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Comparison between low and high buildings
with respect to travel distance

by

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Comparison between low and high buildings with respect to travel distance

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Abstract: In the present paper, we discuss the measure of the point pairs whose distance are less than a distance r at all floor points in a building. By differentiating this measure with respect to r , we get the function $f(r)$ which is called by distance distribution. Using this distance distribution we make a comparison between low and high buildings with respect to travel distance under the condition that the total floor area of a building is constant.

1. Introduction

In the present paper, regarding to space utilization we discuss the difference between low and high buildings under the condition that the total floor area of a building is constant. Between low and high buildings there are differences about structure, building site, unit price of construction, and so on. But we analyze various buildings from a point of movement in each building.

At first we define a function $F(r)$ which is the measure of the point pairs whose distance are less than a distance r at all floor points in a building. By differentiating this measure with respect to r , we get the function $f(r)$ which is called by "distance distribution". This function $f(r)$ indicates as a density the amount of two points whose distance is r in a given building.

Strictly we have to discuss the measure $F(r)$ and distance distribution $f(r)$ in four dimensional space, but we can not draw and recognize some figures in four dimensional space. Therefore we begin the discussion about "distance distribution" in two dimensional space in which we can easily illustrate the figures concerning this distance distribution.



Fig.1 Two points in one dimensional space

Let us consider a segment of length a in which two points are distributed as shown in Fig.1. The two points mean that one point is a origin and the other is a destination of one trip which is a movement of a person or some goods.

When we consider the situation two points in one dimensional apace as shown in Fig.1, we translate the situation to one point in two dimensional space. Consider that x_1 and x_2 denote the distances from the origin O in Fig.1, we can illustrate the situation with one point (x_1, x_2) in two dimensional space as shown in Fig.2.

Now we define that the measure $F(r)$ by

$$F(r) = \iint_{|x_1 - x_2| < r} dx_1 dx_2, \quad (1)$$

namely the measure of two points whose distance is less than a distance r , when the points x_1, x_2 are uniformly distributed in the region $0 < x_1 < a$ and $0 < x_2 < a$ shown as Fig.2.

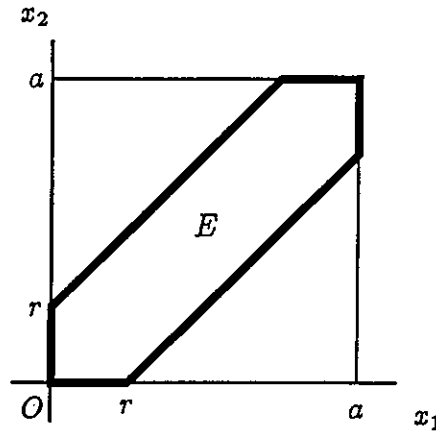


Fig.2 One point in two dimensional space

We can easily calculate integration (1) to get the area of the hexagonal region E drawn by thick line in Fig.2. Therefore

$$F(r) = a^2 - (a - r)^2 \quad (2)$$

is obtained. Differentiating $F(r)$ with respect to r , we obtain the function $f(r)$ which we call "distance distribution". This is because $f(r)\Delta r$ is almost equal to the measure of the point pairs whose distance is just r . In this case, differentiating equation (2) with respect to r , we get the distance distribution

$$f(r) = 2(a - r). \quad (3)$$

Consequently using distance distribution (3), the mean value of the distance \bar{r} is obtained as follows:

$$\bar{r} = \frac{\int_0^a r f(r) dr}{\int_0^a f(r) dr} = \frac{1}{3}a. \quad (4)$$

Distance distribution (3) in one dimensional space is expressed simply in linear form of the distance r , but distance distributions in two dimensional space are complicated as discussed later.

By the way when we get the distance distribution, we assume that the points x_1 and x_2 are uniformly distributed on the segment of distance a . In real world, origins and destinations of movements are not distributed uniformly. Thus the assumption of uniform distribution is often condemned to be too simple. However main subject of the present paper is to discuss not the real activity but the property of physical spaces in which various activities are distributed. Based on the distance between two points distributed uniformly, the distance distribution can show the characteristics of physical spaces composed of many floors(planes).

For example, given a region for planning, we will begin to measure the area of the region. The planned people or the number of facilities of this region does not necessarily depend on the area, because we can vary the density of the people or the facilities in this region. But it is unquestionable that the area is the most fundamental measure of this region. And the area is the measure of the set of points distributed uniformly in this region. In the same sense, we discuss the distance distribution based on uniformity. We do not try some simulations which express real world activities.

The integral from 0 to a of $f(x)$, namely $F(a)$ of equation (2) is equal to a^2 which shows the total measure of point pairs. We do not normalize the distance distribution in such a way that the total measure of point pairs is equal to 1 just like probability density function. This is because we can easily add or subtract distance distributions when we add or subtract the spaces in which x_1, x_2 origins and destinations are distributed. Hence, the mean distance \bar{r} is given by equation (4).

2. Distance distribution in two dimensional space

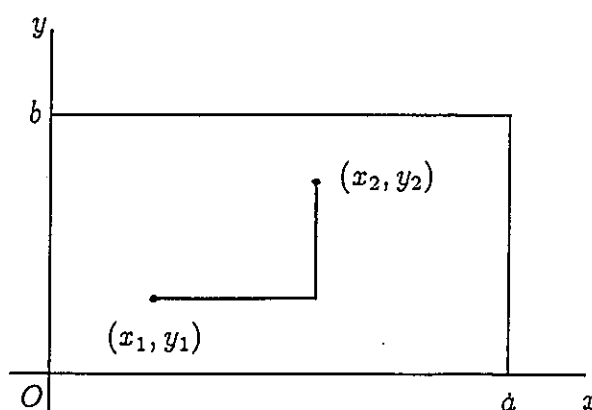


Fig.3 Rectilinear distance in a rectangle

In a building, if it is a office building or a department store, we move from a point to another point by grid system shown as Fig.1 rather than by straight line. The distance by grid system is called rectilinear distance. Let us consider a rectangle which have a long edge of

length a and a short edge of length b , and x axis along the long edge and y axis along the short edge as shown in Fig.3. Then arbitrary two points which mean a origin and a destination are expressed by Cartesian coordinates as (x_1, y_1) and (x_2, y_2) . In this case the measure of point pairs whose rectilinear distance $|x_1 - x_2| + |y_1 - y_2|$ is less than a distance r is

$$F(r) = \iiint\limits_{|x_1 - x_2| + |y_1 - y_2| < r} dx_1 dy_1 dx_2 dy_2. \quad (5)$$

This formulation is an extension of the two dimensional case as Fig. 2 into four dimensional space. The above measure $F(r)$ means the volume of the region which satisfies the inequalities $0 < x_1 < a$, $0 < y_1 < b$, $0 < x_2 < a$, $0 < y_2 < b$ and $|x_1 - x_2| + |y_1 - y_2| < r$ in four dimensional space. Those inequalities are hyperplanes, so that the region of measure $F(r)$ is surrounded by these hyperplanes in four dimensional space and is the extension of the hexagonal space in two dimensional space illustrated by the thick line in Fig.2. But we can not recognize accurately the region of four dimensional space. For this reason, using equation (3) we can reduce the integral dimensionality from 4 to 2.

From equation (3), the measure of the point pair (x_1, x_2) whose distance $|x_1 - x_2| = X$ is $2(a - X)\Delta X$ and in the same manner the measure of the point pair (y_1, y_2) whose distance $|y_1 - y_2| = Y$ is $2(b - Y)\Delta Y$. Accordingly integral (5) can be transformed to

$$F(r) = \iint\limits_{X+Y < r} 2(a - X) \cdot 2(b - Y) dX dY \quad (6)$$

where the domain of definition is $0 < X < a$ and $0 < Y < b$.

Figure 4 illustrates that the region of integral (6) varies with respect to the distance r . If $0 < r \leq b$, the region is expressed by the shaded domain, then integral (6) is obtained by

$$F(r) = \int_0^r \int_0^{-X+r} 4(a - X)(b - Y) dY dX = \frac{1}{6}r^4 - \frac{2}{3}(a + b)r^3 + 2abr^2.$$

Consequently differentiating this $F(r)$ with respect to r , we get the distance distribution

$$f(r) = \frac{2}{3}r^3 - 2(a + b)r^2 + 4abr, \quad (7)$$

where $0 < r \leq b$. In the same manner, paying attention to the domain of integral as shown in Fig.4, we can calculate the measure $F(r)$ and the distance distribution $f(r)$ relating to the range of the distance r .

But our main concern is on the distance distribution $f(r)$ rather than the measure $F(r)$. Here we discuss a method of direct calculation for the function $f(r)$. In Fig.5 the shaded strip is apparently the difference of the measure $F(r + \Delta r) - F(r)$. Substituting $Y - r$ into $-X$ from $X + Y = r$, therefore we get

$$F(r + \Delta r) - F(r) \approx \Delta r \int_0^r 4(a + Y - r)(b - Y) dY.$$

Dividing the both side of the above formula by Δr , we get the equation

$$f(r) = \int_0^r 4(a + Y - r)(b - Y) dY, \quad (8)$$

when $\Delta r \rightarrow 0$. Calculating the above integral, the result is coincide to equation (7). In

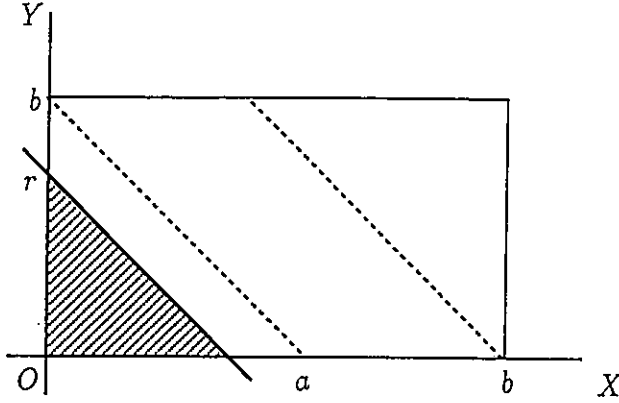


Fig.4 The region of integral (6)

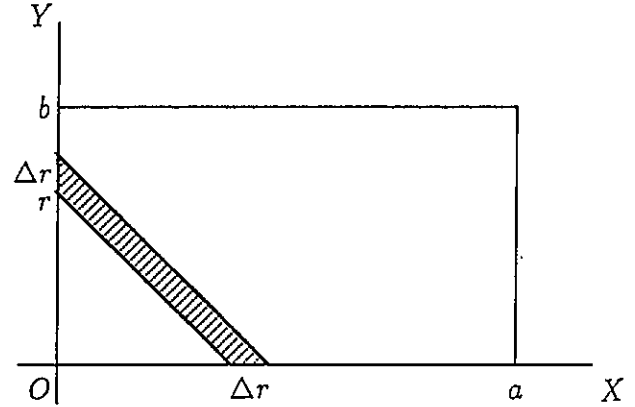


Fig.5 Differential of the measure $F(r)$

the same manner as this, keeping the integral domain in mind as shown Fig.4, we obtain the integrals:

$$\text{when } b < r \leq a \quad f(r) = \int_0^b 4(a + Y - r)(b - Y) dY,$$

$$\text{when } a < r < a + b \quad f(r) = \int_{r-a}^b 4(a + Y - r)(b - Y) dY.$$

We can calculated easily the above integrals and get the rectilinear distance distribution in the rectangle of edges a and b ($a > b$) as follows:

$$\text{if } 0 < r \leq b \quad f(r) = \frac{2}{3}r^3 - 2(a + b)r^2 + 4abr,$$

$$\text{if } b < r \leq a \quad f(r) = -2b^2r + 2ab^2 + \frac{2}{3}b^3,$$

$$\text{if } a < r < a + b \quad f(r) = \frac{2}{3}\{(a + b) - r\}^3 \quad (9)$$

The function $f(r)$ is smooth (c^1 class) at $r = b$ and $r = a$ and is illustrated in Fig.10 later.

3. Distance distribution between different floors

In the previous chapter we discussed the rectilinear distance distribution within a plane (floor). In this chapter we focus our discussion on the distance distribution between different planes (floors).

Let us consider two rectangles which mean one floor and another floor in one building and the rectangles have the same form which has a long edge of length a and a short edge of length b . Suppose that the vertical distance between these two floors (rectangles) is h shown as (1) of Fig.6. From now on, we transfer the vertical distance h to αh , because the velocity of a vertical movement is usually different from that of a horizontal movement. Taking time into account, the coefficient α changes the vertical distance to the horizontal distance.

We can derive the distance distribution between the different floors wherever the point of vertical movement is in the rectangle (floor). For the simplicity, we assume that the vertical movement is possible only at the center of the rectangle (floor) shown as (1) of Fig. 6.

When we calculate the distance distribution, without loss of generality, we can put $h = 0$. Because after the derivation of the distance distribution under the condition $h = 0$, we can get the distance distribution in the case of vertical distance h , substituting $r - \alpha h$ into r of the function $f(r)$ which is the distance distribution.

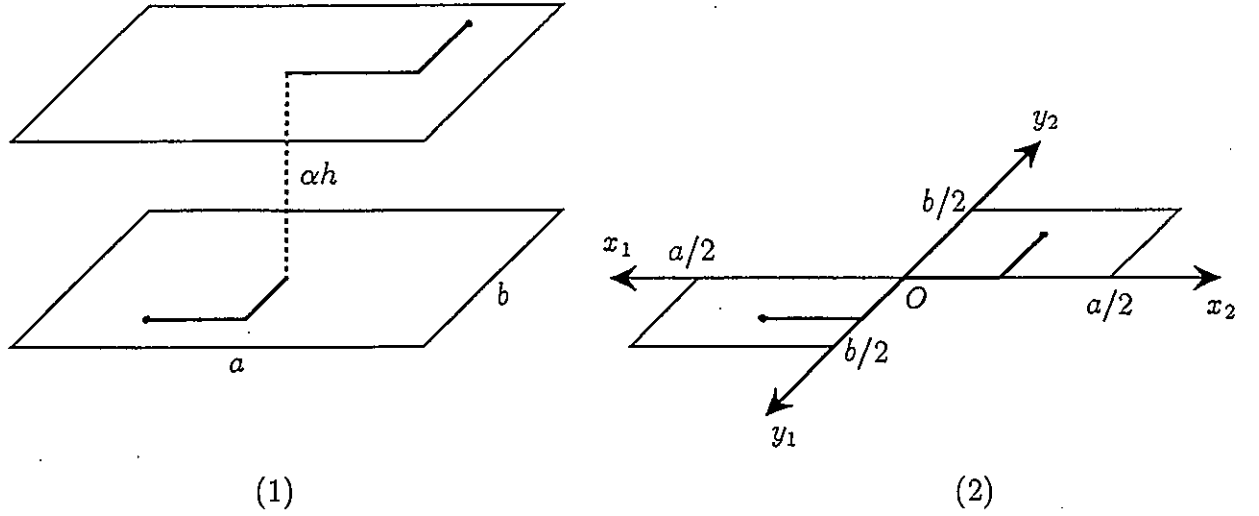


Fig.6 Distance between different floors

Because the point of vertical movement is the center, we can put a point (origin or destination) to the symmetrical point in a rectangle of long edge length $a/2$ and short edge length $b/2$, namely the quarter of the original rectangle (floor) with regard to calculating the distance distribution between different floor as shown in (2) of Fig.6. Thus in this case the distance r is equal to $x_1 + y_1 + x_2 + y_2$, where $0 < x_1 < a/2, 0 < y_1 < b/2, 0 < x_2 < a/2, 0 < y_2 < b/2$.

For the simplicity we divided the distance r into $X = x_1 + x_2$ and $Y = y_1 + y_2$ so as to calculate the distribution of distance X and Y . In order to obtain the distance distribution of X , we use almost the same method as the derivation of equation (3). The measure of point pairs x_1, x_2 whose sum $x_1 + x_2$ is less than X is

$$F(X) = \iint_{x_1+x_2 < X} dx_1 dx_2,$$

where $0 < x_1 < a/2$ and $0 < x_2 < a/2$. Paying attention to the region of the above integral in Fig.7, we get if $0 < X \leq a/2$, $F(X) = X^2/2$ and if $a/2 < X < a$, $F(X) = X^2/2 - (X - a/2)^2$. Hence differentiating $F(X)$ with respect to X , we obtain the distance distribution of X as follows:

$$\begin{aligned} \text{if } 0 < X \leq a/2 & \quad f(X) = X, \\ \text{if } a/2 < X < a & \quad f(X) = a - X. \end{aligned} \tag{10.1}$$

In the same way for the distance distribution of Y , we get

$$\begin{aligned} \text{if } 0 < Y \leq b/2 & \quad f(Y) = Y, \\ \text{if } b/2 < Y < b & \quad f(Y) = b - Y. \end{aligned} \quad (10.2)$$

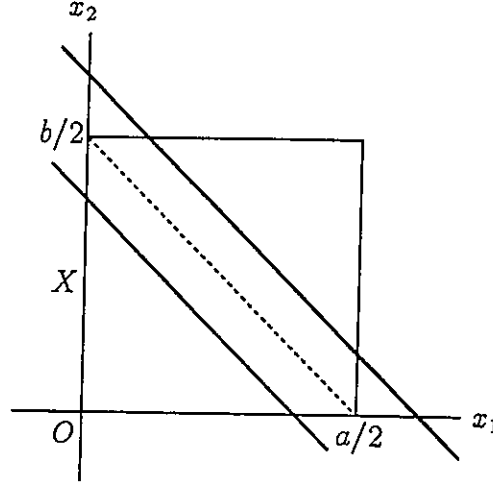


Fig.7 Region of integral $F(X)$

The measure of the point pairs (X, Y) satisfying the condition $X + Y < r$ is

$$F(r) = \iint_{X+Y < r} f(X)f(Y)dXdY,$$

where two dimensional function $f(X)f(Y)$ can be shown as Fig.8 from (10.1) and (10.2). As discussed at Fig.5, we need $f(r)$ the differential of $F(r)$. For this reason, we adopt the direct method discussed at the deviation of equation (8).

At first, if $0 < r \leq b/2$, two dimensional function $f(X)f(Y) = XY$ and substituting $r - Y$ into X because $X + Y = r$, $f(X)f(Y) = (r - Y)Y$. Therefore for the distribution of r we get

$$\frac{1}{16}f(r) = \int_0^r (r - Y)Y dY,$$

where $1/16$ means that we discussed the distance distribution in the quarter region of the original rectangle, as a result we have to multiply by 4^2 the measure obtained in the region of Fig.8 and Fig.9. Calculating above integral, if $0 < r \leq b/2$ we get

$$f(r) = \frac{8}{3}r^3. \quad (11.1)$$

Next, if $b/2 < r \leq a/2$, the region of the integral contains two domains in which the function $f(X)f(Y)$ is equal to XY or $X(b - Y)$. Thus we obtain

$$\frac{1}{16}f(r) = \int_0^{b/2} (r - Y)Y dY + \int_{b/2}^r (r - Y)(b - Y) dY,$$

and calculating the above integral, we get

$$f(r) = -\frac{8}{3}r^3 + 8br^2 - 4b^2r + \frac{2}{3}b^3. \quad (11.2)$$

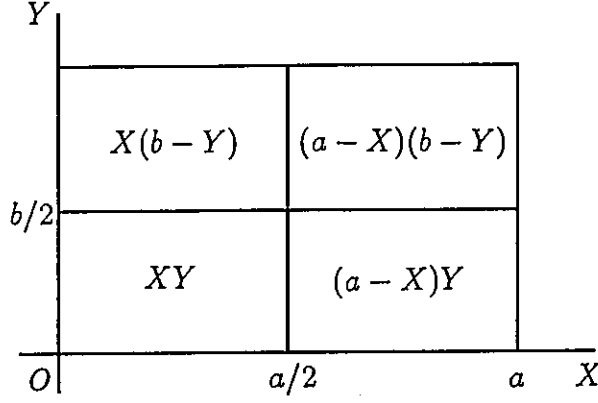


Fig.8 Two dimensional function $f(X)f(Y)$

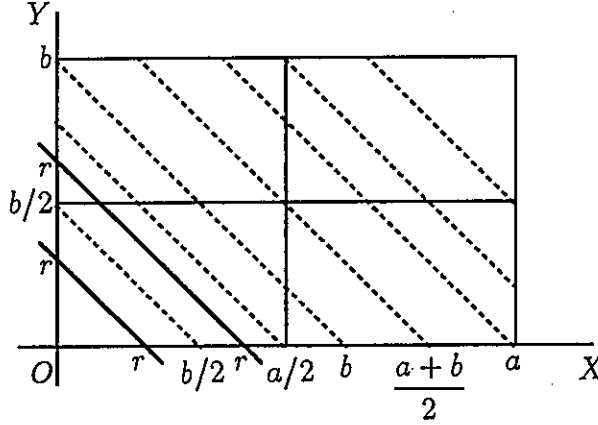


Fig.9 Region of the integrals

In the same manner we obtain the distance distribution in each range of r . The results and their basic integrals are enumerated below.

If $a/2 < r \leq b$

$$\frac{1}{16}f(r) = \int_0^{r-a/2} (Y + a - r)Y dY + \int_{r-a/2}^{b/2} (r - Y)Y dY + \int_{b/2}^r (r - Y)(b - Y) dY,$$

$$f(r) = -8r^3 + 8(a+b)r^2 - 4(a^2 + b^2)r + \frac{2}{3}(a^3 + b^3). \quad (11.3)$$

If $b < r \leq (a+b)/2$

$$\frac{1}{16}f(r) = \int_0^{r-a/2} (Y + a - r)Y dY + \int_{r-a/2}^{b/2} (r - Y)Y dY + \int_{b/2}^b (r - Y)(b - Y) dY,$$

and calculating the above integral, we get

$$f(r) = -\frac{8}{3}r^3 + 8br^2 - 4b^2r + \frac{2}{3}b^3. \quad (11.2)$$

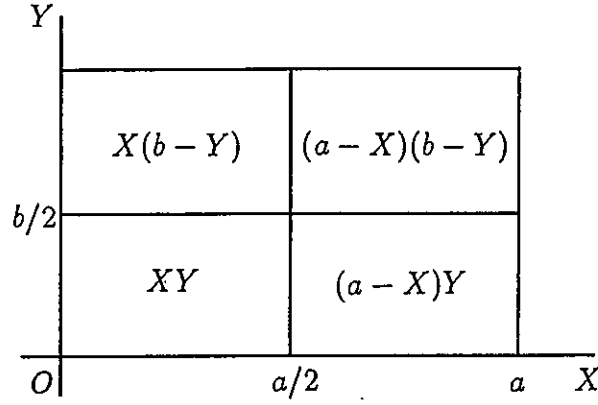


Fig.8 Two dimensional function $f(X)f(Y)$

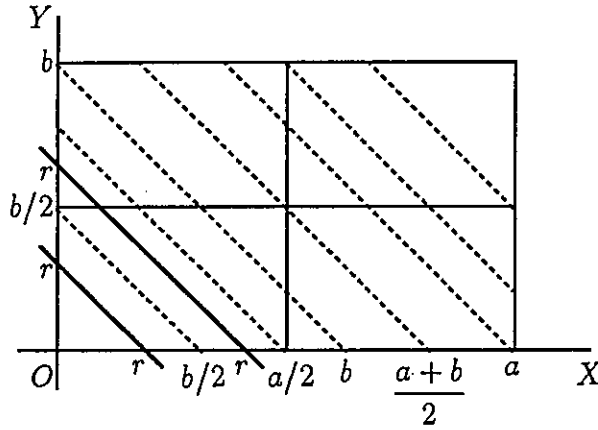


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In the same manner we obtain the distance distribution in each range of r . The results and their basic integrals are enumerated below.

If $a/2 < r \leq b$

$$\frac{1}{16}f(r) = \int_0^{r-a/2} (Y+a-r)Y dY + \int_{r-a/2}^{b/2} (r-Y)Y dY + \int_{b/2}^r (r-Y)(b-Y) dY,$$

$$f(r) = -8r^3 + 8(a+b)r^2 - 4(a^2+b^2)r + \frac{2}{3}(a^3+b^3). \quad (11.3)$$

If $b < r \leq (a+b)/2$

$$\frac{1}{16}f(r) = \int_0^{r-a/2} (Y+a-r)Y dY + \int_{r-a/2}^{b/2} (r-Y)Y dY + \int_{b/2}^b (r-Y)(b-Y) dY,$$

$$f(r) = -\frac{16}{3}r^3 + 8ar^2 - 4(a^2 - b^2)r + \frac{2}{3}a^3 - 2b^3. \quad (11.4)$$

If $(a + b)/2 < r \leq a$

$$\frac{1}{16}f(r) = \int_0^{b/2} (Y + a - r)Y dY + \int_{b/2}^{r-a/2} (Y + a - r)(b - Y) dY + \int_{r-a/2}^b (r - Y)(b - Y) dY,$$

$$f(r) = \frac{16}{3}r^3 - 8(a + 2b)r^2 + 4(a^2 + 4ab + 3b^2)r - \left(\frac{2}{3}a^3 + 4a^2b + 4ab^2 + \frac{10}{3}b^3\right). \quad (11.5)$$

If $a < r \leq a/2 + b$

$$\frac{1}{16}f(r) = \int_{r-a}^{b/2} (Y + a - r)Y dY + \int_{b/2}^{r-a/2} (Y + a - r)(b - Y) dY + \int_{r-a/2}^b (r - Y)(b - Y) dY,$$

$$f(r) = 8r^3 - 16(a + b)r^2 + 4(3a^2 + 4ab + 3b^2)r - \left(\frac{10}{3}a^3 + 4a^2b + 4ab^2 + \frac{10}{3}b^3\right). \quad (11.6)$$

If $a/2 + b < r \leq a + b/2$

$$\frac{1}{16}f(r) = \int_{r-a}^{b/2} (Y + a - r)Y dY + \int_{b/2}^b (Y + a - r)(b - Y) dY,$$

$$f(r) = \frac{8}{3}r^3 - 8ar^2 + (8a^2 - 4b^2)r - \frac{8}{3}a^3 + 4ab^2 + 2b^3. \quad (11.7)$$

If $a + b/2 < r < a + b$

$$\frac{1}{16}f(r) = \int_{r-a}^b (Y + a - r)(b - Y) dY,$$

$$f(r) = \frac{8}{3} \{(a + b) - r\}^3. \quad (11.8)$$

We call the eight equations from (11.1) to (11.8) together by equation (11). This function (11) is smooth (c^2 class) at each point of $r = b/2, a/2, b, (a + b)/2, a, a/2 + b, a + b/2$. And from Fig.8 and Fig.9 it is obvious that the function is symmetrical with respect to the line of $r = (a + b)/2$. Function (11) is derived under the condition that $b \leq a \leq 2b$. If $a/2 > b$, the region of the integral is different from the region shown as Fig.9. For this reason the distance distribution is different from function (11). In the same way as the derivation of (11), if $a/2 > b$ we can obtain the distance distribution. But we omit this function for want of space.

4. Distance distribution in a building

Using equations (9) and (11), we can derive the distance distribution in a building of any number of stories when a floor of the building is the same form of rectangle. Now we calculate

the distance distribution about a simple case. From now on suppose that the total floor area $100 \times 100\text{m}^2$ and the form of a floor is square.

At first we consider a building of one story. In this case we substitute $a = 100\text{m}$ and $b = 100\text{m}$ into equation (9) to calculate the distance distribution illustrated as #1 of Fig.10.

Next, we consider a building of two stories. Within one floor (the first floor and the second floor), substituting $a = 100/\sqrt{2}\text{m}$ and $b = 100/\sqrt{2}\text{m}$ into equation (9), we get the distance distribution. From the first floor to the second floor and from the second floor to the first floor, we can obtain the distance distribution from equation (11), where $h = 4.0\text{m}$ and $\alpha = 5$ because the velocity of horizontal movement (on foot) is 5 times as the velocity of vertical movement (stairs on foot or escalator). Consequently we add the distance distributions in the first floor and in the second floor, and the distance distributions between the first floor and second floor (from the first to the second and from the second to the first) to compute the distance distribution in the two story building shown as #2 of Fig.10.

In the same way, we can compute the distance distribution in a n -story building. But it needs too large space to list the distance distribution function of n -story building. For this reason, we compute the distance distributions from one story to six stories to illustrate in Fig.10, under the condition that the total floor area is $100 \times 100\text{m}^2$ and form of a floor is square.

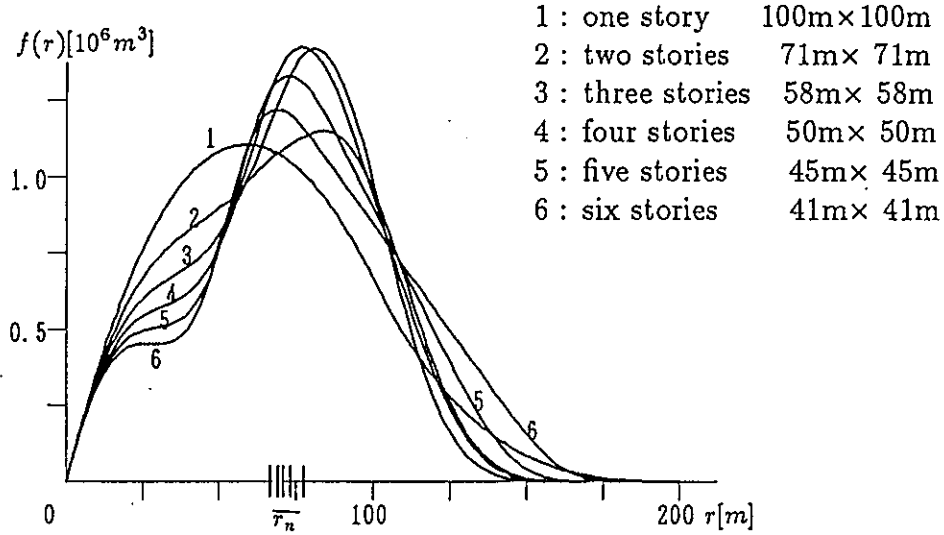


Fig.10 Distance distributions from one story building to six story building

From Fig.10, in the range of small r ($0 < r < 20\text{m}$) there are almost no difference in the six buildings from one story to six stories. As the increase of r , the distance distribution of low buildings are larger than that of high buildings. At the range of a little larger than the mean values of the distance r there are the peaks of the distance distribution of three and four story buildings. Finally in the range of large r (around 150m) the distance distributions of one story and six story buildings are larger than that of the others.

We denote that \bar{r}_n is the mean value of the distance r in a n -story building whose floor is

rectangle of edge a and b ($a > b$). From equation (4) in the case of one story building, we obtain

$$\bar{r}_1 = \frac{1}{3}(a + b).$$

For distance distribution (11), from $r = x_1 + x_2 + y_1 + y_2$ we can easily calculate the mean value of $r = x_1 + x_2 + y_1 + y_2$ to get \bar{r} the mean value of r by

$$\bar{r} = \frac{1}{2}(a + b).$$

Taking into account that the combination number of k flights up and down is $2(n - k)$ in a n -story building, the mean value of distance is

$$\bar{r}_n = \frac{3n - 1}{6n}(a + b) + \frac{(n - 1)(n + 1)}{3n}\alpha h, \quad (12)$$

where one floor is rectangle of edge a and b ($a > b$).

Let us consider that the total floor area is constant and the floor in a n_1 -story building is similar to that in a n_2 -story building. Suppose that the floor rectangle has long edge of length a_n and short edge of length b_n . From the constant area condition, we get

$$a_1 b_1 = n a_n b_n,$$

and from the similarity, we obtain

$$a_n = \frac{a_1}{\sqrt{n}}, \quad b_n = \frac{b_1}{\sqrt{n}}.$$

From the above equations and equation (12), the mean value of distance r in a n -story building is

$$\bar{r}_n = \frac{3n - 1}{6n\sqrt{n}}(a_1 + b_1) + \frac{(n - 1)(n + 1)}{3n}\alpha h, \quad (13)$$

under the condition that the total floor area is equal to $a_1 b_1$ (constant). Computing equation (13), strictly the mean value \bar{r}_n increases slightly with n , but the values are almost the same from the one story building to the six story building in Fig.10.

5. Distance in horizontal space

In the previous chapter, from the one story building to the six story building there are almost no difference in the mean value of the distances. But when the number of stories n

is larger than 6, the mean value \bar{r}_n increases with n and we must use elevators for vertical movement. If we use elevators, the distance distribution is different from equation (11) which is derived in the situation that the vertical movement is done by escalator or by stairs on foot.

As mentioned previously for the mean distance there are almost no differences from one story to six stories. But there are remarkable differences in the distance of horizontal movement. The first term of the right hand side of equation (13) shows the mean distance of horizontal movements and the second term is related to the vertical movements. Let denote that R_n is the ratio of mean distance about horizontal movements in n -story building to that in one story building. From equation (13), we get

$$R_n = \frac{\frac{3n-1}{6n\sqrt{n}}(a_1 + b_1)}{\frac{1}{3}(a_1 + b_1)} = \frac{3n-1}{2n\sqrt{n}}. \quad (14)$$

Substituting $n = 6$ into equation (14), we compute $R_n \approx 0.58$ which means that the mean distance of horizontal movement in a six story building is 58% of that in a one story building. If the number of stories n is large, from (14) we get approximately

$$R_n \sim 3/(2\sqrt{n}), \quad (15)$$

which shows that the ratio R_n is proportional to $1/\sqrt{n}$.

6. Conclusion

If buildings are department stores, the probability of an expected shopping will be proportional to the mean distance of horizontal movements. Formulae (14) and (15) suggest us that the probability decreases with increase of the number of stories n . For example, in a six story department store the probability is 58 percent of the probability in one story department store. Generally in a city man meets many people the number of which will be proportional to the mean distance of horizontal movements. In a high building the number of stories n is large, and the ratio of the mean distance to that in one story is expressed by equation (15). Thus the high buildings lose the function of urbanized area in which man has the possibility to meet many people accidentally.

From the point of the distance distribution, we express the differences between high and low buildings under the condition that the total floor area is constant. In the present paper we consider only escalator and stairs on foot as vertical movement. If the number of stories is large, we have to use elevators in buildings. But in the case of using elevators we must consider waiting times and delays caused by congestions which are difficult to solve as simply as derivation (11).

Therefore these are problems which must be solved in the future.

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