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Abstract—Adomian's decomposition method (ADM) is a nonnumerical method which can be used for solving nonlinear ordinary differential equations. In this paper, first the principle of the decomposition method is described, and its advantages as well as drawbacks are discussed. Then an aftertreatment technique (AT) is proposed, which yields the analytic approximate solution with fast convergence rate and high accuracy by applying Padé approximation to the series solution derived from ADM. Also, some concrete examples are studied to illustrate with numerical results how the AT works efficiently. Finally, the general remarks conclude this study.

Keywords—Adomian's decomposition method, Aftertreatment technique, Ordinary differential equations, Padé approximant, Mathematica

1. Introduction

Mathematical modelling of many frontier physical systems leads to nonlinear ordinary differential equations, e.g. Duffing equation. An effective method is required to analyze

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the mathematical model which provides solutions conforming to physical reality, i.e. the real world of physics. Therefore we must be able to solve nonlinear ordinary differential equations, in space and time, which may be strongly nonlinear. Usual analytic procedures linearize the system or assume nonlinearities are relatively insignificant. Such procedures change the actual problem to make it tractable by the conventional methods. In short the physical problem is transformed to a purely mathematical one for which the solution is readily available. This changes, sometimes seriously, the solution. Generally, the numerical methods such as Runge Kutta method are based on the discretization techniques, and they only permit us to calculate the approximated solutions for some values of time and space variables, which cause us to overlook some important phenomena such as chaos and bifurcations, because generally nonlinear dynamic systems exhibit some delicate structures in very small time amd space intervals. Also, the numerical methods require computer-intensive calculations. The ability to solve nonlinear equations by an analytic method is important because linearization changes the problem being analyzed to a different problem, perturbation is only reasonable when nonlinear effects are very small, and the numerical methods need substantial amount of computation but only get limited informations.

Since the beginning of the 80's, G. Adomian [1-5] has presented and developed a so-called decomposition method for solving linear or nonlinear problems such as ordinary differential equations. Adomian's decomposition method consists of splitting the given equation into linear and nonlinear parts, inverting the highest order derivative operator contained in the linear operator on both sides, identifying the initial and/or boundary conditions and the terms involving the independent variable alone as initial approximation, decomposing the unknown function into a series whose components are to be determined, decomposing the nonlinear function in terms of special polynomials called Adomian's polynomials, and finding the successive terms of the series solution by recurrent relation using Adomian's polynomials. ADM is quantitative rather than qualitative, analytic, requiring neither linearization nor perturbation, and continuous with no resort to discretization and consequent computer-intensive calculations. Some applications [6,7] of the method show its advantages.

However, ADM has some drawbacks. By using ADM, we get a series solution, in practice a truncated series solution. The series often coincides with the Taylor expansion of the true solution at point $x = 0$, in the initial value case. Although the series can be rapidly convergent in a very small region, it has very slow convergence rate in the wider region we concern, and the truncated series solution is a inaccurate solution in that region, which will greatly restrict the application area of the method. Many examples given in [8,9] can be used to support this assertion.

We have proposed an extension of ADM [9], which can improve the convergence rate of the series solution. Because the series solution obtained from the generalized decomposition method is still a Maclaurin series, the method also has the limited accuracy, although it is superior to the ADM. The limitation of the methods available motivated this work.

Venkatarangan and Rajalakshmi [10] presented an alternative technique, which modifies Adomian's series solution and makes it periodic for nonlinear oscillatory systems. They used Laplace transform and Padé approximant to deal with the truncated series. Some examples show that their method yields a more convenient form of the solution compare to the Adomian's series solution for a class of nonlinear oscillatory problems. But their method usually does not work for general ordinary differential equations, because getting the inverse Laplace transform of the complex Padé approximant is not easy, and often fails. In this paper, we will explain why their method works.

Padé approximant [11,12] approximates a function by the ratio of two polynomials. The coefficients of the powers occuring in the polynomials are determined by the coefficients in the Taylor series expansion of the function. Generally the Padé approximant can extend the convergence domain of the truncated Taylor series, and can improve greatly the convergence rate of the truncated Maclaurin series.

In order to improve the accuracy of ADM, we propose a so-called aftertreatment technique (AT) to modify Adomian's series solution for general ordinary differential equations with initial conditions by using the Padé approximant. Generally ADM yields the Taylor series of the true solution. By using the AT, we get the true solution in some cases. Usually the AT can be used to get an analytic approximate solution which will greatly improve the convergence rate and accuracy of Adomian's series. For the oscillatory systems, we use Laplace transformation and Padé approximant, and explain in which cases the AT leads to the true solution and why the technique works. Also, eleven examples are studied carefully, and the numerical results show that the AT enjoys the high precision and is superior to the original ADM and the generalized decomposition method [9]. Finally, the general remarks are given.

2. The Principle of ADM

Consider the equation

$$Fy(x) = g(x), \tag{1}$$

where F represents a general nonlinear ordinary diferential operator involving both linear and nonlinear parts, and $g(x)$ is a given function. The linear terms in Fy are decomposed into $Ly + Ry$, where L is an easily invertible operator, which is taken as the highest order derivative generally for avoiding the difficult integrations when complicated Green's functions are involved, and R is the remainder of the linear operator. Thus the equation (1) is written as

$$Ly + Ry + Ny = g, \tag{2}$$

where Ny represents the nonlinear terms in Fy. Solving for Ly,

$$Ly = g - Ry - Ny. \tag{3}$$

Because L is invertible, operating with its inverse L^{-1} yields

$$L^{-1}Ly = g - L^{-1}Ry - L^{-1}Ny. \tag{4}$$

An equivalent expression is

$$y = \Phi + g - L^{-1}Ry - L^{-1}Ny, \tag{5}$$

where Φ is the integration constant and satisfies $L\Phi = 0$. If this corresponds to an initial-value problem, the operator L^{-1} may be regarded as definite integrations from 0 to x. If

L is a second-order operator, L^{-1} is a two-fold integration, and $\Phi = y(0) + y'(0)x$. Due to Adomian [1,4], the solution y is represented as the infinite sum of series

$$y = \sum_{n=0}^{\infty} y_n, \tag{6}$$

and the nonlinear term Ny, assumed to be an analytic function $f(x)$, is decomposed as follows

$$Ny = f(x) = \sum_{n=0}^{\infty} A_n, \tag{7}$$

where the A_n's are Adomian's polynomials of $y_0, y_1, \ldots, y_n$ and are calculated by the formula

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n} f(\sum_{i=0}^{\infty} \lambda^i y_i)|_{\lambda=0}, n = 0, 1, 2, \ldots \tag{8}$$

Putting (6) and (7) into (5) gives

$$\sum_{n=0}^{\infty} y_n = \Phi + g - L^{-1}R\sum_{n=0}^{\infty} y_n - L^{-1}\sum_{n=0}^{\infty} A_n. \tag{9}$$

Each term of the series (6) is given by the recurrent relation

$$\begin{cases} y_0 = \Phi + g, \\ y_n = -L^{-1}Ry_{n-1} - L^{-1}A_{n-1}, n \geq 1. \end{cases} \tag{10}$$

However, in practice all the terms of the series (6) cannot be determined, and the solution will be approximated by a truncated series $\sum_{n=0}^{N} y_n$.

By using ADM described above, we obtain series solutions for ordinary differential equations. The method reduces significantly the massive computation which may arise in the use of discretization methods for the solution of nonlinear problems. Neither linearization nor perturbation is required. Although the series solutions converge rapidly only in a small region, in the wide region we concern, they have very slow convergence rates, and then their truncations yield inaccurate results. Some examples discussed in [8,9] can be used to support this observation. Here we take the example of Duffing equation given in [5].

Consider the Duffing equation

$$\frac{d^2y}{dx^2} + 3y - 2y^3 = g(x) = \cos x \sin 2x \tag{11}$$

with initial conditions

$$y(0) = 0, y'(0) = 1. \tag{12}$$

The analytic solution of this equation is

$$y^*(x) = \sin x.$$

The Taylor expansion of $g(x)$ at point $x_0 = 0$ is represented as $g(x) = \sum_{n=0}^{\infty} g_n x^n$. By the same way as given in [5], we use an approximation of each term in g up to order x^3, which provides an approximation to y of order x^5. In this case,

$$\cos x = 1 - \frac{x^2}{2},$$
$$\sin 2x = 2x - \frac{8x^3}{3!}.$$

The equation (11) is expressed as

$$Ly = g - 3y + 2y^3.$$

Let $y = \sum_{n=0}^{\infty} y_n$ and $y^3 = \sum_{n=0}^{\infty} A_n$, where the A_n's are Adomian's polynomials for this nonlinearity, and identifying $y_0 = y(0) + xy'(0) + L^{-1}g$, we find

$$y_0 = x + \frac{x^3}{3},$$
$$y_1 = -3L^{-1}y_0 + 2L^{-1}y_0^3 = -\frac{x^3}{2}$$

to order x^3, thus the two-term approximation to y is given by

$$\phi_2(x) = x - \frac{x^3}{3!}.$$

Also, the three-term approximation to y is

$$\phi_3(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}.$$

If an approximation of terms in g to higher order is adopted, then the higher-order-term approximations to y can be obtained. The error function of the truncated series $\phi_3(x)$ for the solution $\sin x$ is denoted by

$$E_3(x) = \sin x - \phi_3(x),$$

which is a strictly decreasing function for $x \geq 0$, and $E_3(0) = 0$, $E_3(0.5) = -1.5447\times10^{-6}$, $E_3(1) = -0.0001957$, $E_3(1.5) = -0.003286$, $E_3(2) = -0.02404$, $E_3(2.5) = -0.1112$, $E_3(3) = -0.3839$, $E_3(3.5) = -1.0818$, $E_3(4) = -2.6235$, $E_3(4.5) = -5.6674$, $E_3(5) = -11.1673$. These results show that $\phi_3(x)$ converges rapidly when $x \in [0, 1.5]$. In the region $(1.5, 2.5]$, $\phi_3(x)$ yields reasonable solution. When $x \geq 3.5$, $\phi_3(x)$ leads to a wrong solution. That means $\phi_3(x)$ converges in a small region but yields a wrong solution in a wider region. All the truncated series solutions have the same problem.

3. The Aftertreatment Technique

If we expand the excitation term g in (1) into Taylor series at point $x_0 = 0$, ADM leads to the Maclaurin series solution, which is equal to a generalized Taylor series about function $y_0(x)$ rather than a point, as claimed in [5,13]. Generally, the truncations of the series solution are the partial sums of the Taylor expansion series of the true solution function at point $x_0 = 0$. For the differential equation in the form

$$\frac{dy}{dx} = f(y) + g, \tag{13}$$

$$y(0) = c_0, \tag{14}$$

where f is the nonlinear term, g is given, and c_0 is a constant, Abbaoui and Cherruault [14] observed the following.

THEOREM. *In the differential system (13)(14), we suppose that $f(y)$ is infinitely differentiable and that g is expandable in Taylor series in the neighborhood of $x_0 = 0$, the series solution $\sum_{n=0}^{\infty} y_n$ of (13) (14) given by the recurrent scheme*

$$\begin{cases} y_0 = c_0, \\ y_{n+1} = L^{-1}A_n + L^{-1}(\alpha_n x^n), \alpha_n = \frac{g^{(n)}(0)}{n!}, n\geq 0 \end{cases} \tag{15}$$

is the Taylor series of its ture solution at point $x_0 = 0$, where the A_n's are calculated by formula (8).

Adomian's recurrent scheme for (13)(14) can be expressed as

$$\begin{cases} y_0 = c_0 + L^{-1}g, \\ y_{n+1} = L^{-1}A_n, n\geq 0. \end{cases} \tag{16}$$

The two schemes (15)(16) in general give different series, but they are identical if $g = 0$. We will use the scheme (15) to serve our purpose.

A Padé approximant [11,12] is the ratio of two polynomials constructed from the coefficients of the Taylor series expansion of a function. The $[L/M]$ Padé approximant to a formal power series $B(x) = \sum_{j=0}^{\infty} b_j x^j$ is given by

$$[L/M] = \frac{P_L(x)}{Q_M(x)},$$

where $P_L(x)$ is a polynomial of degree at most L and $Q_M(x)$ is a polynomial of degree at most M. Without loss of generality, assume $Q_M(0) = 1$. Further, P_L and Q_M have no common factors. This means that the formal power series $B(x)$ equals the $[L/M]$ approximant through $L + M + 1$ terms. In this case, by using the conclusion given in Theorem 1.4.3 [12], we know that the Padé approximant $[L/M]$ is uniquely determined.

Suppose $f(x)$ is the ratio of two polynomials

$$f(x) = \frac{p(x)}{q(x)}, \tag{17}$$

where $p(x) = p_0 + p_1 x + \cdots + p_L x^L, q(x) = 1 + q_1 x + \cdots + q_M x^M$, $p(x)$ and $q(x)$ have no common factors, and the truncated sum $\sum_{i=0}^{K} a_i x^i$ of the Taylor expansion $f(x) = \sum_{i=0}^{\infty} a_i x^i$ is given. Let us denote

$$F_K(x) = \sum_{i=0}^{K} a_i x^i, \tag{18}$$

then clearly

$$\begin{aligned} f(x) &= \sum_{i=0}^{\infty} a_i x^i = \sum_{i=0}^{K} a_i x^i + \sum_{i=K+1}^{\infty} a_i x^i \\ &= F_K(x) + \sum_{i=K+1}^{\infty} a_i x^i \\ &= F_K(x) + O(x^{K+1}). \end{aligned}$$

If we recall that $f(x) = p(x)/q(x)$, this implies that

$$\frac{p(x)}{q(x)} = F_K(x) + O(x^{K+1}),$$

that is

$$F_K(x) = \frac{p(x)}{q(x)} + O(x^{K+1}). \tag{19}$$

If $K+1 \geq L+M+1$, i.e. $K \geq L+M$, (19) is the definition that $f(x) = p(x)/q(x)$ is a Padé approximant of $F_K(x)$. Because $g(0) = 1 \neq 0$, the Padé approximant is unique for given L and M. So, (19) means that for a function equals to the ratio of two polynomials such as (17), the Padé approximant of its truncated Taylor series $F_K(x)$, which is uniquely determined for given L and M, gives the original function $p(x)/q(x) = f(x)$ when $K \geq L+M$.

Suppose ADM yields a truncated Taylor series of the true solution with enough terms, and the solution can be written as the ratio of two polynomials with no common factors. Then the above argument shows the Padé approximant for the truncated series provides the true solution.

Even when the solution cannot be expressed as the ratio of two polynomials, the Padé approximant for the truncated series given by ADM yields a good approximation to the true solution, which usually improves greatly the truncated series in the convergence rate and the accuracy.

When ADM yields a truncated Maclaurin series, which cannot be expressed as the partial sum of the Taylor series of the true solution, the Padé approximant can be used. It yields an approximation to the true solution, which generally has faster convergence rate and higher accuracy than the truncated series has.

For the oscillatory systems of form

$$\frac{d^2y}{dx^2} + \omega^2 y = f(y, \frac{dy}{dx}), \tag{20}$$

where ω is a constant, and f is a linear or nonlinear function, their solutions usually can be written as or can be approximated by the algebraic combination of $\sin x, \cos x, e^x$, polynomials and other functions. Let $T(s) = \mathcal{L}[f(x)]$ stands for the Laplace transformation of function $f(x)$, then we have

$$\begin{aligned}
\mathcal{L}[x^n] &= \frac{n!}{s^{n+1}}, n = 0, 1, 2, \ldots \\
\mathcal{L}[x^n e^{\alpha x}] &= \frac{n!}{(s-\alpha)^{n+1}}, n = 0, 1, 2, \ldots \\
\mathcal{L}[\sin(\alpha x)] &= \frac{\alpha}{s^2+\alpha^2}, \\
\mathcal{L}[\cos(\alpha x)] &= \frac{s}{s^2+\alpha^2},
\end{aligned}$$

$$\mathcal{L}[e^{\alpha x}\sin(\beta x)] = \frac{\beta}{(s-\alpha)^2+\beta^2},$$
$$\mathcal{L}[e^{\alpha x}\cos(\beta x)] = \frac{s-\alpha}{(s-\alpha)^2+\beta^2},$$

where both α and β are constants. Thus Laplace transformation of algebraic combinations of $\sin x, \cos x, e^x$ and polynomial functions can be written as the ratio of two polynomials. Therefor, for many oscillatory systems such as Duffing equation, we apply Laplace transformation to the truncated series obtained by ADM, then convert the transformed series into a meromorphic function by forming its Padé approximant, and finally adopt inverse Laplace transformation to get an analytic solution, which may be the true solution or a better approximate solution than Adomian's truncated series solution, owing to the advantages of the Padé approximant described above. The obtained analytic solution may be periodic, however Adomian's truncated series does not exhibit periodicity. That is why the modification of ADM given in [10] works. Surely, for some osillatory systems, Adomian's truncated series is not the partial sum of the Taylor series of the true solution at point $x_0 = 0$, and it is very difficult to calculate the inverse Laplace transformation of the meromorphic function. In this case, generally part of Adomian's truncated series is the partial sum of the Taylor series of the true solution (see Examples 6 and 10 of Section 4), and the lower order Padé approximant is used to get the ture solution or an approximate analytic solution which improves the accuracy of ADM.

4. Examples

Here we demonstrate how the AT works with eleven numerical examples. All the results are calculated by using the symbolic calculus software Mathematica.

EXAMPLE 1. Consider the equation

$$\frac{dy}{dx} = y^2 \tag{21}$$

with the initial condition

$$y(0) = 1.$$

The analytic solution of this equation is

$$y^*(x) = \frac{1}{1-x}, 0 \leq x < 1.$$

We solve this equation by ADM. Writing $y = \sum_{n=0}^{\infty} y_n$ and $y^2 = \sum_{n=0}^{\infty} A_n\{y^2\}$, we express the recurrent scheme of ADM as

$$\begin{cases} y_0 = 1, \\ y_{n+1} = \int_0^x A_n dx, n \geq 0. \end{cases} \tag{22}$$

The A_n's are calculated by the formula (8), so the partial sum $\phi_n = \sum_{m=0}^{n} y_m$ can be determined by (22). Simple calculations lead to

$$\phi_n = 1 + x + x^2 + \cdots + x^n, n \geq 0,$$

which is the partial sum of the Taylor series of the solution $y^*(x)$ at point $x_0 = 0$. We use the Padé approximant to handle ϕ_n. By using Mathematica, we see that the $[L/M]$ Padé approximant of the series ϕ_n with

$$L = \begin{cases} \frac{n}{2}, & n \text{ is even}, \\ \frac{n-1}{2}, & n \text{ is odd}, \end{cases}$$

$$M = \begin{cases} \frac{n}{2}, & n \text{ is even}, \\ \frac{n+1}{2}, & n \text{ is odd}, \end{cases}$$

leads to the true solution $y^*(x)$ when $n \geq 2$.

EXAMPLE 2. Consider the equation

$$\frac{d^2y}{dx^2} + y = 0 \tag{23}$$

with initial conditions

$$y(0) = 0, y'(0) = 1.$$

The analytic solution of this equation is

$$y^*(x) = \sin x.$$

Writing $y = \sum_{n=0}^{\infty} y_n$, the recurrent scheme of ADM can be expressed as

$$\begin{cases} y_0 = x, \\ y_{n+1} = -L^{-1} y_n, n \geq 0, \end{cases} \tag{24}$$

where L^{-1} stands for the two-fold definite integration from 0 to x. From (24), we get the partial sum $\phi_n = \sum_{m=0}^{n} y_m$

$$\phi_n = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^{n-1}\frac{x^{2n+1}}{(2n+1)!}, n \geq 0,$$

which is the partial sum of the Taylor series of the solution $y^\star(x)$ at point $x_0 = 0$. Because (23) is an oscillatory system, here we apply Laplace transformation to ϕ_n, which yields

$$\mathcal{L}[\phi_n] = \frac{1}{s^2} - \frac{1}{s^4} + \frac{1}{s^6} - \cdots + (-1)^n \frac{1}{s^{2n+2}}, n \geq 0.$$

For the sake of simplicity, let $s = 1/t$, then

$$\mathcal{L}[\phi_n] = t^2 - t^4 + t^6 - \cdots + (-1)^n t^{2n+2}, n \geq 0.$$

Its $[n+1/n+1]$ Padé approximant with $n \geq 1$ yields

$$[n+1/n+1] = \frac{t^2}{1+t^2}.$$

Recalling $t = 1/s$, we obtain $[n+1/n+1]$ in terms of s:

$$[n+1/n+1] = \frac{1}{1+s^2}.$$

By using the inverse Laplace transformation to $[n+1/n+1]$, we obtain the true solution $y^\star(x)$.

EXAMPLE 3. Consider the equation

$$\frac{dy}{dx} = e^{-y} \tag{25}$$

with the initial condition

$$y(0) = 0.$$

The analytic solution of this equation is

$$y^\star(x) = \ln(1+x), x > -1.$$

Writing $y = \sum_{n=0}^{\infty} y_n$ and $e^{-y} = \sum_{n=0}^{\infty} A_n\{e^{-y}\}$, the recurrent scheme of ADM is

$$\begin{cases} y_0 = 0, \\ y_{n+1} = \int_0^x A_n dx, n \geq 0. \end{cases} \tag{26}$$

The A_n's are calculated by the formula (8), so the partial sum $\phi_n = \sum_{m=0}^{n} y_m$ can be determined by (26) as

$$\begin{cases} \phi_0 = 0, \\ \phi_n = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n-1}\frac{x^n}{n}, n\geq 1, \end{cases} \tag{27}$$

which is the partial sum of the Taylor series of the solution $y^*(x)$ at point $x_0 = 0$. By using Mathematica, we find the [5/5] Padé approximant of the truncated series ϕ_{10}

$$[5/5] = \frac{x + 2x^2 + 47/36x^3 + 11/36x^4 + 137/7560x^5}{1 + 5/2x + 20/9x^2 + 5/6x^3 + 5/42x^4 + 1/252x^5}.$$

Let

$$E_1(x) = \ln(1+x) - \phi_{10}(x), E_2(x) = \ln(1+x) - [5/5](x), x > -1,$$

which stands for the error functions determined by ADM and the AT, respectively. The error curves obtained by ADM and the AT are shown in Figure 1 and Figure 2, respectively. Also, $E_2(200) = 1.01016$, $E_2(500) = 1.76545$, $E_2(1000) = 2.40094, E_2(10000) = 4.64976$. From these results, we can see that the AT improves greatly Adomian's truncated series in the convergence rate and the accuracy.

Discussion. We hope to make use of the special structures of the Adomian's series. From (27), we have

$$\phi_n'(x) = 1 - x + x^2 - \cdots + (-1)^{n-1}x^{n-1}, n\geq 1. \tag{28}$$

Then by using Mathematica, we see that the $[L/M]$ Padé approximant of the series $\phi_n'(x)$ with

$$L = \begin{cases} \frac{n-1}{2}, & n \text{ is odd}, \\ \frac{n}{2} - 1, & n \text{ is even}, \end{cases}$$

$$M = \begin{cases} \frac{n-1}{2}, & n \text{ is odd}, \\ \frac{n}{2}, & n \text{ is even}, \end{cases}$$

and $n \geq 3$ is

$$[L/M] = \frac{1}{1+x}.$$

By solving a simple equation

$$\begin{cases} \frac{dy}{dx} = \frac{1}{1+x}, \\ y(0) = 0, \end{cases}$$

we obtain the true solution $y^\star(x)$ of the original equation.

EXAMPLE 4. Consider the equation

$$\frac{d^2y}{dx^2} - y = 0 \tag{29}$$

with initial conditions

$$y(0) = 1, y'(0) = 1.$$

The analytic solution of this equation is

$$y^\star(x) = e^x.$$

Writing $y = \sum_{n=0}^{\infty} y_n$, the recurrent scheme of ADM is

$$\begin{cases} y_0 = 1 + x, \\ y_{n+1} = -L^{-1} y_n, n \geq 0, \end{cases} \tag{30}$$

where L^{-1} stands for the two-fold definite integration from 0 to x. From (30), we have the partial sum $\phi_n = \sum_{m=0}^{n} y_m$

$$\phi_n = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} - \cdots + \frac{x^{2n}}{(2n)!} + \frac{x^{2n+1}}{(2n+1)!}, n \geq 0,$$

which is the partial sums of the Taylor series of the solution $y^\star(x)$ at point $x_0 = 0$. Because equation (23) is an oscillatory system, here we apply Laplace transformation to ϕ_n, which yields

$$\mathcal{L}[\phi_n] = \frac{1}{s} + \frac{1}{s^2} + \frac{1}{s^3} + \frac{1}{s^4} + \cdots + \frac{1}{s^{2n+1}} + \frac{1}{s^{2n+2}}, n \geq 0.$$

For simplicity, let $s = 1/t$, then

$$\mathcal{L}[\phi_n] = t + t^2 + t^3 + t^4 + \cdots + t^{2n+1} + t^{2n+2}, n \geq 0. \tag{31}$$

The $[n+1/n+1]$ Padé approximant of (31) with $n \geq 2$ yields

$$[n+1/n+1] = \frac{t}{1-t}.$$

Recalling $t = 1/s$, we obtain $[n+1/n+1]$ in terms of s:

$$[n+1/n+1] = \frac{1}{s-1}.$$

By using the inverse Laplace transformation to $[n+1/n+1]$, we obtain the true solution $y^\star(x)$.

EXAMPLE 5. Consider the equation

$$\frac{dy}{dx} + y = e^x \tag{32}$$

with the initial condition

$$y(0) = 1.$$

The analytic solution of this equation is

$$y^\star(x) = \cosh x.$$

In order to get the Taylor series of the true solution, we apply Theorem. Writing $y = \sum\limits_{n=0}^{\infty} y_n$ and $e^x = \sum\limits_{n=0}^{\infty} \frac{x^n}{n!}$, we have the recurrent relation

$$\begin{cases} y_0 = 1, \\ y_{n+1} = \int_0^x [x^n/n! - y_n]dx, n \geq 0, \end{cases} \tag{33}$$

and the partial sum $\phi_n = \sum\limits_{m=0}^{n} y_m$ is given by (33)

$$\phi_{2n} = \phi_{2n+1} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + \frac{x^{2n}}{(2n)!}, n \geq 0,$$

which is the partial sum of the Taylor series of the solution $y^\star(x)$ at point $x_0 = 0$. We make use of the special structures of the ϕ_n. Here we apply Laplace transformation to ϕ_{2n}, which leads to

$$\mathcal{L}[\phi_{2n}] = \frac{1}{s} + \frac{1}{s^3} + \frac{1}{s^5} + \cdots + \frac{1}{s^{2n+1}}, n \geq 0.$$

Let $s = 1/t$, then

$$\mathcal{L}[\phi_{2n}] = t + t^3 + t^5 + \cdots + t^{2n+1}, n \geq 0. \tag{34}$$

The $[n/n+1]$ Padé approximant of (34) with $n \geq 1$ yields

$$[n/n+1] = \frac{t}{1-t^2}.$$

Recalling $t = 1/s$, we obtain $[n/n+1]$ in terms of s:

$$[n/n+1] = \frac{s}{s^2-1}.$$

By using the inverse Laplace transformation to $[n/n+1]$, we obtain the true solution $y^\star(x)$.

EXAMPLE 6. Consider the Duffing equation (11) with initial conditions (12). The analytic solution of this equation is

$$y^\star(x) = \sin x.$$

Since the complicated excitation term $g(x)$ can cause difficult integrations and proliferation of terms, we can express $g(x)$ in Taylor series at point $x_0 = 0$, which is truncated for simplification. Suppose we replace $g(x)$ by

$$\widetilde{g}(x) = 2x - \frac{7}{3}x^3 + \frac{61}{60}x^5 - \frac{547}{2520}x^7, \tag{35}$$

then the equation (11) becomes

$$\frac{d^2y}{dx^2} + 3y - 2y^3 = \widetilde{g}(x) \tag{36}$$

with initial conditions (12). Let $L = \frac{d^2}{dx^2}$, then equation (36) becomes

$$Ly = \widetilde{g} + 2y^3 - 3y.$$

Writing $y = \sum\limits_{n=0}^{\infty} y_n$ and $y^3 = \sum\limits_{n=0}^{\infty} A_n\{y^3\}$, the recurrent scheme of ADM is written as

$$\begin{cases} y_0 = x + L^{-1}\widetilde{g}, \\ y_{n+1} = 2L^{-1}A_n - 3L^{-1}y_n, n \geq 0, \end{cases} \tag{37}$$

where L^{-1} stands for the two-fold definite integration from 0 to x. The A_n's are calculated by the formula (8), so the partial sum $\phi_n = \sum\limits_{m=0}^{n} y_m$ is determined by (37). The calculation of $\phi_n(n \geq 1)$ becomes complex rapidly. By using Mathematica software, ϕ_5 is calculated.

Because of the truncation of the excitation term $g(x)$, we get a truncated series $\widetilde{\phi}_5(x)$ to order x^9

$$\widetilde{\phi}_5(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!},$$

which coincides with the first five terms in $\phi_5(x)$, and is a partial sum of the Taylor series of the solution $y^\star(x)$ at point $x_0 = 0$. Because (11)(12) is an oscillatory system, we apply Laplace transformation to $\widetilde{\phi}_5(x)$, which yields

$$\mathcal{L}[\widetilde{\phi}_5(x)] = \frac{1}{s^2} - \frac{1}{s^4} + \frac{1}{s^6} - \frac{1}{s^8} + \frac{1}{s^{10}}.$$

For simplicity, let $s = 1/t$, then

$$\mathcal{L}[\widetilde{\phi}_5(x)] = t^2 - t^4 + t^6 - t^8 + t^{10}. \tag{38}$$

All the $[L/M]$ Padé approximant of (38) with $L \geq 2$, $M \geq 2$ and $L + M \leq 10$ yields

$$[L/M] = \frac{t^2}{1+t^2}.$$

Recalling $t = 1/s$, we obtain $[L/M]$ in terms of s:

$$[L/M] = \frac{1}{1+s^2}.$$

By using the inverse Laplace transformation to $[L/M]$, we obtain the true solution $y^\star(x)$.

EXAMPLE 7. Consider the equation

$$\frac{dy}{dx} + y - y^2 = 0 \tag{39}$$

with the initial condition

$$y(0) = 2.$$

The analytic solution of this equation is

$$y^\star(x) = \frac{2}{2 - e^x}, x < \ln 2.$$

Writing $y = \sum\limits_{n=0}^{\infty} y_n$ and $y^2 = \sum\limits_{n=0}^{\infty} A_n\{y^2\}$, the recurrent scheme of ADM is written as

$$\begin{cases} y_0 = 2, \\ y_{n+1} = \int_0^x [A_n - y_n] dx, n \geq 0. \end{cases} \tag{40}$$

The A_n's are calculated by the formula (8), so the partial sum $\phi_n = \sum_{m=0}^{n} y_m$ can be determined by (40) as

$$\begin{aligned}
\phi_0 &= 2, \\
\phi_1 &= 2 + 2x, \\
\phi_2 &= 2 + 2x + 3x^2, \\
\phi_3 &= 2 + 2x + 3x^2 + \frac{13}{3}x^3, \\
\phi_4 &= 2 + 2x + 3x^2 + \frac{13}{3}x^3 + \frac{25}{4}x^4, \\
\ldots
\end{aligned}$$

As can be seen, the $\phi_n(x)$'s determined by ADM are the partial sums of the Taylor series of the solution $y^\star(x)$ at point $x_0 = 0$. By using Mathematica, we get $\phi_{21}(x)$ from (40) and its [10/10] Padé approximant. Let

$$E_1(x) = \frac{2}{2 - e^x} - \phi_{21}(x), E_2(x) = \frac{2}{2 - e^x} - [10/11](x), x < \ln 2,$$

which stands for the error functions determined by ADM and the AT, respectively. The error curves obtained by ADM and the AT are shown in Figure 3 and Figure 4, respectively. Also, $E_2(-200) = 0.732312$, $E_2(-500) = 0.933523$, $E_2(-1000) = 0.97277$, $E_2(-10000) = 0.997751$. From these results, we can see that the AT leads to accurate results in wide region and that it improves greatly Adomian's series in the convergence rate.

EXAMPLE 8. Consider the equation

$$\frac{dy}{dx} = e^{-y} \tag{41}$$

with the initial condition

$$y(0) = 1.$$

The analytic solution of this equation is

$$y^\star(x) = 1 - \ln(1 - ex), x < \frac{1}{e}.$$

Writing $y = \sum_{n=0}^{\infty} y_n$ and $e^y = \sum_{n=0}^{\infty} A_n\{e^y\}$, the recurrent scheme of ADM is

$$\begin{cases} y_0 = 1, \\ y_{n+1} = \int_0^x A_n dx, n \geq 0. \end{cases} \tag{42}$$

The A_n's are calculated by the formula (8), so the partial sum $\phi_n = \sum_{m=0}^{n} y_m$ can be determined by (42) as

$$\begin{cases} \phi_0 = 1, \\ \phi_n = 1 + ex + \frac{(ex)^2}{2} + \cdots + \frac{(ex)^n}{n}, n \geq 1, \end{cases} \tag{43}$$

which is the partial sum of the Taylor series of the solution $y^*(x)$ at point $x_0 = 0$. By using Mathematica, we find the [5/5] Padé approximant of the truncated series ϕ_{10}

$$[5/5] = \frac{-7560 + 11340ex - 1680(ex)^2 - 3570(ex)^3 + 1410(ex)^4 - 107(ex)^5}{30 \times [-252 + 630ex - 560(ex)^2 + 210(ex)^3 - 30(ex)^4 + (ex)^5]}.$$

Let

$$E_1(x) = 1 - \ln(1 + ex) - \phi_{10}(x), E_2(x) = 1 - \ln(1 + ex) - [5/5](x), x < \frac{1}{e},$$

which stands for the error functions determined by ADM and the AT, respectively. The error curves obtained by ADM and the AT are shown in Figure 5 and Figure 6, respectively. Also, $E_2(-200) = -1.84004$, $E_2(-500) = -2.6922$, $E_2(-1000) = -3.36337$, $E_2(-10000) = -5.64592$. From these results, we see that the AT improves greatly Adomian's truncated series in the convergence rate and that it yields accurate results in wide range.

Discussion. We could make use of the special structures of the Adomian's series ϕ_n in the following way. From (43), we have

$$\phi'_n(x) = e[1 + ex + (ex)^2 + \cdots + (ex)^{n-1}], n \geq 1.$$

Then by using Mathematica, we see that the $[L/M]$ Padé approximant of the series $\phi'_n(x)$ with

$$L = \begin{cases} \frac{n-1}{2}, & n \text{ is odd}, \\ \frac{n}{2} - 1, & n \text{ is even}, \end{cases}$$

$$M = \begin{cases} \frac{n-1}{2}, & n \text{ is odd,} \\ \frac{n}{2}, & n \text{ is even,} \end{cases}$$

and $n \geq 3$ is

$$[L/M] = \frac{e}{1 - ex}.$$

By solving a simple equation

$$\begin{cases} \frac{dy}{dx} = \frac{e}{1-ex}, \\ y(0) = 1, \end{cases}$$

we obtain the true solution $y^\star(x)$ of the original equation.

EXAMPLE 9. Consider the equation

$$\frac{dy}{dx} + x^{m-1}y^2 = 0 \tag{44}$$

with the initial condition

$$y(0) = m,$$

where m is a given positive integer. The analytic solution of this equation is

$$y^\star(x) = \frac{m}{1 + x^m}.$$

Writing $y = \sum\limits_{n=0}^{\infty} y_n$ and $y^2 = \sum\limits_{n=0}^{\infty} A_n\{y^2\}$, the recurrent scheme of ADM is written as

$$\begin{cases} y_0 = m, \\ y_{n+1} = \int_0^m x^{m-1} A_n dx, n \geq 0. \end{cases} \tag{45}$$

The A_n's are calculated by the formula (8), so the partial sum $\phi_n = \sum\limits_{i=0}^{n} y_i$ is determined by (45) as

$$\phi_n = m[1 - x^m + x^{2m} - \cdots + (-1)^n x^{mn}], n \geq 0,$$

which is a partial sum of the Taylor series of the solution $y^\star(x)$ at point $x_0 = 0$. We use the Padé approximant to handle ϕ_n. By using Mathematica, we see that the $[L/M]$ Padé approximant of the ϕ_n with

$$L = \begin{cases} \frac{mn}{2}, & mn \text{ is even,} \\ \frac{mn-1}{2}, & n \text{ is odd,} \end{cases}$$

$$M = \begin{cases} \frac{mn}{2}, & mn \text{ is even}, \\ \frac{mn+1}{2}, & mn \text{ is odd}, \end{cases}$$

leads to the true solution $y^\star(x)$ when $n \geq 2$.

EXAMPLE 10. Consider the equation

$$\frac{d^2y}{dx^2} + 2y = -2\frac{dy}{dx} \tag{46}$$

with initial conditions

$$y(0) = 0, y'(0) = 1.$$

The analytic solution of this equation is

$$y^\star(x) = e^{-x} \sin x.$$

Writing $y = \sum\limits_{n=0}^{\infty} y_n$, the recurrent scheme of ADM can be expressed as

$$\begin{cases} y_0 = x, \\ y_{n+1} = -L^{-1}(2y_n + 2y_n'), n \geq 0, \end{cases} \tag{47}$$

where L^{-1} stands for the two-fold definite integration from 0 to x. From (47), we have the partial sum $\phi_n = \sum\limits_{m=0}^{n} y_m$. By using Mathematica, ϕ_{19} is calculated. Notice that ϕ_{19} is not the partial sum of the Taylor series of the solution $y^\star(x)$ at point $x_0 = 0$. We analyze this series and see that a truncated series $\widetilde{\phi}_{19}(x)$ to order x^{19}, which is the first fifteen terms in ϕ_{19}, is a partial sum of the Taylor series of the solution $y^\star(x)$ at point $x_0 = 0$. $\widetilde{\phi}_{19}(x)$ is expressed as

$$\begin{aligned} \widetilde{\phi}_{19}(x) &= x - x^2 + \frac{x^3}{3} - \frac{4x^5}{5!} + \frac{8x^6}{6!} - \frac{8x^7}{7!} + \frac{16x^9}{9!} - \frac{32x^{10}}{10!} \\ &+ \frac{32x^{11}}{11!} - \frac{64x^{13}}{13!} + \frac{128x^{14}}{14!} - \frac{128x^{15}}{15!} + \frac{256x^{17}}{17!} - \frac{512x^{18}}{18!} + \frac{512x^{19}}{19!}. \end{aligned}$$

Because equation (46) is an oscillatory system, here we apply Laplace transformation to $\widetilde{\phi}_{19}(x)$ and obtain

$$\begin{aligned} \mathcal{L}[\widetilde{\phi}_{19}(x)] &= \frac{1}{s^2} - \frac{2}{s^3} + \frac{2}{s^4} - \frac{4}{s^6} + \frac{8}{s^7} - \frac{8}{s^8} + \frac{16}{s^{10}} - \frac{32}{s^{11}} \\ &+ \frac{32}{s^{12}} - \frac{64}{s^{14}} + \frac{128}{s^{15}} - \frac{128}{s^{16}} + \frac{256}{s^{18}} - \frac{512}{s^{19}} + \frac{512}{s^{20}}. \end{aligned}$$

For simplicity, let $s = 1/t$, then

$$\begin{aligned}\mathcal{L}[\widetilde{\phi}_{19}(x)] &= t^2 - 2t^3 + 2t^4 - 2t^6 + 8t^7 - 8t^8 + 16t^{10} - 32t^{11} \\ &+ 32t^{12} - 64t^{14} + 128t^{15} - 128t^{16} + 256t^{18} - 512t^{19} + 512t^{20}.\end{aligned}$$

All the $[L/M]$ Padé approximant of the $\widetilde{\phi}_{19}(x)$ with $L \geq 2$, $M \geq 2$ and $L+M \leq 20$ yields

$$[L/M] = \frac{t^2}{1 + 2t + 2t^2}.$$

Substituting $t = 1/s$, we obtain $[L/M]$ in terms of s:

$$[L/M] = \frac{1}{s^2 + 2s + 2}.$$

By using the inverse Laplace transformation to $[L/M]$, we obtain the true solution $y^\star(x)$.

EXAMPLE 11. Consider the equation

$$\frac{dy}{dx} + 2y^2 = \frac{1}{1+x^2} \tag{48}$$

with the initial condition

$$y(0) = 0.$$

The analytic solution of this equation is

$$y^\star(x) = \frac{x}{x^2+1}.$$

In order to get the Taylor series of the true solution, we apply the Theorem. Writing $y = \sum\limits_{n=0}^{\infty} y_n$, $y^2 = \sum\limits_{n=0}^{\infty} A_n\{y^2\}$, and $\frac{1}{x^2+1} = \sum\limits_{n=0}^{\infty}(-1)^n x^{2n}$, we have the recurrent relation

$$\begin{cases} y_0 = 1, \\ y_{n+1} = \int_0^x [-2A_n + (-1)^{\frac{n}{2}} \frac{(-1)^n+1}{2} x^n] dx, n \geq 0. \end{cases} \tag{49}$$

The A_n's are calculated by the formula (8), so the partial sum $\phi_n = \sum\limits_{m=0}^{n} y_m$ is given by (49)

$$\begin{cases} \phi_0 = 0, \\ \phi_{2n-1} = \phi_{2n} = x - x^3 + x^5 - \cdots + (-1)^{n-1} x^{2n-1}, n \geq 1, \end{cases}$$

which is the partial sum of the Taylor series of the solution $y^*(x)$ at point $x_0 = 0$. By using Mathematica, we see that $[n-1/n]$ Padé approximant of ϕ_{2n-1} with $n \geq 2$ yields the true solution $y^*(x)$.

All the numerical results given in this section indicate that the AT improves greatly Adomian's truncated series in the convergence rate, and that it often yields the true analytic solution. They support that the AT is powerful and superior to ADM as well as the generalized decomposition method [9].

5. Conclusions

In this paper, we have presented an aftertreatment technique for ADM. Generally ADM yields the Taylor series of the exact solution. Because the Padé approximant usually improves greatly the Maclaurin series in the convergence region and the convergence rate, the AT leads to a better analytic approximate solution from Adomian's truncated series. For the oscillatory systems, Laplace transformation of Adomian's series solution has some specific properties, so we use Laplace transformation and Padé approximant to obtain an analytic solution and to improve the accuracy of ADM. Eleven examples are studied carefully, and the results obtained indicate that the AT is efficient. It really improves the accuracy of ADM. The reason for the powerful aftertreatment is that the AT makes full of the advantages of the Padé approximant. Also, symbolic calculus software Mathematica makes programming the schemes of ADM and the AT very simple and fast. All the figures are drawn by the same software. The AT is appliable to the system of initial-value ordinary differential equations.

In order to obtain more accurate solutions, we suggest first analyzing Adomian's truncated series carefully, then applying some reasonable operations such as Laplace transformation or derivative to the truncated series with some specific structures and making the Padé approximant more efficient, as shown in Examples 3, 5 and 8 of Section 4. Also, for general ordinary differential equations with initial conditions, we suggest calculating Adomian's series solution as well as the AT solution and comparing them to obtain a more accurate analytic solution. The further study of the AT for solving some well-known nonlinear differential equations such as Duffing equation and for discovering some new phenomena such as chaos as well as bifurcations could be an interesting and promising subject.

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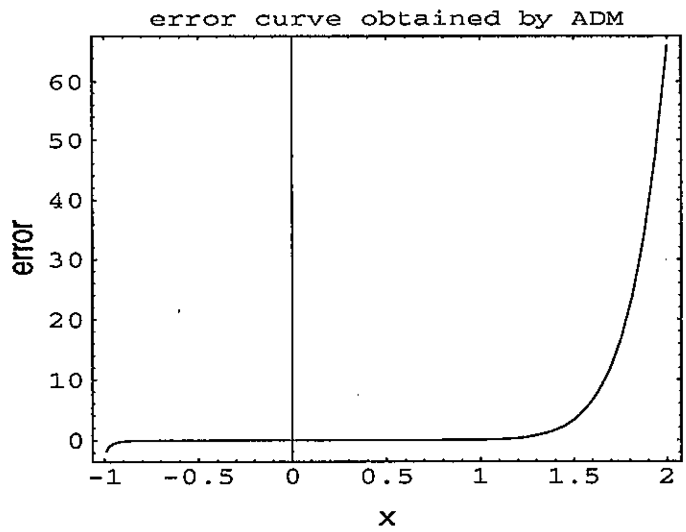

Figure 1: The error function $E_1(x)$ with $E_1(-0.99) = -1.77399$, and $E_1(2) = 65.924$.

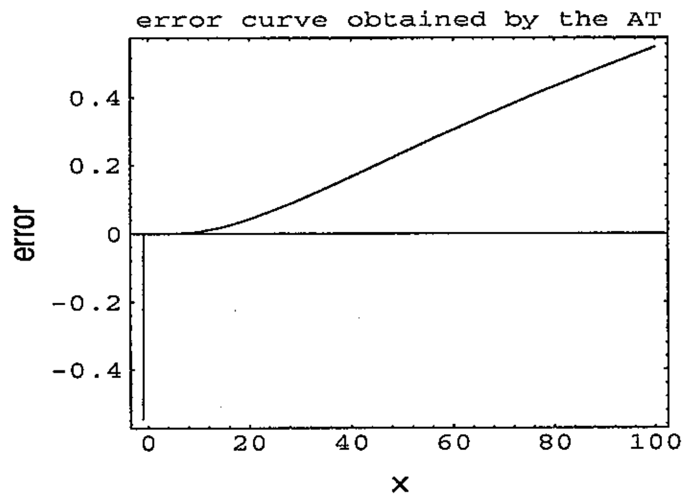

Figure 2: The error function $E_2(x)$ with $E_2(-0.99) = -0.546023$, and $E_2(100) = 0.551705$.

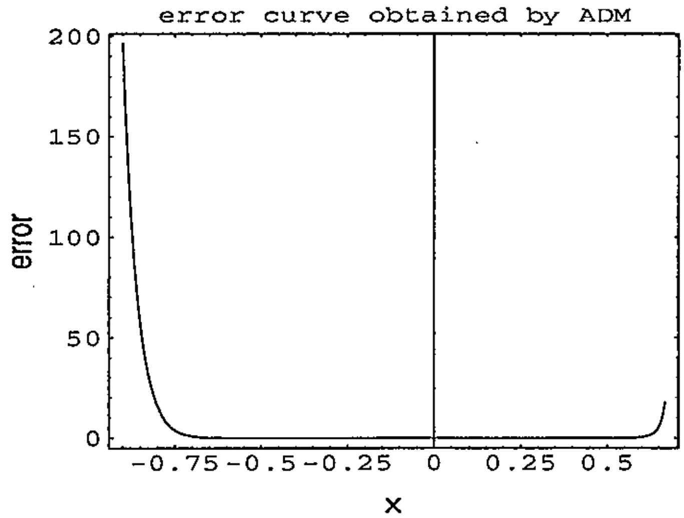

Figure 3: The error function $E_1(x)$ with $E_1(0.668) = 17.6373$, and $E_1(-0.9) = 196.299$.

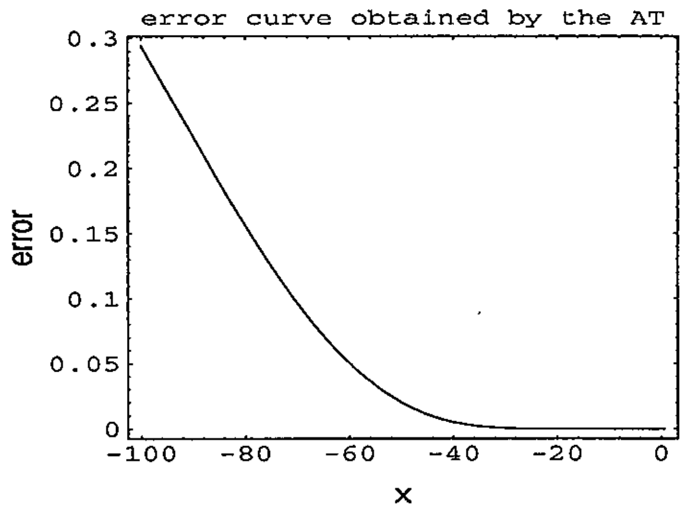

Figure 4: The error function $E_2(x)$ with $E_2(0.69) = -5.68434 \times 10^{-14}$, and $E_2(-100) = 0.294194$.

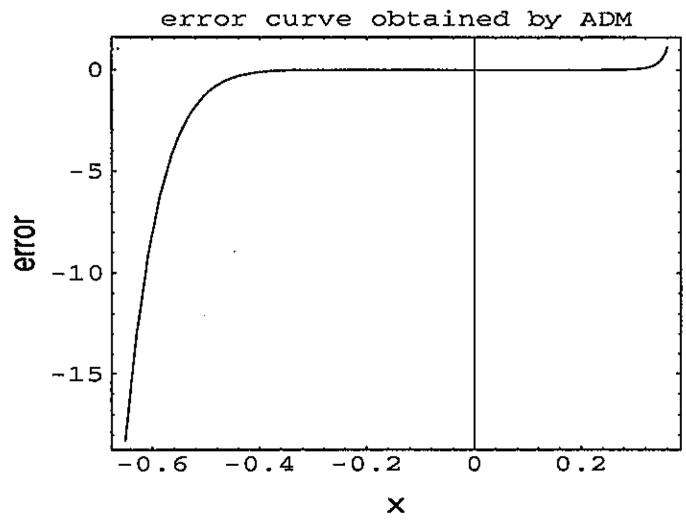

Figure 5: The error function $E_1(x)$ with $E_1(0.36) = 1.11878$, and $E_1(-0.65) = -18.2359$.

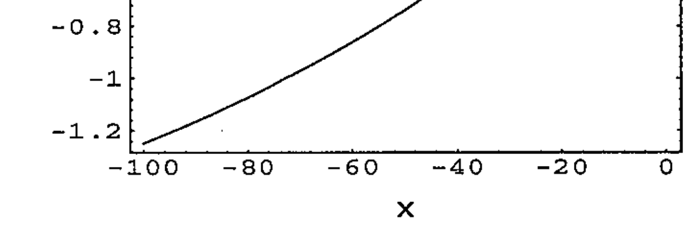

Figure 6: The error function $E_2(x)$ with $E_2(0.36) = 0.20534$, and $E_2(-100) = -1.2482$.