No. 689

Infinite Horizon Equilibria with Convex Production

by

Ning Sun and Sho-Ichiro Kusumoto

June 1996

Infinite Horizon Equilibria with Convex Production

Ning Sun*

and

Sho-Ichiro Kusumoto

Institute of Socio-Economic Planning
University of Tsukuba

June 1996

Abstract This paper extends an *infinite* horizon production model with constant returns to scale (c.r.s.) that Boyd and McKenzie established (1993), to one with a *diminishing* return to scale (d.r.s.) production. There is an example showing that, in an infinite horizon d.r.s. production economy, (i) *their assumptions*† can not ensure the existence of a competitive equilibrium as defined by them and (ii) the existence of a *modified* competitive equilibrium cannot be proved even by introducing "artificial entrepreneurial factors" (McKenzie (1959)). Then, it is needed to establish a theorem that proves the existence of an infinite horizon competitive equilibrium with *convex* (d.r.s. as well as c.r.s.) production, where the rather stringent continuity and monotonicity assumptions are relaxed.

Keywords: Infinite horizon; Edgeworth equilibrium; Competitive equilibrium; Existence; Convex production set

JEL classification: D51

*The first author is a graduate student in Doctoral Program in Socio-Economic Planning. The second author is a Professor of Economics of the Institute. Correspondence: Ning Sun. Doctoral Program in Socio-Economic Planning, University of Tsukuba, Tsukuba, Ibaraki 305, **Japan.** Fax: +81-298-55-3849; E-mail: sunning@aries.sk.tsukuba.ac.jp

†defined in footnote 3.

Infinite Horizon Equilibria with Convex Production

1. Introduction

In the last decade there has been an increasing interest in the studies of general equilibrium models with an infinite number of commodities. Such studies go back to the work of Debreu (1954). In that paper Debreu proposed that the commodity and price spaces for economies with an infinite number of commodities be viewed as dual topological vector spaces. In fact, almost all studies on this topic adopt the Debreu proposition. See, for example, Bewley (1972), Brown and Lewis (1981), Aliprantis and Brown (1983), Araujo (1985), Mas-Colell (1986), Aliprantis, Brown and Burkinshaw (1987a, 1987b, 1989), and Zame (1987). And see the survey article of Mas-Colell and Zame (1991).

Unfortunately, when applying these preceding studies directly to the discrete model, to which we shall confine our attention, we have to employ the space l_∞ of all bounded sequences for the commodity space. This is so, because if we use the space s of real-valued sequences for the commodity space, then the price space will be the dual space s^* of s in the product topology whose elements have only a finite number of nonzero components. To confine prices to s^* means that the prices of all commodities must be zero at almost all periods of time, but this causes a contradictory result when the preferences of a consumer are strictly monotone. On the other hand, to use l_∞ for the commodity space places an artificial restriction on the choices of the consumer. That is, when one consumer faces a price system p, he is not free any more to choose a consumption bundle x which satisfies his budget constraint. He must select a consumption sequence which is bounded (feasible) as well. In fact, in economic growth models, it is often assumed that unbounded paths of growth are feasible and choices over them are allowed.

There are three major exceptions, Peleg and Yaari (1970), Stigum (1973), and Boyd and McKenzie (1993), which are outside the framework of Debreu (1954). They used s or s^n (the space of all sequences of n-dimensional vectors) for their commodity and price spaces. The model of Peleg and Yaari is a pure exchange economy whose consumption sets are assumed to be the nonnegative orthant of the commodity space. The model of Stigum is an economy with convex (diminishing returns to scale (d.r.s.)) production technologies and general consumption sets on which he assumed conditions very difficult to interpret. Then Boyd and McKenzie considered a production economy with constant returns to scale (c.r.s.)[1] and with weaker conditions on the general consumption sets for which they established an infinite horizon existence theorem. Additionally, McKenzie (1993) applied their theorem (we shall say the Boyd-McKenzie theorem) to the Malinvaud model of production and to a generalized Neo-Austrian model of production.

In the *finite* horizon case, McKenzie (1959, 1981) transformed a d.r.s. production economy into a c.r.s. one by introducing "artificial entrepreneurial factors", and proved that if a d.r.s. model satisfies the Arrow-Debreu assumptions then the transformed c.r.s. one satisfies the McKenzie assumptions. In this sense, the Arrow-Debreu theorem is a corollary of the McKenzie theorem. Even in the *infinite* horizon case, McKenzie with Boyd III asserted in their paper that this is still true[2]. Of course, even in an infinite horizon case, every d.r.s. production economy can be changed into a c.r.s. one by his method. However, there is a d.r.s. example which will meet two difficulties. First, the example satisfies all of *their assumptions for a d.r.s. model* [3], but there exists no competitive equilibrium as de-

[1] McKenzie (1981) called the c.r.s. production economy the "activities economy", and called the diminishing returns to scale (d.r.s.) one the "firms economy".

[2] "diminishing returns can be accommodated by introducing artificial entrepreneurial factors (McKenzie 1959)", see Boyd and McKenzie (1993, pp. 5).

[3] In a d.r.s. production model, we cannot use their assumptions for a c.r.s. one as they stand, so we extensively draw the assumptions for a d.r.s. model from theirs for the c.r.s. one, as follows:

(1) *Each* Y_j *is a closed convex set, and* $0 \in Y_j$. *The aggregate production set* Y *is also closed, convex and* $Y \cap s^n_+ = \{0\}$, *and* Y *contains no straight lines.*

(2) *For each* $\bar{y} \in s^n$ *the set* $\{y \in Y : y \geq \bar{y}\}$ *is bounded.*

(3) *Each trading set* $X_i - \omega_i$ *is convex, closed, and bounded below by an element of* l_∞.

(4) *For each* i, *the preference relation* $\succeq_i$ *is reflexive, transitive and convex. Moreover, each preference relation* $\succeq_i$ *is both lower and upper semicontinuous on the whole consumption set* X_i.

(5) *Let* $x \in X_i$. *If* $z > x$ *then* $z \succ_i x$.

(6) *The economy* $\mathcal{E}$ *is strongly irreducible.*

(7) a) *For all* i, *there is* $\bar{x}_i \in X_i - \omega_i - Z_i$ *with* $\bar{x}_i \leq 0$, *where* $Z_i = \sum_{j=1}^{J} \theta_{ij} Y_j$ *is the share set of the* ith *consumer. Moreover,* $\bar{x} = \sum_{i=1}^{I} \bar{x}_i \ll 0$ *and* $\bar{x}(t) = \bar{x}(r)$ *for all* r *and* t.

b) *For any* $x_i \in X_i$, *let* $z_i \in R_i(x_i) - \omega_i - Z_i$ *and* $\delta > 0$, *where* $R_i(x_i) = \{x : x \in X_i \text{ and } x \succeq_i x_i\}$.

fined by them[4]. Therefore, in a d.r.s. production economy, *their assumptions* and/or *their definition* of infinite horizon equilibrium might need further revision. Indeed by newly defining an equilibrium (given in section 3) this d.r.s. production example has the competitive equilibrium under *their assumptions.* Second, the transformed c.r.s. production economy, which is obtained by his method in our infinite horizon d.r.s. example, satisfies their original Assumption 1 through 6 but does not satisfy the first part nor the second part of Assumption 7. This sharply contrasts to the finite horizon case. We found it is because of the *infinite* dimensionality of the commodity space[5]. Then the Boyd-McKenzie theorem cannot apply to this c.r.s.one. In fact, the c.r.s.model does not have any competitive equilibrium as defined in Boyd and McKenzie (1993). Therefore we cannot apply their transforming method directly to solve the existence problem of competitive equilibrium even in our definition.

Thus, it is needed to establish a theorem which will provide a complete proof of the ex-

Then there is a τ_0 such that for each $\tau > \tau_0$, there is an $\alpha > 0$ with $(z_i(1)+\delta e(1), z_i(2), \ldots, z_i(\tau), \alpha\bar{x}_i(\tau+1), \ldots) \in R_i(x_i) - \omega_i - Z_i$.

We shall here call the above assumptions simply *their assumptions (for a d.r.s. model).* It is clear that Assumptions (2), (3), (4), (5) and (6) are completely the same as Assumptions 2, 3, 4, 5 and 6 made in Boyd and McKenzie (1993) for a c.r.s. model. Their Assumptions 1 and 7 are:

Assumption 1. *Y is a closed convex cone with vertex at the origin that contains no straight lines.*

Assumption 7. *For all i, there is $\bar{x}_i \in X_i - \omega_i - Y$ with $\bar{x}_i \leq 0$. Moreover, $\bar{x} = \sum_{i=1}^{I} \bar{x}_i \ll 0$ and $\bar{x}(t) = \bar{x}(r)$ for all r and t. For any x_i, let $z_i \in R_i(x_i) - \omega_i - Y$ and $\delta > 0$. Then there is a τ_0 such that for each $\tau > \tau_0$, there is an $\alpha > 0$ with $(z_i(1) + \delta e(1), z_i(2), \ldots, z_i(\tau), \alpha\bar{x}_i(\tau+1), \ldots) \in R_i(x_i) - \omega_i - Y$, where $e(1) = (1, \ldots, 1) \in \mathcal{R}^n$.*

We can see that Assumption (7) given above is also essentially the same as their Assumption 7.

[4]Similarly to footnote 3, in a d.r.s.production model, we cannot use their own definition of competitive equilibrium in Boyd and McKenzie (1993) directly. In the way of their definition for a c.r.s. model, a competitive equilibrium in a d.r.s. production economy should naturally be defined as follows. We shall call this definition *their definition (for a d.r.s. one).*

A competitive equilibrium for the infinite horizon d.r.s. production economy $\mathcal{E} = (X_i, \succsim_i, \omega_i, Y_j, \theta_{ij}, i = 1, \ldots, I, j = 1, \ldots, J)$ is a set of sequences $(p^*, x_1^*, \ldots, x_I^*, y_1^*, \ldots, y_J^*)$ with $p^* \in s^n$ which satisfies:

(1) $p^*(x_i^* - \omega_i - \sum_{j=1}^{J} \theta_{ij} p^* y_j^*) = 0$, *and* $x_i \succ_i x_i^*$ *implies* $p^*(x_i - \omega_i - \sum_{j=1}^{J} \theta_{ij} p^* y_j^*) > 0$;

(2) $y_j^* \in Y_j$ *and* $\sup p^*(y_j - y_j^*) \leq 0$ *for all* $y_j \in Y_j$;

(3) $\sum_{i=1}^{I} x_i^* = \sum_{i=1}^{I} \omega_i + \sum_{j=1}^{J} y_j^*$;

(4) *Each $p^* y_j^*$ is well-defined and finite.*

The symbol of *sup* and the well-definedness will be given later formally in footnote 7. For the McKenzie method (1959) to be applicable, we add Condition (4) here even in a d.r.s. production model. Without Condition (4), the equilibrium defined here will become more general, but even such an equilibrium won't exist in the example.

[5]Here means "infinite horizon".

istence of an infinite horizon competitive equilibrium in a d.r.s.as well as a c.r.s.production economy. We therefore generalize the c.r.s. production model of Boyd and McKenzie to a convex production model in the two senses that follow :

(i) Boyd and McKenzie defined a competitive equilibrium not only as allowing for sequences with not well-defined values in their definition of profit condition, but also as requiring all possible trading sequences to give well-defined values in their definition of demand condition. We extend slightly the notion of competitive equilibrium by relaxing the demand condition in such a way that trading sequences whose values do not converge are allowed. A similar demand condition (budget constraint) has been considered by Arrow and Kurz (1970), Wilson (1981) and Yi (1989) in different types of infinite horizon models. Clearly, this is a natural extension of the definition of competitive equilibrium in a model whose price systems need not lie in the dual space of the commodity space.

(ii) Peleg and Yaari (1970), Stigum (1973), and Boyd and McKenzie (1993) all suffer from use of the stringent condition that the preferences or utility functions are continuous in the product topology and non-satiated on the *whole* consumption set. This is not usually assumed in economic growth and capital accumulation models. We replace this condition by a weakened one usually implicit in those models. That is, the preferences or utility functions are continuous and non-satiated on the *feasible* consumption sets.

The proof of existence of equilibrium follows that of Debreu and Scarf (1963); see, for example, Peleg and Yaari (1970), Aliprantis et al. (1987a, 1987b), and also see Boyd and McKenzie (1993) too. We first show that the core is nonempty and the Edgeworth equilibria exist, then demonstrate the existence of a price system that supports an Edgeworth equilibrium as a competitive equilibrium. The detailed argument of the first part is omitted because it is essentially the same as those of Aliprantis et al. (1987b, 1989), and Boyd and McKenzie (1993). The continuity assumption is relaxed so that the proofs of the second part—the separation arguments—of Peleg and Yaari, and Boyd and McKenzie are no longer valid. Instead, we offer an elementary separation argument which is much clearer and simpler than theirs.

The rest of the paper is organized as follows. Section 2 presents an example of an infinite horizon d.r.s. production economy which has the two difficulties. In Section 3, we set up the basic model, give definitions and assumptions, and describe the existence theorem of our extended competitive equilibrium. Our existence theorem is proved in Section 4. Concluding remarks are in Section 5.

2. An Example of the Difficulties in the Infinite Case

A d.r.s. production economy that satisfies their assumptions (1) to (7)

We shall consider an infinite horizon d.r.s. production economy with two consumers and one firm. There are two types of goods at each period of time, the first good denotes labor and the second denotes the consumption good. Thus the commodity space is s^2. The consumption sets of the two consumers, X_1 and X_2, are given by

$$X_1 = \Big\{x \in s^2_+ : x_2(\mathrm{t}) \geq 2^t - 1 \quad \text{for } t = 1, 2, \ldots\Big\},$$

and

$$X_2 = \Big\{x \in s^2_+ : x_2(2\mathrm{t}-1) + \tfrac{1}{2}x_2(2\mathrm{t}) \geq 2^{2t} - 2 \text{ and } x_2(2t-1) \geq 2^{2t-1} - 1 \quad \text{for all } t\Big\}.$$

The utility functions of consumers, U_1, U_2 are defined by

$$U_1(x) = \textstyle\sum_{t=1}^{\infty} 2^{-t} \arctan\Big(x_2(t) - 2^t + 1\Big),$$

and

$$U_2(x) = \textstyle\sum_{t=1}^{\infty} 2^{-2t} \arctan\Big(x_2(2t-1) + \tfrac{1}{2}x_2(2t) - 2^{2t} + 2\Big).$$

The endowments of the consumers are $\omega_1 = \omega_2 = \big((0.5, 0), (0.5, 0), \ldots\big)$. The shares of ownership of the firm in terms of products are $\theta_1 = \theta_2 = 0.5$. Finally, the production set is defined by

$$Y = \Big\{\, y \in s^2 : y_2(t) \;\leq\; 2^{t+1}(-y_1(\mathrm{t}))^{\frac{1}{2^t}} \quad \text{for } t = 1, 2, \ldots \Big\}.$$

It is easy to verify that this d.r.s.production example satisfies Assumptions (1), (2), (3), (4), (5) and (6) of footnote 3. Now let us show that this example also satisfies Assumption

(7). Let $f_t(l) = 2^{t+1} l^{\frac{1}{2^t}}$ denote the production function at period t, where l denotes the input of labor. Then we see $f_t'(l)_{l=1} = 2$ for all t, and $f_t'(l) = 2l^{\frac{1}{2^t}-1} \leq 4$ for $l \in [0.5, 1]$ and all t. Thus the production plan $y_0 = ((-0.9, 2^2 - 0.4), (-0.9, 2^3 - 0.4), \ldots) \in Y$. Hence we have that for the first consumer there is

$$\begin{aligned} \bar{x}_1 &= ((-0.05, -0.8), (-0.05, -0.8), \ldots) \\ &= ((0, 2-1), (0, 2^2 - 1), (0, 2^3 - 1), \ldots) - \omega_1 - \theta_1 y_0 \\ &\in X_1 - \omega_1 - \theta_1 Y, \end{aligned}$$

and for the second consumer there is

$$\begin{aligned} \bar{x}_2 = \bar{x}_1 &= ((0, 2-1), (0, 2^2 - 1), (0, 2^3 - 1), \ldots) - \omega_2 - \theta_2 y_0 \\ &\in X_2 - \omega_2 - \theta_2 Y. \end{aligned}$$

Let $\bar{x} = \bar{x}_1 + \bar{x}_2 = ((-0.1, -1.6), (-0.1, -1.6), \ldots)$. We see $\bar{x} \ll 0$ and $\bar{x}(t) = \bar{x}(r)$ for all r and t. Thus we proved that this example satisfies the first part of Assumption (7). In addition, from the continuity of utility functions and the structures of consumption and production, we can show this example also satisfies the second part of Assumption (7).

No competitive equilibrium as defined by them

Let $x_1^* = x_2^* = ((0,2), (0,2^2), (0,2^3), \ldots)$, and $y^* = ((-1, 2^2), (-1, 2^3), \ldots)$. It is easy to verify that (x_1^*, x_2^*, y^*) is a unique Edgeworth equilibrium of the economy. Then, by calculating the marginal utilities and marginal substitution of production, we can see that $p^* = ((1, 2^{-1}), (2^{-1}, 2^{-2}), (2^{-2}, 2^{-3}), \ldots)$ is the only possible candidate of its supporting prices. Recall that Boyd and McKenzie define their competitive equilibrium as one which requires all possible trading sequences to give well-defined values in their definition of demand condition. However, in the state of (p^*, x_1^*, x_2^*, y^*), there are consumption bundles for the second consumer like $x = ((0, 2^2 + 1), (0,0), (0, 2^4), (0,0), (0, 2^6), (0,0), \ldots) \in X_2$, $U_2(x) > U_2(x_2^*)$, and the corresponding trading sequence $x - \omega_2 - \theta_2 y^*$ giving a not well-defined value at the price p^*. Indeed, $\liminf_{T\to\infty} \sum_{t=1}^{T} p^*(t)[x(t) - \omega_2(t) - \theta_2 y^*(t)] = 0.5$, and $\limsup_{T\to\infty}\sum_{t=1}^{T} p^*(t)[x(t)-\omega_2(t) \ -\theta_2 y^*(t)] = 1.5$. So, according to their definition, (p^*, x_1^*, x_2^*, y^*) is not a competitive equilibrium. Furthermore, since the Edgeworth equilibrium is unique, this example has no other competitive equilibrium as defined by them.

The transformed c.r.s. production economy does not satisfy Assumption 7

We show that the c.r.s. production economy obtained from this example by introducing artificial entrepreneurial factors (McKenzie 1959) will not satisfy their original Assumption 7. Actually, this transformed c.r.s. production economy does not have such a competitive equilibrium as is defined in Boyd and McKenzie (1993).

First, it is easy to show that for any small $\delta > 0$, $X_1 \cap (\omega_1 + (\theta_1 - \delta)Y) = X_2 \cap (\omega_2 + (\theta_2 - \delta)Y) = \emptyset$. This implies that in the transformed c.r.s. production economy both consumers can never be the supplier of the entrepreneurial factor. Therefore the c.r.s. production economy does not satisfy the first part of their original Assumption 7. It is exclusively due to the mathematical property of infinite dimensional spaces. For any x and $y \in \mathcal{R}_+^n$ with $x \gg y$, there always exists some $\alpha \in (0,1)$ such that $\alpha x \gg y$. But there is some x and $y \in s_+$ with $x \gg y$ (even with $x \gg y + (1,1,\ldots)$), for which there is no $\alpha \in (0,1)$ satisfying $\alpha x \gg y$. For instance, $(1+2, 2+2, 3+2, \ldots) \gg (1,2,3,\ldots) + (1,1,1,\ldots)$, but observe there is no $\alpha \in (0,1)$ such that $\alpha(1+2, 2+2, 3+2, \ldots) \gg (1,2,3,\ldots)$.

Next, since in the c.r.s. production economy the liability share of the firm as the artificial entrepreneurial factor can be exchanged in the market, the first consumer can always consider such a trading plan that he sells all the liability share he owns θ_1 and his endowments ω_1 to buy a consumption bundle $x_1 \in X_1$. But if such a trading plan is stopped at a period of time, whenever it may be, then from that time he can consume his endowment only because his share of the firm has been sold, and thus his survival is not guaranteed. Therefore the c.r.s. production economy does not satisfy the second part of their Assumption 7.

Thus we can not apply the Boyd-McKenzie theorem to show the existence of equilibrium in a d.r.s. production economy with *their assumptions.*

Note also, the state (p^*, x_1^*, x_2^*, y^*) considered above is in fact a competitive equilibrium where trading sequences with not well-defined values are allowed to choose and infinite profits of firms to prevail (the exact definition will be given in section 3). Moreover, such competitive equilibria can not be consistent with a competitive equilibrium of the transformed c.r.s. production economy. Suppose it could be consistent, then the equilibrium profit of the former competitive equilibrium were equal to the equilibrium price of the entrepreneurial factor, and were finite, leading to a contradiction. Thus, the Boyd-McKenzie theorem cannot apply in general to the existence of such a competitive equilibrium.

3. The Model, Notations, and Assumptions

We define an infinite horizon d.r.s. production economy $\mathcal{E}$ on the topological space $s^n = \Pi_{t=1}^{\infty}\mathcal{R}^n(t)$ with I consumers indexed by $i = 1, \ldots, I$ and J firms indexed by $j = 1, \ldots, J$, where $\mathcal{R}^n(t)$ is the n-dimensional space $\mathcal{R}^n$. The topology of s^n is the product topology. Each consumer i is characterized by a consumption set $X_i \subset s^n$, a preference relation $\succeq_i$ on X_i, an initial endowment $\omega_i \geq 0$, and a liability share θ_{ij} of firm j, where $\theta_{ij} \geq 0$ and $\sum_{i=1}^{I} \theta_{ij} = 1$ for each firm j. According to these liability shares each firm distributes its production — all its inputs and outputs — to its owners[6]. Each firm j is characterized by a production set $Y_j \subset s^n$. Thus, a d.r.s. production economy $\mathcal{E}$ is represented by

$$\mathcal{E} = \Big(X_i,\ \succeq_i,\ \omega_i,\ Y_j,\ \theta_{ij},\ i = 1, \ldots, I,\ j = 1, \ldots, J \Big).$$

Each preference relation $\succeq_i$ is assumed to be reflexive, transitive, and complete on X_i.

[6]Since the price systems are not restricted in the dual space of the commodity space, the profit of a firm may be infinite or not well-defined for a given production plan and a price system. So it is not appropriate that we still interpret θ_{ij} as the ith consumer's share in the profits of firm j. We interpret θ_{ij} as the share of production of firm j for consumer i. In a private correspondence with Professor McKenzie, we reached this interpretation. That is, if firm j chooses a production plan $y_j \in Y_j$, then consumer i obtains $\theta_{ij} y_j$ from firm j. It should be pointed out that since there are markets, consumer i need not consume these outputs nor supply these inputs directly, while he can sell some outputs to buy some inputs in the markets.

For $x, x' \in X_i$, we write $x \succ_i x'$ if $x \succeq_i x'$ and not $x' \succeq_i x$. The preference relation $\succeq_i$ is said to be: (i) *convex,* whenever $x_i \succ_i x_i'$ implies $\alpha x_i + (1-\alpha)x_i' \succ_i x_i'$ holds for all $0 < \alpha < 1$; (ii) *monotone,* whenever[7] $x_i \geq x_i'$ implies $x_i \succeq_i x_i'$. A sequence $x \in s_+^n$ is said to be *extremely desirable on* X (a subset of X_i) for consumer i whenever $x_i + \alpha x \succ_i x_i$ holds for all $x_i \in X$ and all $\alpha > 0$. For any two sets A, $B \subset s^n$, we say that A is open relative to B, whenever for any convergent (in the product topology) sequence $\{x^{(v)}\} \subset B$, if its limit $x^* \in A$, then there exists a positive integer V such that $x^{(v)} \in A$ for all $v \geq V$. Let X be a subset of X_i. The preference relation $\succeq_i$ is said to be *upper semicontinuous on* X, whenever for each $x_i \in X$ the set $\{x : x \in X \text{ and } x_i \succ_i x\}$ is an open set relative to X. The preference relation $\succeq_i$ is said to be *linearly lower semicontinuous (on* X_i*),* whenever for any x' and x'' in X_i, if $x' \succ_i x^*$ for some $x^* \in X_i$ then there is some α_0 $(0 < \alpha_0 < 1)$ such that $\alpha x' + (1-\alpha)x'' \succ_i x^*$ holds for all $\alpha_0 < \alpha \leq 1$.

For each firm j, the production set Y_j is defined on s^n. A sequence $y_j \in Y_j$ is called a production plan or more briefly a production. The negative components of a production represent inputs, and the positive components represent outputs. The *aggregate production set* of the economy $\mathcal{E}$ is defined as $Y = \sum_{j=1}^{J} Y_j$, and $Z_i = \sum_{j=1}^{J} \theta_{ij} Y_j$ is called the *share set* of the ith consumer. It is clear that each Z_i is convex, $\sum_{i=1}^{I} Z_i = Y$, and Z_i can be interpreted as a potential production set of the ith consumer.

An *allocation* is a list $(x_1, \ldots, x_I, y_1, \ldots, y_J)$, where $x_i \in X_i$ for $i = 1, \ldots, I$ and $y_j \in Y_j$ for $j = 1, \ldots, J$. A *feasible allocation* must also satisfy the condition

$$\sum_{i=1}^{I} x_i = \sum_{i=1}^{I} \omega_i + \sum_{j=1}^{J} y_j.$$

[7] For x and $y \in s^n$, $x \geq y$ means $x(t) \geq y(t)$ for all $t = 1, 2, \ldots$, and $x > y$ means $x \geq y$ and there exists some t_0 such that $x(t_0) \gg y(t_0)$, moreover, $x \gg y$ denotes $x(t) \gg y(t)$ for all t. The symbol s_+^n denotes the nonnegative orthant of s^n. For $x \in s^n$ and a natural number τ, $x[\tau]$ represents the "initial segment" of x, i.e., $x[\tau] = (x(1), \ldots, x(\tau), 0, \ldots)$. For p and $x \in s^n$, $\inf px$ denotes the inferior limit of the sequence $\{\sum_{t=1}^{\tau} p(t)x(t)\}$ i.e., $\inf px = \liminf_{\tau \to \infty} \sum_{t=1}^{\tau} p(t)x(t)$, and $\sup px$ denotes the superior limit of $\{\sum_{t=1}^{\tau} p(t)x(t)\}$ i.e., $\sup px = \limsup_{\tau \to \infty} \sum_{t=1}^{\tau} p(t)x(t)$. The inner product px is said to be *well-defined* if and only if $\inf px = \sup px$, and $px = \inf px = \sup px$ when px is well-defined. The inner product px is said to be *not well-defined* whenever $\inf px \neq \sup px$. Since the topological space s^n with the product topology is a metric space, the concept of compactness is equivalent to sequential compactness.

Then the *set of all feasible allocations* for the economy is

$$F = \left\{ (x_1, \dots, x_I, y_1, \dots, y_J) \,:\, x_i \in X_i,\ y_j \in Y_j \quad \text{and} \ \sum_{i=1}^{I} x_i = \sum_{i=1}^{I} \omega_i + \sum_{j=1}^{J} y_j \right\}.$$

If $(x_1, \dots, x_I,\, y_1, \dots, y_J)$ is a feasible allocation, then $(x_1, \dots, x_I)$ is called a feasible consumption allocation.

A commodity bundle $x \in X_i$ is said to be *feasible* for the ith consumer whenever there exists a feasible allocation $(x_1, \dots, x_I,\, y_1, \dots, y_J) \in F$ such that $x_i = x$. The *feasible consumption set* $\hat{X}_i$ of the ith consumer is the set of all his feasible consumption bundles, i.e.,

$$\hat{X}_i = \left\{ x \in X_i : \exists\, (x_1, \dots, x_I,\, y_1, \dots, y_J) \in F \text{ with } x_i = x \right\}.$$

We say that $x_i \in X_i$ is *strongly individually rational* for the ith consumer if $x_i \succ_i x_i'$ for all $x_i' \in X_i \cap (\omega_i + Z_i)$, where $X_i \cap (\omega_i + Z_i)$ represents the possible consumption set of the ith consumer without trade. The economy $\mathcal{E}$ is said to be *strongly irreducible* whenever, for every feasible consumption allocation $(x_1, \dots, x_I)$ and for each consumer i_0, (i) if x_{i_0} is strongly individually rational for consumer i_0, then there exists a feasible allocation $(x_1', \dots, x_I', y_1', \dots, y_J')$ such that $x_i' \succ_i x_i$ for all $i \neq i_0$; (ii) otherwise, there exist an allocation $(x_1', \dots, x_I', y_1', \dots, y_J')$ and a real number $\alpha > 0$ such that $\sum_{i \neq i_0}^{I} (x_i' - \omega_i - \sum_{j=1}^{J} \theta_{ij} y_j') + \alpha (x_{i_0}' - \omega_{i_0} - \sum_{j=1}^{J} \theta_{i_0 j} y_j') = 0$ and $x_i' \succ_i x_i$ for all $i \neq i_0$.

Let r be a natural number. We define a d.r.s.production economy $\mathcal{E}_r$ which is called the rth *replica* of the economy $\mathcal{E}$. In the rth replica economy $\mathcal{E}_r$, there are $I \times r$ consumers indexed by (i,k) $(i = 1, \dots, I,\ k = 1, \dots, r)$, and $J \times r$ firms indexed by (j,l) $(j = 1, \dots, J,\ l = 1, \dots, r)$. The consumers $(i,k)\,(k = 1, \dots, r)$ are of the 'same type' as the ith consumer of the economy $\mathcal{E}$, i.e., they have the same consumption set X_i, the same endowment ω_i, and the same preference relation $\succeq_i$. The production set of the (j,l) firm is Y_j, i.e., $Y_{jl} = Y_j$ for $l = 1, 2, \dots, r$. The (i,k) consumer's share of the (j,l) firm is

$$\theta_{ikjl} = \begin{cases} 0, & \text{if } k \neq l \\ \theta_{ij}, & \text{if } k = l. \end{cases}$$

It is obvious that in the rth replica economy $\mathcal{E}_r$ the same type consumers $(i,k)\,(k = 1,\ldots,r)$ have the same feasible consumption set. Let it be denoted by $\hat{X}_i^{(r)}$. Clearly, we have that $\hat{X}_i = \hat{X}_i^{(1)} \subset \hat{X}_i^{(2)} \subset \hat{X}_i^{(3)} \subset \ldots \subset X_i$ for each i. Having made these preparations, we can now state our assumptions.

Assumptions:

(1′) *Each Y_j is a closed convex set, and $0 \in Y_j$. The aggregate production set is also closed, convex and $Y \cap s_+^n = \{0\}$.*

(2) *For each $\bar{y} \in s^n$ the set $\{y \in Y : y \geq \bar{y}\}$ is bounded.*

(3′) *Each X_i is convex, closed, and bounded from below.*

(4′) *For each i, the preference relation $\succeq_i$ is reflexive, transitive and convex. Moreover, each preference relation $\succeq_i$ is linearly lower semicontinuous on the whole consumption set X_i, and upper semicontinuous on each $\hat{X}_i^{(r)}$ $(r = 1, 2, \ldots)$, where $\hat{X}_i^{(r)}$ is the feasible consumption set of the* ith *type consumers in the* rth *replica economy $\mathcal{E}_r$.*

(5′) a) *Each preference relation $\succeq_i$ is monotone.*

b) *There exists a sequence $\bar{z} \in s_+^n$ which has only a finite number of nonzero components and which is extremely desirable for each consumer on his feasible consumption set.*

(6) *The economy $\mathcal{E}$ is strongly irreducible.*

(7′) a) *For each i, there is $\bar{x}_i \in X_i - \omega_i - Z_i$ with $\bar{x}_i \leq 0$, where Z_i is the share set of the* ith *consumer, and $\bar{x} = \sum_{i=1}^{I} \bar{x}_i \ll 0$.*

b) *For any $x_i \in X_i$, if there is $z_i \in P_i(x_i) - \omega_i - Z_i$, where $P_i(x_i) = \{x : x \in X_i$ and $x \succ_i x_i\}$, then there is a τ_0 such that $z_i[\tau] \in P_i(x_i) - \omega_i - Z_i$ for all $\tau > \tau_0$.*

All of our assumptions are parallel to those of Boyd and McKenzie (1993). But some of ours are weaker than theirs. Peleg and Yaari (1970), Stigum (1973), and Boyd and

McKenzie all assumed that the preferences or utility functions are continuous and non-satiated on the *whole* consumption set. Indeed, both are stringent. We only assume that the preferences are continuous and locally non-satiated on the *feasible* consumption sets. Although such weaker continuity and monotonicity assumptions are introduced on a derived concept, namely, the feasible consumption set, they are usually assumed implicitly in economic growth and capital accumulation models. For instance, in economic growth models there is usually an assumption that the economy is not so productive as to allow the attained utility to become unbounded (see, for example, Barro (1990)). This is equivalent to our assumption that the preference is not satiated on the feasible consumption set.

Boyd and McKenzie have observed that: if one starts from utility functions, not from preferences, then their lower semicontinuity assumption can be replaced by the linearly lower semicontinuity one. Further, since the commodity space s^n with the product topology is a *second-countable space*[8], and then, from the representation theorem of Rader (1963), we observe that upper semicontinuous preferences can be represented by upper semicontinuous utility functions. Therefore, even if we start from preferences, the lower semicontinuity assumption can still be replaced by the linearly lower semicontinuity one. We observe that in the pure exchange model of Peleg and Yaari, the lower semicontinuity assumption can not be relaxed to the linearly lower semicontinuity one. Our Assumption (7′-b) serves to justify the reason for making the linearly lower semicontinuity assumption here.

Our Assumption (7′-a) and (7′-b) are essentially the same as the second part of Assumption 7 of Boyd and McKenzie. Assumption (7′-b) is a joint assumption on consumption and production, and plays a crucial role in proving an Edgeworth equilibrium to be a competitive equilibrium. Roughly speaking, this assumption expresses that given a trading plan for a consumer, the distant future's trades are negligible by the consumer.

[8]See Eisenberg (1974, pp.167). A topological space is said to be *second-countable* if it has some countable base.

Boyd and McKenzie required not only that such trades are negligible, but also that the consumer will be able to supply further some constant commodities in those future periods. In this sense, our assumption (7′-b) is slightly weaker than theirs (the second part of Assumption (7))[9].

Like Stigum, and Boyd and McKenzie, we employ the space s^n for the set of price systems. Given a price system $p \in s^n$, and suppose that firms choose production plans $(y_1, \ldots, y_J)$. We see that $X_i - \omega_i - \sum_{j=1}^{J} \theta_{ij} y_j$ denotes the *set of possible trading sequences* of the ith consumer. Since the price system may not lie in the dual space of the commodity space, the income of a consumer may be infinite or not well-defined. Therefore, for consumer i there may be such a possible trading sequence $z_i \in X_i - \omega_i - \sum_{j=1}^{J} \theta_{ij} y_j$ with $\inf p z_i < 0 < \sup p z_i$. According to the standard budget constraint, we do not know whether or not such a trading sequence z_i is affordable for him. We shall define the *budget set* of consumer i as $B_i(p) = \{x_i : x_i \in X_i$ and $\inf p(x_i - \omega_i - \sum_{j=1}^{J} \theta_{ij} y_j) \leq 0\}$. Such a budget constraint has been considered by Yi (1989) in an infinite-horizon exchange economy where there are finitely many consumers who adopt the overtaking criterion. This budget concept allows consumers to borrow any amount as long as it is paid back. The budget sets defined in such a way may not be convex. However, like the finite horizon case, under the local non-satiation condition (Assumption (5′-b)), the competitive equilibrium defined with the liminf budget constraints is Pareto optimal[10].

[9]Actually, we can further weaken this assumption to

Assumption (7*-b) *For any $x_i \in X_i$, if $z_i \in P_i(x_i) - \omega_i - Z_i$, then there is a τ_0 such that* $(z_i(1), \ldots, z_i(\tau), -\bar{x}(\tau+1), \ldots) \in P_i(x_i) - \omega_i - Z_i$ *for all* $\tau > \tau_0$.

Assumption (7*-b) only requires that for a trading plan of one consumer the distant future's trades can be substituted by some feasible gains in those future times. However, since, under this weakened condition, the proof of existence of a competitive equilibrium like the one defined below needs some topological notions and theorems, further arguments of the implications from this assumption will be done elsewhere.

[10]Alternatively, we can also use "superior limit", i.e., limsup, to define budget sets and demand condition. Such a budget constraint has been used by Arrow and Kurz (1970, pp. 155-156) in a consumer's optimization context, by Wilson (1981) in a dynamic model of pure exchange with a countable number of agents and goods, and by Yi (1989). In Stigum's definition of competitive equilibrium, we find that the budget constraint is also of this type. This form of budget set is still convex. Yi proved, under the local non-satiation condition, a competitive equilibrium with the limsup

On the other hand, the objective of a producer is eventually to choose a production plan in his production set so that he will make his owners have the most advantage in exchange, or he will let the budget sets of his owners be the widest. Then, we can show that the profit condition used by Stigum, and Boyd and McKenzie is still a sufficient condition for the optimal production of firms. Therefore, we can define a competitive equilibrium as follows.

Competitive Equilibrium. *A competitive equilibrium for the infinite horizon d.r.s. production economy* $\mathcal{E} = (X_i, \succeq_i, \omega_i, Y_j, \theta_{ij}, i = 1, \ldots, I, j = 1, \ldots, J)$ *is a set of sequences* $(p^*, x_1^*, \ldots, x_I^*, y_1^*, \ldots, y_J^*)$ *with* $p^* \in s^n$ *which satisfies :*

(1) $p^*(x_i^* - \omega_i - \sum_{j=1}^J \theta_{ij} y_j^*) = 0$, *and* $x_i \succ_i x_i^*$ *implies* $\inf p^*(x_i - \omega_i - \sum_{j=1}^J \theta_{ij} y_j^*) > 0$;

(2) $y_j^* \in Y_j$ *and* $\sup p(y_j - y_j^*) \leq 0$ *for all* $y_j \in Y_j$;

(3) $\sum_{i=1}^I x_i^* = \sum_{i=1}^I \omega_i + \sum_{j=1}^J y_j^*$.

The major difference between our definition and that of Boyd and McKenzie is the demand condition, i.e., Condition (1). If we draw the definition for a d.r.s. model from theirs for a c.r.s. one directly, then the demand condition should be : $x_i \succ_i x_i^*$ implies $p^*(x_i - \omega_i - \sum_{j=1}^J \theta_{ij} y_j^*) > 0$. This means that in the equilibrium state the budget set of the ith consumer is $B_i(p) = \{x_i : x_i \in X_i \text{ and } p(x_i - \omega_i - \sum_{j=1}^J \theta_{ij} y_j) \leq 0\}$. That is, all possible trading sequences must give *well-defined* values at the equilibrium price. However, we allow for trading sequences with *not well-defined* values at the equilibrium price. We observed, in their paper, both that each trading set $X_i - \omega_i$ is bounded below

budget is also Pareto optimal. We found that the overtaking criterion of consumers and the special structure of consumption sets, both serve for him to prove it. In a general case, it will not be true. For example: Consider a two-consumer exchange economy with commodity space s. The consumption sets are given by $X_1 = \{x \in s : x(t) + 2^{-1}x(t+1) \geq 0,\ x(t) \geq -2^t \text{ for all } t\}$ and $X_2 = \{x \in s : x(2t-1) + 2^{-1}x(2t) \geq -1, x(t) \geq -2^t \text{ for all } t\}$. The utility functions are defined by $U_1(x) = x(1)$ and $U_2(x) = \sum_{t=1}^{\infty} 2^{-2t} \arctan(x(2t-1) + 2^{-1}x(2t) + 1)$. And the endowments are $\omega_1 = \omega_2 = (0, 0, \ldots) \in s$. Clearly, both consumers are locally non-satiated on their consumption sets. Let $p^* = (1, 2^{-1}, 2^{-2}, \ldots)$, then we can show that $(p^*, \omega_1, \omega_2)$ is a competitive equilibrium with the lim-sup budget. Indeed it is a Stigum's competitive equilibrium. However, the feasible allocation described by $x_1 = (1, -2, 2^2, -2^3, 2^4, \ldots)$ and $x_2 = (-1, 2, -2^2, 2^3, -2^4, \ldots)$ Pareto dominates the equilibrium allocation.

by an element of l_∞ (in Assumption 3), and that there is a total excess supply $-\bar{x} \gg 0$ satisfying $\bar{x}(t) = \bar{x}(r)$ for all t and r (in Assumption 7), play an important role in the proof of the existence of their competitive equilibrium, whereas, in our d.r.s. production example in Section 2, we observed they do not guarantee their equilibrium. Further, their Assumptions (3) and (7) are slightly stringent for our extended competitive equilibrium, so that they can be replaced by weaker ones; each consumption set is bounded below (in Assumption (3')) and there is a total excess supply $-\bar{x} \gg 0$ (in Assumption (7'-a)).

Our objective is to establish the following theorem :

Theorem *Under Assumptions (1'), (2), (3'), (4'), (5'), (6) and (7'), the infinite horizon d.r.s. production economy $\mathcal{E}$ has a competitive equilibrium with a price system in the nonnegative orthant of the space of all sequences of n-dimensional vectors.*

4. Proof of Theorem

In this section, we shall prove the theorem in two steps. We first show the existence of an Edgeworth equilibrium, next demonstrate the existence of a price system that supports the Edgeworth equilibrium as a competitive equilibrium. Though the continuity and monotonicity assumptions are weakened, and though the consumption sets are no longer the nonnegative orthant of the commodity space, we can prove the existence of Edgeworth equilibrium in a way similar to the proofs of Aliprantis et al. (1987b, 1989), and Boyd and McKenzie (1993). So, the detailed proof of this step is omitted. Because the continuity assumption on preferences is weakened, the convex hull G considered by Boyd and McKenzie is no longer closed so that we can not apply the separation theorem on s^n directly. Instead, we shall make an elementary and simple separation argument. We do not use any advanced theorems of topology, but apply the separation theorem on finite dimensional spaces, and use these finite dimensional separating vectors to construct a competitive equilibrium price sequence.

An allocation $(x_1, \ldots, x_I, y_1, \ldots, y_J)$ is said to be *improved upon* by a coalition of consumers S whenever there exists an allocation $(x'_1, \ldots, x'_I, y'_1, \ldots, y'_J)$ with $\sum_{i \in S} x'_i = \sum_{i \in S} \omega_i + \sum_{j=1}^{J} (\sum_{i \in S} \theta_{ij}) y'_j$ satisfying $x'_i \succ_i x_i$ for all $i \in S$. The *core* of the economy $\mathcal{E}$ is the set of feasible allocations which are not improved upon by any coalitions. For every replicated economy $\mathcal{E}_r$, let $Core(\mathcal{E}_r)$ be the core of $\mathcal{E}_r$. Every allocation $(x_1, \ldots, x_I, y_1, \ldots, y_J)$ of the economy $\mathcal{E}$ gives rise to a natural allocation for the rth replica economy $\mathcal{E}_r$ by letting $x_{ik} = x_i$ for $i = 1, \ldots, I$, $k = 1, \ldots, r$ and $y_{jl} = y_j$ for $j = 1, \ldots, J$, $l = 1, \ldots, r$. Any such allocation is called an *equal treatment allocation* for $\mathcal{E}_r$, and it may be denoted by an allocation of the economy $\mathcal{E}$. The *equal treatment core* K_r is the set of all equal treatment allocations in $Core(\mathcal{E}_r)$. Let $K = \bigcap_{r=1}^{\infty} K_r$. An allocation in K is called an *Edgeworth equilibrium.* See Aliprantis et al. (1987a, 1987b).

Since the commodity space s^n with the product topology is second countable (see footnote 8) and each preference $\succeq_i$ is upper semicontinuous on each $\hat{X}_i^{(r)}$ $(r = 1, 2, \ldots)$, by the representation theorem of Rader (1963), we see that each $\succeq_i$ can be represented by an upper semicontinuous utility function on each $\hat{X}_i^{(r)}$. Thus, by the theorem of Scarf (1967), it can be shown that, under our Assumptions (1′), (2), (3′), (4′), and (7′-a), every replica economy $\mathcal{E}_r$ has a non-empty core. Next, using a similar method of Aliprantis et al. (1987b, 1989) and Boyd and McKenzie (1993), we can show that under our assumptions the economy $\mathcal{E}$ has an Edgeworth equilibrium.

Proposition *Under Assumptions (1′), (2), (3′), (4′), (5′), (6) and (7′-a), the economy $\mathcal{E}$ has, at least, an Edgeworth equilibrium.*

The basic idea for proving an Edgeworth equilibrium to be a competitive allocation originates from Debreu and Scarf (1963), and is classical. In an infinite case, the separation theorem needs some additional conditions which are not satisfied here. We wish to prove it without applying the infinite separation theorem. Therefore our separation argument is not trivial.

Fix an Edgeworth equilibrium $(x_1, \ldots, x_I, y_1, \ldots, y_J) \in K$, let $H = co[\bigcup_{i=1}^{I}(P_i(x_i) - \omega_i - Z_i)]$ be the convex hull of $\bigcup_{i=1}^{I}(P_i(x_i) - \omega_i - Z_i)$. Then, Assumption (5′) implies that each feasible consumption bundle x_i is not satiable for consumer i. Hence H is nonempty. By a similar method of Debreu and Scarf (1963), it is easy to prove that the convex set H does not intersect the negative orthant of s^n.

Lemma 1 *The convex set H does not intersect the negative orthant of s^n. That is, there exists no $h \in H$ with $h \leq 0$.*

Now for each natural number τ, define $H(\tau) = \{x : x \in H, \text{ and } x(t) = 0 \text{ for } t > \tau\} \subset s^n$. It is evident that each $H(\tau)$ is a subset of H. That $\bar{z} \in H$ follows from both Assumption (5′-b) and that $(x_1, \ldots, x_I, y_1, \ldots, y_J)$ is a feasible allocation. Since $\bar{z}$ has only a finite number of nonzero components, $H(\tau)$ is nonempty when τ is large enough. Recall that H is convex and does not intersect the negative orthant of s^n. Then, it is easy to see that the sets $H(\tau)$, $\tau = 1, 2, \ldots$, satisfy the following properties.

Lemma 2 *Each $H(\tau)$ is convex and does not intersect the negative orthant of s^n, and $H(\tau) \subset H(\tau+1)$ holds for all $\tau = 1, 2, \ldots$.*

Note that for any $x \in H(\tau)$ the components $x(t)$ are equal to zero for all $t > \tau$. Hence each $H(\tau)$ can be considered as a subset of $\mathcal{R}^{n\tau}$. In this sense, $H(\tau)$ is also a convex set of $\mathcal{R}^{n\tau}$, and does not intersect the negative orthant of $\mathcal{R}^{n\tau}$. Thus, by using the separation theorem on $\mathcal{R}^{n\tau}$, we can find a non-zero sequence $p^{(\tau)} \in s^n$ with $p^{(\tau)}(t) = 0$ for all $t > \tau$, which separates $H(\tau)$ from the origin of $\mathcal{R}^{n\tau}$, i.e., $p^{(\tau)}h \geq 0$ for all $h \in H(\tau)$. Moreover, notice that $H + s^n_+ \subset H$, we see that for any $h \in H(\tau)$ and $x \in s^n_+$, $h + x[\tau]$ belongs to $H(\tau)$ also. We therefore have that $p^{(\tau)} \geq 0$ for all τ.

Next since each x_i is a feasible consumption bundle for consumer i, Assumption (5′) implies that $x_i + \bar{z} \succ_i x_i$ for all i. Notice that $\sum_{j=1}^{J} \theta_{ij} y_j \in Z_i$, then we see that $g_i = x_i + \bar{z} - \omega_i - \sum_{j=1}^{J} \theta_{ij} y_j \in P_i(x_i) - \omega_i - Z_i$. By Assumption (7′-a), we see that $\bar{x}_i \in X_i - \omega_i - Z_i$

for all i. Then, by linearly lower semicontinuity of the preferences and the convexity of Z_i, we may choose a real number α $(1 > \alpha > 0)$ such that $(1-\alpha)g_i + \alpha\bar{x}_i \in P_i(x_i) - \omega_i - Z_i$ for all i. Thus we have that

$$\begin{aligned} d &= (1-\alpha)\bar{z} + \frac{\alpha}{I}\bar{x} \\ &= \frac{(1-\alpha)}{I}\sum_{i=1}^{I}(x_i + \bar{z} - \omega_i - \sum_{j=1}^{J}\theta_{ij}y_j) + \frac{\alpha}{I}\sum_{i=1}^{I}\bar{x}_i \\ &= \sum_{i=1}^{I}\frac{1}{I}[(1-\alpha)g_i + \alpha\bar{x}_i] \in H. \end{aligned}$$

Choose a natural number τ^* such that $\bar{z} = \bar{z}[\tau^*]$. Then, in the light of $\bar{x} \ll 0$, we see that $d[\tau] = (1-\alpha)\bar{z}[\tau] + \frac{\alpha}{I}\bar{x}[\tau] = (1-\alpha)\bar{z} + \frac{\alpha}{I}\bar{x}[\tau] > (1-\alpha)\bar{z} + \frac{\alpha}{I}\bar{x} = d$ for all $\tau \geq \tau^*$. Hence $d[\tau] \in H$ and $d[\tau] \in H(\tau)$ for all $\tau \geq \tau^*$. Thus $p^{(\tau)}[(1-\alpha)\bar{z} + \frac{\alpha}{I}\bar{x}] = p^{(\tau)}d = p^{(\tau)}d[\tau] \geq 0$ for all $\tau \geq \tau^*$. We therefore have $p^{(\tau)}\bar{z} \geq \frac{\alpha}{I(1-\alpha)}p^{(\tau)}(-\bar{x}) > 0$. We shall choose $p^{(\tau)}$ for $\tau \geq \tau^*$ satisfying $p^{(\tau)}\bar{z} = 1$.

For the argument about constructing d, Boyd and McKenzie used their stronger Assumption 7 (the second part), while we did not use any such one. This motivated weakening the second part of their Assumption 7.

Lemma 3 *The sequence $\{p^{(\tau)}\}$ has, at least, a convergent sub-sequence whose limit point is nonzero.*

PROOF. First, we have that $p^{(\tau)}(-\bar{x}) \leq \frac{I(1-\alpha)}{\alpha}p^{(\tau)}\bar{z} = \frac{I(1-\alpha)}{\alpha}$ for all $\tau \geq \tau^*$. Next, since $-\bar{x}$ is strictly positive, we know that the set $A = \{q \in s^n : q \geq 0 \text{ and } q(-\bar{x}) \leq \frac{I(1-\alpha)}{\alpha}\}$ is compact. It follows from $\{p^{(\tau)} : \tau \geq \tau^*\} \subset A$ that the sequence $\{p^{(\tau)}\}$ has at least a convergent sub-sequence[11]. Let p^* be the limit point of such a convergent sub-sequence. Then p^* is also in A, i.e., $p^* \geq 0$ and $p^*(-\bar{x}) \leq \frac{I(1-\alpha)}{\alpha}$. Moreover, recall that $p^{(\tau)}\bar{z} = p^{(\tau)}[\tau^*]\bar{z}[\tau^*] = 1$ holds for all $\tau \geq \tau^*$, and note that p^* is an accumulation point

[11] Without using the compactness of the set of s^n, we can also construct such a convergent sub-sequence by a direct method.

of $\{p^{(\tau)}\}$ implies $p^*[\tau^*]$ is also an accumulation point of $\{p^{(\tau)}[\tau^*]\}$. We therefore see that $p^*\bar{z} = p^*[\tau^*]\bar{z}[\tau^*] = 1$. That is, p^* is a nonzero sequence. □

We shall show in Lemma 4 that this p^* separates H from the origin of s^n. Note that our Assumption (7′-b) plays a crucial role here.

Lemma 4 *Inf* $p^*h \geq 0$ *for all* $h \in H$.

PROOF. For any arbitrary $h \in H$, by the definition of H, there exist $z_i \in P_i(x_i) - \omega_i - Z_i$ and $\lambda_i \geq 0$ with $\sum_{i=1}^I \lambda_i = 1$ such that $h = \sum_{i=1}^I \lambda_i z_i$. Next, by Assumption (7′-b), $z_i \in P_i(x_i) - \omega_i - Z_i$ implies that there exists a τ_0^i such that $z_i[\tau] \in P_i(x_i) - \omega_i - Z_i$ for all $\tau > \tau_0^i$. Let $\tau_0 = \max\{\tau_0^1, \dots, \tau_0^I\}$. Then $z_i[\tau] \in P_i(x_i) - \omega_i - Z_i$ for all i and $\tau > \tau_0$. Hence, $h[\tau] = \sum_{i=1}^I \lambda_i z_i[\tau] \in H$ for all $\tau > \tau_0$. Thus $h[\tau] \in H(\tau) \subset H(v)$ holds for all $v \geq \tau > \tau_0$. We therefore have $p^{(v)}h[\tau] \geq 0$ for all $v \geq \tau > \tau_0$. Moreover, p^* is an accumulation point of $\{p^{(v)}\}$ implies that $p^*[\tau]$ is also an accumulation point of $\{p^{(v)}[\tau]\}$, so that $p^*h[\tau] = p^*[\tau]h[\tau] \geq 0$ must hold for all $\tau > \tau_0$. Thus, we proved that $\inf p^*h \geq 0$ for all $h \in H$. □

We shall provide a proof in a way parallel to that of Boyd and McKenzie to establish that $(p^*, x_1, \dots, x_I, y_1, \dots, y_J)$ is a competitive equilibrium for the economy $\mathcal{E}$. However, in the process of our proof that will follow below, we have to pay attention to the inequality relation between *liminf* and *limsup*.

First, it is clear that Condition (3) for competitive equilibrium of economy $\mathcal{E}$ is satisfied because $(x_1, \dots, x_I, y_1, \dots, y_J)$ is a feasible allocation. Second, to show that Condition (2) holds, we must show that for each producer j, $\sup p^*(y_j' - y_j) \leq 0$ holds for all production plan $y_j' \in Y_j$. Note that $\bar{z}$ is extremely desirable for each consumer i on his feasible consumption set $\hat{X}_i$. We see that $x_i + \delta\bar{z} \succ_i x_i$ holds for each i and all $\delta > 0$. Pick an arbitrary producer j, and suppose that each producer k except j fixes the production plan y_k, and producer j chooses an arbitrary production plan $y_j' \in Y_j$. We see that

$x_i + \delta\bar{z} - \omega_i - \sum_{k\neq j}\theta_{ik}y_k - \theta_{ij}y'_j \in P_i(x_i) - \omega_i - Z_i$ for all i, and therefore

$$\delta\bar{z} + \frac{1}{I}(y_j - y'_j) = \sum_{i=1}^{I}\frac{1}{I}(x_i + \delta\bar{z} - \omega_i - \sum_{k\neq j}\theta_{ik}y_k - \theta_{ij}y'_j) \in H.$$

Thus, from Lemma 4 we obtain that $\inf p^*[\delta\bar{z} + \frac{1}{I}(y_j - y'_j)] \geq 0$ for all $\delta > 0$. Furthermore, since $p^*\bar{z}$ is well defined and gives a finite value, so we have that $\inf p^*(y_j - y'_j) \geq 0$, i.e., $\sup p^*(y'_j - y_j) \leq 0$. Thus Condition (2) holds.

Lastly, note that $x_i + \delta\bar{z} - \omega_i - \sum_{j=1}^{J}\theta_{ij}y_j \in H$ holds for each i and all $\delta > 0$, Lemma 4 implies that $\inf p^*(x_i + \delta\bar{z} - \omega_i - \sum_{j=1}^{J}\theta_{ij}y_j) \geq 0$ for all $\delta > 0$. So we have that $\inf p^*(x_i - \omega_i - \sum_{j=1}^{J}\theta_{ij}y_j) \geq 0$ for all i. Recall that $\sum_{i=1}^{I}x_i - \sum_{i=1}^{I}\omega_i - \sum_{j=1}^{J}y_j = 0$. We obtain that

$$\begin{aligned} 0 &= \sup p^*[\sum_{i=1}^{I}(x_i - \omega_i - \sum_{j=1}^{J}\theta_{ij}y_j)] \\ &\geq \sup p^*(x_i - \omega_i - \sum_{j=1}^{J}\theta_{ij}y_j) + \sum_{k\neq i}^{I}\inf p^*(x_k - \omega_k - \sum_{j=1}^{J}\theta_{kj}y_j) \geq 0 \end{aligned}$$

for each i. This implies that $\sup p^*(x_i - \omega_i - \sum_{j=1}^{J}\theta_{ij}y_j) \leq 0$ for each i. Therefore, $p^*(x_i - \omega_i - \sum_{j=1}^{J}\theta_{ij}y_j) = 0$ holds for $i = 1, \ldots, I$. Thus, the first part of Condition (1) is proved.

Lemma 4 ensures that $\inf p(x - \omega_i - \sum_{j=1}^{J}\theta_{ij}y_j) \geq 0$ for all $x \succ_i x_i$. Therefore, to complete the proof that Condition (1) holds, we have only to show that $x \succ_i x_i$ implies $\inf p(x - \omega_i - \sum_{j=1}^{J}\theta_{ij}y_j) > 0$. Namely, any consumption bundle that is preferred by consumer i to x_i must lie outside his budget set. In this connection, we shall first have a weaker result.

Lemma 5 *If there is a consumer i and an $\tilde{x}_i \in X_i$ with $\sup p(\tilde{x}_i - \omega_i - \sum_{j=1}^{J}\theta_{ij}y_j) < 0$ and $\inf p(x'_i - \omega_i - \sum_{j=1}^{J}\theta_{ij}y_j) \geq 0$ for all $x'_i \succ_i x_i$, then $\inf p(x'_i - \omega_i - \sum_{j=1}^{J}\theta_{ij}y_j) > 0$ for all $x'_i \succ_i x_i$.*

PROOF. Suppose that there is an $x'_i \succ_i x_i$ and $\inf p(x'_i - \omega_i - \sum_{j=1}^{J}\theta_{ij}y_j) = 0$. By linearly lower semicontinuity of the preference $\succeq_i$, there is a consumption bundle $x''_i = \alpha\tilde{x}_i + (1-\alpha)x'_i$ $(0 < \alpha < 1)$ such that $x''_i \succ_i x_i$. However, we see that

$$\begin{aligned}
&\inf p(x_i'' - \omega_i - \textstyle\sum_{j=1}^{J} \theta_{ij} y_j) \\
&= \inf p[\alpha(\tilde{x}_i - \omega_i - \textstyle\sum_{j=1}^{J} \theta_{ij} y_j) + (1-\alpha)(x_i' - \omega_i - \textstyle\sum_{j=1}^{J} \theta_{ij} y_j)] \\
&\leq \alpha \sup p(\tilde{x}_i - \omega_i - \textstyle\sum_{j=1}^{J} \theta_{ij} y_j) + (1-\alpha) \inf p(x_i' - \omega_i - \textstyle\sum_{j=1}^{J} \theta_{ij} y_j) \\
&= \alpha \sup p(\tilde{x}_i - \omega_i - \textstyle\sum_{j=1}^{J} \theta_{ij} y_j) < 0.
\end{aligned}$$

This contradicts the hypothesis. Therefore, such an x_i' cannot exist. □

From Lemma 5 we see that the proof will be completed if it is proven that every consumer i has a consumption bundle $\tilde{x}_i$ in his consumption set X_i such that $\sup p^*(\tilde{x}_i - \omega_i - \sum_{j=1}^{J} \theta_{ij} y_j) < 0$. Recall that $\bar{x}_i \in X_i - \omega_i - Z_i$. Then there exist $\tilde{x}_i \in X_i$ and $y_{ij} \in Y_j$ $(i = 1, \ldots, I; j = 1, \ldots, J)$ such that $\bar{x}_i = \tilde{x}_i - \omega_i - \sum_{j=1}^{J} \theta_{ij} y_{ij}$ for all $i = 1, \ldots, I$. So, we see that

$$\begin{aligned}
&\sup p^*(\tilde{x}_i - \omega_i - \textstyle\sum_{j=1}^{J} \theta_{ij} y_j) \\
&= \sup p^*(\bar{x}_i + \textstyle\sum_{j=1}^{J} \theta_{ij} y_{ij} - \textstyle\sum_{j=1}^{J} \theta_{ij} y_j) \\
&\leq \sup p^* \bar{x}_i + \textstyle\sum_{j=1}^{J} \theta_{ij} \sup p^*(y_{ij} - y_j) \leq p^* \bar{x}_i
\end{aligned}$$

for all i. Thus, we have that

$$\sum_{i=1}^{I} \sup p^*(\tilde{x}_i - \omega_i - \sum_{j=1}^{J} \theta_{ij} y_j) \leq \sum_{i=1}^{I} p^* \bar{x}_i = p^* \bar{x} < 0.$$

Therefore, there exists at least one consumer i^* with $\sup p^*(\tilde{x}_{i^*} - \omega_{i^*} - \sum_{j=1}^{J} \theta_{i^*j} y_j) < 0$.

To show that every consumer i has a consumption bundle $\tilde{x}_i$ in his consumption set X_i such that $\sup p^*(\tilde{x}_i - \omega_i - \sum_{j=1}^{J} \theta_{ij} y_j) < 0$, suppose, by way of contradiction, that there existed a consumer $i_0 (\neq i^*)$ with $\sup p^*(x - \omega_{i_0} - \sum_{j=1}^{J} \theta_{i_0 j} y_j) \geq 0$ for all $x \in X_{i_0}$. Thus, it follows from the strong irreducibility assumption that there exist an allocation $(x_1', \ldots, x_I', y_1', \ldots, y_J')$ and a real number $\alpha > 0$ such that $\sum_{i \neq i_0}^{I} (x_i' - \omega_i - \sum_{j=1}^{J} \theta_{ij} y_j') + \alpha(x_{i_0}' - \omega_{i_0} - \sum_{j=1}^{J} \theta_{i_0 j} y_j') = 0$ and $x_i' \succ_i x_i$ for all $i \neq i_0$. We see that $\inf p^*(x_i' - \omega_i - \sum_{j=1}^{J} \theta_{ij} y_j) \geq 0$ for all $i \neq i_0$ and $\inf p^*(x_{i^*}' - \omega_{i^*} - \sum_{j=1}^{J} \theta_{i^*j} y_j) > 0$. Next, note that $\sup p^*(y_j' - y_j) \leq 0$ for all j. Then, we obtain that

$$0 \;\geq\; \sum_{i \neq i_0}^{I} \sum_{j=1}^{J} \theta_{ij} \sup p^*(y_j' - y_j) + \alpha \sum_{j=1}^{J} \theta_{i_0 j} \sup p^*(y_j' - y_j)$$

$$
\begin{aligned}
&\geq \sup p^*[\textstyle\sum_{i\neq i_0}^{I}\sum_{j=1}^{J}\theta_{ij}(y_j' - y_j) + \alpha\sum_{j=1}^{J}\theta_{i_0 j}(y_j' - y_j)] \\
&= \sup p^*[\textstyle\sum_{i\neq i_0}^{I}(x_i' - \omega_i - \sum_{j=1}^{J}\theta_{ij}y_j) + \alpha(x_{i_0}' - \omega_{i_0} - \sum_{j=1}^{J}\theta_{i_0 j}y_j)] \\
&\geq \sum_{i\neq i_0}^{I}\inf p^*(x_i' - \omega_i - \sum_{j=1}^{J}\theta_{ij}y_j) + \alpha \sup p^*(x_{i_0}' - \omega_{i_0} - \sum_{j=1}^{J}\theta_{i_0 j}y_j) \\
&\geq \sum_{i\neq i_0}^{I}\inf p^*(x_i' - \omega_i - \sum_{j=1}^{J}\theta_{ij}y_j) \\
&\geq \inf p^*(x_{i^*}' - \omega_{i^*} - \sum_{j=1}^{J}\theta_{i^* j}y_j) > 0
\end{aligned}
$$

This contradiction shows such a consumer i_0 cannot exist. Therefore, for each consumer there exists a consumption bundle $\tilde{x}_i$ in his consumption set X_i such that $\sup p^*(\tilde{x}_i - \omega_i - \sum_{j=1}^{J}\theta_{ij}y_j) < 0$. Then, by Lemma 5, $x \succ_i x_i$ implies $\inf p^*(x - \sum_{j=1}^{J}\theta_{ij}y_j - \omega_i) > 0$ for all i. Thus, the second part of Condition (1) is also proved.

Collecting Lemmata (1)–(5) and other remarks, we have completed the proof that $(p^*, x_1, \ldots, x_I, y_1, \ldots, y_J)$ is a competitive equilibrium for the economy $\mathcal{E}$.

The question of optimality of the equilibrium remains. It is easy to show that under Assumption (5′-b) every competitive equilibrium is Pareto optimal, and is an Edgeworth equilibrium. So we have the same corollary as Boyd and McKenzie had.

Corollary *Under Assumptions (1′), (2), (3′), (4′), (5′), (6) and (7′), an allocation is an Edgeworth equilibrium of economy $\mathcal{E}$ if and only if there is a price sequence $p^* \in s_+^n$ for which it is a competitive equilibrium.*

5. Concluding Remarks

We have generalized an infinite horizon c.r.s. production model to a d.r.s. production one, where a new notion of competitive equilibrium is necessarily introduced and the existence conditions are weakened. We have established a general version of the Boyd-McKenzie theorem.

We might as well get an equilibrium as defined by them in footnote 4 by adding some other assumptions to their assumptions in footnote 3. To make the profits of our extended equilibrium well-defined, we need to assume

Assumption (8) *For each firm j, if $y_j \in Y_j$ then $y_j[\tau] \in Y_j$ for all τ.*

This is called *Exclusion Assumption* by Bewley (1972). The example in Section 2, which does not have their equilibrium, does satisfy this assumption. Hence, Assumption (8), with Assumptions (1)–(7), is not sufficient to get their equilibrium. To ensure further the equilibrium profits not to be infinite, we need to assume

Assumption (7*-a) *There exists a real number $k\,(0 < k < 1)$ such that, for all i there is $\bar{x}_i \in X_i - \omega_i - kZ_i$ with $\bar{x}_i \leq 0$. Moreover, $\bar{x} = \sum_{i=1}^{I} \bar{x}_i \ll 0$ and $\bar{x}(t) = \bar{x}(r)$ for all r and t.*

This is slightly stronger than Assumption (7-a). It is further assumed that each firm's production capacity is surplus for consumers' survival. It implies that the transformed c.r.s. production model, obtained by introducing artificial entrepreneurial factors, satisfies the first part of their original Assumption 7.

Under Assumptions (1)–(6), (7-b), (7*-a), and (8), applying our existence theorem to the d.r.s. production model, we can prove that there exists their equilibrium. However, these assumptions can not imply that the transformed c.r.s. production model satisfies the second part of their Assumption 7. Therefore, we can not apply the Boyd-McKenzie theorem to the transformed c.r.s. production one. With the general consumption sets, we have not found an assumption yet in a d.r.s. production model, such that the transformed c.r.s. production one would meet the second part.

Finally, we remark that adding Assumption (7*-a) alone is not sufficient either to get their equilibrium. To show this we raise below an example.

Example Consider an infinite horizon d.r.s. production economy with one consumer and two firms. The commodity space is s. The consumption set of the consumer is s_+, and the utility function U of the consumer is given by

$$U(x) = 10\sqrt{x(1)} + \sum_{t=1}^{\infty} 2^{-t} \arctan\Big(x(3t-1) + x(3t) + x(3t+1)\Big).$$

The endowment of the consumer is $\omega = (\,0,\ 0,0,2,\ 0,0,2,\ 0,0,2,\ \ldots)$.

The production sets of the two firms are defined by

$$Y_1 = \Big\{\ y \in s : y(3t-1)\ \leq\ \min\big\{2^t+1,\ -\tfrac{2^t+1}{2^t} y(3t)\big\}\ \text{ and }\ y(r) \leq 0,$$
$$\text{for all }\ t = 1,2,\ldots\ \text{ and }\ r \neq 3t-1\Big\};$$

and

$$Y_2 = \Big\{\ y \in s : y(1)\ \leq\ \min\big\{1, -\tfrac{y(3t-1)}{2^t}, -y(3t+1),\, t = 1,2,\ldots\big\},$$
$$y(3t-1) \leq 0,\ y(3t+1) \leq 0 \text{ and }\ y(3t)\ \leq\ \min\big\{2^t+1,$$
$$-\tfrac{2^t+1}{2^t} y(3t-1), -(2^t+1)y(3t+1)\big\}\ \text{ for all }\ t = 1,2,\ldots\ \Big\}.$$

We can see this example satisfy their Assumptions (1)-(7) and Assumption (7*-a), but not Assumption (8). Let

$$p^* = (50,\ \tfrac{1}{2^1}, \tfrac{1}{2^1}, \tfrac{1}{2^1},\ \tfrac{1}{2^2}, \tfrac{1}{2^2}, \tfrac{1}{2^2},\ \ldots),$$
$$x^* = (1,\ 1,1,1,\ 1,1,1,\ 1,1,1,\ \ldots),$$
$$y_1^* = (0,\ 2^1+1, -2^1, 0,\ 2^2+1, -2^2, 0,\ 2^3+1, -2^3, 0,\ \ldots),$$
$$y_2^* = (1,\ -2^1, 2^1+1, -1,\ -2^2, 2^2+1, -1,\ -2^3, 2^3+1, -1,\ \ldots).$$

Then we can show $(\,p^*,\ x^*,\ y_1^*,\ y_2^*)$ is a unique extended competitive equilibrium. However, it is not a competitive equilibrium as defined in footnote 4 because the profits of firms are not well-defined.

Reference

Aliprantis, C. D. and D. J. Brown, 1983, Equilibrium in markets with a Riesz space of commodities, *Journal of Mathematical Economics* 11, 189-207.

Aliprantis, C. D., D. J. Brown and O. Burkinshaw, 1987a, Edgeworth equilibria, *Econometrica* 55, 1109-1137.

——, —— and ——, 1987b, Edgeworth equilibria in production economies, *Journal of Economic Theory* 43, 252-291.

——, —— and ——, 1989, *Existence and Optimality of Competitive Equilibria* (Springer-Verlag, New York).

Araujo, A., 1985, Lack of Pareto optimal allocations in economies with infinitely many commodities: The need for impatience, *Econometrica* 53, 455-461.

Arrow, K. J. and G. Debreu, 1954, Existence of an equilibrium for a competitive economy, *Econometrica* 22, 265-290.

Arrow, K. J. and M. Kurz, 1970, *Public investment, the rate of return and optimal public policy* (Johns Hopkins Press, Baltimore).

Barro, R. J., 1990, Government spending in a simple model of endogenous growth, *Journal of Political Economy* 98, 103-125.

Bewley, T., 1972, Existence of equilibria in economies with infinitely many commodities, *Journal of Economic Theory* 4, 514-540.

Boyd, J. H., III and L. W. McKenzie, 1993, The existence of competitive equilibrium over an infinite horizon with production and general consumption sets, *International Economic Review* 34, 1-20.

Brown, D. J. and L. M. Lewis, 1981, Myopic economic agents, *Econometrica* 49, 359-368.

Debreu, G., 1954, Valuation equilibrium and Pareto optimum, *Proceedings of the National Academy of Sciences* 40, 588-594.

Debreu, G. and H. Scarf, 1963, A limit theorem on the core of an economy, *International Economic Review* 4, 235-46.

Eisenberg, M., 1974, *Topology* (New York: Holt, Rinehart and Winston).

Yi G., 1989, Classical welfare theorems in economies with the overtaking criterion, *Jour-*

nal of Mathematical Economics 18, 57-75.

Mas-Colell, A., 1986, The price equilibrium existence problem in topological vector lattices, *Econometrica* 54, 1039-1054.

Mas-Colell, A. and W. R. Zame, 1991, Equilibrium theory in infinite dimensional space, *Handbook of Mathematical Economics,* Vol. 4, Chapter 34 (North-Holland, Amsterdam).

McKenzie, L. W., 1959, On the existence of general equilibrium for a competitive market, *Econometrica* 27, 54-71.

——, 1981, The classical theorem on the existence of competitive equilibrium, *Econometrica* 49, 819-941.

——, 1993, Achieving a general consumption set in an infinite model of competitive equilibrium, University of Rochester, Working Paper No. 346.

Peleg, B., and M. E. Yaari, 1970, Markets with countably many commodities, *International Economic Review* 11, 369-377.

Rader, T., 1963, The existence of a utility function to represent preferences, *Review of Economic Studies* 30, 229-232.

Scarf, H., 1967, The core of an n-game, *Econometrica* 35, 50-69.

Stigum, B. P., 1973, Competitive equilibria with infinitely many commodities (II), *Journal Economic Theory* 6, 415-445.

Zame, W. R., 1987, Competitive equilibria in production economies with an infinite dimensional commodity space, *Econometrica* 55, 1075-1108.

Wilson, C., 1981, Equilibrium in dynamic models with an infinity of agents, *Journal of Economic Theory* 24, 95-111.