No. 703

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November 1996

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Abstract

By developing an algorithm, Herings, Talman and Yang [6] recently proved the following interesting theorem: A correspondence ζ from the n-dimentional unit cube U^n to n-dimentional Euclidean space R^n has a continuum of zero points containing the origin and the vector of all-ones if the correspondence satisfies certain conditions. In this note we give an alternative concise proof of the theorem in the case where the correspondence $\zeta : U^n \to R^n$ is single-valued.

1 Introduction

Since the appearance of the celebrated Brouwer's fixed point theorem, various fixed point theorems have been established. Over the last thirty years, there is a significant development in the field of fixed point computation(see Scarf [9], Eaves [3], van der Laan and Talman [8], and Kojma and Yamamoto [7]). However, most of the existing fixed point theorems only guarantee the existence of a single fixed point. As far as we are aware of, existence results for multiple fixed points are very rare.

By developing a simplicial algorithm, Herings, Talman and Yang [6] recently demonstrated the following theorem in a constructive manner: there exists a conneted set of zero-points of a correspondence $\zeta : U^n \to R^n$ containing the origin and the vector of all-ones if the correspondence satisfies some suitable conditions (see Theorem 4.3 of [6]). For the above theorem, Herings and Talman [5] tried to provide an alternative existence proof but their proof has a serious mistake which we think is hard to be repaired. In this note we present a concise proof of this

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theorem based on a theorem of Browder [1]. Our framework also gives a natural path-following interpretation of the algorithm of Herings, Talman and Yang [6].

This paper is organized as follows. In Section 2, we give some preliminaries. In Section 3, we present the main results.

2 Preliminaries

We denote the n-dimensional Euclidean space by R^n, the set of all real numbers by R, the closed interval $[0,1]$ by I, set of integers $\{1,\ldots,n\}$ by I_n, and the n-dimensional unit cube by U^n, i.e.,

$$U^n = \{x \in R^n \mid 0 \leq x_i \leq 1 \quad \forall i \in I_n\}.$$

For any $x,\ y \in R^n, x \geq y$ means $x_i \geq y_i,\ \forall i \in I_n;\ x > y$ means $x_j \geq y_j$ with $\exists j \in I_n$ such that $x_j > y_j$, and $x \gg y$ means $x_i > y_i,\ \forall i \in I_n$. We define two vectors $0 = (0,\ldots,0) \in R^n$, $\mathrm{e} = (1,\ldots,1) \in R^n$ and write $R^n_{++} = \{x \in R^n \mid x \gg 0\}$. For a set $A \subset R^n$ ∂A denotes the boundary of A in R^n.

Definition 1 *A map $P : R^n \to U^n$ is said to be the orthogonal projection from R^n onto U^n if*

$$P(x) = \operatorname{argmin}\{\|x - y\| \mid y \in U^n\},$$

where $\|\cdot\|$ denotes the Euclidean norm.

Let $f : U^n \to R^n$ be a continuous function and X a convex open subset of R^n containing U^n. We define a map $\theta : X \to R^n$ by

$$(1) \qquad \theta(x) = P(x) + f(P(x)).$$

Then it is clear that P is continuous and retraction on U^n, so that θ is continuous on X.

For each $t \in I$, we denote the hyper-plane $H(t)$ by

$$(2) \qquad H(t) = \{x \in X \mid \Sigma_{i=1}^n x_i = nt\}$$

and r_t is the orthogonal retraction from R^n onto $U^n \cap H(t)$.

Definition 2 *The topological space Y is called connected if it is not the union of two nonempty disjoint closed sets. A subset C of Y is called connected if it is connected as a subspace of Y.*

It is well-known that the connectedness is a topological invariant. In particular, the continuous image of a connected set is connected.

Definition 3 *A set of Z is called a component of a point x in Z if it is the union of all connected subsets of Z containing x.*

It is not difficult to see that each component is closed in Z and the collection of all distinct components in Z forms a partition of Z.

Assumption. *The function $f : U^n \to R^n$ satisfies:*

$$
\begin{aligned}
&(a) \quad f \quad \textit{is continuous;}\\
&(b) \quad f_j(x) \geq 0 \quad \textit{if} \quad j \in J_0 = \{i \mid x_i = 0\} \quad \textit{and}\\
&\qquad\; f_j(x) \leq 0 \quad \textit{if} \quad j \in J_1 = \{i \mid x_i = 1\} \quad \textit{for} \quad \forall x \in \partial U^n\\
&(c) \quad \forall x \in U^n,\ \exists p(x) \in R^n_{++} \quad s.t. \quad p^t(x) f(x) = 0.
\end{aligned}
$$

By Z_f we denote the set of zero points of f, i.e.,

$$Z_f = \{x \in U^n \mid f(x) = 0\}.$$

Remark. *According to the condition (c), we see that either $f(x) \geq 0$ or $f(x) \leq 0$ implies $f(x) = 0$, that is, $x \in Z_f$.*

3 Main results

In this section, we will prove our main result. To this end, we require the following theorem which is a special case of Theorem 2 in Browder [1].

Theorem 1 *Let X be a convex, open subset of R^n. If $h : X \times I \to X$ is a continuous function and exists a compact subset K of X such that $h(X \times I) \subset K$. Then there exists a component D of C_h such that*

$$D \cap (X \times \{0\}) \neq \emptyset \textit{ and } D \cap (X \times \{1\}) \neq \emptyset,$$

where $C_h = \{(x, t) \in X \times I \mid x = h(x, t)\}$.

Now we give the following two lemmas for the proof of the following theorem 2.

Lemma 1 *For the orthogonal retraction $r_t : R^n \to U^n \cap H(t)$, it holds that*

$$\|r_t(x) - r_t(x')\| \leq \|x - x'\|, \quad \forall x, x' \in R^n, \forall t \in I.$$

Lemma 1 is well-known, we omit the proof. The following lemma is Corollary 8.1 of Hogan [4].

Lemma 2 *Let $\Omega : T \to Y$ be a point-to-set map, $g : T \times Y \to R$ be a real-valued function and let*

$$M(t) = \{y \in \Omega(t) \mid g(t,y) = \inf_{z \in \Omega(t)} g(t,z)\} \quad \textit{for any} \quad t \in T.$$

Suppose Ω is continuous at $\bar{t}$, g is continuous on $\{\bar{t}\} \times \Omega(\bar{t})$, M is nonempty and uniformly compact near $\bar{t}$, and $M(\bar{t})$ is single-valued. Then M is continuous at $\bar{t}$.

According to Lemma 1 and 2, we obtain the following theorem:

Theorem 2 *Let X be a convex open subset of R^n containing U^n, $P : R^n \to U^n$ be the orthogonal retraction and $f : U^n \to R^n$ be a continuous function. The function $\theta : X \to R^n$, the hyper-plane $H(t)$ and the orthogonal retraction $r_t (t \in I)$ are defined as above (1) and (2), respectively. Then the function $h : X \times I \to U^n$ defined by*

$$h(x,t) = r_t(\theta(x)) \quad \textit{for all } (x,t) \in X \times I$$

is continuous on $X \times I$.

Proof. First we show the continuity of h with respect to t. To this purpose, we define $\Omega : I \to U^n$ and $g : I \times U^n \to R$ by

$$\Omega(t) = H(t) \cap U^n \quad \text{and} \quad g(t,y) = \|\theta(x) - y\|^2 \quad x \in R^n,$$

respectively, and also let

$$M(t) = \{y \in H(t) \cap U^n \mid \|\theta(x) - y\|^2 = \inf_{z \in H(t) \cap U^n} \|\theta(x) - z\|^2\}.$$

Since $\Omega(t)$ is a compact convex set and $g(t,y)$ is a strictly convex function in y, $M(t)$ is single-valued. Other conditions of Lemma 2 can be easily checked. Note $M(t) = h(x,t)$ for each $x \in R^n$. Therefore we see that h is continuous with respect to t by Lemma 2.

Next we show the continuity of h in (x,t). Let $\epsilon > 0$ and $(x,t) \in X \times I$. According to the continuity of θ on X, there exists $\delta_1 > 0$ such that

$$(3) \quad \|\theta(x') - \theta(x)\| < \varepsilon/2 \quad \text{if } \|x' - x\| < \delta_1, \ \forall x' \in X.$$

For $x \in X$ using the continuity of h on t, we can choose $\delta_2 > 0$ such that

$$(4) \quad \|h(x,t') - h(x,t)\| \leq \varepsilon/2 \ \text{ if } |t' - t| < \delta_2, \ \forall t' \in I.$$

Put $\delta = \min\{\delta_1, \delta_2\}$. For any $(x',t') \in X \times I$ with $\|x' - x\| < \delta$ and $|t' - t| < \delta$, using (3), (4) and Lemma 1, we have

$$\begin{aligned}
&\|h(x',t') - h(x,t)\| \\
&\leq \|h(x',t') - h(x,t')\| + \|h(x,t') - h(x,t)\| \\
&< \|r_{t'}(\theta(x')) - r_{t'}(\theta(x))\| + \varepsilon/2 \\
&\leq \|\theta(x') - \theta(x)\| + \varepsilon/2 \\
&< \varepsilon.
\end{aligned}$$

Therefore h is continuous at (x,t). Since $(x,t) \in X \times I$ is arbitrary, the theorem has been proved. ∎

Now we are ready to prove the existence theorem(i.e., Theorem 4.3) of Herings, Talman and Yang [6]. It should be noted that their theorem is more general and holds for point-to-set mappings.

Theorem 3 *Let $f : U^n \to R^n$ be a function satisfying Assumption. Then the set Z_f contains a component S containing* 0 *and* e.

To prove the theorem, we first give the following lemma:

Lemma 3 *Let the function $h : X \times I \to U^n$ be defined as in Theorem 2. Then for any $(x,t) \in X \times I$*

$$(x,t) \in C_h \iff x \in Z_f \cap H(t).$$

Proof. First we prove $\Rightarrow$): Let $(x,t) \in C_h$, we have

$$x = h(x,t) = r_t(\theta(x)) \in H(t) \cap U^n.$$

Then $P(x) = x$ and $x = r_t(x + f(x))$. We denote $J_2 = \{j \in I_n \mid 0 < x_j < 1\}$ and J_0, J_1 are defined as in Assumption. By the Karush-Kuhn-Tucker condition for the minimization problem

$$\min\{\|y - (x + f(x))\|^2 \mid y \in H(t) \cap U^n\},$$

we can write

$$\begin{aligned} f(x) = \lambda_0 \mathrm{e} + \quad & \Sigma_{j \in J_0} \lambda_j (-\mathrm{e}^j) + \Sigma_{j \in J_1} \lambda_j \mathrm{e}^j, \\ & \lambda_0 \in R, \lambda_j \geq 0 \text{ for all } j \in J_0 \cup J_1. \end{aligned}$$

where $\mathrm{e}^j = (0, \cdots, 0, \overset{j}{1}, 0, \cdots, 0)$ for all $j \in I_n$.

We consider the following three cases:

Case1. $J_0 \neq \emptyset$ *and* $J_1 \neq \emptyset$. For any $j \in J_0$ and $j' \in J_1$, we have from (b) of Assumption

$$\lambda_0 - \lambda_j \geq 0 \text{ and } \lambda_0 + \lambda'_j \leq 0,$$

so that $\lambda_0 = 0$. Then

$$\lambda_j = 0 \ \ \forall j \in J_0 \text{ and } \lambda'_j = 0 \ \ \forall j' \in J_1.$$

Hence $f(x) = 0$, i.e., $x \in Z_f$.

Case 2. $J_0 \neq \emptyset$ *but* $J_1 = \emptyset$. In this case

$$f(x) = \lambda_0 \mathrm{e} + \Sigma_{j \in J_0} \lambda_j (-\mathrm{e}^j).$$

We see from the condition (b) of Assumption

$$\lambda_0 - \lambda_j \geq 0 \quad \forall j \in J_0.$$

Hence $f(x) \geq 0$. Applying Remark we obtain $x \in Z_f$.

Case 3. $J_0 = \emptyset$ *but* $J_1 \neq \emptyset$. We have $\lambda_0 + \lambda_j \leq 0 \ \ \forall j \in J_1$, then

$$f(x) = \lambda_0 \mathrm{e} + \Sigma_{j \in J_1} \lambda_j (\mathrm{e}^j) \leq 0$$

Applying Remark we obtain $x \in Z_f$. Therefore $x \in Z_f \cap H(t)$.

Next we prove $\Leftarrow$) : Let $x \in Z_f \cap H(t)$. Since $x \in U^n$, $P(x) = x$ and $f(x) = 0$. Thus

$$\theta(x) = P(x) + f(P(x)) = x.$$

Then

$$h(x,t) = r_t(\theta(x)) = r_t(x)$$

which coincides with x because $x \in H(t)$. Therefore $(x,t) \in C_h$. ∎

Now we give the proof of Theorem 3.

Proof of Theorem 3. We define the function $h : X \times I \to U^n$ by

$$h(x,t) = r_t(\theta(x)), \quad \forall (x,t) \in X \times I.$$

Then h is continuous by Theorem 2. According to Theorem 1, there exists a connected subset D of C_h such that

$$D \cap (U^n \times \{0\}) \neq \emptyset \ \text{ and } \ D \cap (U^n \times \{1\}) \neq \emptyset.$$

For any $(x,0), (x',1) \in D$, we have

$$x = h(x,0) = r_0(\theta(x)) = 0$$

and

$$x' = h(x',1) = r_1(\theta(x')) = \mathrm{e},$$

that is, $(0,0)$, $(\mathrm{e},1) \in D$.

Let $P_x : X \times I \to X$ be the projection onto the first coordinate. Then P_x is continuous. Note that D is a connected set contained by $U^n \times I$. By Lemma 1 and the continuity of P_x, we have $S = P_x(D) \subset Z_f$ which contains two points

$$0 = P_x((0,0)) \ \text{ and } \ \mathrm{e} = P_x((\mathrm{e},1)).$$

Now the proof is completed. ∎

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