in the context of dynamic programming.

8. Other rules for allocating principal payments include planned amortization class (PAC), targeted amortization class (TAC), interest only (IO) and principal only (PO).

9. The extent to which imbedded options affect the value of a MCABS depends on the type of assets in the collateral. Carron, Olson and Soares (1989), for example,

state that "because of the relatively short maturity and small balance of the typical automobile receivable, refinancing of a high rate obligation by a lower rate one is less common in the automobile market than in the mortgage market."

10. The HL model can be replaced by any other discrete time model of uncertainty, including those for multiple sources of risk.

11. If all loans in the collateral were identical in all respects, they would be prepaid at the same time.

12. The proposed model can be modified to accommodate multiple statistics.

13. We define h_n so that $h_0 = 0$.

14. Assumption (iii) of (A5) holds if and only if (i_n, h_n) are Markov.

15. The values of N and $\bar{h}$ are taken from Johnston and Van Drunen (1988).

16. Although 5.7×10^6 sounds a large number, an alternative method based on the event tree of Debrue (1959) would require a far larger number of nodes to compute model prices of MCABS's — 4.70×10^{108} nodes in the case of the above CMO example.

17. In Section 5, this model for two-class MCABS's is applied to MCABS's with more than two classes.

18. There are two reasons for principal payments: scheduled principal payments specified by $\{F_k^{(n)} | n = 1, \cdots, N\}$ and prepayments.

19. This is a trivial case that h_{n+1} is a function of i_{n+1}.

20. Alternatively, the residual can be valued by the direct application of the ET method.

21. Certainly, ET's can be defined on any arbitrary partition of the collateral.

22. In fact, if ET's are defined on a sufficiently fine partition of the collateral, they truly become coupon bonds with stochastic maturities.

23. This is true as long as the market interest rate exceeds the contract interest rates of the ET's.

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Figure 1(a)
Cash Flow to R-1

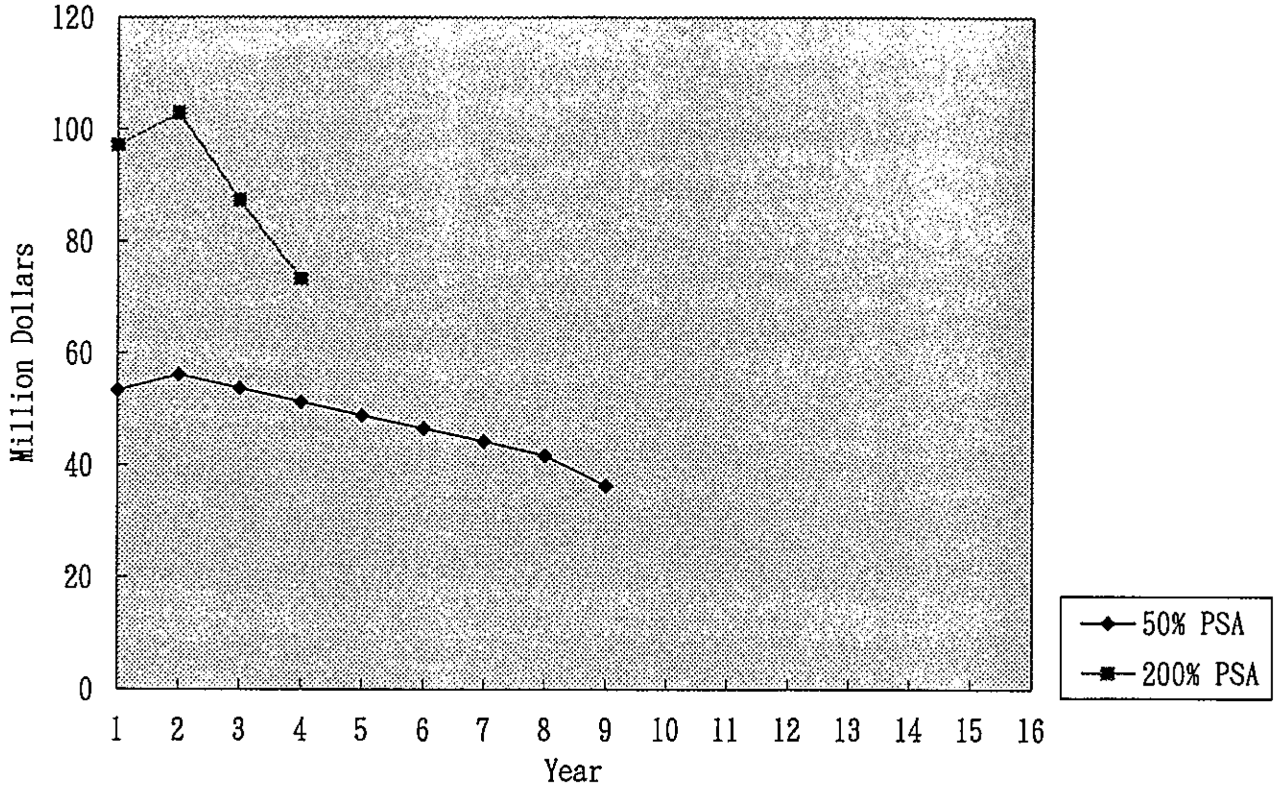


Figure 1(b)
Cash Flow to R-2

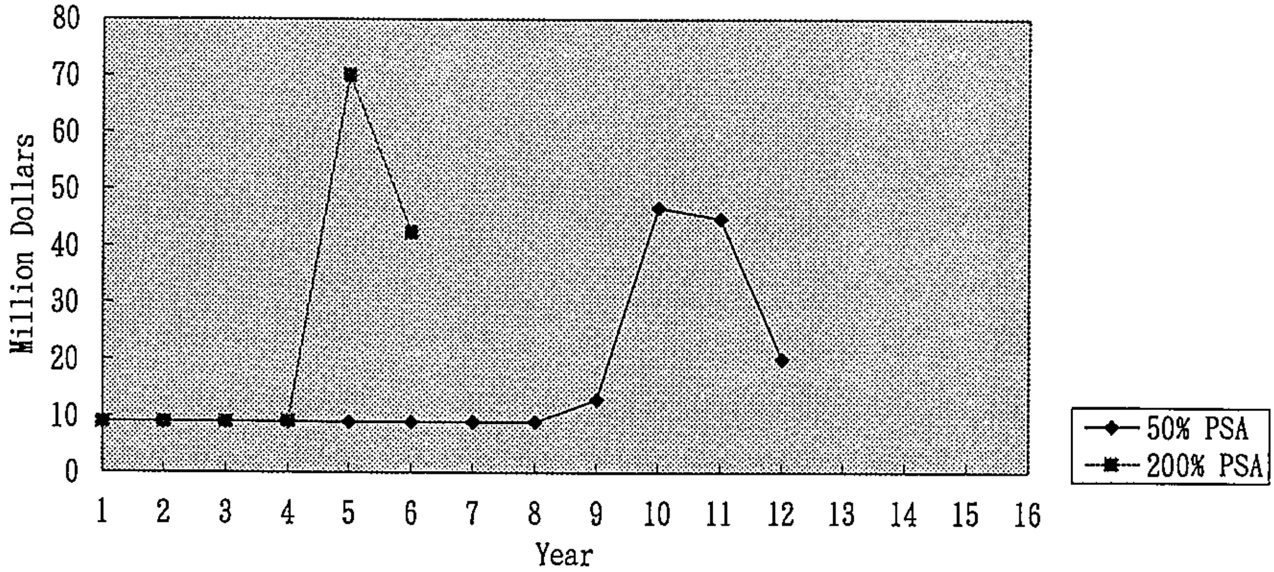
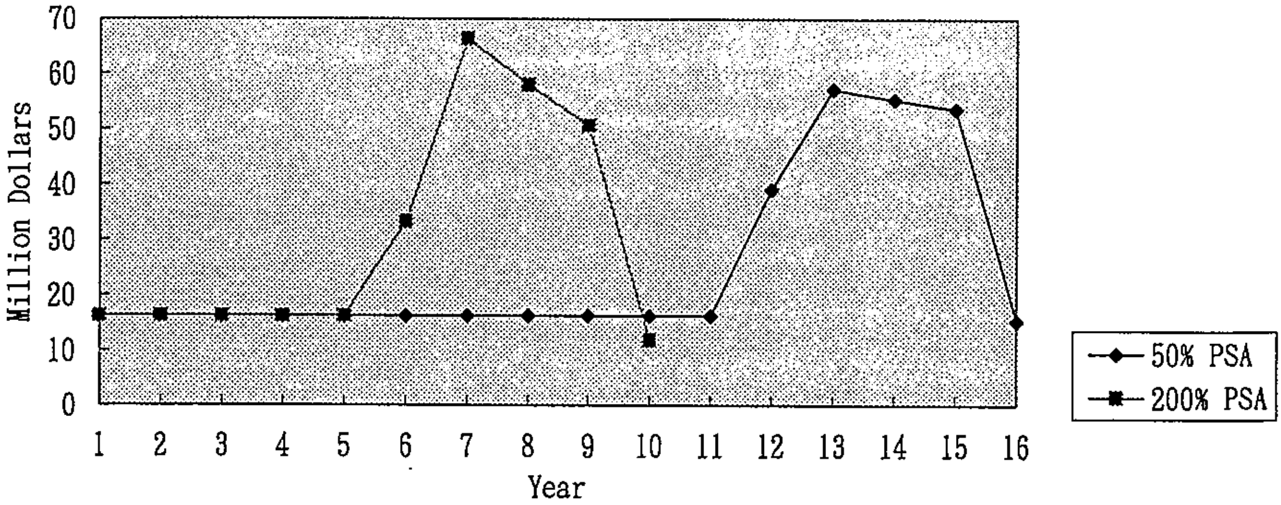


Figure 1(c)
Cash Flow to R-3



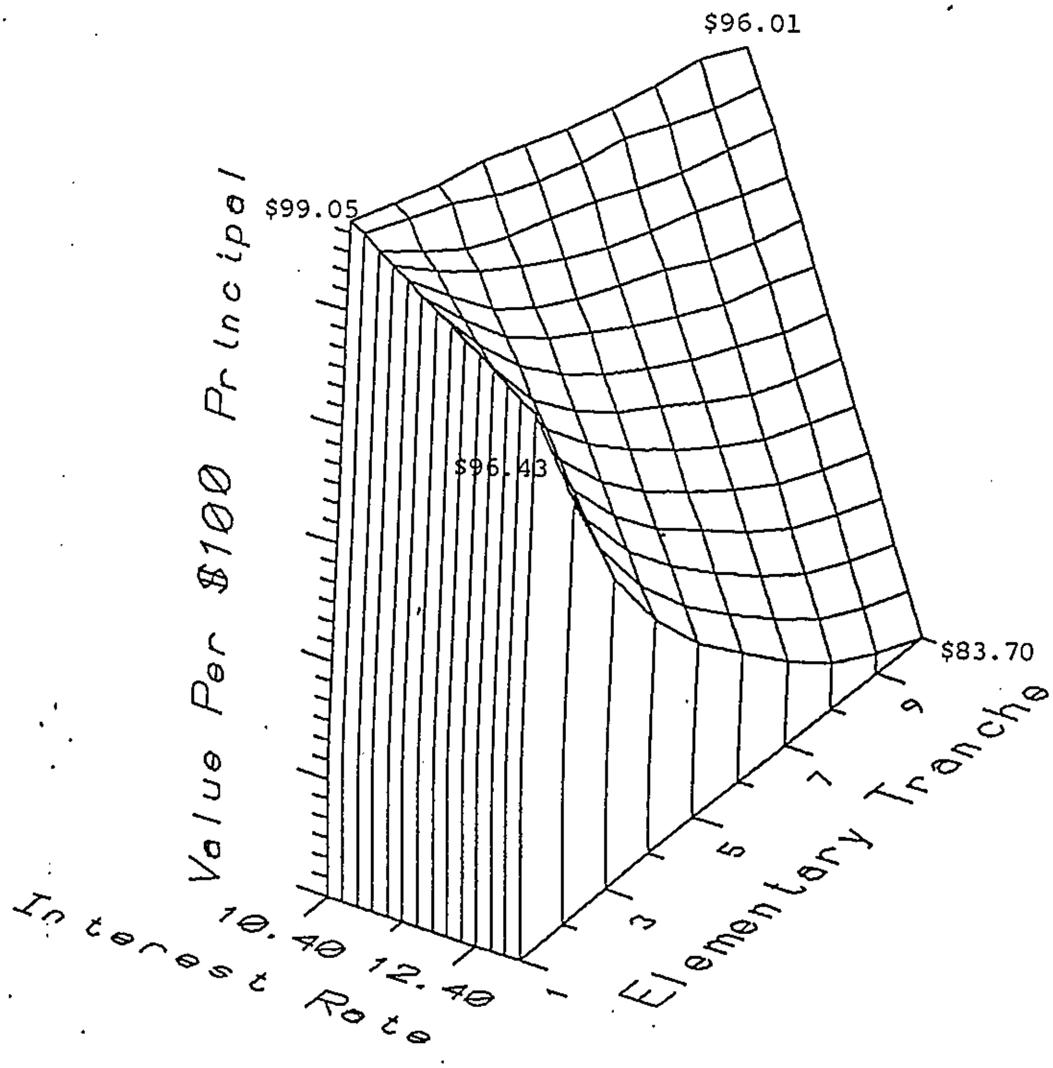


Figure 2
Value of the elementary tranche

The prices of elementary tranches with $100 principals are plotted for interest rates 10.4%-13.0% . The four numbers at the corners of the figure represent the prices of elementary tranches 1 and 10 at interest rates 10.4% and 13.0%.

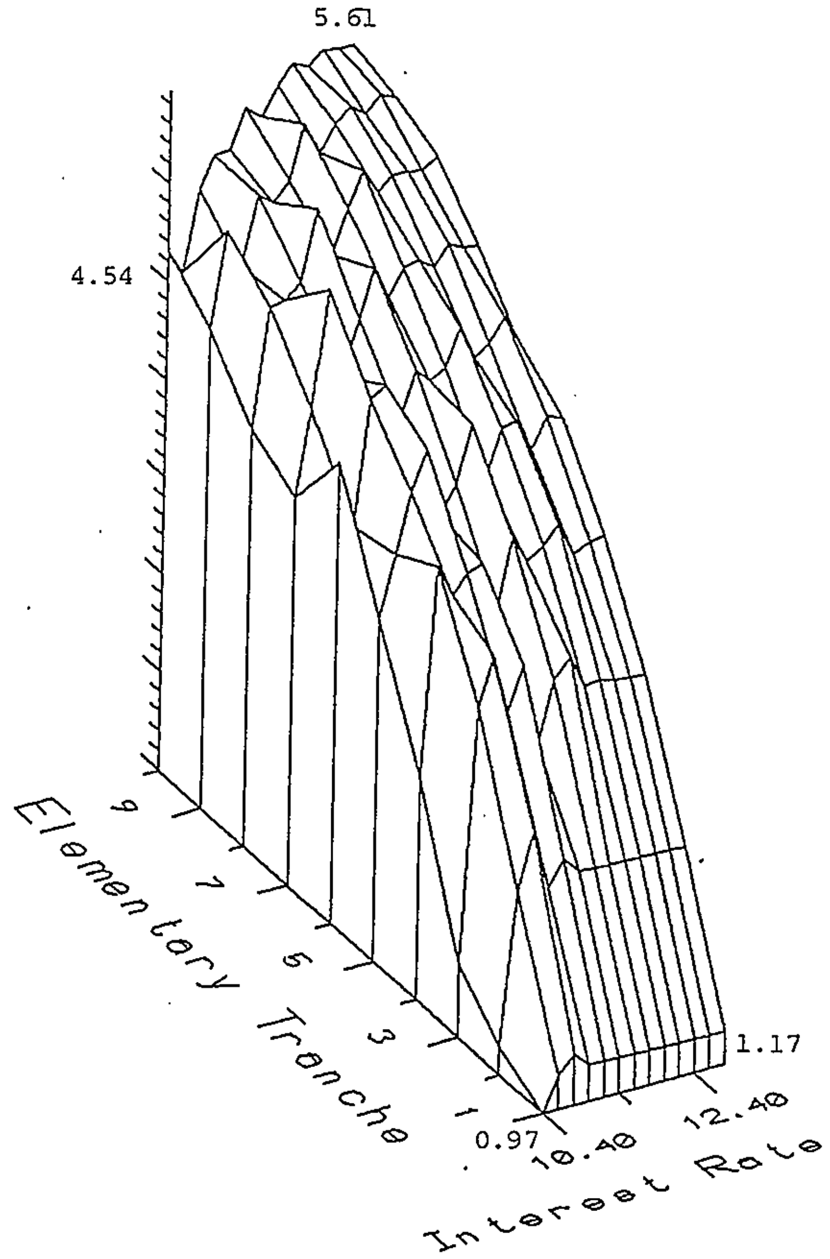


Figure 3
Duration of the elementary tranche

The duration is plotted for elementary tranches 1 to 10 at interest rates 10.4%-13.0%. The four numbers at the corners of the figure represent the duration for elementary tranches 1 and 10 at interest rates 10.4% and 12.8%.

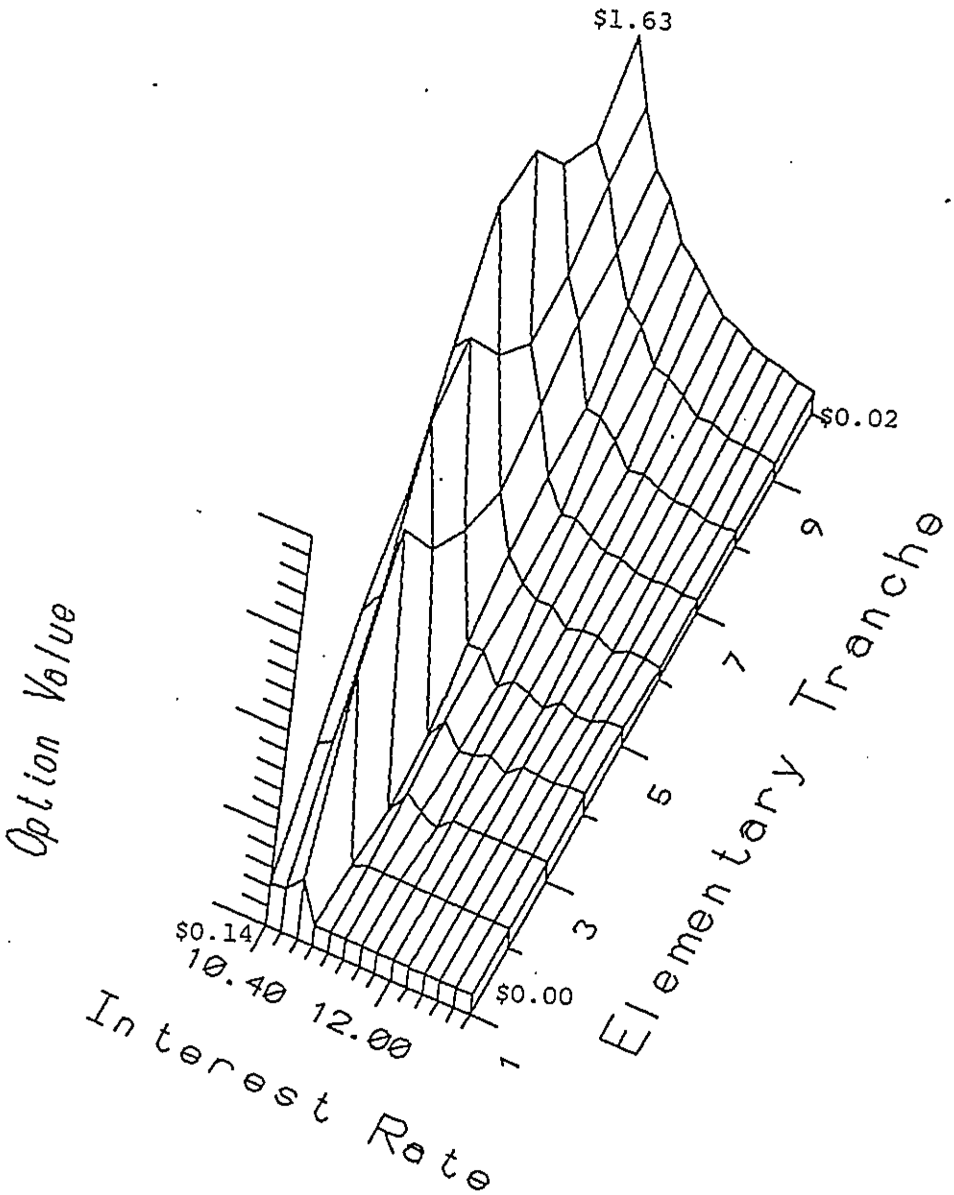


Figure 4
Distribution of option values

The option values of elementary tranches with $100 principals are plotted at interest rates 10.4%-13.0%. The four numbers at the corners of the figure represent the option values of elementary tranches 1 and 10 at interest rates 10.4% and 13.0%.

Table 1
Terms of Morgan Stanley Mortgage Trust R

Class	Original Principal Amount	Contract Interest Rate	Stated Maturity
R-1	$295,000,000	9.0%	August 1, 2001
R-2	$100,000,000	9.0%	February 1, 2004
R-3	$173,000,000	9.4%	May 1, 2007