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Optimal Stopping Problem with Recall Cost

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OPTIMAL STOPPING PROBLEM WITH RECALL COST

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This paper presents an optimal stopping problem with recall where some cost must be paid to accept the best offer y which has so far produced. In this model, the optimal decision rule may have a property, called the DRV (Double Reservation Value) property, that for at least one current offer w there exist two different critical values y_* and y^* such that, if $y_* \leq y \leq y^*$, continuing is optimal, or alternatively stopping by accepting the better offer between the current offer w and the best offer y up to the previous point in time is optimal. We examine conditions for the optimal decision rule to have the DRV property and its limiting behavior as the time horizon tends to infinity.

1 Introduction

The following example conveys the flavor of our model. Suppose a board of directors in a manufacturing company decides to put a new product on sale at some point in the future and call on the manager of the R&D department to propose an idea for a new product at the beginning of every year with an estimation of the long-run net profit that could be expected from it. If proposals with low long-run net profit are successively submitted, they will be passed up. If, further to these, no proposals of sufficiently high long-run net profit is submitted, the board of directors will have no choice but to adopt the best of the proposals that have already been tabled. In this case, the value of a past offer with intent to recall may have decreased for some reason. For example, some of the engineers involved may have already quit the company, and thus some additional cost may be incurred for the reexamination of the proposal. Furthermore, a competitor may have put a product based on a similar idea on the market in the interim. From this point of view, we consider the optimal stopping problem with recall where some cost must be paid to accept the best offer y so far produced. In this paper, we refer to the difference between the original value of the past offer and its value at the present time on recall as the *recall cost*.

Many models of optimal stopping problems have been posed and investigated [1-29]. However, in none of the models, a concept that requires some cost to accept a past offer has been taken into consideration. In the above example, the recall cost is an important factor and it is reasonable to approach it from this practical point of view. Generally speaking, it should be noted that the value of a past offer changes as time goes by.

The previous studies on the optimal stopping problem can be classified into three main groups from the aspect of whether an offer inspected and passed up once remains available or not at the next point in time: 1) model with no recall where the offer inspected and passed up once becomes instantly and forever unavailable, 2) model with recall where it remains forever available, 3) model with uncertain recall where its availability is stochastic.

The model described in this paper is related to 2) the conventional model with recall [18], and differs only from the point that some cost must be paid to accept a past offer. In this respect, to put it more precisely, this paper deals with a model where the value of a past offer decreases in substance due to the recall cost. The purpose of this paper is to propose a model with such a recall cost and examine the properties of its optimal stopping rules. One of the most distinctive

results obtained in this paper is that; by introducing recall cost to the conventional model with recall, the DRV property which is known to appear in the Ikuta's uncertain recall model [4] may appear in the conventional model with recall where only one critical value exists. For further details of this, see Section 5.

In Section 2 we define our model. In Section 3 we introduce a function, called the K-function [7], defined to be used for the mathematical development of the model in Section 8. In Section 4 the objective function is derived and the functional equation that satisfies the objective function is provided. In Section 5 the optimal stopping rules are stated and a simple example is given to show the rule may have the DRV property. In Section 6, as a conclusion to this paper, the conditions for the DRV property to appear are stated. The conditions are ascertained by a numerical example in Section 7 and verified in Section 8. Finally, in Section 9 some variations of our model are presented.

2 Model

Consider the following discrete-time stochastic decision process with a finite planning horizon. For convenience, let points in time be numbered backward from the final point of the planning horizon as 0, 1,$\cdots$, and so on. The interval between time t and time $t-1$ is called the period t. In our discussion from now on, an offer of value w will be expressed as offer w. Subsequent offers $w, w', \cdots$ produced at successive points in time are assumed to be stochastically independent random samples from a known probability distribution function $F(w)$ with a finite expectation μ where, for certain given numbers a and b such as $0 \leq a < b < \infty$, let $F(w) = 0$ for $w < a$, $0 < F(w) < 1$ for $a \leq w < b$, and $F(w) = 1$ for $b \leq w$, hence $a < \mu < b$. Now, if some fixed cost $c \geq 0$, referred to as the *search cost*, is paid at a point in time, it is assumed that an offer can be produced at the next point in time. Furthermore we denote the per-period discount factor by β $(0 < \beta \leq 1)$. Here let $\alpha = \beta\mu - c > 0$, hence $\alpha \leq \mu < b$.

Assuming that an offer inspected and passed up once is forever available, our model can be said to be a model with recall. Here, being available means that the offer once inspected and passed up can be accepted at any time in the future at the searcher's convenience. If one of the past offers is accepted, it is of course the best offer y so far. In this case, we postulate that ry, referred to as the *recall cost*, must be paid to accept the best offer where r is a given constant with $0 < r < 1$. Hence, if it is accepted, the value gained will be $y - ry = sy$ $(< y)$ with $s = 1 - r$, so $0 < s < 1$. Let us call r and s, respectively, the *recall cost factor* and *value deterioration factor*. As for the treatment of the recall cost, some variations can be considered (Section 9). However, as we have mentioned before, due to the importance of introducing the recall cost concept to the conventional optimal stopping problem, we will concentrate on the model with recall cost ry which is proportion to the value y of the best offer so far.

Finally, we assume that an offer must be accepted up to time 0 (deadline) and the searcher can terminate the search process by accepting one offer at any time before the deadline. Hence, the searcher has the following three choices for a current offer w at every point in time except on the deadline; 1) pass up and continue the search process, 2) stop by accepting the current offer w, 3) stop by accepting the best offer y so far where its value becomes sy by paying the recall cost ry.

If $r = 0$ (or $r = 1$), then our model is substantially reduced to the conventional model with recall (or with no recall) [18]. Landsberger et al. [11] examined a variation of the model with

recall in which the best offer y up to a point is available at the next point in time with a known probability p. Such a model is usually called a model with uncertain recall. Our model can be said to be a Landsberger model with probability $p = 1$ and recall cost factor r.

We refer to the rule prescribing when to stop the search by accepting an offer as the stopping rule. The objective of this paper is to find the optimal stopping rule maximizing the expected present discounted net value, which is the expectation of the present discounted value of an accepted offer minus the total present discounted value of the search costs paid up to the termination of the search with its acceptance.

3 Preliminaries

For any real number x define

$$K(x) = \beta \int_0^\infty \max\{\xi, x\} dF(\xi) - x - c, \tag{3.1}$$

and let h^* and h° be the smallest solutions of $K(x) = 0$ and $K(sx) + (s-1)x = 0$, respectively, if they exist, i.e.,

$$h^* = \min\{x \mid K(x) = 0\}, \tag{3.2}$$

$$h^\circ = \min\{x \mid K(sx) + (s-1)x = 0\}. \tag{3.3}$$

Lemma 3.1

(a) $K(x) = \alpha - x$ *for* $x \le a$ *and* $K(x) = -(1-\beta)x - c \le 0$ *for* $b \le x$.

(b) $K(x)$ *is nonincreasing in* x *on* $-\infty < x < \infty$ *and strictly decreasing on* $x < b$, *so* $K(x) > K(b)$ *for any* $x < b$.

(c) $K(x) + x$ *is nondecreasing in* x *on* $-\infty < x < \infty$ *and strictly increasing in* $x \ge a$ *with* $K(x) + x = \alpha$ *for* $x \le a$ *and* $K(x) + x = \beta x - c \le x$ *for* $b \le x$.

(d) $K(x)$ *is convex in* x.

(e) $K(x) - K(y) \le (y - x)(1 - \beta F(x))$ *for any* x *and* y.

(f) h^* *exists with* $h^* \ge \alpha$.

(g) *If* $\alpha > a$, *then* $a < h^*/s$.

(h) *If* $K(sb) < (\ge)\ 0$, *then* $h^*/b < (\ge)\ s$.

(i) *If* $(1-\beta)^2 + c^2 = 0$, *then* $h^* = b$, *so* $K(sb) > 0$ *and* $h^*/s > b$.

(j) *If* $(1-\beta)^2 + c^2 \ne 0$, *then*

1. h^* *is the unique solution of* $K(x) = 0$ *where* $h^* < b$, *hence* $h^*/b < 1$,
2. *if* $\alpha > a$, *then* $a < h^* < b$,
3. *if* $\alpha \le a$, *then* $h^* = \alpha \le a$.

Proof See [7] for the proofs of the remain statements excluding (g), (h), and the inequality $K(sb) > 0$ of (i).

(g) This is immediate from $K(sa) = \alpha - sa > a - sa > 0$.

(h) Assume $K(sb) < (\ge)\ 0$. Then we have $h^* < (\ge)\ sb$, hence $h^*/s < (\ge)\ b$.

(i) The fact $h^* = b$ implies that $K(b) = K(h^*) = 0$. Noting $sb < b$ due to the assumption of $s < 1$, we obtain $K(sb) > K(b) = 0$ from (b). ■

Lemma 3. 2

(a) $K(sx)+(s-1)x$ *is strictly decreasing in* x *on* $(-\infty,\infty)$ *and takes a positive value at* $x<\min\{a,\alpha\}$ *and a negative value at* $x=b$.

(b) $K(sx)+(s-1)x=0$ *has the unique solution* h° *with* $\min\{a,\alpha\}\le h^\circ<b$, $0<h^\circ$, *and* $h^\circ<h^*/s$.

(c) *If* $\alpha>(\le)\ a$, *then* $K(sa)+(s-1)a>(\le)\ 0$.

(d) *If* $\alpha>a$, *then* $a\le h^\circ<b$, *or else* $h^\circ=h^*=\alpha\le a$.

Proof (a) For any $x>x'$ from Lemma 3. 1 (e) we obtain

$$\begin{aligned}(K(sx)+(s-1)x)-(K(sx')+(s-1)x')&\\ =K(sx)-K(sx')+(s-1)(x-x')&\\ \le s(x'-x)(1-\beta F(sx))+(s-1)(x-x')&\\ =(x'-x)(1-\beta sF(sx))<0&\end{aligned}\tag{3.4}$$

due to the assumption $s<1$. Suppose $x<\min\{a,\alpha\}$, so $x<a$ and $x<\alpha$. Then $K(sx)+(s-1)x=\alpha-x>0$ from Lemma 3. 1 (a). At $x=b$ we have $K(sb)+(s-1)b=(K(sb)+sb)-b<(K(b)+b)-b=K(b)\le 0$ from Lemma 3. 1 (c,a).

(b) The existence of the unique solution is clear from (a). The inequality $\min\{a,\alpha\}\le h^\circ<b$ is clear from (a) and $K(s\min\{a,\alpha\})+(s-1)\min\{a,\alpha\}=\alpha-\min\{a,\alpha\}\ge 0$ due to $s\min\{a,\alpha\}\le sa<a$. Now, $K(s\times 0)+(s-1)\times 0=K(0)=\alpha>0$ by assumption, hence $h^\circ>0$. By definition we have $0=K(sh^\circ)+(s-1)h^\circ$, hence $K(sh^\circ)=(1-s)h^\circ>0$, which implies $sh^\circ<h^*$, so $h^\circ<h^*/s$.

(c) It is clear that $K(sa)+(s-1)a=\alpha-sa+sa-a=\alpha-a$.

(d) If $\alpha>(\le)\ a$, then the former half of the assertion is obvious from (b). If $\alpha\le a$, then $h^*=\alpha$ from Lemma 3. 1 (j3) and $K(s\alpha)+(s-1)\alpha=\alpha-s\alpha+s\alpha-\alpha=0$, which implies $h^\circ=\alpha$. Hence we obtain $h^\circ=h^*=\alpha$. ■

4 Functional Equation

Let $u_t(w,y)$ be the maximum total expected present discounted net value starting from time $t\ge 0$ with a present offer w and a best offer y up to the previous time. Then, clearly

$$u_0(w,y)=\max\{w,sy\}\tag{4.1}$$

and for $t\ge 1$

$$u_t(w,y)=\max\{w,sy,U_t(w,y)\}\ge w\tag{4.2}$$

where $\max\{w,sy\}$ is the reward from stopping the search and $U_t(w,y)$ is the maximum total expected present discounted net value from continuing the search, expressed as, for $t\ge 1$,

$$U_t(w,y)=\beta\int_0^\infty u_{t-1}(\xi,\max\{w,y\})dF(\xi)-c\tag{4.3}$$

$$=\beta\int_0^\infty \max\{\xi,s\max\{w,y\},U_{t-1}(\xi,\max\{w,y\})\}dF(\xi)-c\tag{4.4}$$

with $U_0(w,y) = 0$ for all w and y. Now, from Eq. (4.4) and Eq. (3.1) we have

$$U_1(w,y) = K(s\max\{w,y\}) + s\max\{w,y\}. \tag{4.5}$$

Remark 4.1 *$U_t(w,y)$ is independent of y for $w \geq y$ and of w for $y \geq w$.*

5 Optimal Stopping Rule

Let us define

$$G_t(w,y) = U_t(w,y) - \max\{w, sy\}, \quad t \geq 1, \tag{5.1}$$

where

$$G_1(w,y) = \begin{cases} K(sw) + (s-1)w, & w \geq y, \\ K(sy) + sy - w, & y \geq w \geq sy, \\ K(sy), & sy \geq w. \end{cases} \tag{5.2}$$

The optimal stopping rule of time t can then be prescribed as follows: If $G_t(w,y) \leq 0$, stop the search by accepting the better offer between the current offer w and the best offer y so far, or else continue the search. Here let us call the set $\mathcal{S}_t = \{(w,y) \mid G_t(w,y) \leq 0\}$ the stop region and its compliment $\mathcal{C}_t = \mathcal{S}_t^c$ the continuation region.

Below, taking the same example of the R&D problem as stated in Section 1, let us demonstrate a distinctive property of the optimal stopping rule in this model with recall cost. Suppose we must decide on a proposal to accept until the next year, that is, the present time $t = 1$. Assume that the value of the best proposal so far submitted is y. The recall cost ry must be paid to accept it, and the value of this year's proposal is w. The problem now is to decide on whether to stop the R&D activity by accepting this year's proposal w or the best proposal y so far or to continue the R&D activity in the hope of getting a better proposal next year by paying the R&D cost c (search cost). Here let $c = 0.01$, $\beta = 0.9$, $s = 0.8$ ($r = 0.2$), and $F(w)$ is a uniform distribution on $[0,1]$.

When it is decided to stop future R&D activity at this time, we have the choice of the better offer between this year's proposal w and the best proposal y so far where its value becomes sy by paying the recall cost. Therefore, the gain in this case is

$$S(w,y) = \max\{w, sy\}. \tag{5.3}$$

On the other hand, if it is decided to continue the R&D activity, the gain is

$$C(w,y) = U_1(w,y), \tag{5.4}$$

which can be calculated by Eq. (4.5) where $K(x) = \beta(x + 0.5(1-x)^2) - x - c$ in this case. Now suppose $w = 0.58$. Then we can write the following scenarios.

Case 1 Suppose the value of the best proposal so far is $y = 0.90$. In this case, $S(0.58, 0.90) = 0.72 > 0.6732800 = C(0.58, 0.90)$, implying that the optimal decision is to stop the R&D activity by accepting the best offer y so far. The result can be said to be quite natural because the value of the best proposal y ($= 0.90$) is very high, close to the upper bound of possible value ξ.

Table 1: An Example at $t = 1$

case	w	y	sy	$S(w,y)$		$C(w,y)$	decision
	0.58	1.00	<u>0.80</u>	0.80	>	0.7280000	stop
case 1	0.58	0.90	<u>0.72</u>	0.72	>	0.6732800	stop
	0.58	0.80	<u>0.64</u>	0.64	>	0.6243200	stop
case 2	0.58	0.75	<u>0.60</u>	0.60	<	0.6020000	continue
case 3	<u>0.58</u>	0.70	0.56	0.58	<	0.5811200	continue
	<u>0.58</u>	0.60	0.48	0.58	>	0.5436800	stop
	<u>0.58</u>	0.50	0.40	0.58	>	0.5338832	stop
	<u>0.58</u>	0.40	0.32	0.58	>	0.5338832	stop
	<u>0.58</u>	0.30	0.24	0.58	>	0.5338832	stop
	<u>0.58</u>	0.20	0.16	0.58	>	0.5338832	stop
case 4	<u>0.58</u>	0.10	0.08	0.58	>	0.5338832	stop
	<u>0.58</u>	0.00	0.00	0.58	>	0.5338832	stop

Case 2 Consider another situation in which the value of the best proposal $y = 0.75$. Then we have $S(0.58, 0.75) = 0.60 < 0.6020000 = C(0.58, 0.75)$, implying that continuing the R&D activity is optimal.

Case 3 Furthermore, consider another situation in which the value of the best proposal so far $y = 0.70$. Since the value of best offer y is smaller than in Case 2, it can be anticipated that continuing the R&D activity will become optimal. In fact, it is so because $S(0.58, 0.70) = 0.58 < 0.5811200 = C(0.58, 0.70)$.

Case 4 Finally, consider the situation in which the value of the best proposal so far $y = 0.10$, close to the lower-bound of possible value ξ. Since the value of the best offer y is smaller than in Case 3, it can be naturally also be expected that continuing the R&D activity will become optimal. However, we have $S(0.58, 0.10) = 0.58 > 0.5338832 = C(0.58, 0.10)$, implying that stopping by accepting this year's proposal w is optimal.

The above scenario is seemingly counter-intuitive; however, it can be interpreted as follows. When the best proposal y so far is sufficiently small, if the R&D activity is continued, the value of this year's proposal w that becomes the best proposal at the next year decreases due to the existence of the recall cost. Hence it will be better to stop the R&D activity by accepting this year's proposal w rather then continuing the R&D activity by going so far as to pay the recall cost ry.

Table 1 shows the twelve cases of $y = 0.0, 0.1, \ldots, 1.0$ where the underlined numbers of columns w and sy represent the larger of the corresponding two numbers. Furthermore the numbers in column $C(w, y)$ do not include any computational error. From Table 1 we can see that there exist two critical values y_* and y^* in terms of y where y_* is between 0.60 and 0.70, and y^* between 0.75 and 0.80. More precisely the two critical values are, respectively, $y_* = 0.6972$ and $y^* = 0.7554$, which are the solutions to the equation $S(w, y) = C(w, y)$. This implies that the optimal decision changes from "stop" to "continue" at $y_* = 0.6972$ and returns again to "stop" at $y^* = 0.7554$.

This is the distinctive property of the optimal stopping problem that may appear in our model. The remainder of this paper will be devoted to the examination of the conditions in which the property appears.

6 DRV Property

If the optimal stopping rule has the following property, then it is said to have the double reservation value property, DRV property, for short.

• *DRV Property* For at least one current offer w with $a < w < b$, there exist two different critical values y_* and y^* with $a < y_* < y^* < b$ such that, for the best offer y so far, if $y_* \leq y \leq y^*$, continuing is optimal, or alternatively stopping by accepting the better offer between the current offer w and the best offer y so far is optimal.

From Eq. (5.2) and Lemmas 3. 1, and 3. 2 , we can depict the continuation region for $t = 1$ as the shaded region in Figure 7.1 where note $h^\circ < h^*/s$ (Lemma 3. 2 (b)). The figure shows that the DRV property appears for $\tilde{w}$ on the w-axis. It can be easily recognized from the figure that all three statements below are true.

S1: If $h^*/s \geq b$, the DRV property does not appear,
S2: If $h^* \leq a$, the DRV property does not appear,
S3: If $h^*/s < b$ and $a < h^\circ < b$, the DRV property appears.

In general, it is verified in Section 8 that the same figure can be depicted for any t. In this case it is verified that h° in S3 is replaced with $h_t(h_1 = h^\circ)$ defined by Eq. (8.15), $h_t^*\ (= h^*/s)$ which is defined by Eq. (8.16) and correspond to y^* is fixed for all t, and if $\alpha > a$ and $h^*/s < b$, then h_t strictly increases in t (Theorem 8. 1 (a)) and converges to h^* (Theorem 8. 2). Consequently the above three statements are also true for any t, as proved in Section 8. From the above considerations and the Lemmas and Theorems in Section 8, we can describes the conditions for the optimal stopping rule to have or not to have the DRV property as follows.

• *Conditions for DRV Property*

(a) If $(1-\beta)^2 + c^2 = 0$, then $h^*/s \geq b$ (Lemma 3. 1 (i)). In this case the DRV property does not appear for any $t \geq 1$. $\cdots$ (S1)
(b) If $(1-\beta)^2 + c^2 \neq 0$, and
  i. if $\alpha \leq a$, then $h^* \leq a$ (Lemma 3. 1 (j3)). In this case the DRV property does not appear for any $t \geq 1$. $\cdots$ (S2)
  ii. if $\alpha > a$, then $a < h_t < b$, and, in addition,
    1. if $h^*/s \geq b$, then the DRV property does not appear for any $t \geq 1$.$\cdots$ (S1)
    2. if $h^*/s < b$, then he DRV property appear for any $t \geq 1$. $\cdots$ (S3)

From the above, eventually we get the following theorem.

Theorem 6. 1 *The necessary and sufficient condition for the DRV property to appear for all $t \geq 1$ is given by $(1-\beta)^2 + c^2 \neq 0$, $\alpha > a$, and $s > h^*/b$.*

It has been already proved that although it does not appear in the model with uncertain recall by Lansberger [11], the DRV property does appear in that by Ikuta [4]. Although it does not appear in the conventional model with recall[18], in our model as defined by introducing the recall cost to the conventional model with recall, the DRV property may appear if the conditions given in Theorem 6. 1 hold. This tells us that the DRV property may appear not only in the uncertain recall model but also in the recall model.

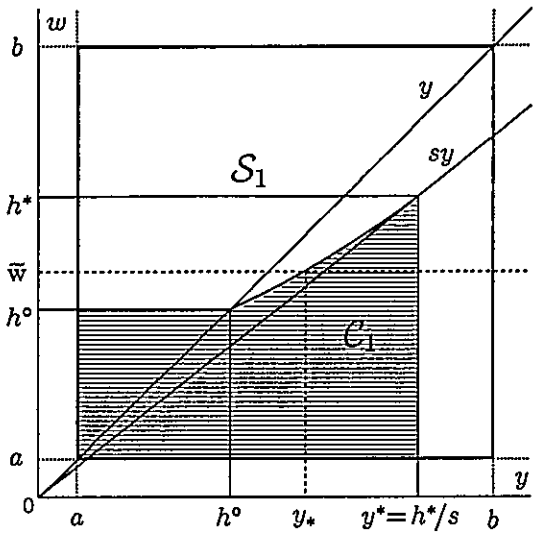

Figure 7.1: DRV property at $t = 1$

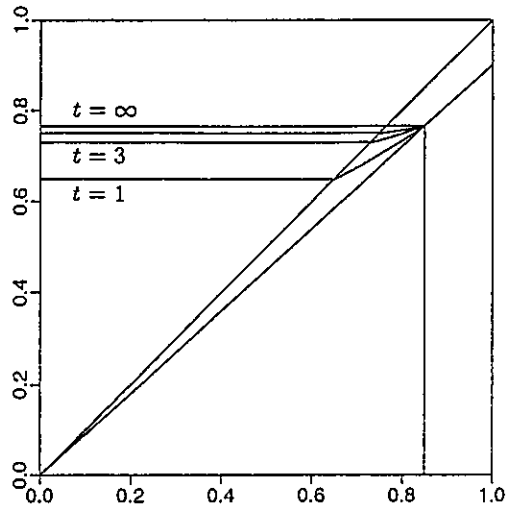

Figure 7.2: Continuation Regions at $t \geq 1$

7 Numerical Example

This section illustrates the properties of h_t and $h_t^* (= h^*/s)$ by a numerical example with $\beta = 0.97$, $s = 0.9$, and $c = 0.005$. Here the offer w is assumed to follow a discrete uniform distribution function with 101 mass points, equally spaced on $[0, 1]$ where $\alpha = 0.48 > 0 = a$. In this case we have $h^* = 0.7605$ and $h^\circ = 0.6418$, hence $h^*/s = 0.8450$. Therefore, all the conditions for the DRV property to appear (Theorem 6. 1) hold true in this case. Figure 7.2 shows how the continuation region C_t increases in t and converges as $t \to \infty$ to a rectangle, height h^*, width h^*/s. Furthermore, it can be seen that $h_t < h_t^* = h^*/s$ of Theorem 8. 1 (b) in the following section.

8 Analysis

For expressional simplicity, throughout the section, "for any w, y, and ξ" means "for any w with $a \leq w \leq b$, any y with $a \leq y \leq b$, and any ξ with $a \leq \xi \leq b$".

Lemma 8. 1

(a) $\alpha \leq U_t(w, y) \leq \beta b - c \leq b$ *for any* $t \geq 1, w$, *and* y.

(b) $U_t(w, y)$ *is nondecreasing in* w *and* y *for any* t.

(c) $U_t(w, y)$ *is convex in* w *and* y *for any* t.

(d) $U_t(w, y)$ *is nondecreasing in* t *for any* w *and* y.

(e) $U_t(w, y) - y$, *hence* $\beta U_t(w, y) - y$ *are strictly decreasing in* y *for any* t *and* w.

(f) $U_t(w, y) - w$ *is strictly decreasing in* w *for any* t *and* y.

(g) *If* $\beta < (=)\ 1$, *then* $U_t(w, y) - sy$, *hence* $\beta U_t(w, y) - sy$ *is strictly decreasing (nonincreasing) in* y *for any* t *and* w.

(h) $U_t(z, z) - z$ *is strictly decreasing in* z *for any* t.

(i) *If* $K(sb) < 0$, *then* $U_t(w, b) < sb$ *for any* t *and* w.

Proof (a) From Eq. (4.3) we have immediately for any $t \geq 1$, w and y

$$U_t(w,y) \geq \beta \int_0^\infty \xi \, dF(\xi) - c = \beta\mu - c = \alpha. \tag{8.1}$$

It is clear that $u_0(w,y) \leq \max\{b, sb\} = b$ for all w and y. Suppose $u_{t-1}(w,y) \leq b$ for any w and y. Then, from Eq. (4.3) we have for any w and y

$$U_t(w,y) \leq \beta \int_0^\infty b \, dF(\xi) - c = \beta b - c \leq b, \tag{8.2}$$

hence $u_t(w,y) \leq \max\{b, sb, b\} = b$ for any w and y.

(b) The proof is easily obtained by induction.

(c) Clearly $U_0(w,y)$ is convex in w and y. Suppose $U_{t-1}(w,y)$ is convex in w and y, so also is $U_{t-1}(\xi, \max\{w,y\})$. Consequently, $U_t(w,y)$ is also convex in w and y from Eq. (4.4).

(d) The proof is easily obtained by induction starting with $U_1(w,y) \geq 0 = U_0(w,y)$.

(e) Since $U_0(w,y) - y = -y$ and $\beta U_0(w,y) - y = -y$, the assertion is true for $t = 0$. Suppose it is true for $t-1$. Then we have

$$U_t(w,y) - y = \begin{cases} \beta \int_0^\infty \max\{\xi, sw, U_{t-1}(\xi, w)\} dF(\xi) - c - y, & w \geq y, \quad (8.3) \\ \int_0^\infty \max\{\beta\xi - y, (\beta s - 1)y, \beta U_{t-1}(\xi, y) - y\} dF(\xi) - c. & y \geq w. \quad (8.4) \end{cases}$$

By induction hypothesis, Eqs. (8.3) and (8.4) are both strictly decreasing in y from the assumption $s < 1$. Thus, $U_t(w,y) - y$, hence $\beta U_t(w,y) - y$ is strictly decreasing in y for any w.

(f) If $y \geq w$, then $U_t(w,y)$ is independent of w from Remark 4. 1 , hence $U_t(w,y) - w$ is strictly decreasing in w. If $w \geq y$, then

$$U_t(w,y) - w = \int_0^\infty \max\{\beta\xi - w, (\beta s - 1)w, \beta U_{t-1}(\xi, w) - w\} dF(\xi) - c, \tag{8.5}$$

which is strictly decreasing in $w \geq y$ from (e) and the assumption $s < 1$.

(g) Since $U_0(w,y) - sy = -sy$ and $\beta U_0(w,y) - sy = -sy$, the assertion is true for $t = 0$. Suppose it is true for $t-1$. Then we have

$$U_t(w,y) - sy = \begin{cases} \beta \int_0^\infty \max\{\xi, sw, U_{t-1}(\xi, w)\} dF(\xi) - c - sy, & w \geq y, \quad (8.6) \\ \int_0^\infty \max\{\beta\xi - sy, (\beta - 1)sy, \beta U_{t-1}(\xi, y) - sy\} dF(\xi) - c. & y \geq w. \quad (8.7) \end{cases}$$

If $\beta < (=) \, 1$, then, by induction hypothesis, Eqs. (8.6) and (8.7) are both strictly decreasing (nonincreasing) in y. Consequently it follows that $U_t(w,y) - sy$, hence $\beta U_t(w,y) - sy$ are strictly decreasing (nonincreasing) in y.

(h) Since $U_0(z,z) - z = -z$, the assertion is true for $t = 0$. Suppose it is true for $t-1$. Then

we have

$$U_t(z,z) - z = \int_0^\infty \max\{\beta\xi - z, (\beta s - 1)z, \beta U_{t-1}(\xi, z) - z\} dF(\xi) - c, \tag{8.8}$$

which is strictly decreasing in z from (e) and the assumption $s < 1$.

(i) From Eq. (4.5) and the assumption of $K(sb) < 0$ we obtain $U_1(w,b) = K(sb) + sb < sb$. Since $U_0(w,b) = 0 < sb$ for all w, the assertion is true for $t = 0$. Suppose the assertion is true for $t-1$. Then, by assumption $K(sb) < 0$ we get

$$\begin{aligned} U_t(w,b) &= \beta \int_0^\infty \max\{\xi, sb, U_{t-1}(\xi, b)\} dF(\xi) - c \\ &= \beta \int_0^\infty \max\{\xi, sb\} dF(\xi) - c \\ &= K(sb) + sb < sb. \quad \blacksquare \end{aligned} \tag{8.9}$$

Now $G_t(w,y)$ is independent of y if $w \geq y$ and of w if $sy \geq w$, hence let them be written, respectively, $G_t^1(w)$ and $G_t^2(y)$, i.e.,

$$G_t^1(w) = U_t(w,y) - w = \beta \int_0^\infty \max\{\xi, sw, U_{t-1}(\xi, w)\} dF(\xi) - c - w, \quad w \geq y, \tag{8.10}$$

$$G_t^2(y) = U_t(w,y) - sy = \beta \int_0^\infty \max\{\xi, sy, U_{t-1}(\xi, y)\} dF(\xi) - c - sy, \quad sy \geq w \tag{8.11}$$

where from Eq. (5.2) we have for $t = 1$

$$G_1^1(w) = K(sw) + (s-1)w, \quad sy \geq w. \tag{8.12}$$

When $y \geq w \geq sy$, a locus of (w,y) satisfying $G_t(w,y) = U_t(w,y) - w = 0$ is given by the function

$$H_t(y) = U_t(w,y) = \beta \int_0^\infty \max\{\xi, sy, U_{t-1}(\xi, y)\} dF(\xi) - c, \quad y \geq w \geq sy \tag{8.13}$$

where $U_t(w,y)$ is independent of w (Remark 4. 1) and we have from Eq. (5.2)

$$H_1(y) = K(sy) + sy, \quad y \geq w \geq sy. \tag{8.14}$$

Let h_t and h_t^* be the smallest solutions of $G_t^1(w) = 0$ and $G_t^2(y) = 0$ for any $t \geq 1$, respectively, if they exist, i.e.,

$$h_t = \min\{w \mid G_t^1(w) = 0\}, \tag{8.15}$$

$$h_t^* = \min\{y \mid G_t^2(y) = 0\}. \tag{8.16}$$

Here note that for any $t \geq 1$

$$h_t = U_t(h_t, h_t), \tag{8.17}$$

$$sh_t^* = U_t(sh_t^*, h_t^*), \tag{8.18}$$

from which it is easily seen that the two points (h_t, h_t) and (sh_t^*, h_t^*) lie on the locus given by Eq. (8.13).

Lemma 8. 2

(a) *For any* $t \geq 1$, *if* $\alpha > a$, *then* $G_t^1(a) > 0$, *and if* $w < \alpha$, *then* $G_t^1(w) > 0$.
(b) *For any* $t \geq 1$, *we have* $G_t^1(b) \leq 0$, *and if* $K(sb) < 0$, *then* $G_t^1(b) < 0$.
(c) $G_t^1(w)$ *is strictly decreasing and convex in* w *for any* $t \geq 1$.
(d) *If* $K(sb) < 0$, *then* $G_t^1(w) = 0$ *has the unique solution* h_t *where* $h_1 = h^\circ$.
(e) $G_t^1(w)$ *and* h_t *are nondecreasing in* t *for any* w.

Proof (a) It is clear from $G_t^1(w) \geq \alpha - w$ for any $t \geq 1$ due to Lemma 8. 1 (a).

(b) For any $t \geq 1$ and w we have $G_t^1(w) \leq \beta b - c - w$ from Lemma 8. 1 (a). Hence $G_t^1(b) \leq (\beta - 1)b - c \leq 0$. Suppose $K(sb) < 0$. Then, noting Lemma 8. 1 (i), for all $t \geq 1$ we have

$$\begin{aligned} G_t^1(b) &= \beta \int_0^\infty \max\{\xi, sb, U_{t-1}(\xi, b)\} dF(\xi) - c - b \\ &= \beta \int_0^\infty \max\{\xi, sb\} dF(\xi) - c - b \\ &= K(sb) + sb - b \\ &< sb - b < 0. \end{aligned} \tag{8.19}$$

(c) It is clear from Lemma 8. 1 (f,c).

(d) The unique existence of the solution is clear from (a), (b), and (c). Since $G_1^1(h_1) = K(sh_1) + (s-1)h_1 = 0$ by definition, we have $h_1 = h^\circ$ from Eq. (3.3).

(e) The monotonicity of $G_t^1(w)$ in t is clear from Lemma 8. 1 (d). The monotonicity of h_t in t is immediate from this and (c) . ■

Lemma 8. 3

(a) *For any* $t \geq 1$, *if* $\alpha > a$, *then* $G_t^2(a) > 0$, *and if* $y < \alpha/s$, *then* $G_t^2(y) > 0$.
(b) *If* $K(sb) < 0$, *then* $G_t^2(b) < 0$.
(c) *If* $\beta < (=)\ 1$, *then* $G_t^2(y)$ *is strictly decreasing(nonincreasing) and convex in* y *for all* t.
(d) *If* $K(sb) < 0$, *then* $G_t^2(y) = 0$ *has a solution* h_t^*, *which is unique if* $\beta < 1$ *where* $h_1^* = h^*/s$.
(e) $G_t^2(y)$ *is nondecreasing in* t *for all* y.

Proof (a) It is clear from $G_t^2(y) \geq \alpha - sy$ for any $t \geq 1$ due to Lemma 8. 1 (a).

(b) If $K(sb) < 0$, then from Lemma 8. 1 (i) we have $G_t^2(b) < sb - sb = 0$.

(c) It is clear from Lemma 8. 1 (g,c).

(d) It is clear from (a), (b) and equality $h_1^* = h^*/s$ is clear from $0 = G_1^2(h_1^*) = K(sh_1^*)$.

(e) It is clear from Lemma 8. 1 (d). ■

Lemma 8. 4 $H_t(y)$ *is nondecreasing and convex in* y *for all* t.

Proof It is clear from Lemma 8. 1 (b,c). ■

Lemma 8. 5 *Suppose* $\alpha > a$ *and* $K(sb) < 0$. *Then,*

(a) *for any* $\xi^* \in (a, b)$ *and* $y \in (a, b)$ *such that* $y \leq \xi^* < h_1$, *we have* $U_t(\xi^*, y) > \max\{\xi^*, sy\} = \xi^*$ *for all* $t \geq 1$,

(b) *if* $w \geq y$, *then* $U_t(w,y)$ *is strictly increasing in* t.

Proof (a) Consider any $\xi^* \in (a,b)$ and any $y \in (a,b)$ such that $y \leq \xi^* < h_1$ where $h_t > a$ for all $t \geq 1$ from Lemma 8. 2 (a,b,c). Then, noting $sy \leq \xi^*$, we have $U_1(\xi^*,y) - \max\{\xi^*, sy\} = U_1(\xi^*,y) - \xi^* = G_1^1(\xi^*) > 0 = G_1^1(h_1)$. Hence the assertion is true for $t = 1$. Now $U_t(\xi^*,y)$ is nondecreasing in t from Lemma 8. 1 (d), hence it follows that it is also true for all $t \geq 1$.

(b) In general, we get $\max\{\xi, s\max\{w,y\}, U_1(\xi, \max\{w,y\})\} \geq \max\{\xi, s\max\{w,y\}\}$ for all ξ. Suppose $w \geq y$. Then, from (a) we obtain the following for any $\xi^* \in (a,b)$ and $w \in (a,b)$ such that $w \leq \xi^* < h_1$.

$$\begin{aligned}
&\max\{\xi^*, s\max\{w,y\}, U_1(\xi^*, \max\{w,y\})\} \\
&\quad = \max\{\xi^*, sw, U_1(\xi^*, w)\} \\
&\quad = \max\{\max\{\xi^*, sw\}, U_1(\xi^*, w)\} \\
&\quad = U_1(\xi^*, w) \\
&\quad > \max\{\xi^*, sw\} \\
&\quad = \max\{\xi^*, s\max\{w,y\}\}. \qquad (8.20)
\end{aligned}$$

Hence, we have

$$\begin{aligned}
U_2(w,y) &= \beta \int_0^\infty \max\{\xi, s\max\{w,y\}, U_1(\xi, \max\{w,y\})\} dF(\xi) - c \\
&> \beta \int_0^\infty \max\{\xi, s\max\{w,y\}\} dF(\xi) - c \\
&= U_1(w,y). \qquad (8.21)
\end{aligned}$$

Suppose $U_{t-1}(w,y) > U_{t-2}(w,y)$ for $w \geq y$. Now we have for $w \geq y$

$$U_t(w,y) = \beta \int_0^\infty g_t(\xi) dF(\xi) - c, \qquad g_t(\xi) = \max\{\xi, sw, U_{t-1}(\xi, w)\}, \qquad (8.22)$$

$$U_{t-1}(w,y) = \beta \int_0^\infty g_{t-1}(\xi) dF(\xi) - c, \quad g_{t-1}(\xi) = \max\{\xi, sw, U_{t-2}(\xi, w)\}. \qquad (8.23)$$

Therefore, it follows from (a) and the induction hypothesis that for any $\xi^* \in (a,b)$ and $w \in (a,b)$ such that $w \leq \xi^* < h_1$, we have $g_t(\xi^*) = U_{t-1}(\xi^*, w) > U_{t-2}(\xi^*, w) = g_{t-1}(\xi^*)$. Thus we have $U_t(w,y) > U_{t-1}(w,y)$. ■

Theorem 8. 1

(a) *If* $\alpha > a$ *and* $K(sb) < 0$, *then* h_t *is strictly increasing in* t *with* $a < h_t < b$ *for all* $t \geq 1$.
(b) $h_t \leq sh_t^* < h_t^* = h^*/s$ *for any* $t \geq 1$.

Proof (a) We have $h_t = U_t(h_t, h_t) > U_{t-1}(h_t, h_t)$ for all $t \geq 2$ from Eq. (8.17) and Lemma 8. 5 (b). Now since h_t is nondecreasing in t from Lemma 8. 2 (e), we obtain $U_{t-1}(h_t, h_t) \geq U_{t-1}(h_{t-1}, h_{t-1}) = h_{t-1}$ from Lemma 8. 1 (b). Hence we get $h_t > h_{t-1}$, i.e., h_t is strictly increasing in t. The inequality $a < h_t < b$ for all t is clear from Lemma 8. 2 (a,b).

(b) From Eqs. (8.17) and (8.18) we have $h_t = U_t(h_t, h_t)$ and $sh_t^* = U_t(sh_t^*, h_t^*) \geq U_t(sh_t^*, sh_t^*)$ from Lemma 8. 1 (b), that is, $U_t(h_t, h_t) - h_t = 0$ and $U_t(sh_t^*, sh_t^*) - sh_t^* \leq 0$, implying that

$h_t \le sh_t^* < h_t^*$ from Lemma 8. 1 (h). Now we have

$$sh_t^* = \beta \int_0^\infty \max\{\xi, sh_t^*, U_{t-1}(\xi, h_t^*)\} dF(\xi) - c. \tag{8.24}$$

If $\xi \le sh_t^*$, then we get $U_{t-1}(\xi, h_t^*) - sh_t^* \le U_t(\xi, h_t^*) - sh_t^* = G_t^2(h_t^*) = 0$ by definition, hence $U_{t-1}(\xi, h_t^*) \le sh_t^*, \cdots (1^*)$. If $\xi > sh_t^*$, then from Lemma 8. 1 (f) we obtain $U_{t-1}(\xi, h_t^*) - \xi < U_{t-1}(sh_t^*, h_t^*) - sh_t^* \le U_t(sh_t^*, h_t^*) - sh_t^* = G_t^2(h_t^*) = 0$, hence $U_{t-1}(\xi, h_t^*) \le \xi \cdots (2^*)$. Thus from (1^*) and (2^*) we have $U_{t-1}(\xi, h_t^*) \le \max\{\xi, sh_t^*\}$ for all ξ. Consequently, Eq. (8.24) can be rewritten as follows.

$$sh_t^* = \beta \int_0^\infty \max\{\xi, sh_t^*\} dF(\xi) - c, \tag{8.25}$$

from which $K(sh_t^*) = 0$, hence we have $sh_t^* = h^*$, i.e., $h_t^* = h^*/s$. ■

Below we shall examine the limiting natures of $U_t(w, y)$ and h_t. For the model with no recall, by $\widehat{u}_t(w)$ we shall denote the maximum expected present discounted net value starting from time t with a present offer w. Then, it is clear that $\widehat{u}_0(w) = w$, and for all $t \ge 1$

$$\widehat{u}_t(w) = \max\{w, \widehat{U}_t\}, \quad \widehat{U}_t = \beta \int_0^\infty \widehat{u}_{t-1}(\xi) dF(\xi) - c. \tag{8.26}$$

Similarly, for the model with recall, by $\widetilde{u}_t(y)$ we shall denote the maximum expected present discounted net value starting from time t with the best offer y up to time t. Then, it is clear that $\widetilde{u}_0(y) = y$, and for $t \ge 1$

$$\widetilde{u}_t(y) = \max\{y, \widetilde{U}_t(y)\}, \quad \widetilde{U}_t(y) = \beta \int_0^\infty \widetilde{u}_{t-1}(\max\{\xi, y\}) dF(\xi) - c. \tag{8.27}$$

Then we have the following lemma.

Lemma 8. 6

(a) *$\widehat{U}_t$ is nondecreasing in t and converges to $\widehat{U} = h^*$ as $t \to \infty$.*
(b) *$\widetilde{U}_t(y)$ is nondecreasing in t and y and converges to $\widetilde{U}(y) = \max\{K(y) + y, h^*\}$ as $t \to \infty$.*
(c) *$\widetilde{u}_t(\max\{w, y\}) \ge u_t(w, y) \ge \widehat{u}_t(w)$ for all t, w, and y.*
(d) *$\widetilde{U}_t(\max\{w, y\}) \ge U_t(w, y) \ge \widehat{U}_t$ for all t, w, and y.*

Proof (a,b) See [7].

(c,d) It is clear that $\widetilde{u}_0(\max\{w, y\}) = \max\{w, y\} \ge \max\{w, sy\} = u_0(w, y) \ge w = \widehat{u}_0(w)$, hence (c) holds true for $t = 0$. Suppose it also holds for $t - 1$. Then we have

$$\begin{aligned} U_t(w, y) &\ge \beta \int_0^\infty \widehat{u}_{t-1}(\xi) dF(\xi) - c = \widehat{U}_t, \\ U_t(w, y) &\le \beta \int_0^\infty \widetilde{u}_{t-1}(\max\{\xi, w, y\}) dF(\xi) - c = \widetilde{U}_t(\max\{w, y\}), \end{aligned} \tag{8.28}$$

from which we obtain

$$\begin{aligned} u_t(w, y) &\ge \max\{w, \widehat{U}_t\} = \widehat{u}_t(w), \\ u_t(w, y) &\le \max\{w, y, \widetilde{U}_t(\max\{w, y\})\} = \widetilde{u}_t(\max\{w, y\}). \end{aligned} \tag{8.29}$$

Consequently, (c) and (d) are both true for all t by induction. ■

Theorem 8. 2 *h_t converges to h^* as $t \to \infty$.*

Proof The convergency of h_t is immediate from Lemma 8. 2 (e) and Theorem 8. 1 (b), so let its limit be denoted by h_∞. From Eq. (8.17) we have

$$h_t = \beta \int_0^\infty \max\{\xi, sh_t, U_{t-1}(\xi, h_t)\} dF(\xi) - c. \tag{8.30}$$

Thus, from Lemma 8. 6 (d) we have

$$h_t \geq \beta \int_0^\infty \max\{\xi, sh_t, \widehat{U}_{t-1}\} dF(\xi) - c, \tag{8.31}$$

$$h_t \leq \beta \int_0^\infty \max\{\xi, sh_t, \widetilde{U}_{t-1}(\max\{\xi, h_t\})\} dF(\xi) - c. \tag{8.32}$$

Hence, as $t \to \infty$, we have from Lemma 8. 6 (a)

$$\begin{aligned} h_\infty &\geq \beta \int_0^\infty \max\{\xi, sh_\infty, h^*\} dF(\xi) - c \\ &\geq \beta \int_0^\infty \max\{\xi, h^*\} dF(\xi) - c \\ &= K(h^*) + h^* = h^* \cdots (1^*). \end{aligned} \tag{8.33}$$

Noting that $\max\{\xi, h_\infty\} \geq h_\infty \geq h^*$ for all ξ, so $K(\max\{\xi, h_\infty\}) + \max\{\xi, h_\infty\} \leq \max\{\xi, h_\infty\}$ due to $K(\max\{\xi, h_\infty\}) \leq 0$ and that $\lim_{t\to\infty} \widetilde{U}_{t-1}(\max\{\xi, h_t\}) = \widetilde{U}(\max\{\xi, h_\infty\})$, we obtain from Eq. (8.32) and Lemma 8. 6 (b)

$$\begin{aligned} h_\infty &\leq \beta \int_0^\infty \max\{\xi, sh_\infty, \max\{K(\max\{\xi, h_\infty\}) + \max\{\xi, h_\infty\}, h^*\}\} dF(\xi) - c \\ &\leq \beta \int_0^\infty \max\{\max\{\xi, h_\infty\},\ K(\max\{\xi, h_\infty\}) + \max\{\xi, h_\infty\}, h^*\} dF(\xi) - c \\ &= \beta \int_0^\infty \max\{\max\{\xi, h_\infty\}, h^*\} dF(\xi) - c \\ &= \beta \int_0^\infty \max\{\xi, h_\infty\} dF(\xi) - c \\ &= K(h_\infty) + h_\infty, \end{aligned} \tag{8.34}$$

from which $0 \leq K(h_\infty)$. Suppose $(1-\beta)^2 + c^2 \neq 0$. Then $K(x) = 0$ has a unique solution from Lemma 3. 1 (j1), hence $h_\infty \leq h^* \cdots (2^*)$. Thus, in this case we have $h_\infty = h^*$ from (1^*) and (2^*). Suppose $(1-\beta)^2 + c^2 = 0$. Then $h^* = b$ from Lemma 3. 1 (i). Furthermore, from Lemma 8. 2 (b) we have $h_t \leq b$, hence $h_\infty \leq b = h^* \cdots (3^*)$. Consequently also in this case we have $h_\infty = h^*$ from (1^*) and (3^*). ■

9 Some Variations

Some variations of our model should be considered:

1. The recall cost is given by a nondecreasing function of y, $r(y)$.

2. The recall cost factor r is dependent on the age j of past offers, i,e., r_j; more generally the recall cost is $r_j(y)$. In our model the recall cost is assumed to be independent of the age of

the past offers, so it suffices to consider only the best offer y so far. However, on the above assumption, $U_t(w,y)$ is written as $U_t(w_0, s_1w_1, s_2w_2, \cdots)$ where w_j is the value of an offer that appeared j periods ago and $s_j(=1-r_j)$ is the value deterioration factor for offer w_j.

3. A Landsberger model [11] to which recall cost is introduced.

4. An Ikuta model [4] to which recall cost is introduced. In the Ikuta model an offer inspected and passed up once j periods ago is assumed to be unavailable at the next point in time with known probability p_j, $j=0,1,\cdots$, provided that it still remains available at present. The model is a generalization of Landsberger model, the mathematical treatment of which is expected however to become extraordinarily intractable.

Some other related models to which our recall cost should be introduced include: A model with controlled recall [17] which is a model with no recall where if some fixed reserving cost is paid for a present offer, the offer becomes forever available, A model with several search area [6], A model with a finite search budget [5] where the search cost invested at each point in time can be controlled within a given search budget and the offer distribution depends on the cost of search invested.

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References

[1] Chow, Y.S., Robbins, H., & Siegmund, D.(1971). *Great Expectations: The Theory of Optimal Stopping.* Boston: Houghton Mifflin Company.

[2] Gilbert, J.P. & Mosteller, F. (1966). Recognizing the Maximum of a Sequence. *Journal of the American Statistic Association* 61: 35-73.

[3] Hayes, R.H. (1967). Optimal Strategies for Divestiture. *Operations Research* 17(2): 292-310.

[4] Ikuta, S. (1988). Optimal Stopping Problem with Uncertain Recall. *Journal of the Operations Research Society of Japan* 31(2): 145-170.

[5] Ikuta, S. (1992). The Optimal Stopping Problem in Which the Sum of the Accepted Offer's Value and the Remaining Search Budget is an Objective Function. *Journal of the Operations Research Society of Japan* 35(2): 172-193.

[6] Ikuta, S. (1995). The Optimal Stopping Problem with Several Search Areas. *Journal of the Operations Research Society of Japan* 38(1): 89-106.

[7] Ikuta, S. (1996). K-function and Optimal Stopping Problems. *Discussion paper Series* 705, Institute of Policy and Planning Sciences, University of Tsukuba.

[8] Karlin, S. (1962). Stochastic Models and Optimal Policy for Selling an Asset. In Arrow, J.A., Karlin, S., & Scarf. H. (eds.) *Studies in applied probability and management science.* Stanford, Calif: Stanford University Press, pp.148-158.

[9] Kohn, M.G. & Shavell, S. (1974). The Theory of Search. *Journal of Economic Theory* 9: 93-123.

[10] Karni, E. & Schwartz, A. (1977). Search Theory: The Case of Search with Uncertain Recall. *Journal of Economic Theory* 16: 38-52.

[11] Landsberger, M. & Peled, D. (1977). Duration of Offers, Price Structure, and the Gain from Search. *Journal of Economic Theory* 16: 17- 37.

[12] Lippman, S.A. & McCardle, K.F. (1991). Uncertain Search: A Model of Search Among Technologies of Uncertain Values.. *Management Science* 37(11): 1474-1490.

[13] Lippman, S.A. & McCall, J.J. (1976). The Economics of Job Search: A Survey. *Economic Inquiry* 14: 155-189.

[14] MacQueen, J. & Miller Jr, R.G. (1960). Optimal Persistence Policies. *Operations Research* 8(3): 362-380.

[15] Morgan, P. & Manning, R. (1985). Optimal Search. *Econometrica* 53(4): 923-944.

[16] Rosenfield, D.B., Shapiro, R.D., & Butler, D.A. (1983). Optimal Strategies for Selling an Asset. *Management Science* 29(9): 1051-1061.

[17] Saito, T. (1996). Optimal stopping problem with controlled recall. *Discussion Paper Series* 682, Institute of Policy and Planning Sciences, University of Tsukuba.

[18] Sakaguchi, M. (1961). Dynamic Programming of Some Sequential Sampling Design. *Journal of Mathematical Analysis and Applications* 2: 446-466.

[19] Sargent, T.J. (1987). *Dynamic Macroeconomic Theory.* Cambridge, Mass.: Harvard University Press, pp.57-91

[20] Shiryayev, A.N. (1978). *Optimal Stopping Rules.* New York: Springer-Verlag.

[21] Weitzman, M.L. (1979). Optimal Search for the Best Alternative. *Econometrica* 47(3): 641-654.

[22] Zuckerman, D. (1984). On Preserving the Reservation Wage Property in a Continuous Job Search Model. *Journal of Economic Theory* 34: 175-179.