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K-function

and

Optimal Stopping Problems

by

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K–FUNCTION AND OPTIMAL STOPPING PROBLEMS

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Abstract

In this paper a certain function of the real number x, referred to as the K–function, is defined and its some properties are examined. We reveal the close relation of the function to some optimal stopping problems by demonstrating that it plays important roles in the examination of the structures of their optimal decision rules.

1 Introduction

In this paper we define a certain function of the real number x, $K(x)$, and reveal its some properties. Let us refer to it as the K–function, which is defined in Section 2. In the twelve subsections of Section 3 we demonstrate that, in some optimal stopping problems and some other types of decision problems, how the function is used and applied in order to describe the optimal equations of them and examine the natures of their optimal decision rules.

For the most typical and conventional models of optimal stopping problems presented in the first two subsections, the model with no recall and model with recall, a complete examination has already been made, especially it is to be appreciated that Sakaguchi [26] showed the first and clear formulation of the models using the dynamic programming technique in order to reemphasize the importance of its roles in optimal stopping problems. However, we provide here the complete reexamination of them by use of the K–function in order to reemphasize the importance of its role in the treatment of optimal stopping problems. In the subsequent six subsections we show the applications of the function to other types of optimal stopping problems; the models with uncertain recall [8][15][16], with controlled recall [25], with recall cost [14], with multiple search areas [11], with finite search budget [9], and the model of Pandora type [29]. By these it will be realized that the K–function and the solution of the equation $K(x) = 0$, denoted by h^*, have a close relation with these models, especially in their mathematical treatment to investigate the structures and properties of their optimal decision rules.

Furthermore, in the last four subsections we show that the T–function and S–function which are defined as the relatives of the K–function have also the close relations with other types of decision problems: the sequential assignment problems [5], the Markovian decision process with random observation [10] which is a general model including all the above problems, and the well-known newsboy problem.

From all these considerations above, a fragment of the effectiveness of the K–function in the discussions of different decision problems such as been stated above could be recognized.

It can be expected that some or many models of decision problems that will be posed in the future, although they might be such types or classes as been discussed in this paper, are well defined and examined by use of the K–function, T-function, and S-function, and their properties.

2 K–Function

Let $F(w)$ be a one dimensional distribution function, discrete or continuous, with a finite expectation μ. Here for certain given numbers a and b such that $0 \le a < b < \infty$ let $F(w) = 0$ for $w < a$, $0 < F(w) < 1$ for $a \le w < b$, and $F(w) = 1$ for $b \le w$, hence $a < \mu < b$. Let β and c be certain given numbers such that $0 \le \beta \le 1$ and $c \ge 0$, and let $\alpha = \beta\mu - c \le b$. Then for any real number x define the function

$$K(x) = \beta \int_0^\infty \max\{w, x\} dF(w) - x - c, \tag{2.1}$$

referred to as the K–function, which can be rewritten as follows.

$$K(x) = \beta \int_x^\infty \{w - x, 0\} dF(w) - (1-\beta)x - c \tag{2.2}$$

$$= \beta \int_x^\infty (w - x) dF(w) - (1-\beta)x - c. \tag{2.3}$$

Let us define

$$T(x) = \int_0^\infty \max\{w - x, 0\} dF(w) = \int_x^\infty (w - x) dF(w), \tag{2.4}$$

$$S(x) = \int_0^\infty \max\{w, x\} dF(w) = T(x) + x, \tag{2.5}$$

referred to as the T–function and S–function, respectively. The $T(x)$ is usually called the shortage function. Now here note that $K(x)$, $T(x)$, and $S(x)$ are all continuous functions even if $F(w)$ is discrete. For convenience of later discussions, let us define

$$T(-\infty) = \infty, \quad T(\infty) = 0, \quad S(-\infty) = \mu, \quad S(\infty) = \infty. \tag{2.6}$$

Then the K-function can be expressed as

$$K(x) = \beta T(x) - (1-\beta)x - c \tag{2.7}$$

$$= \beta S(x) - x - c. \tag{2.8}$$

Furthermore for any given real numbers γ and δ let us define

$$L(x, \gamma, \delta) = \gamma x + \delta T(x). \tag{2.9}$$

It is immediately seen from Eq. (2.1) that

$$K(x) = \begin{cases} \alpha - x, & x \leq a, \\ -(1-\beta)x - c \leq 0, & b \leq x, \end{cases} \tag{2.10}$$

from which we have

$$\lim_{x \to -\infty} K(x) = \infty, \tag{2.11}$$

$$\lim_{x \to \infty} K(x) = \left\{ \begin{array}{ll} -\infty & \text{if } \beta < 1 \\ -c & \text{if } \beta = 1 \end{array} \right\} \leq 0, \tag{2.12}$$

implying that the equation $K(x) = 0$ has at least one solution, so let h^* be the minimum solution, that is, $h^* = \min\{x \mid K(x) = 0\}$. Now clearly for any y we have

$$K(x) \geq \beta \int_y^\infty (w - x) dF(w) - (1-\beta)x - c. \tag{2.13}$$

$$T(x) \geq \int_y^\infty (w - x) dF(w). \tag{2.14}$$

$$\tag{2.15}$$

From Eq. (2.3) and Eq. (2.13) we have the following inequality for any x and y.

$$K(x) - K(y) \leq (y - x)(1 - \beta F(x)), \tag{2.16}$$

$$K(x) + x - K(y) - y \leq -\beta(y - x)F(x). \tag{2.17}$$

If $F(w)$ is a discrete distribution function defined on $w = 0, \pm 1, \cdots$ where a and b are both integers, then $T(x)$ for any real number x can be expressed as follwos.

$$T(x) = \sum_{w>x}(w-x)f(w) \tag{2.18}$$

$$= \sum_{w\geq[x]+1}(w-x)f(w) \tag{2.19}$$

where $[x]$ represents the maximum integer less than or equal to x, usually called the Gauss's symbol. Then for any integer x let $\Delta K(x) = K(x) - K(x-1)$, $\Delta T(x) = T(x) - T(x-1)$, and $\Delta S(x) = S(x) - S(x-1)$.

Lemma 2. 1

(a) *$K(x)$ is nonincreasing in x on $(-\infty, \infty)$, strictly decreasing on $(-\infty, b)$, so $K(x) > K(b)$ for any $x < b$, and convex in x. If $\beta < 1$, then $K(x)$ is strictly decreasing in x on $(-\infty, \infty)$.*

(b) *$K(x) + x$ is nondecreasing in x on $(-\infty, \infty)$, strictly increasing in x on $[a, \infty)$, and convex in x. Furthermore $K(x) + x > \alpha$ for $a < x$ with $K(a) + a = \alpha$ and $K(x) + x < b$ for $x < b$ with $K(b) + b = \beta b - c \leq b$,*

(c) *For any x and y we have*
1. $|K(x) - K(y)| \leq |y - x|$,
2. $|K(x) + x - K(y) - y| \leq \beta|y - x|$.

PROOF (a) It is clear from Eq. (2.2) that $K(x)$ is nonincreasing in x on $(-\infty, \infty)$. From Eq. (2.16), if $y < x < b$, then $1 - \beta F(x) > 0$ due to $F(x) < 1$, hence $K(y) - K(x) < 0$, that is, $K(x) < K(y)$, implying that $K(x)$ is strictly decreasing in $x < b$. If $\beta < 1$, then $1 - \beta F(x) > 0$ for all x on $(-\infty, \infty)$, hence it follows that $K(x)$ is strictly decreasing in x on $(-\infty, \infty)$. The convexity is immediate from the fact that $\max\{w, x\}$ is convex in x on $(-\infty, \infty)$ for any given w.

(b) It is clear from Eq. (2.1) that $K(x) + x$ is nondecreasing in x on $(-\infty, \infty)$. From Eq. (2.17) , if $a \leq x < y$, then $K(x) + x - K(y) - y < 0$ due to $F(x) > 0$, hence $K(x) + x < K(y) + y$, that is, $K(x) + x$ is strictly increasing in $x \geq a$. The convexity is clear from (a). If $a < x$, then $K(x)+x > K(a)+a = \alpha - a + a = \alpha$. If $x < b$, then $K(x)+x < K(b)+b = -(1-\beta)b - c + b = \beta b - c \leq b$.

(c) Interchanging x and y in Eq. (2.16) and Eq. (2.17) and then multiplying the both sides by -1 yield

$$K(x) - K(y) \geq (y - x)(1 - \beta F(y)), \tag{2.20}$$

$$K(x) + x - K(y) - y \geq -\beta(y - x)F(y). \tag{2.21}$$

It is immediate from Eq. (2.16), Eq. (2.17), Eq. (2.20), and Eq. (2.21) that $|K(x) - K(y)| \leq |y - x|$ and $|K(x) + x - K(y) - y| \leq \beta|y - x|$. ■

Lemma 2. 2

(a) *$h^* \geq \alpha$.*

(b) *If $(1 - \beta)^2 + c^2 = 0$, then $h^* = b$.*

(c) *If $(1 - \beta)^2 + c^2 \neq 0$, then*
1. *h^* is given by the unique solution of $K(x) = 0$ where $h^* < b$,*
2. *if $\alpha \leq a$, then $h^* = \alpha \leq a$,*
3. *if $\alpha > a$, then $a < h^* < b$.*

(d) *h^* is strictly increasing in β and strictly decreasing in c.*

PROOF (a) By definition we have $0 = K(h^*) \geq \beta \int_0^\infty w dF(w) - h^* - c = \alpha - h^*$, hence $h^* \geq \alpha$.

(b) Since $\beta = 1$ and $c = 0$ in this case, we have $K(x) = 0$ for all $x \geq b$ from Eq. (2.10) and $K(x) > K(b) = 0$ for all $x < b$ from Lemma 2. 1 (a). Hence by definition we have $h^* = b$.

(c1) The existence of the solutions has already been shown. Its uniqueness is clear from Eqs. (2.11), (2.12) and Lemma 2. 1 (a). The inequality $h^* < b$ is clear from $K(b) < 0$ in this case due to Eq. (2.10).

(c2) If $\alpha \leq a$, then $K(\alpha) = \alpha - \alpha = 0$ from Eq. (2.10), hence $h^* = \alpha \leq a$ from (c1).

(c3) If $\alpha > a$, then $K(a) = \alpha - a > 0$, hence $a < h^*$.

(d) The statement is immediate from the fact that $K(x)$ is strictly increasing in β and strictly decreasing in c for any given x (Fig. 2.1). ■

From Lemmas 2. 1 and 2. 2 it can be easily understood that $K(x)$ is graphed as in Fig. 2.2. From Lemma 2. 2 we have the following corollary.

Corollary 2. 1 *The h^* can be classified into the following three cases.*

$$\begin{aligned} \alpha = h^* \leq a &\iff (1-\beta)^2 + c^2 \neq 0 \text{ and } \alpha \leq a, \\ a < h^* < b &\iff (1-\beta)^2 + c^2 \neq 0 \text{ and } a < \alpha, \\ h^* = b &\iff (1-\beta)^2 + c^2 = 0. \end{aligned}$$

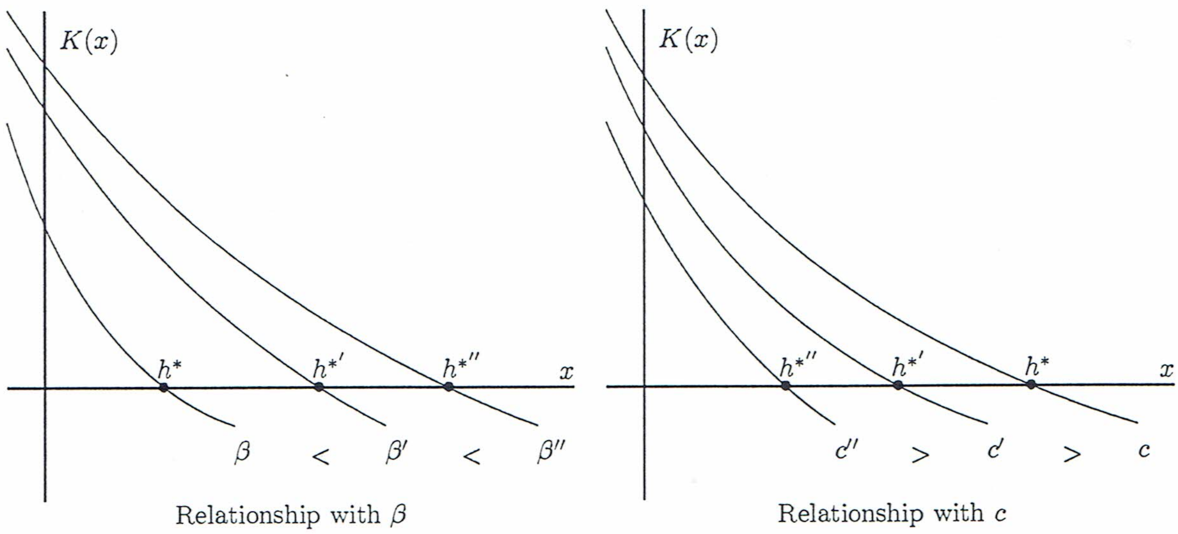

Figure 2.1: Relationship of h^* with β and c ($h^* = \bullet$)

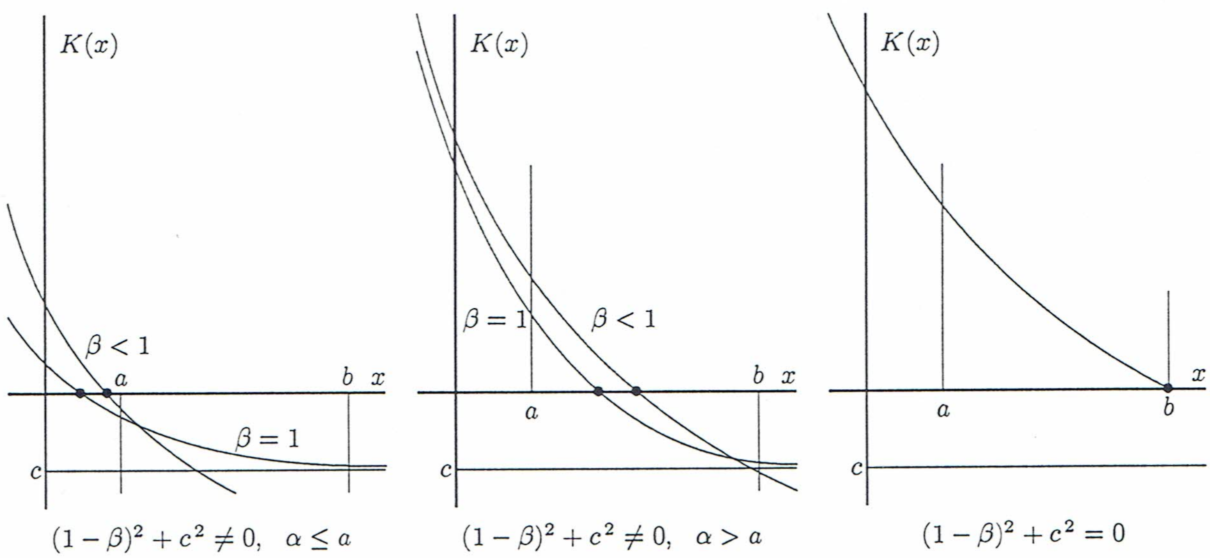

Figure 2.2: The function $K(x)$ ($h^* = \bullet$)

The following corollary that is immediately derived from the above discussions will be sometimes useful.

Corollary 2. 2

(a) *$K(x) > 0$ for $x < h^*$ and $K(x) \le 0$ for $x \ge h^*$.*

(b) *If $(1-\beta)^2 + c^2 \neq 0$, then $K(x) < 0$ for $x > h^*$.*

(c) *If $(1-\beta)^2 + c^2 = 0$, then $K(x) = 0$ for $x \ge h^*$.*

Lemma 2. 3

(a) *$T(x)$ is nonincreasing in x with $T(x) = \mu - x$ for $x \le a$ and $T(x) = 0$ for $x \ge b$.*

(b) *$S(x)$ is nondecreasing in x with $S(x) = \mu$ for $x \le a$ and $S(x) = x$ for $x \ge b$.*

(c) *If $(1-\beta)^2 + c^2 = 0$, then $K(x) = T(x) = S(x) - x$.*

(d) *If $T(x)$ is differentiable with respect to x, then $dT(x)/dx = F(x) - 1$ and $dK(x)/dx = \beta F(x) - 1$.*

(e) *If $F(w)$ is discrete, then $\Delta T(x) = F(x-1) - 1$ and $\Delta K(x) = \beta F(x-1) - 1$ for any integer x.*

(f) *Suppose $\gamma < \delta$. Then $L(x, \gamma, \delta)$ is minimized at the real number x^* such that $F(x^*) = 1 - \gamma/\delta$ if $F(w)$ is continuous and at the integer x^* such that $F(x^*-1) < 1 - \gamma/\delta \le F(x^*)$ if $F(w)$ is discrete.*

PROOF (a) to (e) Easy.

(f) Noting Eq. (2.14), for any x and y we have

$$\begin{aligned} L(y,\gamma,\delta) - L(x,\gamma,\delta) &= \gamma(y-x) + \delta(T(y) - T(x)) \\ &\le \gamma(y-x) + \delta\Big(\int_y^\infty (w-y)dF(w) - \int_y^\infty (w-x)dF(w)\Big) \\ &= (y-x)(\gamma - \delta(1 - F(y))) \\ &= \delta(y-x)(F(y) - 1 + \gamma/\delta). \end{aligned}$$

Hence, if $F(x)$ is continuous, then $L(x^*,\gamma,\delta) - L(x,\gamma,\delta) \le 0$ for any x, hence it follows that $L(x,\gamma,\delta)$ is minimized at $x = x^*$. If $F(x)$ is discrete, then from (e) we have

$$L(x,\gamma,\delta) - L(x-1,\gamma,\delta) = \delta(F(x-1) - 1 + \gamma/\delta), \tag{2.22}$$

which is nondecreasing in x. Thus we have $L(x^*,\gamma,\delta) - L(x^*-1,\gamma,\delta) = \delta(F(x^*-1) - 1 + \gamma/\delta) < 0$ and $L(x^*+1,\gamma,\delta) - L(x^*,\gamma,\delta) = \delta(F(x^*) - 1 + \gamma/\delta) \ge 0$, hence $L(x^*+1,\gamma,\delta) \ge L(x^*,\gamma,\delta) < L(x^*-1,\gamma,\delta)$, implying that $L(x,\gamma,\delta)$ is minimized at $x = x^*$. ■

As a generalized one of the K–function we shall define

$$K(M(-),x) = \beta \int_0^\infty (M(w) - x)dF(w) - x - c, \tag{2.23}$$

referred to as the generalized K–function. In almost the same way as been stated above it can be easily shown that $K(M(-),x)$ is nonincreasing in x with $K(M(-),x) \to \infty$ as $t \to -\infty$ and $K(M(-),x) \le 0$ as $t \to \infty$, hence the equation $K(M(-),x) = 0$ has a solution. An example of application of the generalized K–function is given in Section 3.10.

3 Applications of K–Function

Here we give some examples of decision problems where the K–function is well used to examine the properties of their decision rules. They have all already been completely investigated so far. Through the subsequent subsections it could be realized that all the statements in the theorems stated there are all closely related to the K–function and h^*, the solution of $K(x) = 0$, and that the properties of the K–function which were verified in Section 2 play important roles in the proofs of not only these statements but also other theorems that are described in the original papers. In Theorems 3. 1 and 3. 2 for the model with no recall in Section 3.1 and the model with recall in Section 3.2 we state all the necessary statements characterizing the properties of the optimal decision rules and give in Appendix their complete proofs using the K–function. Although some of these statements are not seemingly connected to the K–function, the reader could know that the function plays essential roles in their proofs.

3.1 Model with No Recall [26]

Consider the following discrete-time stochastic decision process with a finite planning horizon. For convenience let points in time be numbered backward from the final point in time of the horizon, time 0, as $0, 1, \cdots$, and so on, and a time interval between two successive points in time, say time t and time $t-1$, is called the period t. If some fixed cost $c \geq 0$, called the search cost, is paid at a point in time, then an offer can be obtained at the next point in time. Offers $w, w', \cdots$ obtained at successive points in time are assumed to be independent identically distributed random variables with a known distribution function $F(w)$, called the offer distribution, where for certain given numbers a and b such that $0 < a < b < \infty$ let $F(w) = 0$ for $w < a$, $0 < F(w) < 1$ for $a \leq w < b$, and $F(w) = 1$ for $b \leq w$, hence $a < \mu < b$. An offer must be necessarily accepted up to time 0, assumed that an offer once inspected and passed up becomes instantly and forever unavailable. Here being available means that an offer once inspected and passed up can be accepted at any time in the future. In general, the following three models can be considered in terms of the future availability of a past offer: 1. the model with no recall where it becomes instantly and forever unavailable, 2. the model with recall where it remains forever available, and 3. the model with uncertain recall where the availability is stochastic. What we are going to examine in this section is the model with no recall. The other two will be deal with in the subsequent two sections. Let a per-period discount factor be denoted by $\beta \leq 1$, and let $\alpha = \beta\mu - c \leq \mu$ and $\alpha > 0$. We shall refer to the rule prescribing when to stop the search by accepting an offer as the stopping rule. The objective here is to find the optimal stopping rule maximizing the expected present discounted net value, the expectation of the present discounted value of an offer w accepted minus the total present discounted value of search costs paid up to the termination of the search with its acceptance.

Let $u_t(w)$ represent the maximum expected present discounted net value starting from time t with an offer w. Then clearly $u_0(w) = w$ and

$$u_t(w) = \max\{w, U_t\} \geq w, \quad t \geq 1, \tag{3.1}$$

where w and U_t in the right hand side are, respectively, the gain from stopping the search and the maximum expected present discounted net value from continuing the search by using the optimal stopping rule over the remaining planning horizon, expressed as

$$U_t = \beta \int_0^\infty u_{t-1}(\xi) dF(\xi) - c, \quad t \geq 1, \tag{3.2}$$

where ξ is the value of an offer that will be obtained at the next point in time. Since $u_0(\xi) = \xi$ by definition, we have $U_1 = \alpha > 0$. Then the optimal stopping rule of time t can be prescribed as follows: If $w > U_t$, stop the search by accepting the present offer w, or else continue the search. The critical value U_t, at which whether to stop or not becomes indifferent, is usually, especially in economics, called the reservation value of time t. Now, using the K–function, we can express Eq. (3.2) as follows.

$$U_t = K(U_{t-1}) + U_{t-1}, \quad t \geq 1. \tag{3.3}$$

If U_t, hence $u_t(w)$ converges as $t \to \infty$, then let their limits be denoted by U and $u(w)$, respectively. Then we have the following theorem.

Theorem 3.1

(a) *U_t is nondecreasing and concave*[‡] *in t.*
(b) *U_t converges to a limit $U = h^*$ as $t \to \infty$ with $\alpha \leq U_t < b$ for all t.*
(c) $u(w) = \max\{w, h^*\}$
(d) *If $\alpha \leq a$, then $U_t = \alpha$ $(\leq a)$ for all $t \geq 1$,*
(e) *If $\alpha > a$, then U_t is strictly increasing and strictly concave in t, hence $U_t < h^*$ for all $t \geq 1$.*

PROOF See [26]. For the proof using K–function, see Appendix. ■

3.2 Model with Recall [26]

This is the model where even if an offer is once inspected and passed up, it can be accepted at any time in the futuer.

Let $u_t(y)$ represent the maximum expected present discounted net value starting from time t with the best offer y so far appeared. Then, clearly $u_0(y) = y$, and

$$u_t(y) = \max\{y, U_t(y)\} \geq y, \quad t \geq 1, \tag{3.4}$$

where y and $U_t(y)$ in the right hand side are, respectively, the gain from stopping the search by accepting the best offer y and the maximum expected present discounted net value from continuing the search by using the optimal stopping rule over the remaining planning horizon, expressed as

$$U_t(y) = \beta \int_0^\infty u_{t-1}(\max\{y, \xi\}) dF(\xi) - c. \tag{3.5}$$

Arranging $U_1(y)$ by substituting $u_0(\max\{y, \xi\}) = \max\{y, \xi\}$ into yields

$$U_1(y) = K(y) + y. \tag{3.6}$$

Then the optimal stopping rule can be prescribed as follows: If $y > U_t(y)$, stop the search by accepting the best offer y, or else continue the search. If $U_t(y)$, hence $u_t(w)$ converges as $t \to \infty$, let their limits be denoted $U(y)$ and $u(y)$, respectively. Then we have from Eq. (3.5)

$$U(y) = \beta \int_0^\infty u(\max\{y, \xi\}) dF(\xi) - c. \tag{3.7}$$

Then we have the following theorem.

Theorem 3. 2
(a) *$U_t(y)$, hence $u_t(y)$ is nondecreasing in t and y.*
(b) *If $y \geq h^*$, then $U_t(y) \leq y$, hence $u_t(y) = y$, and if $y \leq h^*$, then $y \leq U_t(y) \leq h^*$, hence $u_t(y) = U_t(y) \leq h^*$.*
(c) *If $y \geq h^*$, then $U_t(y) = K(y) + y$ for all $t \geq 1$, and if $y \leq h^*$, then $U_t(y)$ converges to $U(y)$ as $t \to \infty$.*
(d) *If $y \geq h^*$, then $U(y) = K(y) + y \leq y$, hence $u(y) = y$, and if $y \leq h^*$, then $y \leq U(y) \leq h^*$, hence $u(y) = U(y) \leq h^*$.*
(e) *$U(y) = h^*$ for $y \leq h^*$.*
(f) *For any y we have $u(y) = \max\{y, h^*\}$ and $U(y) = \max\{K(y) + y, h^*\}$.*

PROOF See [26]. ■

The statement (b) of the theorem implies that the optimal stopping rule can be restated as follows. If $y \geq h^*$, stop the search by accepting the best offer y, or else continue the search. This means that the

[‡]Here in general a function $g(x)$ defined on $x = 0, \pm 1, \pm 2, \cdots$ is said to be concave (convex) in x if the difference $\Delta g(x) = g(x) - g(x-1)$ is nonincreasing (nondecreasing) in x.

model becomes completely identical to the model with recall which has the infinite planning horizon (Theorem 3. 1 (b)).

3.3 Model with Uncertain Recall [8]

This is the same as the model with recall only except that the recall is uncertain. Here the uncertainty of recall is defined as follows. An offer once inspected and passed up j periods ago becomes unavailable at the next point in time with probability p_j, $j = 0, 1, \cdots$, provided that it still remains available at the present time, so p_0 is the probability that an offer obtained at the present point in time becomes unavailable at the next point in time. If $p_j = 0\,(1)$ for all $j \geq 0$, then it is reduced to the model with recall (with no recall*).

In this section let us assume that there exists a fixed integer $N > 0$ such that $0 < p_j < 1$ for $0 \leq j \leq N-1$ and $p_j = 1$ for $N \leq j$. This implies that every offer has at most N periods of age. The case of $N = 0$ is reduced to the model with no recall. Now suppose the search starts from time t. Let w_j denote the offers of time $t+j$ (j periods ago), and let $y_0 = w_0$ and $y_j = 0$ (w_j) if the offer w_j is unavailable (available) at the present time t. Then a state of the search process at time t can be described by the vector $\boldsymbol{y} = (y_0, y_1, \cdots, y_N)$.

Let $u_t(\boldsymbol{y})$ denote the maximum expected present discounted net value, starting from time t when in state $\boldsymbol{y}$. Then clearly $u_0(\boldsymbol{y}) = \max \boldsymbol{y}$ and

$$u_t(\boldsymbol{y}) = \max\{\max \boldsymbol{y}, U_t(\boldsymbol{y})\}, \quad t \geq 1, \tag{3.8}$$

where $\max \boldsymbol{y}$ is the gain from stopping the search and $U_t(\boldsymbol{y})$ is the maximum expected present discounted net value from continuing the search. If $N = 1$, so $\boldsymbol{y} = (y_0, y_1)$, then $U_t(\boldsymbol{y})$ can be expressed as

$$U_t(y_0, y_1) = \beta\Big(p_0 \int_0^\infty u_{t-1}(\xi, 0) dF(\xi) + (1-p_0) \int_0^\infty u_{t-1}(\xi, y_0) dF(\xi)\Big) - c. \tag{3.9}$$

where ξ is an offer obtained at time $t-1$ (next time). Then we have

$$U_1(y_0, y_1) \;=\; p_0 \alpha + (1-p_0)(K(y_0) + y_0). \tag{3.10}$$

In general let $C_t = \{\boldsymbol{y} \mid \max \boldsymbol{y} \leq U_t(\boldsymbol{y})\}$. Then if $\boldsymbol{y} \in C_t$, it is optimal to continue the search, or else stop, so let us refer to C_t as the continuation region and its compliment $S_t = C_t^c$ as the stopping region. Let $\boldsymbol{z} = (y_1, y_2, \cdots, y_N)$. Then, if the optimal stopping rule has the following property, it is said to have a double reservation property, DRV property, for short.

DRV Property For at least one $\boldsymbol{z}$ with $a < y_j < b$ for $j = 1, 2, \cdots, N$, there exist two different critical values w_* and w^* with $a < w_* < w^* < b$ such that, for a present offer w_0, if $w_* \leq w_0 \leq w^*$, then continuing is optimal, or else stopping is optimal.

Then we can prove the following theorem.

Theorem 3. 3

(a) *The necessary and sufficient condition for the optimal stopping rule to have DRV property for any $t \geq 1$ and any $N \geq 1$ is given by $\alpha > a$.*

(b) *As t tends to ∞, the continuation region C_t increases and converges to the cube $H^* = \{(y_0, y_1, \cdots, y_N) \mid 0 \leq y_j \leq h^*, 0 \leq j \leq N\}$.*

Proof See [8]. ■

*The case of $p_0 = 1$, even if $p_j > 0$ for all $j \geq 1$, is also substantially reduced to the model with no recall.

The statement (b) implies that DRV property gradually fades and completely disappears in its limit, implying that if the planning horizon is infinite, then the model with uncertain recall becomes completely identical to the model with no recall which has the infinite planning horizon (Theorem 3. 1 (b)).

3.4 Model with Controlled Recall [25]

This is the model with no recall in which it is assumed that that an offer y once inspected and passed up can be recalled and accepted if some deposit d is paid for it. For convenience let us call an offer appearing at the present time and the best of the offers reserved so far, respectively, the current offer and leading offer. In this model there exist the following four possible decisions: AS: accept the current offer and stop the search, RC: reserve the current offer and continue the search, PS: pass up the current offer, accept the leading offer, and stop the search, and PC: pass up the current offer and continue the search. Available decisions at the time 0 are only AS and PS.

Let $u_t(x, w)$ denote the maximum expected present discounted net value starting from time t with a leading offer x and a current offer w, and let $v_t(x) = \int_0^\infty u_t(x, w)dF(w)$. Then clearly $u_0(x, w) = \max\{w, x\}$ and

$$u_t(x, w) = \max\{w, -d - c + \beta v_{t-1}(\max\{x, w\}), x, -c + \beta v_{t-1}(x)\}, \quad t \geq 1, \tag{3.11}$$

where the four terms inside the braces corresponds to the decisions AS, RC, PS, and PC, respectively, and the two terms inside the braces of $u_0(x, w)$ to the decisions AS and PS, respectively. Then, using the K–function, we can express Eq. (3.11) for $t = 1$ as follwos.

$$u_1(x, w) = \max\{w, -d + K(\max\{x, w\}) + \max\{x, w\}, x, K(x) + x\} \tag{3.12}$$

By $\mathcal{AS}_t$, $\mathcal{RC}_t$, $\mathcal{PS}_t$, and $\mathcal{PC}_t$ let us denote the sets of (x, w) at which decisions AS, RC, PS, and PC becomes the optimal at time t. Then it can be shown that these sets are depicted as in Fig. 3.1, and the following theorem can be proved.

Theorem 3. 4

(a) *If it is optimal to accept a leading offer, then it is only at time 0 (deadline).*

(b) *The reserving region of time $t = 1$, $\mathcal{RC}_1$, is not empty if and only if $d < \max\{a - a, K(a)\}$.*

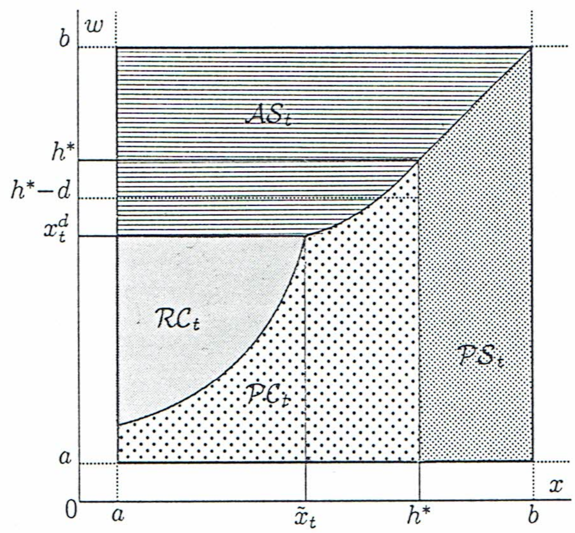

Figure 3.1: $\mathcal{AS}_t$, $\mathcal{RC}_t$, $\mathcal{PS}_t$, and $\mathcal{PC}_t$

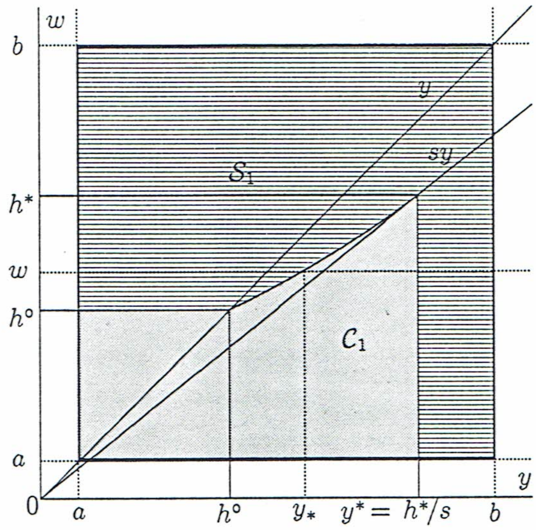

Figure 3.2: DRV at $t = 1$

(c) *As t tends to ∞, the reserving region $\mathcal{RC}_t$ gradually fades and completely disappears in its limit and converges to the rectangle $\{(x, w) \mid a \leq x \leq h^*, a \leq w \leq h^* - d\}$.*

PROOF See [25]. ■

3.5 Model with Recall Cost [14]

This is the model with recall in which it is assumed that the best offer y so far appeared can be recalled and accepted if some cost ry is paid with $0 \leq r \leq 1$. Then the value obtained from it, if it is accepted, is $y - ry = sy$ where $s = 1 - r$.

Let $u_t(w, y)$ be the maximum total expected present discounted net value starting from time $t \geq 0$ with a present offer w and a best offer y so far appeared. Then clearly $u_0(w, y) = \max\{w, sy\}$ and

$$u_t(w, y) = \max\{\max\{w, sy\}, U_t(w, y)\}, \quad t \geq 1, \tag{3.13}$$

$$U_t(w, y) = \beta \int_0^\infty u_{t-1}(\xi, \max\{w, y\}) dF(\xi) - c \tag{3.14}$$

where $\max\{w, sy\}$ is the gain from stopping the search and $U_t(w, y)$ is the maximum total expected present discounted net value from continuing the search. Then clearly we have

$$U_1 = K(s \max\{w, y\}) + s \max\{w, y\}. \tag{3.15}$$

Let $\mathcal{C}_t = \{(w, y) \mid \max\{w, sy\} \leq U_t(w, y)\}$. Then if $(w, sy) \in \mathcal{C}_t$, it is optimal to continue the search, or else stop, so let $\mathcal{C}_t$ be called the continuation region, and its compliment $\mathcal{S}_t = \mathcal{C}_t^c$ the stopping region. If the optimal stopping rule has the following property, let us say that it has a double reservation value property, DRV property, for short.

DRV Property For at least one w with $a < w < b$, there exist two different critical values y_* and y^* with $a < y_* < y^* < b$ such that, for a present offer w, if $y_* \leq y \leq y^*$, then continuing is optimal, or else stopping is optimal.

Then it can be shown that the continuation region $\mathcal{C}_t$ is depicted as Fig. 3.2 where h_t is the solution of $U_t(w, y) - y = 0$ with $w \geq y$, and the following theorem can be proved.

Theorem 3.5

(a) *The necessary and sufficient condition for the DRV property to appear for all* $t \geq 1$ *is given by* $(1-\beta)^2 + c^2 \neq 0$, $s > h^*/b$, *and* $\alpha > a$.

(b) *As* t *tends to* ∞, *the continuation region* C_t *increases and converges to the rectangle* $C = \{(w, y) \mid a \leq w \leq h^*,\ a \leq w \leq h^*/s\}$.

PROOF See [14]. ∎

The statement (b) means that as t tends to ∞, the DRV property gradually fades and completely disappears in its limit.

3.6 Model with Multiple Search Areas [11]

This is the model with no recall only except that there exist multiple search areas where a search area to conduct the search must be determined every point in time. Suppose there exist $N \geq 1$ possible search areas, and let the set of them be $\mathcal{S} = \{1, 2, \ldots, N\}$. When the searcher moves from search area i to j, a travel cost $d_{ij} \geq 0$ is incurred with $d_{ii} = 0$. If paying $s_i \geq 0$ in search area i, then an offer can be obtained. Below, for all $i, j \in \mathcal{S}$ define $c_{ij} = d_{ij} + s_j$, called a travel and search cost. An offer w obtained in search area j is a random variable having a known distribution $F_j(w)$ with a finite expectation μ_j. Sequentially obtained offers $w, w', \cdots$ are assumed to be stochastically independent. Throughout the paper let $\alpha_{ij} = \beta\mu_j - c_{ij} > 0$ for all $i, j \in \mathcal{S}$. The objective here is to maximize the expected present discounted net value, the expected value of an offer accepted minus the total expected travel and search cost. The optimal decision rule in the model consists of the following two rules: *optimal stopping rule*, prescribing how to stop the search by accepting an offer and *optimal selection rule*, stating, if continuing the search, whether or not to conduct the search by staying in the current search area or, if not, which search area to move to. Let

$$K_{ij}(x) = \beta \int_0^\infty \max\{w, x\} dF_j(w) - x - c_{ij}, \tag{3.16}$$

and let the minimum solution of the equation $K_{ij}(x) = 0$, if it exists, be denoted by h_{ij}, and let $h_i^* = \max_{j \in \mathcal{S}} h_{ij}$ and $h^* = h_{i\cdot}^* = \max_{i \in \mathcal{S}} h_i^* = \max_{i,j \in \mathcal{S}} h_{ij}$.

Let $u_t(w, i)$ denote the maximum expected present discounted net value starting from time t in search area i with a current offer w. Then, clearly $u_0(w, i) = w$, and

$$u_t(w, i) = \max\{w, U_t(i)\}, \quad t \geq 1, \tag{3.17}$$

where $U_t(i)$ is the maximum expected present discounted net value when continuing the search, written

$$U_t(i) = \max_{j \in \mathcal{S}} \{\beta \int_0^\infty u_{t-1}(\xi, j) dF_j(\xi) - c_{ij}\} \tag{3.18}$$

in which ξ is the value of an offer that will be obtained in search area j at the next point in time. Rearranging Eq. (3.18) by substituting Eq. (3.17) into yields

$$U_t(i) = \max_{j \in \mathcal{S}} \{K_{ij}(U_{t-1}(j)) + U_{t-1}(j)\}, \quad t \geq 1, \tag{3.19}$$

where $U_1(i) = \max_{j \in \mathcal{S}} \alpha_{ij}$. Supposing an offer w has been obtained at time t in search area i, we can prescribe the optimal decision rules as follows. If $w > U_t(i)$, stop the search by accepting the offer w, or else continue the search. If it is decided to continue the search, the optimal search area of the next point in time is given by the j attaining the maximum of the right hand of Eq. (3.19). Let the j be denoted by $\nu_t(i)$. Then if $\nu_t(i) = i$, it is optimal to continue the search by staying in the current search area i. If d_{ij} is independent of i, then $U_t(i)$, $\nu_t(i)$, and h_{ij} are all independent of i, so let them be denoted by, respectively, U_t, ν_t, and h_j. Then let the limits of U_t and ν_t in t, if they exist, be denoted by U and ν, and let $h_{j^*} = \max_{j\in S} h_j \,(= h^*)$.

Then the following theorem can be proved.

Theorem 3. 6

(a) *$U_t(i)$ is nondecreasing in t and converges as $t \to \infty$ to a limit $U(i) \le h^*$ with $h_{ii} \le U(i)$,*

(b) *If d_{ij} is independent of i, then $U = h^*$ and $\nu = j^*$.*

PROOF See [11]. ■

The statement (b) means that if d_{ij} is independent of i, once entering into search area j^*, it is optimal to continue the search staying in search area j^* till an offer $w \ge h_j^* \,(= h^*)$ appears and it is accepted.

- **Recall Model**

In this model we can also consider the model with recall. Then let $u_t(y, i)$ denote the maximum expected present discounted net value, starting from time t in search area i with the best offer y so far. Then clearly $u_0(y, i) = y$, and $u_t(y, i) = \max\{y, U_t(y, i)\}$ for $t \ge 1$, in which $U_t(y, i)$ is the maximum expected present discounted net value when continuing the search, expressed by

$$U_t(y, i) = \max_{j\in S}\{\beta \int_0^\infty u_{t-1}(\max\{\xi, y\}, j)dF_j(\xi) - c_{ij}\}, \quad t \ge 1, \tag{3.20}$$

where $U_1(y, i) = \max_{j\in S} K_{ij}(y) + y$. Let $z_t(i)$ be the minimum solution of the equation $U_t(y, i) - y = 0$ with unknown y, and let the limit of $U_t(y, i)$ in t be denoted by $U(y, i)$, if it exists. If d_{ij} is independent of i, then $U_t(y, i)$, $u_t(y, i)$, and $U(y, i)$ are all also independent of i, so let them be represented $U_t(y)$, $u_t(y)$, and $U(y)$, respectively. Then the following theorem can be proved.

Theorem 3. 7

(a) *There exists the minimum solution of $U_t(y, i) - y = 0$, $z_t(i) \ge 0$.*

(b) *$u_t(y, i) = U_t(y, i) > y$ for $y < z_t(i)$ and $u_t(y, i) = y \ge U_t(y, i)$ for $z_t(i) \le y$.*

(c) *$z_1(i) = h_i^*$ and $h_i^* \le z_t(i) \le h^*$ for all t, i, hence $z_t(i^*) = h^*$ for all t.*

(d) *If d_{ij} is independent of i, then $u_t(y) = U_t(y) > y$ if $y < h^*$, and $u_t(y) = y \ge U_t(y)$ if $h^* \le y$, and $u(y) = U(y) > y$ if $y < h^*$, and $u(y) = y \ge U(y)$ if $h^* \le y$.*

PROOF See [11] ■

The statement (b) means that the optimal stopping rule can be stated as follows. If $y \ge z_t(i)$, then stop the search by accepting the current best offer y, or else continue the search. The statement (d) implies that, if d_{ij} is independent of i, then, even if the search is conducted in any search areas, it is optimal to accept the current offer w if $w \ge h^*$, or else to continue the search.

3.7 Model with Finite Search Budget [9]

This is the model with recall where a search cost invested at each point in time can be controlled within a given search budget. Suppose that if c dollars out of the search budget is invested in a search activity of each point in time, then at the next point in time an offer can be obtained whose value is a random variable having a known distribution function $F(w|c)$ with a finite expectation $\mu(c)$ where $F(w|c) = 0$ on $w \le 0$ for all $c \ge 0$. Sequentially obtained offers $w, w', \cdots$ are assumed to be stochastically independent. The objective is to maximize an expected present discounted revenue, the expected present discounted value of the sum of the offer w accepted and the search budget remaining at that point.

Let $u_t(i, w)$ denote the maximum of the total expected present discounted revenue starting from time t with a remaining search budget i and a current offer w. Then, clearly $u_0(i, w) = i + w$ and

$$u_t(i, w) = \max\{i + w,\ U_t(i)\}, \quad t \ge 1, \tag{3.21}$$

where $i + w$ is the gain from stopping the search by accepting the offer w and $U_t(i)$ is the maximum of the total expected present discounted revenue, expressed as

$$U_t(i) = \max_{0 \le c \le i} \beta \int_0^\infty u_{t-1}((i-c)/\beta, \xi) dF(\xi|c). \tag{3.22}$$

For any real numbers $i \ge 0$ and x define

$$K(i, x) = \max_{0 \le c \le i} \{\beta \int_0^\infty \max\{w, x\} dF(w|c) - c\} - x. \tag{3.23}$$

Then we have $U_1(i) = K(i, 0) + i$. Now define $V_t(i) = U_t(i) - i$ and $V_t = \lim_{i \to \infty} V_t(i)$. Then the decision strategy in this model can be stated as follows. If $w > V_t(i)$, stop the search by accepting the offer w, or else continue the search. The optimal investment $c_t(i)$ is provided by the c attaining the maximum of the left hand of Eq. (3.22). Fig. 3.3 is a numerical example of $V_t(i)$ and $c_t(i)$. Then the following theorem can be proved.

Theorem 3. 8
(a) *If $K(\infty, 0) = 0$, then $V_t(i) = 0$ for all t and i.*
(b) *If $\beta < 1$, then V_t converges to $h^* < \infty$ as $t \to \infty$,*

Proof See [9]. ■

- **Model with recall**

We can also consider the model with recall. In this case let $u_t(i, y)$ denote the maximum expected present discounted value starting from time t with a remaining search budget i and a present best offer y. Then, clearly $u_0(i, y) = i + y$ and

$$u_t(i, y) = \max\{i + y, U_t(i, y)\}, \quad t \ge 1, \tag{3.24}$$

where $i+y$ is the gain from stopping the search by accepting the best offer y and $U_t(i, y)$ is the maximum of the total expected present discounted revenue, expressed as

$$U_t(i, y) = \beta \max_{0 \le c \le i} \int_0^\infty u_{t-1}\big((i-c)/\beta, \max\{y, \xi\}\big) dF(\xi|c), \tag{3.25}$$

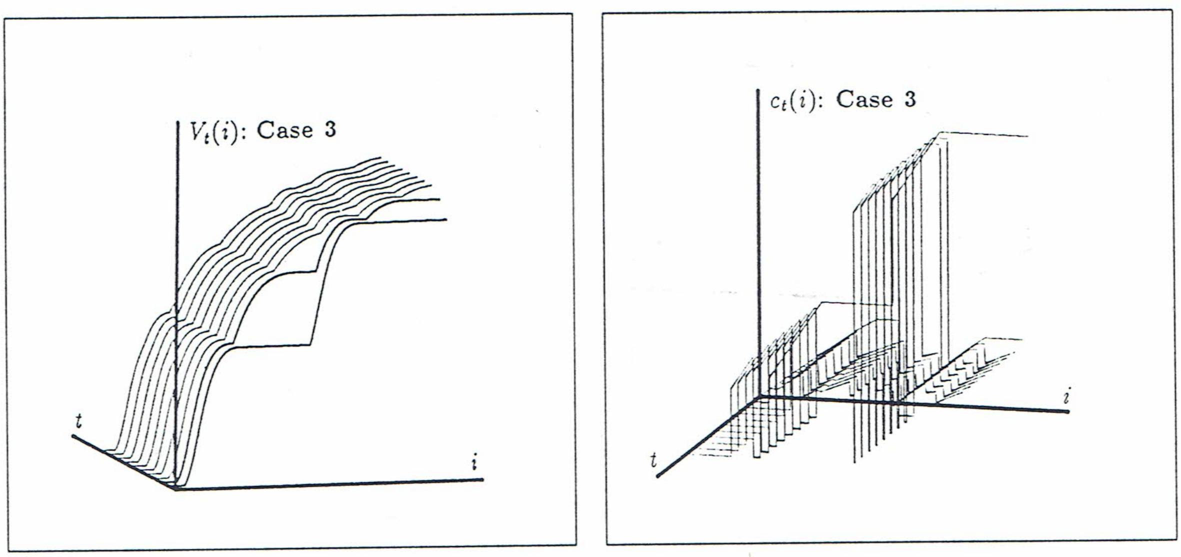

Figure 3.3: Reservation value $V_t(i)$ and Optimal investment $c_t(i)$

where $U_1(i,y) = K(i,y) + y + i$. Now define $V_t(i,y) = U_t(i,y) - i$ and $V_t(y) = \lim_{i\to\infty} V_t(i,y)$. Then we have the following theorem.

Theorem 3. 9

(a) *Suppose $K(\infty, 0) = 0$. Then $V_t(i,y) \le y$ for all t, i, and y.*

(b) *Suppose $K(\infty, 0) > 0$. If $\beta = 1$, then $V_t(i,y) \ge y$ for all t, i, and y, and if $\beta < 1$, $V_t(i,y)$ is upper-bounded in t and i for all y.*

(c) *Suppose $K(\infty, 0) > 0$ and $\beta < 1$. Then we have $V(y) = h^* < \infty$ for $y \le h^*$, and if $h^* \le y$, then $V_t(y) \le y$, hence $v_t(y) = y$, and if $y \le h^*$, then $V_t(y) \ge y$, hence $v_t(y) = V_t(y)$.*

PROOF See [9]. ■

3.8 Model of Pandora Type [29]

Suppose there are N closed boxes. Each box i contains an unknown reward w with a known distribution function $F_i(w)$ having a finite expectation μ_i. The cost c_i is incurred in opening box i, and the reward in it becomes known after the time lag t_i since opening box i. Both cost and reward are continuously discounted by an instant discount rate $\alpha > 0$, implying that one unit of monetary value after time periods t is equivalent to $e^{-\alpha t}$ at that time. Below let $\beta_i = e^{-\alpha t_i}$. If y is the best of rewards in boxes opened so far and is accepted, then the process terminates. If the search is terminated after opening some or all boxes, then the maximum rewards in boxes opened up to that point, including the initial reward y_0, must be accepted as a gain.

The objective is to find the optimal decision strategy so as to maximize the expected present discounted net value, the expectation of the present value of the maximum rewards gained after having terminated the search minus the total present value of costs incurred in opening boxes to that point in time. The optimal decision rule in the model consists of the following two rules: *optimal stopping rule*, prescribing when to stop the search by accepting an offer and *optimal selection rule*, stating, if continuing the search, which box should be opened.

Let us denote a set of the N boxes by $\mathcal{S} = \{1, 2, \cdots, N\}$, and by $u(\mathcal{S}, y)$ we shall denote the maximum expected present discounted net value starting with the set of boxes $\mathcal{S}$ and a maximum reward y sampled so far. Then, clearly $u(\Phi, y) = y$, and for $\mathcal{S} \ne \Phi$

$$u(\mathcal{S}, y) = \max_{i\in\mathcal{S}}\{y, \beta_i \int_{-\infty}^{\infty} u(\mathcal{S}_i, \max\{y,\xi\}) dF_i(\xi) - c_i\}. \tag{3.26}$$

where $\mathcal{S}_i = \mathcal{S} - \{i\}$. For any $\mathcal{S}$ letting

$$U_i(\mathcal{S}_i, y) = \beta_i \int_{-\infty}^{\infty} u(\mathcal{S}_i, \max\{y,\xi\}) dF_i(\xi) - y - c_i, \tag{3.27}$$

we can express Eq. (3.26) as

$$u(\mathcal{S}, y) = y + \max_{i\in\mathcal{S}}\{0, U_i(\mathcal{S}_i, y)\}. \tag{3.28}$$

Then clearly

$$U_i(\Phi, y) = K_i(y) \tag{3.29}$$

where for any real number y

$$K_i(y) = \beta_i \int_{-\infty}^{\infty} \max\{y,\xi\} dF_i(\xi) - y - c_i. \tag{3.30}$$

Let the minimum solution of $K_i(y) = 0$ be h_i. Here without loss of generality let $h_1 \ge h_2 \ge \cdots \ge h_N$. Then the optimal decision strategy is given by the following theorem.

Theorem 3.10

Stopping rule The search is to be terminated whenever the maximum sampled reward so far is greater than or equal to the maximum among the h_j's of closed boxes.

Selection rule If a box is to be opened, it should be that box with the maximum h_i among the boxes that have not yet been opened.

PROOF The complete proof of the theorem is given by Weitzman [29], the way of which is very technical and complicated, so in Appendix II the author rewrite the proof with supplying omitted interpretations in the original paper. In Appendix I let us show a conventional way of proof using the K–function. ■

3.9 Sequential Assignment Problem I [1]

Assume that i jobs arrive at a group of i workers one by one. If worker x, $1 \le x \le i$, is assigned to an arriving job of value w, the group can obtain the reward of $r_x w$ where $r_1 \le r_2 \le \cdots \le r_i$. Let $\boldsymbol{r}_i = (r_1, r_2, \cdots, r_i)$, i–vector. The value of each arriving job is assumed to be an independent identically distributed random variable having a known distribution $F(w)$ with an expectation $\mu < \infty$. A worker assigned to a job is unavailable for future assignment. The objective is to find the optimal assignment rule maximizing the total expected reward obtained. For convenience let us refer to the problem as the $\boldsymbol{r}_i$-problem.

By $v(w, \boldsymbol{r}_i)$ we shall denote the maximum total expected reward, provided that a job of value w has arrived. Then, clearly $u(w, \boldsymbol{r}_1) = r_1 w$ and

$$u(w, \boldsymbol{r}_i) = \max_{1 \le x \le i} \{ r_x w + \int_0^\infty v(\xi, \boldsymbol{r}_i(x)) dF(\xi) \}, \quad i \ge 2, \tag{3.31}$$

where if $i \ge 2$, then $\boldsymbol{r}_i(x)$ is the $(i-1)$-vector resulting from removing the x-th element of the i-vector $\boldsymbol{r}_i$. Let

$$v(\boldsymbol{r}_i) = \int_0^\infty v(w, \boldsymbol{r}_i) dF(w). \tag{3.32}$$

Then Eq. (3.31) can be expressed as

$$u(w, \boldsymbol{r}_i) = \max_{1 \le x \le i} \{ r_x w + v(\boldsymbol{r}_i(x)) \}. \tag{3.33}$$

The optimal decision rule is given by the following theorem.

Theorem 3.11 *For $i \ge 2$ we have*

(a) *The optimal assignment rule is to assign the worker x if $c_i(x) < w \le c_i(x+1)$ with $c_i(1) = -\infty$ and $c_i(i+1) = \infty$.*

(b) *The $c_i(x)$ satisfy the following equations where $c_i(x)$ is nondecreasing in x for all $i \ge 2$.*

$$c_i(x) = S(c_{i-1}(x-1)) - T(c_{i-1}(x)), \quad 2 \le x \le i. \;\; i \ge 2, \tag{3.34}$$

PROOF See [5]. Another proof by use of T-function and S-function is given in Appendix. ■

3.10 Sequential Assignment Problem II

Let us consider the following variation of the sequential assignment problem I in Section 3.9. The decision process proceeds on the discrete time axis without deadline, that is, its planning horizon is infinite. If a fixed cost c (search cost) is paid at each point in time, then a job can be obtained, and if its value is not so high, it can be rejected. Then

$$u(w, r_i) = \max\{M(w, r_i), \beta v(r_i) - c\} \tag{3.35}$$

where $M(w, r_i)$ and $\beta v(r_i) - c$ are the maximum expected total present discounted rewards, respectively, from accepting an arriving job w and assigning a worker to it and from rejecting it. Clearly $M(w, r_1) = r_1 w$ and

$$M(w, r_i) = \max_{1 \le x \le i}\{r_x w + \beta v(r_i(x)) - c\}, \quad i \ge 2. \tag{3.36}$$

Let $V(r_i) = \beta v(r_i) - c$. Then Eqs. (3.35) and (3.36)can be written

$$u(w, r_i) = \max\{M(w, r_i), V(r_i)\}, \quad i \ge 1, \tag{3.37}$$

$$M(w, r_i) = \max_{1 \le x \le i}\{r_x w + V(r_i(x))\}, \quad i \ge 2. \tag{3.38}$$

Suppose $i \ge 2$. Then, using the generalized K–function defined by Eq. (2.23), from Eq. (3.37) we have

$$K(M(-, r_i), V(r_i)) = 0. \tag{3.39}$$

Consequently, letting the minimum solution of $K(M(-, r_i), z) = 0$ be denoted by $h(r_i)$, we have

$$V(r_i) = h(r_i). \tag{3.40}$$

Here note that $h(r_i(x, y)) = h(r_i(y, x))$ for any x and y with $x \ne y$. Then Eqs. (3.37) and (3.38)can be rewritten, respectively,

$$u(w, r_i) = \max\{M(w, r_i), h(r_i)\}, \tag{3.41}$$

$$M(w, r_i) = \max_{1 \le x \le i}\{r_x w + h(r_i(x))\} \tag{3.42}$$

where $M(w, r_i)$ is nondecreasing and concave in w. Then we have the following theorem.

Theorem 3. 12
(a) *$M(w, r_i(x))$ is nonincreasing in x for all w and $i \ge 2$.*
(b) *$h(r_i(x))$ is nonincreasing in x.*

PROOF See Appendix. ■

When $i = 1$, Eq. (3.39) can be rewritten as follows.

$$r_1 \beta \int_0^\infty \max\{w, V(r_1)/r_1\} dF(w) - V(r_1) - c = 0. \tag{3.43}$$

Let us define

$$K_1(z) = \beta \int_0^\infty \max\{w, z\} dF(w) - z - c/r_1. \tag{3.44}$$

Then Eq. (3.43) becomes $K_1(V(r_1)/r_1) = 0$. Letting the minimum solution of $K_1(x) = 0$ be denoted by h_1, we have $V(r_1)/r_1 = h_1$, or $V(r_1) = r_1 h_1$. Consequently, if $i \ge 2$, then we have

$$M(w, r_2) = \max\{r_1 w + r_2 h_2, r_2 w + r_1 h_1\}. \tag{3.45}$$

Let $c_2(2, r_2) = (r_2 h_2 - r_1 h_1)/(r_2 - r_1)$ if $r_2 > r_1$. Then, if $M(w, r_2) > h(r_2)$, it is optimal to accept the arriving job w, and in this case, if $w < c_2(r_2)$, it is optimal to assign the worker 1 to it, or else worker 2. If $M(w, r_2) \le h(r_2)$, it is optimal to reject it.

3.11 Markovian Decision Process with Random Observations [10]

Consider a decision process whose state at each point in time is characterized by a pair (i, w) of two vectors i and w, called the first-state and second-state, respectively. The second-state w is a random observation, which is a random sample from an i-dependent distribution $F_i(w)$ with a sample space Ω_i and a finite expectation μ_i. Let the action that can be taken when in first-state i be designated by $x \in \mathcal{A}(i) = \{1, 2, \cdots, k_i\}$ with $k_i \geq 1$ is a given integer. If an action x is taken in state (i, w), then an immediate reward $r(i, w, x)$ is obtained. If an action x is taken in state (i, w), the current first-state i changes into j at the next point in time with probability $p(j|i, x)$. Let a discount factor be $\beta \in (0, 1]$.

The objective here is to find the optimal decision policy attaining the maximum of the total expected present discounted reward, the expectation of the sum of immediate rewards obtained at each point in time over the finite planning horizon.

By $u_t(i, w)$ we shall denote the maximum of the total expected present discounted reward starting from time t when in state (i, w). The $u_0(i, w)$ are usually appropriately defined; in many cases, $u_0(i, w) = \max_{x \in \mathcal{A}(i)} r(i, w, x)$, and for $t \geq 1$ we have

$$u_t(i, w) = \max_{x \in \mathcal{A}(i)} r(i, w, x) + \beta \sum_{j \in \mathcal{I}} p(j|i, x) v_{t-1}(j), \quad t \geq 1, \tag{3.46}$$

where

$$v_t(i) = \int_{w \in \Omega_i} u_t(i, w) dF_i(w). \tag{3.47}$$

Let w be a scalar random variable with a distribution function $F_i(w)$ where $F_i(w) = 0$ for $w \leq 0$, and let $r(i, w, x) = r(i, x)w + e(i, x)^\dagger$ where $r(i, x)$ is assumed to be either strictly increasing or strictly decreasing in x for all i, and define

$$z_t(i, x) = e(i, x) + \beta \sum_{j}^{N} p(j|i, x) v_{t-1}(j), \tag{3.48}$$

$$c_t(i, x) = -\Delta z_t(i, x)/\Delta r(i, x), \tag{3.49}$$

$$c_t(i) = -\tilde{\Delta} z_t(i)/\tilde{\Delta} r(i), \tag{3.50}$$

$$\Delta z_t(i, x) = z_t(i, x) - z_t(i, x-1), \tag{3.51}$$

$$\Delta r_t(i, x) = r_t(i, x) - r_t(i, x-1), \tag{3.52}$$

$$\tilde{\Delta} z_t(i) = z_t(i, k_i) - z_t(i, 1), \tag{3.53}$$

$$\tilde{\Delta} r_t(i) = r_t(i, k_i) - r_t(i, 1), \tag{3.54}$$

Then we have the following theorem.

Theorem 3. 13 *For certain given t and i we have*

(a) *Suppose $\Delta r(i, x)$ and $\Delta z(i, x)$ are both nonincreasing in x.*

1. *Assume $r(i, x)$ is strictly increasing in x and let $c_t(i, 1) = -\infty$ and $c_t(i, k_i + 1) = \infty$. Then $c_t(i, x)$ is nondecreasing in x, and if $c_t(i, x) < w \leq c_t(i, x+1)$, then the optimal solution is x, and*

$$v_t(i) = r(i, 1)\mu_i + z_t(i, 1) + \sum_{x=2}^{k_i} \Delta r(i, x) T_i(c_t(i, x)). \tag{3.55}$$

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† In the original paper [10], a general case is discussed.

2. *Assume* $r(i,x)$ *is strictly decreasing in* x *and let* $c_t(i,1) = \infty$ *and* $c_t(i,k_i+1) = -\infty$. *Then* $c_t(i,x)$ *is nonincreasing in* x, *and if* $c_t(i,x+1) \le w < c_t(i,x)$, *the optimal solution is* x, *and*

$$v_t(i) = r(i,k_i)\mu_i + z_t(i,k_i) - \sum_{x=2}^{k_i} \Delta r(i,x) T_i(c_t(i,x)). \tag{3.56}$$

(b) *Suppose* $\Delta r(i,x)$ *and* $\Delta z(i,x)$ *are both nondecreasing in* x.

1. *Assume* $r(i,x)$ *is strictly increasing in* x. *Then, if* $w \le c_t(i)$, *the optimal solution is* $x=1$, *or else* $x = k_i$, *and*

$$v_t(i) = r(i,1)\mu_i + z_t(i,1) + \widetilde{\Delta} r(i) T_i(c_t(i)). \tag{3.57}$$

If $c_t(i,x) < w \le c_t(i,x+1)$, *then the optimal solution is* x.

2. *Assume* $r(i,x)$ *is strictly decreasing in* x. *Then, if* $w \le c_t(i)$, *the optimal solution is* $x=1$, *or else* $x = k_i$, *and*

$$v_t(i) = r(i,k_i)\mu_i + z_t(i,k_i) - \widetilde{\Delta} r(i) T_i(c_t(i)). \tag{3.58}$$

PROOF See [10]. ■

3.12 Newsboy Problem

Consider the following conventional newsboy problem. Let c be a purchasing price per copy, p a selling price per copy, d a disposing price per copy left-over, and s a shortage cost per copy where $p > c > d$. Let the number θ of customers who come to buy the papers to him every morning be identically distributed random variable having a known distribution $F(w)$ with a finite expectation μ. The objective is to find the optimal purchasing quantity attaining the maximum of the every day's expected profit. Suppose he has decided to purchase x papers from a newspaper office every morning. Then, the expected profit of every day can be expressed by

$$v(x) = \int_0^\infty \{p\min\{\theta,x\} - cx - s\max\{\theta - x, 0\} + d\max\{x-\theta,0\}\} dF(\theta). \tag{3.59}$$

Arranging Eq. (3.59) by using the formula $\max\{x-\theta,0\} = \max\{\theta - x, 0\} + x - \theta$ and $\min\{\theta,x\} = \theta - \max\{\theta - x, 0\}$ yields

$$v(x) = (p-d)\mu - T(x,\gamma,\delta) \tag{3.60}$$

where $\gamma = c - d$ and $\delta = p + s - d$. Clearly $\gamma < \delta$ due to the assumption of $c < p$. Then we have

Theorem 3.14 *$v(x)$ is maximized at x^* such that* $F(x^*-1) < (p+s-c)/(p+s-d) \le F(x^*)$.

PROOF Immediate from Lemma 2. 3 (f). ■

Appendix I

■ Proof of Theorem 3. 1 (Model with Recall)

(a) From Eq. (3.2) we have $U_t \geq \beta \int \xi dF(\xi) - c = \alpha = U_1$ for any $t \geq 1$, hence $U_2 \geq U_1$. Suppose $U_t \geq U_{t-1}$. Then, from Lemma 2. 1 (b) we have $U_{t+1} = K(U_t) + U_t \geq K(U_{t-1}) + U_{t-1} = U_t$. Thus it follows by induction that U_t is nondecreasing in $t \geq 1$. From Eq. (3.3) we have $U_t - U_{t-1} = K(U_{t-1})$, which is nonincreasing in t from the above result and Lemma 2. 1 (a), so U_t is concave in t.

(b) First note $U_1 \geq \alpha$, hence $U_t \geq \alpha$ for all t from (a). Next clearly $U_1 = \alpha \leq \mu < b$. If $U_{t-1} < b$, then from Eq. (3.3) and Lemma 2. 1 (b) we have $U_t < K(b) + b \leq b$ due to $K(b) \leq 0$ from Eq. (2.10) . Hence it follows by induction that $U_t < b$ for all $t \geq 1$. Consequently since U_t is upper-bounded in t, the U_t converges to a finite number $U \leq b$ as $t \to \infty$, so we get $0 = K(U)$ from Eq. (3.3). Hence, if $(1-\beta)^2 + c^2 \neq 0$, then $U = h^*$ from Lemma 2. 2 (c1). Suppose $(1-\beta)^2 + c^2 = 0$. Then, if $U < b$, we have the contradiction of $0 = K(U) > K(b) = 0$ from Lemma 2. 1 (a) and Eq. (2.10). Therefore, from Lemma 2. 2 (b) it must be $U = b = h^*$.

(c) It is immediate from Eq. (3.1) and (b).

(d) Suppose $\alpha \leq a$. Then clearly $U_1 = \alpha \leq a$. If $U_{t-1} = \alpha \leq a$, then from Eq. (3.3) and Eq. (2.10) we have $U_t = \alpha - U_{t-1} + U_{t-1} = \alpha \leq a$.

(e) Assume $\alpha > a$. Then $a < U_t < b$ for all t from (b). Since $u_1(\xi) > \xi$ for $a < \xi < U_1 < b$ and $u_1(\xi) = \xi$ for $a < U_1 \leq \xi \leq b$, we have $U_2 > \beta \int_0^\infty \xi dF(\xi) - c = \alpha = U_1$. Suppose $U_{t-1} > U_{t-2}$ $(> a)$. Then $U_t > K(U_{t-2}) + U_{t-2} = U_{t-1}$ from Lemma 2. 1 (b). Therefore, it follows by induction that U_t is strictly increasing in t, from which we have $U_t < U = h^*$ for all $t \geq 1$. The strict concavity is immediate from the fact that $U_t - U_{t-1} = K(U_t)$ is strictly decreasing in t from the above result and Lemma 2. 1 (a).

■ Proof of Theorem 3. 2 (Model with no Recall)

(a) Easily proved by induction starting with $U_1(y)$ being nondecreasing in y from Lemma 2. 1 (b) and $U_2(y) \geq \beta \int \max\{y, \xi\} dF(\xi) - c = K(y) + y = U_1(y)$.

(b) The assertions for $t = 1$ are immediate from Eq. (3.6), Corollary 2. 2 (a), and $U_1(y) \leq K(h^*) + h^* = h^*$ for $y \leq h^*$ due to Lemma 2. 1 (b). Hence, if $y \leq h^*$, then $y \leq U_t(y)$ for all t from (a). Assume that the assertions are true for $t-1$, hence $u_{t-1}(y) \leq \max\{y, h^*\}$ for all y. Thus we have $u_{t-1}(\max\{y, \xi\}) \leq \max\{\max\{y, \xi\}, h^*\} = \max\{\xi, \max\{y, h^*\}\}$ for all y and ξ. Arranging Eq. (3.5) by substituting the inequality yields $U_t(y) \leq K(\max\{y, h^*\}) + \max\{y, h^*\}$ for all y. Hence, if $h^* \leq y$, then $U_t(y) \leq K(y) + y \leq y$ due to $K(y) \leq 0$, and if $y \leq h^*$, then $U_t(y) \leq K(h^*) + h^* = h^*$.

(c) If $y \geq h^*$, then $\max\{y, \xi\} \geq h^*$ for all ξ, hence $u_{t-1}(\max\{y, \xi\}) = \max\{y, \xi\}$ for all ξ from (b). Therefore arranging Eq. (3.5) by substituting this yields $U_t(y) = K(y) + y$. If $y \leq h^*$, then $U_t(y)$ is upper bounded in t from (b), hence $U_t(y)$ converges as $t \to \infty$ from (a).

(d) Immediate from (b).

(e) 1. First suppose $h^* < b$, hence $F(h^*) < 1$. Now, if $y \leq h^* \leq \xi$, then $\max\{y, \xi\} = \xi \geq h^*$, hence Eq. (3.7) can be expressed as

$$U(y) = \beta \int_0^{h^*} u(\max\{y, \xi\}) dF(\xi) + \beta \int_{h^*}^\infty u(\max\{y, \xi\}) dF(\xi) - c \tag{3.61}$$

$$= \beta \int_0^{h^*} u(\max\{y, \xi\}) dF(\xi) + \beta \int_{h^*}^\infty \xi dF(\xi) - c. \tag{3.62}$$

In order to prove the assertion, it suffices to show that (i) $U(y) = h^*$ with $y \leq h^*$ is the solution of the equation Eq. (3.62) and (ii) the solution is unique.

First, let us prove (i). For this, we show that, when arranging the right hand side of Eq. (3.62) by substituting $U(y) = h^*$ with $y \leq h^*$, the resulting expression becomes equal to h^*. Let the right hand side of Eq. (3.62) be designated by $R(y)$. If $\xi \le h^*$, then since $\max\{y,\xi\} \le h^*$, we have $u(\max\{y,\xi\}) = h^*$. Thus we have

$$\begin{aligned} R(y) &= \beta \int_0^{h^*} h^* I(\xi \le h^*) dF(\xi) + \beta \int_{h^*}^{\infty} \xi dF(\xi) - c \\ &= \beta \int_0^{\infty} \max\{h^*, \xi\} dF(\xi) - c \\ &= K(h^*) + h^* = h^*. \end{aligned}$$

Thus, the proof of (i) is complete.

Next, let us prove (ii). Suppose there exist another solution $Z(y)$ such that $|Z(y)| < \infty$ for $y \le h^*$, and let $\Delta = \sup_{y \le h^*} |U(y) - Z(y)|$ where $0 < \Delta < \infty$. Then

$$Z(y) = \beta \int_0^{h^*} z(\max\{y,\xi\}) dF(\xi) + \beta \int_{h^*}^{\infty} \xi dF(\xi) - c \tag{3.63}$$

where $z(\max\{y,\xi\}) = \max\{\max\{y,\xi\}, Z(\max\{y,\xi\})\}$. Taking the difference of Eq. (3.62) and Eq. (3.63) leads to

$$\begin{aligned} |U(y) - Z(y)| &\le \beta \int_0^{h*} |u(\max\{y,\xi\}) - z(\max\{y,\xi\})| dF(\xi) \\ &\le \beta \int_0^{h^*} \Delta dF(\xi) \le \Delta F(h^*), \end{aligned}$$

from which we get $\Delta \le \Delta F(h^*)$, yielding the contradiction of $1 \le F(h^*)$. Consequently, the solution must be unique.

2. Next assume $h^* = b$, so $\beta = 1$ and $c = 0$ from Corollary 2.1 . Note that for any y and ξ we have $u(\max\{y,\xi\}) = \max\{\max\{y,\xi\}, U(\max\{y,\xi\})\} \ge \max\{\xi, U(y)\}$ because $\max\{y,\xi\} \ge \xi$ and $U(\max\{y,\xi\}) \ge U(y)$ from $\max\{y,\xi\} \ge y$ and (a). Hence we have $U(y) \ge \int_0^{\infty} \max\{\xi, U(y)\} dF(\xi)$, from which we get $0 \ge \int_0^{\infty} \max\{\xi - U(y), 0\} dF(\xi) \ge 0$; that is, $\int_0^{\infty} \max\{\xi - U(y),\ 0\} dF(\xi) = 0$ for all y. This implies $\xi \le U(y)$ for all ξ on $[a,b]$; hence it must be $U(y) \ge b = h^*$ for all y. From this and $U(y) \le h^*$ for $y \le h^*$ due to (d) it must follows that $U(y) = h^*$.

(f) From (d) and (e) if $y \ge h^*$, then $u(y) = y$ and $U(y) = K(y) + y \ge K(h^*) + h^* = h^*$, and if $y \le h^*$, then $u(y) = \max\{y, h^*\} = h^*$ and $U(y) = h^* = K(h^*) + h^* \ge K(y) + y$. Therefore we have $u(y) = \max\{y, h^*\}$ and $U(y) = \max\{K(y) + y, h^*\}$.

■ Another Proof of Theorem 3.10 (Model of Pandora Type)

For convenience let $F_i(w)$ be a continuous distribution function for all i. In order to prove the theorem, it suffices to show the following two points.

(a) *If* $y \ge h_1$, *then* $U_i(\mathcal{S}_i, y) \le 0$ *for all* $i \in \mathcal{S}$,

(b) *If* $y < h_1$, *then* $\max_{i \in \mathcal{S}} U_i(\mathcal{S}_i, y) = U_1(\mathcal{S}_1, y) > 0$.

For the proof of the theorem, the following two supplementary statements must be also proved.

(c) $U_1(\mathcal{S}_1, h_i) - U_i(\mathcal{S}_i, h_i) \ge 0$ *for all* i.

(d) $U_1(\mathcal{S}_1, y)$ *is nonincreasing in* y.

The statement (d) can be easily proved by induction. Here note that if (a) and (b) are both are true, then Eq. (3.28) can be expressed as follows.

$$u(\mathcal{S}, y) = y + \max\{0, U_1(\mathcal{S}_1, y)\}. \tag{3.64}$$

If $\mathcal{S}$ consists of more than one boxes, then by $\mathcal{S}_{ij}$, $i \ne j$, let us denote the set resultant from removing element i and then j from $\mathcal{S}$. Then from Eq. (3.64) we have

$$u(\mathcal{S}_1, y) = y + \max\{0, U_2(\mathcal{S}_{12}, y)\}, \tag{3.65}$$
$$u(\mathcal{S}_i, y) = y + \max\{0, U_1(\mathcal{S}_{i1}, y)\}, \quad 2 \le i \le N. \tag{3.66}$$

Arranging Eq. (3.27) by substituting the above yields

$$U_1(\mathcal{S}_1, y) = K_1(y) + \beta_1 \int_{-\infty}^{\infty} \max\{0, U_2(\mathcal{S}_{12}, \max\{y, \xi\})\} dF_1(\xi), \tag{3.67}$$
$$U_i(\mathcal{S}_i, y) = K_i(y) + \beta_i \int_{-\infty}^{\infty} \max\{0, U_1(\mathcal{S}_{i1}, \max\{y, \xi\})\} dF_i(\xi), \quad 2 \le i \le N. \tag{3.68}$$

If $y < h_1$, then it is clear from Corollary 2. 2 (a) that $U_1(\mathcal{S}_1, y) > 0$. Before proceeding to the proof, note that without proof clearly $U_1(\widetilde{\mathcal{S}}, y) \le U_1(\mathcal{S}, y)$ if $\widetilde{\mathcal{S}} \subseteq \mathcal{S}$.

• First from Eq. (3.29) we have for any y

$$U_1(\{1\}_1, y) = K_1(y), \tag{3.69}$$
$$U_2(\{2\}_2, y) = K_2(y). \tag{3.70}$$

Thus, in this case it is clear from Corollary 2. 2 (a) that (a) and (b) hold true. Therefore from Eq. (3.64) we have

$$u(\{1\}, y) = y + \max\{0, U_1(\{1\}_1, y)\}, \tag{3.71}$$
$$u(\{2\}, y) = y + \max\{0, U_2(\{2\}_2, y)\}. \tag{3.72}$$

• Suppose $\mathcal{S}$ consists of more than one boxes. Then, noting $\{1,2\}_{12} = \{2\}_2$ and $\{1,2\}_{21} = \{1\}_1$, we have

$$U_1(\{1,2\}_1, y) = K_1(y) + \beta_1 \int_{-\infty}^{\infty} \max\{0, U_2(\{2\}_2, \max\{y, \xi\})\} dF_1(\xi), \tag{3.73}$$
$$U_2(\{1,2\}_2, y) = K_2(y) + \beta_2 \int_{-\infty}^{\infty} \max\{0, U_1(\{1\}_1, \max\{y, \xi\})\} dF_2(\xi). \tag{3.74}$$

where from Eq. (3.69)

$$U_1(\{1\}_1, \max\{y, \xi\}) = K_1(\max\{y, \xi\}), \tag{3.75}$$
$$U_2(\{2\}_2, \max\{y, \xi\}) = K_2(\max\{y, \xi\}). \tag{3.76}$$

□ *Proof of* (a) Suppose $y \ge h_1$ $(\ge h_2)$. Then, since $\max\{y, \xi\} \ge h_1$ $(\ge h_2)$ for all ξ, from Corollary 2. 2 (a) we have $U_1(\{1\}_1, \max\{y, \xi\}) \le 0$ and $U_2(\{2\}_2, \max\{y, \xi\}) \le 0$ for all ξ. Thus it follows from Eqs. (3.73) and (3.74)that if $y \ge h_1$, then

$$U_1(\{1,2\}_1, y) = K_1(y) \le 0, \tag{3.77}$$
$$U_2(\{1,2\}_2, y) = K_2(y) \le 0. \tag{3.78}$$

□ *Proof of* (b) Suppose $y < h_1$. If $y \ge h_2$, then since $\max\{y, \xi\} \ge h_2$ for all ξ, we have $U_2(\{2\}_2, \max\{y, \xi\}) \le 0$. If $y < h_2$, then $\xi \le y$ leads to $U_2(\{2\}_2, \max\{y, \xi\}) = K_2(y) > 0$, $y < \xi \le h_2$ to $U_2(\{2\}_2, \max\{y, \xi\}) = K_2(\xi) \ge 0$, and $h_2 < \xi$ to $U_2(\{2\}_2, \max\{y, \xi\}) = K_2(\xi) \le 0$. Hence Eq. (3.73) can be arranged as

$$U_1(\{1,2\}_1, y) = K_1(y) + \beta_1 \Big(K_2(y) F_1(y) + \int_y^{h_2} K_2(\xi) dF_1(\xi) \Big) I(y < h_2). \tag{3.79}$$

If $\xi \le y$ $(< h_1)$, then $U_1(\{1\}_1, \max\{y, \xi\}) = K_1(y) > 0$, and if $y < \xi \le h_1$, then $U_1(\{1\}_1, \max\{y, \xi\}) =$

$K_1(\xi) \geq 0$, and if $h_1 < \xi$, then $U_1(\{1\}_1, \max\{y, \xi\}) = K_1(\xi) \leq 0$. Hence Eq. (3.74) can be expressed as

$$U_2(\{1,2\}_2, y) \;=\; K_2(y) + \beta_2\Big(K_1(y)F_2(y) + \int_y^{h_1} K_1(\xi)dF_2(\xi)\Big). \tag{3.80}$$

Differentiating Eqs. (3.79) and (3.80)with respect to y by using Lemma 2. 3 (d) produces, respectively,

$$\begin{aligned} U_1'(\{1,2\}_1, y) &= \beta_1 F_1(y) + \beta_1 F_1(y)(\beta_2 F_2(y) - 1)I(y < h_2) - 1 & (3.81)\\ &= \beta_1 F_1(y) I(h_2 \leq y) + \beta_1 F_1(y)\beta_2 F_2(y) I(y < h_2) - 1, & (3.82)\\ U_2'(\{1,2\}_2, y) &= \beta_1\beta_2 F_1(y)F_2(y) - 1, & (3.83)\end{aligned}$$

from which we have

$$U_1'(\{1,2\}_1, y) - U_2'(\{1,2\}_2, y) = \beta_1 F_1(y)(1 + (\beta_2 F_2(y) - 1)I(y < h_2) - \beta_2 F_2(y)) \tag{3.84}$$

where the right hand side is equal to 0 on $y < h_2$ and nonnegative on $h_2 \leq y\,(< h_1)$. Hence the difference $U_1(\{1,2\}_1, y) - U_2(\{1,2\}_2, y)$ is constant on $y < h_2$ and nondecreasing on $h_2 \leq y\,(< h_1)$. Consequently, to complete the proof of (b) it suffices to verify (c), i.e., $U_1(\{1,2\}_1, h_2) - U_2(\{1,2\}_2, h_2) \geq 0$.

□ *Proof of* (c) From Eq. (3.74), noting that $\beta_2 \leq 1$, $U_1(\{1\}_1, y)$ is nonincreasing in y, and $U_1(\{1\}_1, h_2) = K_1(h_2) \geq K_1(h_1) = 0$, we get

$$\begin{aligned} U_2(\{1,2\}_2, h_2) &= \beta_2 \int_{-\infty}^{\infty} \max\{0, U_1(\{1\}_1, \max\{h_2, \xi\})\} dF_2(\xi)\\ &\leq \int_{-\infty}^{\infty} \max\{0, U_1(\{1\}_1, h_2)\} dF_2(\xi)\\ &= U_1(\{1\}_1, h_2)\\ &\leq U_1(\{1,2\}_1, h_2). \end{aligned} \tag{3.85}$$

Therefore we obtain $U_1(\{1,2\}_1, h_2) - U_2(\{1,2\}_2, h_2) \geq 0$.

- Suppose (a) to (d) hold true for $\mathcal{S}$'s consisting of 2 box, 3 boxes, ..., and $N-1$ boxes.

□ *Proof of* (a) Suppose $y \geq h_1$ ($\geq h_2$). Then, since $\max\{y, \xi\} \geq h_1 \geq h_2$ for all ξ, by assumption we have $U_2(\mathcal{S}_{12}, \max\{y, \xi\}) \leq 0$ and $U_1(\mathcal{S}_{i1}, \max\{y, \xi\}) \leq 0$ for all ξ. Thus it follows from Eqs. (3.67) and Eq. (3.68) that $U_i(\mathcal{S}_i, y) = K_i(y) \leq 0$ for all $i \in \mathcal{S}$.

□ *Proof of* (b) Suppose $y < h_1$. If $y \geq h_2$, then since $\max\{y, \xi\} \geq h_2$ for all ξ, $U_2(\mathcal{S}_{12}, \max\{y, \xi\}) \leq 0$. If $y < h_2$, then $\xi \leq y\,(< h_2)$ leads to $U_2(\mathcal{S}_{12}, \max\{y, \xi\}) = U_2(\mathcal{S}_{12}, y) > 0$, $y < \xi < h_2$ to $U_2(\mathcal{S}_{12}, \max\{y, \xi\}) = U_2(\mathcal{S}_{12}, \xi) > 0$, and $(y <)\, h_2 < \xi$ to $U_2(\mathcal{S}_{12}, \max\{y, \xi\}) = U_2(\mathcal{S}_{12}, \xi) \leq 0$. Therefore Eq. (3.67) can be expressed as

$$U_1(\mathcal{S}_1, y) \;=\; K_1(y) + \beta_1\Big(U_2(\mathcal{S}_{12}, y)F_1(y) + \int_y^{h_2} U_2(\mathcal{S}_{12}, \xi)dF_1(\xi)\Big)I(y < h_2), \tag{3.86}$$

If $\xi \leq y\,(< h_1)$, then $U_1(\mathcal{S}_{i1}, \max\{y, \xi\}) = U_1(\mathcal{S}_{i1}, y) > 0$, and if $y < \xi < h_1$, then $U_1(\mathcal{S}_{i1}, \max\{y, \xi\}) = U_1(\mathcal{S}_{i1}, \xi) > 0$, and if $(y <)\, h_1 \leq \xi$, then $U_1(\mathcal{S}_{i1}, \max\{y, \xi\}) = U_1(\mathcal{S}_{i1}, \xi) \leq 0$. Hence Eq. (3.74) can be expressed as

$$U_i(\mathcal{S}_i, y) \;=\; K_i(y) + \beta_i\Big(U_1(\mathcal{S}_{i1}, y)F_i(y) + \int_y^{h_1} U_1(\mathcal{S}_{i1}, \xi)dF_i(\xi)\Big), \quad 2 \leq i \leq N. \tag{3.87}$$

Differentiating Eqs. (3.86) and Eq. (3.87) with respect to y produces

$$U_1'(\mathcal{S}_1, y) \;=\; \beta_1 F_1(y)\Big(1 + U_2'(\mathcal{S}_{12}, y)I(y < h_2)\Big) - 1, \tag{3.88}$$

$$U_i'(\mathcal{S}_i, y) = \beta_i F_i(y)\Big(1 + U_1'(\mathcal{S}_{i1}, y)\Big) - 1, \quad 2 \le i \le N. \tag{3.89}$$

Now define $\Gamma_k(y) = \prod_{i=1}^k \beta_i F_i(y)$. Then from Eq. (3.82) we have

$$U_1'(\{1,2\}_1, y) = \Gamma_1(y) I(h_2 \le y) + \Gamma_2(y) I(y < h_2) - 1. \tag{3.90}$$

$$\begin{aligned} U_2'(\{2,3\}_2, y) &= \beta_2 F_2(y) I(h_3 \le y) + \beta_2 F_2(y) \beta_3 F_3(y) I(y < h_3) - 1 \\ &= \Gamma_2(y) I(h_3 \le y)/\beta_1 F_1(y) + \Gamma_3(y) I(y < h_3)/\beta_1 F_1(y) - 1. \end{aligned} \tag{3.91}$$

Therefore, noting $\{1,2,3\}_{12} = \{2,3\}_2$, we have from Eq. (3.88)

$$\begin{aligned} U_1'(\{1,2,3\}_1, y) &= \beta_1 F_1(y)\big(1 + U_2'(\{2,3\}_2, y) I(y < h_2)\big) - 1 \\ &= \beta_1 F_1(y)\big(1 + \big(\Gamma_2(y) I(h_3 \le y)/\beta_1 F_1(y) + \Gamma_3(y) I(y < h_3)/\beta_1 F_1(y) - 1\big) I(y < h_2)\big) - 1 \\ &= \beta_1 F_1(y) + \Gamma_2(y) I(h_3 \le y) I(y < h_2) + \Gamma_3(y) I(y < h_3) I(y < h_2) - \beta_1 F_1(y) I(y < h_2) - 1 \\ &= \Gamma_1(y) I(h_2 \le y) + \Gamma_2(y) I(h_3 \le y < h_2) + \Gamma_3(y) I(y < h_3) - 1. \end{aligned}$$

Repeating the same operation leads to in general

$$U_1'(\mathcal{S}_1, y) = \Gamma_1(y) I(h_2 \le y) + \sum_{k=2}^{N-1} \Gamma_k(y) I(h_{k+1} \le y < h_k) + \Gamma_N(y) I(y < h_N) - 1. \tag{3.92}$$

Using this, we have from Eq. (3.89)

$$\begin{aligned} U_2'(\mathcal{S}_2, y) &= \beta_2 F_2(y)(1 + U_1(\{1,3,4,\cdots,N\}_1, y) - 1 \\ &= \Gamma_2(y) I(h_3 \le y) + \sum_{k=3}^{N-1} \Gamma_k(y) I(h_{k+1} \le y < h_k) + \Gamma_N(y) I(y < h_N) - 1. \end{aligned} \tag{3.93}$$

$$\begin{aligned} U_i'(\mathcal{S}_i, y) &= \beta_i F_i(y)(1 + U_1(\{1,2,\cdots,i-,i+1,\cdots,N\}_1, y) - 1 \\ &= \beta_i F_i(y)\big(\Gamma_1(y) I(h_2 \le y) \\ &\quad + \sum_{k=2}^{i-2} \Gamma_k(y) I(h_{k+1} \le y < h_k) + \Gamma_{i-1}(y) I(h_{i+1} \le y < h_{i-1})\big) \\ &\quad + \sum_{k=i+1}^{N-1} \Gamma_k(y) I(h_{k+1} \le y < h_k) + \Gamma_N(y) I(y < h_N) - 1, \qquad N-1 \ge i \ge 3, \end{aligned} \tag{3.94}$$

$$\begin{aligned} U_N'(\mathcal{S}_N, y) &= \beta_N F_N(y)\big(\Gamma_1(y) I(h_2 \le y) \\ &\quad + \sum_{k=2}^{N-2} \Gamma_k(y) I(h_{k+1} \le y < h_k) + \Gamma_{N-1}(y) I(y < h_{N-1})\big) - 1. \end{aligned} \tag{3.95}$$

Now arranging Eq. (3.94) by substituting

$$\Gamma_{i-1}(y) I(h_{i+1} \le y < h_{i-1}) = \Gamma_i(y) I(h_{i+1} \le y < h_i)/\beta_i F_i(y) + \Gamma_{i-1}(y) I(h_i \le y < h_{i-1}) \tag{3.96}$$

leads to

$$\begin{aligned} U_i'(\mathcal{S}_i, y) &= \beta_i F_i(y)\big(\Gamma_1(y) I(h_2 \le y) + \sum_{k=2}^{i-1} \Gamma_k(y) I(h_{k+1} \le y < h_k)\big) \\ &\quad + \sum_{k=i}^{N-1} \Gamma_k(y) I(h_{k+1} \le y < h_k) + \Gamma_N(y) I(y < h_N) - 1. \end{aligned} \tag{3.97}$$

Taking the difference of Eq. (3.92) and Eq. (3.96) yields, for $2 \le i \le n$,

$$U_1'(\mathcal{S}_1, y) - U_i'(\mathcal{S}_i, y) = \big(1 - \beta_i F_i(y)\big)\Big(\Gamma_1(y) I(h_2 \le y) + \sum_{k=2}^{i-1} \Gamma_k(y) I(h_{k+1} \le y < h_k)\Big), \quad (3.98)$$

the right hand side of which is nonnegative for any y and is equal to 0 on $y < h_i$. This implies that the difference $U_1(\mathcal{S}_1, y) - U_i(\mathcal{S}_i, y)$ is constant on $y \le h_i$ and nondecreasing on $h_i \le y < h_1$. Consequently, in order to complete the proof of (b), it suffices to verify (c).

Proof of (c) From Eq. (3.68) we have

$$\begin{aligned} U_i(\mathcal{S}_i, h_i) &= K_i(h_i) + \beta_i \int_{-\infty}^{\infty} \max\{0, U_1(\{1,2,3,\cdots,i-1,i+1,\cdots,N\}_1, \max\{h_i, \xi\})\} dF_i(\xi) \\ &\le \int_{-\infty}^{\infty} \max\{0, U_1(\{1,2,3,\cdots,i-1,i+1,\cdots,N\}_1, h_i)\} dF_i(\xi) \\ &= \max\{0, U_1(\{1,2,3,\cdots,i-1,i+1,\cdots,N\}_1, h_i)\} \\ &= U_1(\{1,2,3,\cdots,i-1,i+1,\cdots,N\}_1, h_i) \\ &\le U_1(\{1,2,3,\cdots,i-1,i,i+1,\cdots,N\}_1, h_i) \\ &= U_1(\mathcal{S}_1, h_i). \end{aligned} \quad (3.99)$$

Therefore, we have $U_1(\mathcal{S}_1, h_i) - U_i(\mathcal{S}_i, h_i) \ge 0$, implying that (b) holds true.

■ Another Proof of Theorem 3.11 (Sequential Assignment Problem I)

In order to prove the theorem it must verify also the following statement.

(c) *For $2 \le x \le i$, there exists $c_i(x)$ such that* $c(x, r_i) = v(r_i(x-1)) - v(r_i(x)) = (r_x - r_{x-1})c_i(x)$.

First, clearly $u(w, r_1) = r_1 w$, so $v(r_1) = r_1\mu$, hence $v(r_2(1)) = r_2\mu$ and $v(r_2(2)) = r_1\mu$. Therefore, for r_2-problem we have $v(r_2(1)) - v(r_2(2)) = (r_2 - r_1)\mu$, so $c_2(2) = \mu$. If $w \le c_2(2)$, then since $r_2 w + v(r_2(2)) - r_1 w - v(r_2(1)) = (r_2 - r_1)(w - c_2(2))$, it is optimal to assign the worker 1, or else worker 2. Let $c_1(1) = -\infty$ and $c_1(2) = \infty$, so $S(c_1(1)) = \mu$ and $T(c_1(2)) = 0$ due to Eq. (2.6). Furthermore let $c_2(1) = -\infty$ and $c_2(3) = \infty$. Then clearly $c_2(1) \le c_2(2) \le c_2(3)$, the inequalities $w \le c_2(2)$ and $c_2(2) < w$ can be written, respectively, $c_2(1) < w \le c_2(2)$ and $c_2(2) < w \le c_2(3)$, and $c_2(2)$ can be expressed as $c_2(2) = S(c_1(1)) - T(c_1(2))$. Therefore the statements (a), (b), and (c) are all true for the r_2-problem.

Second, assume that the three statements are true for r_i-problem, hence for $2 \le x \le i$ we have

$$r_x w + v(r_i(x)) - r_{x-1} w - v(r_i(x-1)) = (r_x - r_{x-1})(w - c_i(x)),$$

from which the following three points can be said.

1. If $w \le c_i(2)$, so $w \le c_i(2) \le c_i(3) \le \cdots \le c_i(i)$, then $u(w, r_i) = r_1 w + v(r_i(1))$,
2. For $3 \le x \le i-1$, if $c_i(x) < w \le c_i(x+1)$, so $c_i(2) \le \cdots \le c_i(x-1) \le c_i(x) < w \le c_i(x+1) \le c_i(x+2) \le \cdots \le c_i(i)$, then $u(w, r_i) = r_x w + v(r_i(x))$,
3. If $c_i(i) < w$, so $c_i(2) \le \cdots \le c_i(i-1) \le c_i(i) < w$, then $u(w, r_i) = r_i w + v(r_i(i))$,

which implies that the maximum of the right hand side of Eq. (3.33) is attained at x for w such that $c_i(x) < w \le c_i(x+1)$ with $c_i(1) = -\infty$ and $c_i(i+1) = \infty$. In other words, it is optimal to assign the worker x if $c_i(x) < w \le c_i(x+1)$. Hence $u(w, r_i)$ can be expressed as

$$u(w, r_i) = \sum_{x=1}^{i} (r_x w + v(r_i(x))) I(c_i(x) < w \le c_i(x+1)). \quad (3.100)$$

Noticing $I(c_i(x) < w) = I(c_i(x) < w \le c_i(x+1)) + I(c_i(x+1) < w)$, the above expression can be rearranged as follows.

$$u(w, \boldsymbol{r}_i) = r_1 w + v(\boldsymbol{r}_i(1)) + \sum_{x=2}^{i} (r_x - r_{x-1})(w - c_i(x)) I(c_i(x) < w). \tag{3.101}$$

Hence, by using T–function, we have

$$v(\boldsymbol{r}_i) = r_1 \mu + v(\boldsymbol{r}_i(1)) + \sum_{x=2}^{i} (r_x - r_{x-1}) T(c_i(x)). \tag{3.102}$$

Now by $\boldsymbol{r}_{i+1}(i, j)$, $i \neq j$, let us denote the $(i-1)$-vector resulting from removing the i-th and j-th elements of $\boldsymbol{r}_{i+1}$ where $\boldsymbol{r}_{i+1}(i, j) = \boldsymbol{r}_{i+1}(j, i)$. Then from Eq. (3.102) we have

$$v(\boldsymbol{r}_{i+1}(1)) = r_2 \mu + v(\boldsymbol{r}_{i+1}(1, 2)) + \sum_{x=2}^{i} (r_{x+1} - r_x) T(c_i(x)), \tag{3.103}$$

$$\begin{aligned} v(\boldsymbol{r}_{i+1}(y)) &= r_1 \mu + v(\boldsymbol{r}_{i+1}(y, 1)) + \sum_{x=2}^{y-1} (r_x - r_{x-1}) T(c_i(x)) + (r_{y+1} - r_{y-1}) T(c_i(y)) \\ &\quad + \sum_{x=y+1}^{i} (r_{x+1} - r_x) T(c_i(x)), \quad 2 \leq y \leq i, \end{aligned} \tag{3.104}$$

$$v(\boldsymbol{r}_{i+1}(i+1)) = r_1 \mu + v(\boldsymbol{r}_{i+1}(i+1, 1)) + \sum_{x=2}^{i} (r_x - r_{x-1}) T(c_i(x)). \tag{3.105}$$

From the above expressions we obtain

$$v(\boldsymbol{r}_{i+1}(y-1)) - v(\boldsymbol{r}_{i+1}(y)) = (r_y - r_{y-1})(S(c_i(y-1)) - T(c_i(y))), \quad 2 \leq y \leq i+1. \tag{3.106}$$

Therefore we have

$$c_{i+1}(y) = S(c_i(y-1)) - T(c_i(y)), \quad 2 \leq y \leq i+1. \tag{3.107}$$

Now clearly $c_{i+1}(2) > c_{i+1}(1) = -\infty$ and $\infty = c_{i+1}(i+2) > c_{i+1}(i+1)$. Next for $3 \leq x \leq i+1$

$$c_{i+1}(y) - c_{i+1}(y-1) = \big(S(c_i(y-1)) - S(c_i(y-2))\big) - \big(T(c_i(y)) - T(c_i(y-1))\big). \tag{3.108}$$

The differences of S–function and T–function in the above expression are, respectively, nonnegative and nonpositive from the induction hypothesis and Lemma 2. 3 (a,b). Hence it follows that $c_{i+1}(x)$ is nondecreasing on $2 \leq x \leq i+2$.

■ Proof of Theorem 3. 12 (Sequential Assignment Problem II)

Clearly $M(w, \boldsymbol{r}_2(1)) = r_2 w \geq r_1 w = M(w, \boldsymbol{r}_2(2))$ for all w, hence the statement (a) is true for $i = 2$. Suppose the statement (a) is true for $i - 1$, hence $K(M(-, \boldsymbol{r}_{i-1}(x)), z)$ is also nonincreasing in x for all z. This implies that $h(\boldsymbol{r}_i(x))$ is nonincreasing in x, hence the statement (b) holds. Here note that be assumption we have $h(\boldsymbol{r}_i(x, y)) = h(\boldsymbol{r}_i(y, x) \geq h(\boldsymbol{r}_i(y, x+1) = h(\boldsymbol{r}_i(x+1, y)$ and $h(\boldsymbol{r}_i(x, x+1) = h(\boldsymbol{r}_i(x+1, x)$ for all x and y, $x \neq y$. Then for all w

$$\begin{aligned} & M(w, \boldsymbol{r}_i(x)) \\ &= \max\{\max_{1 \leq y \leq x-1} \{r_y w + h(\boldsymbol{r}_i(x, y))\}, r_{x+1} w + h(\boldsymbol{r}_i(x, x+1)), \max_{x+2 \leq y \leq i+1} \{r_y w + h(\boldsymbol{r}_i(x, y))\} \\ &\geq \max\{\max_{1 \leq y \leq x-1} \{r_y w + h(\boldsymbol{r}_i(x+1, y))\}, r_x w + h(\boldsymbol{r}_i(x+1, x)), \max_{x+2 \leq y \leq i+1} \{r_y w + h(\boldsymbol{r}_i(x+1, y))\} \\ &= M(w, \boldsymbol{r}_i(x+1)) \end{aligned}$$

Hence $K(M(-, r_i(x)), z)$ is nonincreasing in x for all z. This completes the proof. ■

Appendix II

■ Weitzman's Proof of Theorem 3.10 [29]

Here for convenience let us denote a reward in box i by w_i. First from Eq. (3.28) we have

$$u(\{k\}, y) = y + \max\{0, K_k(y)\}$$

for every $k \in \mathcal{S}$. Hence it follows that if $y \le h_k$, opening box k is optimal due to $K_k(y) \ge 0$, or else stopping with accepting the maximum reward y is optimal due to $K_k(y) \le 0$. This implies that Pandora's rule is optimal when starting with a single closed box.

Assuming that Pandora's rule is optimal with any n closed boxes and any maximum reward y, we shall start with any $n+1$ closed boxes $\mathcal{L} = \{i_1, i_2, \cdots, i_{n+1}\}$ and any maximum reward y where let j and g be boxes with, respectively, the highest and the second highest reservation value in $\mathcal{L}$, that is,

$$h_j \ge h_g \ge \max_{\substack{l=1,2,\cdots,n+1 \\ l\ne j, l\ne g}} h_l.$$

Then, by O_0 we shall denote the expected present discounted net value from opening no box and stopping with accepting the current maximum reward y; clearly $O_0 = y$.

i. Suppose $y \ge h_j$. Then since $\max\{y, w_k\} \ge h_j \ge h_g$ for any w_k, the expected present discounted value from opening any box $k \in \mathcal{L}$ becomes

$$O_k = \beta_k \int_{-\infty}^{\infty} \max\{y, w_k\} dF_k(w_k) - c_k$$

from hypothesis. Therefore, since $O_k - O_0 = K_k(y) \le 0$ because of $y \ge h_j \ge h_k$, it follows that opening no box, or stopping with accepting the current maximum reward y is optimal.

ii. Suppose $h_j > y$. Then the expected present discounted value from opening box $j \in \mathcal{L}$ and then stopping becomes

$$O_j = \beta_j \int_{-\infty}^{\infty} \max\{y, w_j\} dF_j(w_j) - c_j.$$

Hence, since $O_j - O_0 = K_j(y) > 0$, it follows that opening no box can not become optimal. Here a question of which box to be opened arises. In order to answer the question, let us consider the three alternatives:

A: Open box j and proceed by Pandora's rule thereafter,

B: Open any box k ($\ne j$) and proceed by Pandora's rule thereafter,

C: Open box j. If $w_j > h_g$, stop, or else open any box k and proceed by Pandora's rule thereafter,

Let the expected present discounted value for each alternative be designated by A, B, and C, respectively. What should be proved here is $A \ge B$. Here, by hypothesis, clearly $A \ge C$. Hence, if $C \ge B$ can be verified, it follows that $A \ge B$. Weitzman proved the inequality by showing the difference $C - B \ge 0$.

First, by definition, we have

$$\begin{aligned} C = -c_j + \beta_j \Big\{ & Pr(w_j \ge h_g) E[\max\{y, w_j\} | w_j \ge h_g] \\ & + Pr(h_g > w_j)\Big(-c_k + \beta_k E[u(\mathcal{S} - \{j\} - \{k\}, \max\{y, w_j, w_k\}) | h_g > w_j]\Big)\Big\}, \end{aligned}$$

$$\begin{aligned} B = -c_k + \beta_k \Big\{ & Pr(w_k \ge h_j) E[w_k | w_k \ge h_j] \\ & + Pr(h_j > w_k)\Big(-c_j + \beta_j E[u(\mathcal{S} - \{k\} - \{j\}, \max\{y, w_k, w_j\}) | h_j > w_k]\Big)\Big\}, \end{aligned}$$

which can be rewritten as follows:

$$
\begin{aligned}
C = -c_j + \beta_j \Big\{ & Pr(w_j \geq h_j) E[w_j | w_j \geq h_j] \\
& + Pr(h_j > w_j \geq h_g) E[\max\{y, w_j\} | h_j > w_j \geq h_g] \\
& + Pr(h_g > w_j) \Big(-c_k \\
& \quad + \beta_k \big\{ Pr(w_k \geq h_j) E[w_k | w_k > h_j] \\
& \qquad + Pr(h_j > w_k \geq h_g) E[\max\{y, w_k\} | h_j > w_k \geq h_g] \\
& \qquad + Pr(h_g > w_k) E[u(\mathcal{S} - \{j\} - \{k\}, \max\{y, w_j, w_k\}) | h_g > w_j,\ h_g > w_k] \big\} \Big) \Big\},
\end{aligned}
$$

$$
\begin{aligned}
B = -c_k + \beta_k \Big\{ & Pr(w_k \geq h_j) E[w_k | w_k \geq h_j] \\
& + Pr(h_j > w_k \geq h_g) \Big(-c_j \\
& \quad + \beta_j \big\{ Pr(w_j \geq h_j) E[w_j | w_j \geq h_j] \\
& \qquad + Pr(h_j > w_j \geq h_g) E[\max\{y, w_k, w_j\} | h_j > w_k \geq h_g,\ h_j > w_j \geq h_g] \\
& \qquad + Pr(h_g > w_j) E[\max\{y, w_k\} | h_j > w_k \geq h_g] \big\} \Big) \\
& + Pr(h_g > w_k) \Big(-c_j \\
& \quad + \beta_j \big\{ Pr(w_j \geq h_j) E[w_j | w_j \geq h_j] \\
& \qquad + Pr(h_j > w_j \geq h_g) E[\max(y, w_j) | h_j > w_j \geq h_g] \\
& \qquad + Pr(h_g > w_j) E[u(\mathcal{S} - \{k\} - \{j\}, \max\{y, w_k, w_j\}) | h_g > w_k,\ h_g > w_j] \big\} \Big) \Big\}.
\end{aligned}
$$

Now define

$$
\begin{aligned}
\pi_j &= Pr\{w_j \geq h_j\}, \\
\pi_k &= Pr\{w_k \geq h_j\}, \\
\lambda_j &= Pr\{h_j > w_j \geq h_g\}, \\
\lambda_k &= Pr\{h_j > w_k \geq h_g\}, \\
\mu_k &= Pr\{h_g > w_k \geq h_k\}, \\
e_j &= E[w_j | w_j \geq h_j], \\
e_k &= E[w_k | w_k \geq h_j], \\
a_j &= E[w_j | h_j > w_j \geq h_g], \\
a_k &= E[w_k | h_j > w_k \geq h_g], \\
b_k &= E[w_k | h_g > w_k \geq h_k], \\
\tilde{a}_j &= E[\max\{y, w_j\} | h_j > w_j \geq h_g], \\
\tilde{a}_k &= E[\max\{y, w_k\} | h_j > w_k \geq h_g], \\
d &= E[\max\{y, w_k, w_j\} | h_j > w_k \geq h_g,\ h_j > w_j \geq h_g], \\
\Phi &= E[u(\mathcal{S} - \{j\} - \{k\}, \max\{y, w_k, w_j\}) | h_g > w_j,\ h_g > w_k].
\end{aligned}
$$

By using the above symbols, C and B are expressed as follows:

$$
C = -c_j + \beta_j \Big\{ \pi_j e_j + \lambda_j \tilde{a}_j + (1 - \pi_j - \lambda_j) \Big(-c_k + \beta_k \big\{ \pi_k e_k + \lambda_k \tilde{a}_k + (1 - \pi_k - \lambda_k) \Phi \big\} \Big) \Big\},
$$

$$
\begin{aligned}
B = -c_k + \beta_k \Big\{ & \pi_k e_k + \lambda_k \Big(-c_j + \beta_j \big\{ \pi_j e_j + \lambda_j d + (1 - \pi_j - \lambda_j) \tilde{a}_k \big\} \Big) \\
& + (1 - \pi_k - \lambda_k) \Big(-c_j + \beta_j \big\{ \pi_j e_j + \lambda_j \tilde{a}_j + (1 - \pi_j - \lambda_j) \Phi \big\} \Big) \Big\},
\end{aligned}
$$

which furthermore can be arranged as follows:

$$
C = \underline{-c_j + \pi_j \beta_j e_j} + \lambda_j \beta_j \tilde{a}_j + (1 - \pi_j - \lambda_j) \beta_j \Big(\underline{-c_k + \pi_k \beta_k e_k} + \lambda_k \beta_k \tilde{a}_k \Big)
$$

$$+\underline{(1-\pi_j-\lambda_j)(1-\pi_k-\lambda_k)\beta_j\beta_k\Phi},$$

$$\begin{aligned}B = \underline{-c_k+\pi_k\beta_k e_k} + \lambda_k\beta_k\Big(\underline{-c_j+\pi_j\beta_j e_j} + \lambda_j\beta_j d + (1-\pi_j-\lambda_j)\beta_j\tilde{a}_k\Big)\\ +(1-\pi_k-\lambda_k)\beta_k\Big(\underline{-c_j+\pi_j\beta_j e_j} + \lambda_j\beta_j\tilde{a}_j\Big)\\ +\underline{(1-\pi_k-\lambda_k)(1-\pi_j-\lambda_j)\beta_k\beta_j\Phi}.\end{aligned}$$

Taking the difference of C and B with noting the underlined terms produces

$$\begin{aligned}C-B = -(c_j-\pi_j\beta_j e_j)(1-\beta_k+\pi_k\beta_k) + (c_k-\pi_k\beta_k e_k)(1-\beta_j+\beta_j\pi_j+\beta_j\lambda_j)\\ +\lambda_j\beta_j\tilde{a}_j - \lambda_k\beta_k\lambda_j\beta_j d - (1-\pi_k-\lambda_k)\beta_k\lambda_j\beta_j\tilde{a}_j.\end{aligned}$$

Now $K_j(h_j)=0$ and $K_k(h_k)=0$ can be expressed as follows, respectively,

$$\begin{aligned}c_j &= \beta_j\Big(h_j\int_{-\infty}^{h_j} dF(w_j) + \int_{h_j}^{\infty} w_j dF(w_j)\Big) - h_j\\ &= \beta_j\Big(h_j(1-\Pr\{w_j\ge h_j\}) + \Pr\{w_j\ge h_j\}E[w_j|w_j\ge h_j]\Big) - h_j\end{aligned}$$

$$\begin{aligned}c_k &= \beta_k(h_k\int_{-\infty}^{h_k} dF(w_k) + \int_{h_k}^{\infty} w_k dF(w_k)) - h_k\\ &= \beta_k\Bigg(h_k\Big(1-\Pr\{w_k\ge h_j\} - \Pr\{h_j>w_k\ge h_g\}) - \Pr\{h_g>w_k\ge h_k\}\Big)\\ &\qquad +\Pr\{w_k\ge h_j\}E[w_k|w_k\ge h_j]\\ &\qquad +\Pr\{h_j>w_k\ge h_g\})E[w_k|h_j>w_k\ge h_g]\\ &\qquad +\Pr\{h_g>w_k\ge h_k\})E[w_k|h_g>w_k\ge h_k]\Bigg) - h_k,\end{aligned}$$

which can be expressed as, respectively

$$\begin{aligned}&c_j = \beta_j\pi_j(e_j-h_j) - (1-\beta_j)h_j,\\ &c_k = \beta_k\Big(\pi_k(e_k-h_k) + \lambda_k(a_k-h_k) + \mu_k(b_k-h_k)\Big) - (1-\beta_k)h_k,\end{aligned}$$

from which we obtain

$$\begin{aligned}&c_j - \beta_j\pi_j e_j = -h_j(1-\beta_j+\beta_j\pi_j),\\ &c_k - \beta_k\pi_k e_k = -h_k(1-\beta_k+\beta_k\pi_k) + \beta_k\lambda_k(a_k-h_k) + \beta_k\mu_k(b_k-h_k).\end{aligned}$$

If substituting the above into the right hand side of Eq. (3.109), canceling some terms, and grouping others, then the difference C-B can be arranged as follows:

$$\begin{aligned}C-B = {}& h_j(1-\beta_j+\beta_j\pi_j)(1-\beta_k+\pi_k\beta_k)\\ &+\Big(-h_k(1-\beta_k+\beta_k\pi_k) + \beta_k\lambda_k(a_k-h_k) + \beta_k\mu_k(b_k-h_k)\Big)(1-\beta_j+\beta_j\pi_j+\beta_j\lambda_j)\\ &+\lambda_j\beta_j\tilde{a}_j - \lambda_k\beta_k\lambda_j\beta_j d - (1-\pi_k-\lambda_k)\beta_k\lambda_j\beta_j\tilde{a}_j.\end{aligned}$$

Now we have $\tilde{a}_j \ge E[w_j|h_j>w_j\ge h_g] \ge E[h_g|h_j>w_j\ge h_g] = h_g \ge h_k$. Similarly we get $a_k \ge h_k$ and $b_k \ge h_k$. Furthermore we obtain

$$\begin{aligned}d &= h_g + E[\max\{\max\{y,w_j\}-h_g,\ w_k-h_g\}|h_j>w_k\ge h_g,\ h_j>w_j\ge h_g]\\ &\le h_g + E[\max\{y,w_j\}-h_g+w_k-h_g|h_j>w_k\ge h_g,\ h_j>w_j\ge h_g]\\ &= E[\max\{y,w_j\}+w_k|h_j>w_k\ge h_g,\ h_j>w_j\ge h_g]\\ &= \tilde{a}_j + a_k - h_g\\ &\le \tilde{a}_j + a_k - h_k.\end{aligned}$$

Thus we have the following inequalities:

$$h_j - h_k \geq 0, \quad \tilde{a}_j - h_k \geq 0, \quad a_k - h_k \geq 0, \quad b_k - h_k \geq 0, \quad \tilde{a}_j + a_k - h_k - d \geq 0.$$

Weitzman states $C - B \geq 0$ without proof in his paper; this is quite puzzling. The author confirmed this by transforming the difference $C - B$ step by step as follows:

$$\begin{aligned}
C - B = {} & h_j(1 - \beta_j + \beta_j\pi_j)(1 - \beta_k + \pi_k\beta_k) \\
& -h_k(1 - \beta_k + \beta_k\pi_k)(1 - \beta_j + \beta_j\pi_j + \beta_j\lambda_j) \\
& +\beta_k\lambda_k(a_k - h_k)(1 - \beta_j + \beta_j\pi_j + \beta_j\lambda_j) \\
& +\beta_k\mu_k(b_k - h_k)(1 - \beta_j + \beta_j\pi_j + \beta_j\lambda_j) \\
& +\lambda_j\beta_j\tilde{a}_j - \lambda_k\beta_k\lambda_j\beta_j d - (1 - \pi_k - \lambda_k)\beta_k\lambda_j\beta_j\tilde{a}_j \\
= {} & h_j(1 - \beta_j + \beta_j\pi_j)(1 - \beta_k + \pi_k\beta_k) \\
& -h_k(1 - \beta_k + \beta_k\pi_k)(1 - \beta_j + \beta_j\pi_j) \\
& -h_k(1 - \beta_k + \beta_k\pi_k)\beta_j\lambda_j \\
& +(a_k - h_k)\beta_k\lambda_k\beta_j\lambda_j \\
& +(a_k - h_k)\beta_k\lambda_k(1 - \beta_j + \beta_j\pi_j) \\
& +(b_k - h_k)\beta_k\mu_k(1 - \beta_j + \beta_j\pi_j + \beta_j\lambda_j) \\
& +\lambda_j\beta_j\tilde{a}_j - \lambda_k\beta_k\lambda_j\beta_j d - (1 - \pi_k - \lambda_k)\beta_k\lambda_j\beta_j\tilde{a}_j \\
= {} & (h_j - h_k)(1 - \beta_j + \beta_j\pi_j)(1 - \beta_k + \pi_k\beta_k) \\
& -h_k(1 - \beta_k + \beta_k\pi_k)\beta_j\lambda_j \\
& +(a_k - h_k)\beta_k\lambda_k\beta_j\lambda_j \\
& +(a_k - h_k)\beta_k\lambda_k(1 - \beta_j + \beta_j\pi_j) \\
& +(b_k - h_k)\beta_k\mu_k(1 - \beta_j + \beta_j\pi_j + \beta_j\lambda_j) \\
& +\lambda_j\beta_j\tilde{a}_j - \lambda_k\beta_k\lambda_j\beta_j d - (1 - \pi_k - \lambda_k)\beta_k\lambda_j\beta_j\tilde{a}_j \\
= {} & (h_j - h_k)(1 - \beta_j + \beta_j\pi_j)(1 - \beta_k + \pi_k\beta_k) \\
& +(\tilde{a}_j - h_k)(1 - \beta_k + \beta_k\pi_k)\beta_j\lambda_j \\
& +(a_k - h_k)\beta_k\lambda_k(1 - \beta_j + \beta_j\pi_j) \\
& +(b_k - h_k)\beta_k\mu_k(1 - \beta_j + \beta_j\pi_j + \beta_j\lambda_j) \\
& -\tilde{a}_j(1 - \beta_k + \beta_k\pi_k)\beta_j\lambda_j \\
& +(a_k - h_k)\beta_k\lambda_k\beta_j\lambda_j \\
& +\lambda_j\beta_j\tilde{a}_j - \lambda_k\beta_k\lambda_j\beta_j d - (1 - \pi_k - \lambda_k)\beta_k\lambda_j\beta_j\tilde{a}_j
\end{aligned}$$

in which the last three terms can be reduced to the single term $(\tilde{a}_j + a_k - h_k - d)\lambda_k\beta_k\lambda_j\beta_j$. Thus, eventually the difference $C - B$ becomes

$$\begin{aligned}
C - B = {} & (h_j - h_k)(1 - \beta_j + \beta_j\pi_j)(1 - \beta_k + \pi_k\beta_k) \\
& +(\tilde{a}_j - h_k)\beta_j\lambda_j(1 - \beta_k + \beta_k\pi_k) \\
& +(a_k - h_k)\beta_k\lambda_k(1 - \beta_j + \beta_j\pi_j) \\
& +(b_k - h_k)\beta_k\mu_k(1 - \beta_j + \beta_j\pi_j + \beta_j\lambda_j) \\
& +(\tilde{a}_j + a_k - h_k - d)\lambda_k\beta_k\lambda_j\beta_j.
\end{aligned} \tag{3.109}$$

Therefore it follows that the right hand side is nonnegative, hence $C \geq B$. This completes the proof.

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