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Optimal Stopping Problem with Controlled Recall

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OPTIMAL STOPPING PROBLEM WITH CONTROLLED RECALL

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In this paper we deal with an optimal stopping problem without recall where an offer once inspected and passed up becomes available if a reserving cost is paid. One of the most distinctive results is that the accepting of the reserved offer should occur only at the end of the process.

1. Introduction

Suppose we have an asset, say a house, a lot of land, or some such item that must be sold by a certain day in the future (deadline). The price offered by each potential buyer will be a random variable with known distribution and some cost must be paid to find a buyer. Now postulate that none of the prices offered by the buyers appearing before the deadline are high enough to be acceptable and the deadline has arrived without the asset being sold. In such a case, if we cannot go back and sell to any of the past buyers, then we will have to sell to any buyer who appears at the deadline, no matter how low the price offered by him may be. In this situation we have left ourselves open to a risk. This type of decision problem is usually called an optimal stopping problem [1-17]. Here consider what should have been the case if we had paid a certain reserving cost to favored buyers. We would have pended the decision as to whether to sell to him or not and at the same time avoided the risky situation.

In this paper, we pose a general model for the class of decision process called the optimal stopping problem with controlled recall, and examine some of the properties of the optimal decision rule.

2. Model

To begin with, let us explain the standard model for the discrete-time optimal stopping problem without recall and with a finite planning horizon. Here, for convenience, let points in time be numbered backward from the deadline as $0, 1, \cdots$, equally spaced, where the interval between two successive points in time, say time t and $t-1$, is called period t.

Paying some fixed cost $s > 0$, called search cost, at any time except on the deadline, we can obtain an offer at the next time. In our discussion from now on, an offer of value w will simply be called offer w. Subsequent offers w, w', $\cdots$ are assumed to be stochastically independent random samples from a known distribution function $F(w)$ with a finite mean μ where, for given numbers a and b such as $0 \leq a < b < \infty$, let $F(w) = 0$ for $w < a$, $0 < F(w) < 1$ for $a \leq w < b$, and $F(w) = 1$ for $b \leq w$. Here let us postulate that one of the offers received during the given planning horizon will have to be accepted. Of course, we can terminate the process at any time by accepting any offer. Once an offer is passed up however, it becomes henceforth unavailable. Furthermore, a per-period discount factor β $(0 < \beta \leq 1)$ should be considered, that is, a monetary value of one unit obtained a period after is regarded as β at the present time.

In addition to these requirements, we here assume that, at any time except on the deadline, if we pay some fixed cost $d > 0$, called reserving cost, for an offer obtained at that time, we can make the offer forever available, that is, it is reserved.

For convenience, we use the two terms "current offer" and "leading offer". The former means an offer appearing at the present time t and the latter is the best of the offers reserved so far. Where it is taken that there are no reserved offers before the start of the process.

Now, let an offer be obtained at a time except on the deadline. We have the following four possible decisions to make: accept the current offer (A), reserve the current offer (R), pass up the current offer and accept the leading offer (PS), or pass up the current offer and continue the search (PC). Here the notations A, R, P, S, and C designate, respectively, Accept, Reserve, Pass up, Stop, and Continue. By definition of the model, available decisions at the deadline are only A and PS.

The objective here is to find the optimal decision rule to maximize the expected present discounted net value, that is, the expectation of the present discounted value of an accepted offer minus that of the total present discounted value of the search costs and reserving costs paid up to the termination of the search with its acceptance.

If $d = 0$, or reserving an offer costs us no money, then our model is virtually reduced to a conventional model with recall. On the other extreme, if $d = \infty$, that is, we must pay a huge amount to reserve an offer. Obviously we will not reserve in this case. This implies that our model is eventually identical to a conventional model without recall[15].

3. Preliminaries

For convenience in later discussion, we shall define the following two functions: For any real number x, let

$$T(x) = \int_a^b \max\{w-x, 0\}dF(w), \tag{3.1}$$

$$K(x) = \beta\int_a^b \max\{w, x\}dF(w)-x-s \;=\; \beta T(x)+(\beta-1)x-s \tag{3.2}$$

where β and s are certain given numbers such as $0 < \beta \leq 1$ and $s > 0$. Let the maximum solution of the equation $K(x) = 0$, if it exists, be denoted by x^o.

Lemma 3.1

(a) *$T(x)$ and $K(x)$ are continuous and nonincreasing in x and strictly decreasing in $x \leq b$.*
(b) *$T(x)+x$ is continuous, convex, and nondecreasing in x and strictly increasing in $x \geq a$.*
(c) *$T(x) = \mu-x$ for $x \leq a$ and $T(x) = 0$ for $x \geq b$.*
(d) *x^o is continuous and nondecreasing in β.*

Proof: See [5] for the proofs of (a) to (c). The assertion (d) is clear from the fact that $K(x)$ is continuous and strictly increasing in β for any x. ∎

4. Functional Equation

Let $u_t(x, w)$ denote the maximum expected present discounted net value starting from time t with a leading offer x and a current offer w, and $v_t(x)$ the expectation of $u_t(x, w)$ in terms of w, that is,

$$v_t(x) = \int_a^b u_t(x, w)dF(w), \qquad t \geq 0. \tag{4.1}$$

We can then express $u_t(x, w)$ as follows:

$$u_t(x, w) = \max\left\{\begin{array}{lcl} \mathrm{A} & : & w, \\ \mathrm{R} & : & -d-s+\beta v_{t-1}(\max\{x, w\}), \\ \mathrm{PS} & : & x, \\ \mathrm{PC} & : & -s+\beta v_{t-1}(x) \end{array}\right\}, \qquad t \geq 1, \tag{4.2}$$

$$u_0(x, w) = \max\left\{\begin{array}{lcl} \mathrm{A} & : & w, \\ \mathrm{PS} & : & x \end{array}\right\}. \tag{4.3}$$

Each of the four expressions in Eq.(4.2) represents the maximum expected present discounted net value for, respectively, decisions A, R, PS, and PC.

The reason why $u_t(x,w)$ can be expressed as Eq.(4.2) is as follows. When the decision R is taken as an example, we must pay the search cost s and the reserving cost d. In addition, because the leading offer the next time $t-1$ becomes $\max\{x,w\}$ due to reserving the offer w at time t, the maximum expected present discounted net value starting from time $t-1$ is expressed as $v_{t-1}(\max\{x,w\})$. Similar reasoning holds for the other expressions.

Lemma 4.1

(a) *$v_t(x)$ is continuous, convex, and nondecreasing in x for any t.*

(b) *$v_t(x)$ is nondecreasing in t for any x.*

Proof: (a) For $t=0$, from Eqs.(4.1) and (4.3), we get

$$v_0(x) = \int_a^b \max\{w,x\}dF(w) = T(x)+x. \tag{4.4}$$

Hence, from Lemma 3.1(b), the statement (a) holds true at $t=0$. Assume (a) holds for t–1. Then clearly it also does for t.

(b) It is clear from Eqs.(4.2) and (4.3) that $u_0(x,w) \le u_1(x,w)$ for any x and any w. Now, assuming that $u_{t-1}(x,w) \le u_t(x,w)$, thus $v_{t-1}(x) \le v_t(x)$ holds for any x and any w, we obtain the inequality below for any x and any w.

$$\begin{aligned} u_t(x,w) &= \max\{w,\ -d-s+\beta v_{t-1}(\max\{x,w\}),\ x,\ -s+\beta v_{t-1}(x)\} \\ &\le \max\{w,\ -d-s+\beta v_t(\max\{x,w\}),\ x,\ -s+\beta v_t(x)\} \\ &= u_{t+1}(x,w), \end{aligned} \tag{4.5}$$

hence, $v_t(x) \le v_{t+1}(x)$ for any x. ∎

Lemma 4.2 *For any $t \ge 1$,*

$$\max\left\{\begin{array}{l} w, \\ -d-s+\beta v_{t-1}(\max\{x,w\}), \\ x, \\ -s+\beta v_{t-1}(x) \end{array}\right\} = \max\left\{\begin{array}{l} w, \\ -d-s+\beta v_{t-1}(w), \\ x, \\ -s+\beta v_{t-1}(x) \end{array}\right\}. \tag{4.6}$$

Proof: Suppose that the left hand side of Eq.(4.6) is equal to w, that is,

$$\max\{-d-s+\beta v_{t-1}(\max\{x,w\}),\ x,\ -s+\beta v_{t-1}(x)\} \le w. \tag{4.7}$$

Noting $v_{t-1}(w) \le v_{t-1}(\max\{x,w\})$ from Lemma 4.1(a), we find from Eq.(4.7) that $\max\{-d-s+\beta v_{t-1}(w),\ x,\ -s+\beta v_{t-1}(x)\} \le w$, hence the right hand side of Eq.(4.6) is equal to w.

Similarly, the assumption that the left hand side of Eq.(4.6) is equal to x or $-s+\beta v_{t-1}(x)$ produces the fact that the right hand side is equal to x or $-s+\beta v_{t-1}(x)$, respectively.

Suppose that the left hand side of Eq.(4.6) is equal to $-d-s+\beta v_{t-1}(\max\{x,w\})$, that is,

$$\max\{w,\ x,\ -s+\beta v_{t-1}(x)\} \le -d-s+\beta v_{t-1}(\max\{x,w\}). \tag{4.8}$$

Here if $x \geq w$, then $-s+\beta v_{t-1}(x) \leq -d-s+\beta v_{t-1}(x)$. This contradicts $d > 0$. Hence $x < w$ must hold, and it follows from Eq.(4.8) that $\max\{w,\ x,\ -s+\beta v_{t-1}(x)\} \leq -d-s+\beta v_{t-1}(w)$. This implies that the both sides of Eq.(4.6) become equal to $-d-s+\beta v_{t-1}(w)$. ∎

According to Lemma 4.2, Eq.(4.2) can be expressed as

$$u_t(x,w) = \max \left\{ \begin{array}{lcl} A & : & w, \\ R & : & -d-s+\beta v_{t-1}(w), \\ PS & : & x, \\ PC & : & -s+\beta v_{t-1}(x) \end{array} \right\}, \qquad t \geq 1. \tag{4.9}$$

Note the following two points: One is that the expression corresponding to R of Eq.(4.9) differs from that of Eq.(4.2), and the other is that, as seen in the proof of Lemma 4.2, decision R becomes by no means optimal when the current offer is inferior to the leading offer.

Lemma 4.3 *For any t,*
(a) $v_t(x) = v_t(a)$ *for* $x \leq a$.
(b) $v_t(x) > x$ *for* $x < b$.
(c) $v_t(x) = x$ *for* $b \leq x$.
(d) $v_t(x) \geq \mu$ *for all* x.

Proof: (a) We have $v_0(x) = v_0(a)$ for $x \leq a$ from Eq.(4.4) and Lemma 3.1(c). Assume $v_{t-1}(x) = v_{t-1}(a)$ for $x \leq a$. Then it holds that $u_t(x,w) = \max\{w,\ -d-s+\beta v_{t-1}(w),\ x,\ -s+\beta v_{t-1}(a)\}$ for $x \leq a$. Hence if $a \leq w \leq b$ and $x \leq a$, then

$$u_t(x,w) = \max\{w,\ -d-s+\beta v_{t-1}(w),\ a,\ -s+\beta v_{t-1}(a)\} = u_t(a,w). \tag{4.10}$$

Consequently we get $v_t(x) = v_t(a)$ for $x \leq a$ and any t.

(b) For $x < b$, from Lemma 3.1(a,c), clearly we get $0 < T(x)$, thus $x < T(x)+x = v_0(x)$ from Eq.(4.4). From this and Lemma 4.1(b), we have $x < v_t(x)$ for any $x < b$ and any t.

(c) Clearly $v_0(x) = x$ for $b \leq x$ from Eq.(4.4) and Lemma 3.1(c). Assume $v_{t-1}(x) = x$ for $b \leq x$. Then for $a \leq w \leq b$, if $x \geq b$, the following three things are all true: (1) $w \leq x$, (2) $v_{t-1}(w) \leq v_{t-1}(x)$, and (3) $-s+\beta v_{t-1}(x) < v_{t-1}(x) = x$, from which we have $u_t(w,x) = x$, thus $v_t(x) = x$.

(d) The fact that $u_t(x,w) \geq w$ from Eq.(4.9) becomes clear. ∎

Lemma 4.4 *For any t, $\beta v_t(x)-x$ is nonincreasing in x and strictly decreasing in $x \leq b$.*

Proof: As a preliminary to the proof, let us prove that $v_t(x)-x$ is nonincreasing in x and strictly decreasing in $x \leq b$. Owing to Lemma 4.3(a), $v_t(x)-x$ is strictly decreasing in $x \leq a$. In order to prove that $v_t(x)-x$ is strictly decreasing in x on $[a,b]$, it suffices to show that the inequality $v_t(x_2) - v_t(x_1) < x_2 - x_1$ holds for any x_1 and any x_2 such that $a \leq x_1 < x_2 \leq b$. Since $v_t(x)$ is convex in x, we obtain

$$\frac{v_t(x_2) - v_t(x_1)}{x_2 - x_1} \leq \frac{v_t(b) - v_t(x_1)}{b - x_1}. \tag{4.11}$$

Noting that $x_1 < v_t(x_1)$ and $v_t(b) = b$ from Lemma 4.3(b,c), we have

$$\frac{v_t(b) - v_t(x_1)}{b - x_1} = \frac{b - v_t(x_1)}{b - x_1} < \frac{b - x_1}{b - x_1} = 1. \tag{4.12}$$

Applying Eqs.(4.11) and (4.12) will complete the proof of $v_t(x_2)-v_t(x_1) < x_2-x_1$. Therefore $v_t(x)-x$ is strictly decreasing in $x \le b$. Furthermore, we get $v_t(x)-x = 0$ for $x \ge b$ from Lemma 4.3(c), thus, $v_t(x)-x$ is nonincreasing in x. Consequently, the assertion becomes true because of $\beta v_t(x)-x = \beta(v_t(x)-x)+(\beta-1)x$. ∎

Here, we introduce the following two functions:

$$z_t^d(x) = \max\{x,\ -d-s+\beta v_{t-1}(x)\}, \qquad t \ge 1, \tag{4.13}$$
$$z_t^o(x) = \max\{x,\ -s+\beta v_{t-1}(x)\}, \qquad t \ge 1 \tag{4.14}$$

where we let $z_0^d(x) = x$ and $z_0^o(x) = x$. Note that $z_t^d(x)$ is composed of the first and second expressions of Eq.(4.9) and $z_t^o(x)$ the third and fourth. By using $z_t^d(x)$ and $z_t^o(x)$, we can rewrite $u_t(x,w)$ as follows.

$$u_t(x,w) = \max\{z_t^d(w),\ z_t^o(x)\}, \qquad t \ge 0. \tag{4.15}$$

The following lemmas are immediate from Lemmas 4.1 and 4.3.

Lemma 4.5
(a) *$z_t^o(x)$ and $z_t^d(x)$ are continuous, convex, and nondecreasing in x for any t.*
(b) *$z_t^o(x)$ and $z_t^d(x)$ are nondecreasing in t for any x.*

Lemma 4.6 *For any t, $z_t^d(x) \le z_t^o(x)$ for any x and $z_t^d(x) = z_t^o(x) = x$ for any $x \ge b$.*

5. Optimal Decision Rule

Let us define the following two functions:

$$g_t^d(x) = -d-s+\beta v_{t-1}(x)-x = 0, \qquad t \ge 1, \tag{5.1}$$
$$g_t^o(x) = -s+\beta v_{t-1}(x)-x = 0, \qquad t \ge 1, \tag{5.2}$$

which are the differences of the two terms inside the braces of Eqs.(4.13) and (4.14), respectively. Let x_t^d, $t \ge 1$, be a value of x such as $g_t^d(x) = 0$ and x_t^o, $t \ge 1$, be an x such as in $g_t^o(x) = 0$. Furthermore, let $w_t(x) = \max\{w \mid z_t^d(w) = z_t^o(x)\}$, $t \ge 0$.

Lemma 5.1
(a) *x_t^d uniquely exists on $[-d-s+\beta\mu, b)$.*
(b) *x_t^o uniquely exists on $[-s+\beta\mu, b)$ and satisfies $x_t^d < x_t^d+d \le x^o$.*
(c) *$w_t(x)$ exists for any x and satisfies $x \le w_t(x)$. In particular, $w_t(x) = x$ for $x \ge x_t^o$.*

Proof: (a) We get $g_t^d(-d-s+\beta\mu) \ge 0$ because $g_t^d(x) \ge -d-s+\beta\mu-x$ for any x from Lemma 4.3(d). We also get $g_t^d(b) = -d-s+(\beta-1)b < 0$ from Lemma 4.3(c). Hence, since $g_t^d(x)$ is continuous and strictly decreasing in $x \le b$ from Lemmas 4.1(a) and 4.4. From these, it follows that there exists an unique solution x_t^d of the equation $g_t^d(x) = 0$ on $[-d-s+\beta\mu, b)$.

(b) Since $v_{t-1}(x_t^d+d) \ge v_{t-1}(x_t^d)$ from Lemma 4.1(a), we obtain

$$g_t^o(x_t^d+d) = -s+\beta v_{t-1}(x_t^d+d)-(x_t^d+d) \ge -s+\beta v_{t-1}(x_t^d)-x_t^d-d = g_t^d(x_t^d) = 0. \tag{5.3}$$

Furthermore, similarly to (a), it can be shown that $g_t^o(b) < 0$ and that $g_t^o(x)$ is continuous and strictly decreasing in $x \le b$. Thus the solution of the equation $g_t^o(x) = 0$, or x_t^o, uniquely exists on $[x_t^d+d, b)$. From this and (a), we get $x_t^o \geq x_t^d+d \geq -s+\beta\mu$. Consequently, $x_t^d < x_t^d+d \leq x_t^o$ holds because of $d > 0$.

(c) If $x \geq x_t^d$, then $g_t^d(x) \leq 0$, that is, $-d-s+\beta v_{t-1}(x) \leq x$, from which we have for $x \geq x_t^d$ that

$$z_t^d(x) = x. \tag{5.4}$$

Similarly, it follows for $x \geq x_t^o$ ($> x_t^d$) that

$$z_t^o(x) = x. \tag{5.5}$$

Let us prove (c) by first showing $w_t(x) = x$ for $x \geq x_t^o$, then confirming the existence of $w_t(x)$ for $x \leq x_t^o$, and finally verifying $x \leq w_t(x)$ for any x.

First, we shall show $w_t(x) = x$ for $x \geq x_t^o$ ($> x_t^d$). For any fixed $x^* \geq x_t^o$, we have $z_t^d(x^*) = x^* = z_t^o(x^*)$ from Eqs.(5.4) and (5.5). Hence, the x^* satisfies the equation $z_t^d(w) = z_t^o(x^*)$. Furthermore, no $w' > x^*$ satisfies $z_t^d(w') = z_t^o(x^*)$ because such w' satisfies $z_t^d(w') = w' > x^* = z_t^o(x^*)$. As a result, the maximum solution of the equation is x^*, that is, $w_t(x^*) = x^*$, hence $w_t(x) = x$ holds for any $x \geq x_t^o$.

Next, let us show the existence of $w_t(x)$ for $x \leq x_t^o$. For any fixed $x^* \leq x_t^o$, consider the sets $\mathcal{W}_1 = \{w \mid z_t^d(w) = z_t^o(x^*),\ x_t^o < w\}$, $\mathcal{W}_2 = \{w \mid z_t^d(w) = z_t^o(x^*),\ w \leq x_t^o\}$, and $\mathcal{W} = \{w \mid z_t^d(w) = z_t^o(x^*)\} = \mathcal{W}_1 \cup \mathcal{W}_2$. For any $w' > x_t^o$, it holds from Eqs.(5.4), (5.5), and Lemma 4.5(a) that $z_t^o(x^*) \leq z_t^o(x_t^o) = x_t^o < w' = z_t^d(w')$. This means that such a w' does not satisfy the equation $z_t^d(w) = z_t^o(x^*)$, that is, $\mathcal{W}_1$ is empty. Hence, we have $\mathcal{W} = \mathcal{W}_2$. By virtue of Lemma 4.3(a), we know that $z_t^d(w)$ takes the minimum value $-d-s+\beta v_{t-1}(a)$ at a certain w. Moreover, Eq.(5.4) implies $z_t^d(x_t^o) = x_t^o$, thus $-d-s+\beta v_{t-1}(a) \leq z_t^d(w) \leq x_t^o$ for $w \leq x_t^o$ from Lemma 4.5(a). In the same way, we get $-s+\beta v_{t-1}(a) \leq z_t^o(x) \leq x_t^o$ for $x \leq x_t^o$, hence, $-d-s+\beta v_{t-1}(a) \leq z_t^o(x^*) \leq x_t^o$. Therefore, there exists the maximum element of $\mathcal{W}_2$ from Lemma 8.1. Hence, $\mathcal{W}$ also has the maximum element, which is $w_t(x^*)$ by definition. Consequently, there exists $w_t(x)$ for any $x \leq x_t^o$.

Finally, we shall show that $x \leq w_t(x)$ holds for any x. Assuming that $w_t(x^*) < x^*$ for any given x^*, we get $z_t^d(w_t(x^*)) \leq z_t^d(x^*)$ from Lemma 4.5(a). Furthermore, since $z_t^d(w_t(x^*)) = z_t^o(x^*)$ by definition of $w_t(x^*)$, the inequality $z_t^d(w_t(x^*)) \leq z_t^d(x^*)$ is equivalent to $z_t^o(x^*) \leq z_t^d(x^*)$. From this and Lemma 4.6, immediately we have $z_t^d(x^*) = z_t^o(x^*)$, that is, the x^* satisfies the equation $z_t^d(w) = z_t^o(x^*)$. This contradicts the definition of $w_t(x^*)$. Therefore, $x^* \leq w_t(x^*)$, thus $x \leq w_t(x)$ for any x. ∎

The following two lemmas are immediate from the definitions of x_t^d, x_t^o, and $w_t(x)$.

Lemma 5.2

(a) $z_t^d(x_t^d) = x_t^d = -d-s+v_{t-1}(x_t^d)$ *for* $t \geq 1$.

(b) $z_t^o(x_t^o) = x_t^o = -s+v_{t-1}(x_t^o)$ *for* $t \geq 1$.

(c) $z_t^d(w_t(x)) = z_t^o(x)$ *for* $t \geq 0$.

Lemma 5.3

(a) $z_t^d(x) = \begin{cases} -d-s+\beta v_{t-1}(x) & \text{if } x \leq x_t^d, \\ x & \text{if } x_t^d \leq x, \end{cases} \quad$ for $t \geq 1$.

(b) $z_t^o(x) = \begin{cases} -s+\beta v_{t-1}(x) & \text{if } x \leq x_t^o, \\ x & \text{if } x_t^o \leq x, \end{cases} \quad$ for $t \geq 1$.

(c) $u_t(x,w) = \begin{cases} z_t^o(x) & \text{if } w \le w_t(x), \\ z_t^d(w) & \text{if } w_t(x) \le w, \end{cases}$ for $t \ge 0$.

From Lemma 5.3, the optimal decision rule for any time t can be expressed as follows.

Optimal Decision Rule Assume that the process starts from time t with a current offer w and a leading offer x. In the case of $w_t(x) < w$, if $x_t^d < w$, then accept the w, or else reserve the w. In the case of $w_t(x) \ge w$, if $x_t^o < x$, then accept the x, or else continue the search.

We shall reveal some properties of $w_t(x)$, x_t^d, and x_t^o characterizing the optimal decision rule. Let $\tilde{x}_t$ be the minimum solution of the equation $z_t^o(x) = x_t^d$, if it exists, that is,

$$\tilde{x}_t = \min\{x \mid z_t^o(x) = x_t^d\}, \qquad t \ge 1. \tag{5.6}$$

If $-s+\beta v_{t-1}(a)$, the minimum value of $z_t^o(x)$, is greater than x_t^d, then there exists $\tilde{x}_t$ from Lemma 8.2, hence $z_t^o(\tilde{x}_t) = x_t^d$. Let $\tilde{x}_t = -\infty$ if the equation $z_t^o(x) = x_t^d$ does not have the minimum solution.

Lemma 5.4 *$\tilde{x}_t < x_t^d < x_t^o < b$ holds for any t.*

Proof: We have from Lemma 5.2(a),

$$\begin{aligned} z_t^o(x_t^d) &= \max\{x_t^d,\ -s+\beta v_{t-1}(x_t^d)\} \\ &= \max\{x_t^d-d,\ -d-s+\beta v_{t-1}(x_t^d)\} + d \\ &= \max\{x_t^d-d,\ x_t^d\} + d \\ &= x_t^d+d. \end{aligned} \tag{5.7}$$

Assuming $x_t^d \le \tilde{x}_t$, we have $z_t^o(x_t^d) \le z_t^o(\tilde{x}_t)$ from Lemma 4.5(a). From this, Eq.(5.7), and $z_t^o(\tilde{x}_t) = x_t^d$, we get $x_t^d+d \le x_t^d$, which contradicts $d > 0$. Hence $\tilde{x}_t < x_t^d$ must hold. The other inequalities have already been shown in Lemma 5.1(b). ∎

Lemma 5.5 *$w_t(x) < x_t^d$ if and only if $z_t^o(x) < x_t^d$ for any t and any x.*

Proof: Suppose $z_t^o(x) < x_t^d$, which is equivalent to $z_t^d(w_t(x)) < z_t^d(x_t^d)$ from Lemma 5.2(a,c). Furthermore from Lemma 4.5(a), we get $w_t(x) < x_t^d$ when $z_t^d(w_t(x)) < z_t^d(x_t^d)$.

If $w_t(x) < x_t^d$, then we have $z_t^d(w_t(x)) \le z_t^d(x_t^d)$ from Lemma 4.5(a). Here, assuming $z_t^d(w_t(x)) = z_t^d(x_t^d)$, we obtain $z_t^o(x) = z_t^d(x_t^d)$ from Lemma 5.2(c), implying that x_t^d satisfies the equation $z_t^d(w) = z_t^o(x)$. This contradicts the definition of $w_t(x)$. Therefore it must be that $z_t^d(w_t(x)) < z_t^d(x_t^d)$, that is, $z_t^o(x) < x_t^d$. ∎

Suppose that $\tilde{x}_t$ is finite. Then $z_t^o(\tilde{x}_t) = x_t^d$ holds. From this and Lemma 5.3(a), we have $z_t^d(x_t^d) = x_t^d = z_t^o(\tilde{x}_t)$ and all $w > x_t^d$ satisfy $z_t^d(w) = w > x_t^d = z_t^o(\tilde{x}_t)$. Hence, x_t^d is the maximum solution of the equation $z_t^d(w) = z_t^o(\tilde{x}_t)$, that is, $w_t(\tilde{x}_t) = x_t^d$. Furthermore by definition of $\tilde{x}_t$ and Lemma 4.5(a), we have $z_t^o(x) < x_t^d$ for any $x < \tilde{x}_t$, hence $w_t(x) < x_t^d$ holds for any $x < \tilde{x}_t$ from Lemma 5.5. For the reasons stated above, if $\tilde{x}_t$ is finite, then it is the minimum solution of the equation $w_t(x) = x_t^d$. Consequently, $\tilde{x}_t$, defined by Eq.(5.6), can be expressed as follows.

$$\tilde{x}_t = \min\{x \mid w_t(x) = x_t^d\}, \qquad t \ge 1. \tag{5.8}$$

Lemma 5.6 *For any $t \geq 1$,*
(a) $z_t^o(x) \leq x_t^d$ *and* $w_t(x) \leq x_t^d$ *for* $x \leq \tilde{x}_t$,
(b) $x_t^d \leq z_t^o(x)$ *and* $x_t^d \leq w_t(x)$ *for* $\tilde{x}_t \leq x$.

Proof: Easy from $z_t^o(\tilde{x}_t) = x_t^d$, $w_t(\tilde{x}) = x_t^d$, and Lemma 5.5. ∎

According to the optimal decision rule, when the leading offer is x, a current offer w to reserve must satisfy $w_t(x) < w \leq x_t^d$. Hence, such w does not exist if $x_t^d \leq w_t(x)$. Therefore, Lemma 5.6 means that if $x < \tilde{x}_t$, there then exists an offer w that should be reserved, or else it does not exist.

From the above and the fact that any actual offer is in a sample space $[a, b]$ of the distribution F, we obtain the following corollary.

Corollary 5.7 *For any t, if and only if $\tilde{x}_t \leq a$, there exists no current offer w to be reserved for any leading offer x.*

Theorem 5.8
(a) x_t^o *is a constant, which is equal to* x^o*, the maximum solution of the equation* $K(x) = 0$.
(b) x_t^d *is nondecreasing in t with* $x_t^d < b$ *for all t.*

Proof: (a) First of all, we shall show that $x_t^o = x_{t+1}^o$ if $v_{t-1}(x_t^o) = v_t(x_t^o)$. If $v_{t-1}(x_t^o) = v_t(x_t^o)$, then clearly $g_{t+1}^o(x_t^o) = g_t^o(x_t^o) = 0$. Since the equation $g_{t+1}^o(x) = 0$ has an unique solution x_{t+1}^o from Lemma 5.1(b), we get $x_t^o = x_{t+1}^o$. Thus, in order to prove that x_t^o is a constant, it suffices to show that $v_{t-1}(x_t^o) = v_t(x_t^o)$ for any t.

(i) Assume $a \leq x_1^o$ ($< b$ due to Lemma 5.1(b)). Then, we can rewrite $v_0(x_1^o)$ and $v_1(x_1^o)$ as follows, respectively,

$$
\begin{aligned}
v_0(x_1^o) &= \int_a^b \max\{w, x_1^o\} dF(w) \\
&= \int_a^{x_1^o} x_1^o \, dF(w) + \int_{x_1^o}^b w \, dF(w), \qquad (5.9)
\end{aligned}
$$

$$
\begin{aligned}
v_1(x_1^o) &= \int_a^b \max\{z_1^d(w),\ z_1^o(x_1^o)\} dF(w) \\
&= \int_a^{w_1(x_1^o)} z_1^o(x_1^o) dF(w) + \int_{w_1(x_1^o)}^b z_1^d(w) dF(w) \\
&= \int_a^{x_1^o} x_1^o \, dF(w) + \int_{x_1^o}^b z_1^d(w) dF(w) \\
&= \int_a^{x_1^o} x_1^o \, dF(w) + \int_{x_1^o}^b w \, dF(w), \qquad (5.10)
\end{aligned}
$$

in which we applied Lemmas 5.3(a,c), 5.2(b), and $a \leq w_1(x_1^o) = x_1^o < b$ from Lemma 5.1(b,c). Therefore, it follows that $v_0(x_1^o) = v_1(x_1^o)$, thus we get $a \leq x_1^o = x_2^o$.

Next, assume $v_{t-1}(x_t^o) = v_t(x_t^o)$ and $a \leq x_t^o$, so that, $a \leq x_t^o = x_{t+1}^o$. Then, in the same way as in Eq.(5.10), we have

$$
\begin{aligned}
v_t(x_{t+1}^o) &= \int_a^b \max\{z_t^d(w),\ z_t^o(x_{t+1}^o)\} dF(w) \\
&= \int_a^b \max\{z_t^d(w),\ z_t^o(x_t^o)\} dF(w)
\end{aligned}
$$

$$
\begin{aligned}
&= \int_a^{w_t(x_t^o)} z_t^o(x_t^o)dF(w) + \int_{w_t(x_t^o)}^b z_t^d(w)dF(w) \\
&= \int_a^{x_t^o} x_t^o \; dF(w) + \int_{x_t^o}^b w \; dF(w), \qquad (5.11)
\end{aligned}
$$

$$
\begin{aligned}
v_{t+1}(x_{t+1}^o) &= \int_a^b \max\{z_{t+1}^d(w),\; z_{t+1}^o(x_{t+1}^o)\}dF(w) \\
&= \int_a^{w_{t+1}(x_{t+1}^o)} z_{t+1}^o(x_{t+1}^o)dF(w) + \int_{w_{t+1}(x_{t+1}^o)}^b z_{t+1}^d(w)dF(w) \\
&= \int_a^{x_{t+1}^o} x_{t+1}^o \; dF(w) + \int_{x_{t+1}^o}^b w \; dF(w) \\
&= \int_a^{x_t^o} x_t^o \; dF(w) + \int_{x_t^o}^b w \; dF(w). \qquad (5.12)
\end{aligned}
$$

Therefore, it follows that $v_t(x_{t+1}^o) = v_{t+1}(x_{t+1}^o)$, thus we have $a \le x_{t+1}^o = x_{t+2}^o$.

(ii) Even if $x_1^o < a$, in the same way as (i), we can show $v_{t-1}(x_t^o) = v_t(x_t^o)$ for all t .

Consequently, $v_{t-1}(x_t^o) = v_t(x_t^o)$, hence $x_t^o = x_{t+1}^o$, for any t. Furthermore, since $g_1^o(x) = -s+\beta v_0(x)-x = K(x)$ from Eq.(4.4), x_1^o is equal to x^o. Therefore, we conclude that x_t^o is a constant which is equal to x^o.

(b) For any t, we get $g_{t+1}^d(b) < 0$ (see the proof of Lemma 5.1(a)) and $0 = g_t^d(x_t^d) \le g_{t+1}^d(x_t^d)$ from Lemma 4.1(b), In addition to these, since $g_{t+1}^d(x)$ is continuous and strictly decreasing in $x \le b$, there exists x_{t+1}^d on $[x_t^d, b)$, hence $x_t^d \le x_{t+1}^d$. ∎

From now on, we use x^o instead of x_t^o.

Theorem 5.9 *$w_t(x)$ is nondecreasing in x for any t.*

Proof: If $x_1 < x_2$, then we have $z_t^o(x_1) \le z_t^o(x_2)$ from Lemma 4.5(a). Suppose $w_t(x_1) > w_t(x_2)$. Then $z_t^o(x_1) \le z_t^o(x_2) = z_t^d(w_t(x_2)) \le z_t^d(w_t(x_1)) = z_t^o(x_1)$ from Lemma 5.2(c), from which we get $z_t^d(w_t(x_1)) = z_t^o(x_2)$. This means that $w_t(x_1)$ satisfies the equation $z_t^d(w) = z_t^o(x_2)$, which contradicts the definition of $w_t(x_2)$. Hence, if $x_1 < x_2$, then $w_t(x_1) \le w_t(x_2)$ must hold, that is, $w_t(x)$ is nondecreasing in x. ∎

Lemma 5.10 *For any $t \ge 1$,*

(I) *if $a < \tilde{x}_t$, then*
- (a) *for $a \le x \le \tilde{x}_t$, Eqs.(5.13) and (5.16) hold,*
- (b) *for $\tilde{x}_t \le x \le x^o$, Eqs.(5.14) and (5.17) hold,*
- (c) *for $x^o \le x \le b$, Eqs.(5.15) and (5.18) hold,*

(II) *if $\tilde{x}_t \le a$ and furthermore,*
- (i) *if $a < x^o$, then*
  - (a) *for $a \le x \le x^o$, Eqs.(5.14) and (5.17) hold,*
  - (b) *for $x^o \le x \le b$, Eqs.(5.15) and (5.18) hold,*
- (ii) *if $x^o \le a$, then*
  - (a) *for $a \le x \le b$, Eqs.(5.15) and (5.18) hold*

where

$$
v_t(x)=\int_a^{w_t(x)}\{-s+\beta v_{t-1}(x)\}dF(w)+\int_{w_t(x)}^{x_t^d}\{-d-s+\beta v_{t-1}(w)\}dF(w)+\int_{x_t^d}^b w \; dF(w), \qquad (5.13)
$$

$$
v_t(x)=\int_a^{w_t(x)}\{-s+\beta v_{t-1}(x)\}dF(w)+\int_{w_t(x)}^b w \; dF(w), \qquad (5.14)
$$

$$
v_t(x)=T(x)+x, \qquad (5.15)
$$

$$-d-s+\beta v_{t-1}(w_t(x)) = -s+\beta v_{t-1}(x), \tag{5.16}$$

$$w_t(x) = -s+\beta v_{t-1}(x), \tag{5.17}$$

$$w_t(x) = x. \tag{5.18}$$

Proof: (I) In the case of $a < \tilde{x}_t$, it follows that $a < \tilde{x}_t < x_t^d < x^o < b$ from Lemma 5.4 and that $a \le w_t(x) \le b$ for $a \le x \le b$ from Lemma 5.1(c) and Theorem 5.9.

(a) Suppose $a \le x \le \tilde{x}_t$. Note that $w_t(x) \le x_t^d$ in this case from Lemma 5.6(a). First, we have $z_t^o(x) = -s+\beta v_{t-1}(x)$ from Lemma 5.3(b). This and Lemma 5.3(c) implies that $u_t(x,w) = -s+\beta v_{t-1}(x)$ for $w \le w_t(x)$. Secondly, assuming $w_t(x) \le w \le x_t^d$ produces by Lemma 5.3(a,c) that $u_t(x,w) = -d-s+\beta v_{t-1}(w)$. Finally, we get $u_t(x,w) = w$ for $x_t^d \le w$ from Lemma 5.3(a,c). Consequently, Eq.(5.13) holds for $a \le x \le \tilde{x}_t$. As stated above, $z_t^o(x) = -s+\beta v_{t-1}(x)$ for $a \le x \le \tilde{x}_t$, and we have $z_t^d(w_t(x)) = -d-s+\beta v_{t-1}(w_t(x))$ from Lemma 5.3(a) due to $w_t(x) \le x_t^d$. Eventually Eq.(5.16) holds for $a \le x \le \tilde{x}_t$ from Lemma 5.2(c).

(b) In the case of $\tilde{x}_t \le x \le x^o$, since $x_t^d \le w_t(x)$, it suffices to consider the two regions, $w \le w_t(x)$ and $w_t(x) \le w$. The remaining discussions are almost the same as (a).

(c) Suppose $x^o \le x \le b$, so that Eq.(5.18) holds from Lemma 5.1(c). In the same way as (a), we get for $x^o \le x \le b$,

$$v_t(x) = \int_a^{w_t(x)} x\, dF(w) + \int_{w_t(x)}^b w\, dF(w), \tag{5.19}$$

which can be rearranged as follows by use of Eq.(5.18).

$$\begin{aligned} v_t(x) &= \int_a^x x\, dF(w) + \int_x^b w\, dF(w) \\ &= \int = a^b \max\{x,w\} dF(w) \\ &= T(x)+x. \end{aligned} \tag{5.20}$$

(II) If $\tilde{x}_t \le a$, we must consider the two additional cases: $a < x^o$ and $x^o \le a$. The remaining arguments are almost the same as (I). ∎

Lemma 5.11 *$v_t(x)-v_{t-1}(x)$ is nonincreasing in x for any $t \ge 1$.*

Proof: For convenience, let $\delta_t(x_1,x_2) = v_t(x_2)-v_t(x_1)$. Then, any x_a, x_b, and x_c, clearly

$$\delta_t(x_a,x_c) = \delta_t(x_a,x_b) + \delta_t(x_b,x_c). \tag{5.21}$$

To prove this lemma, it suffices to show that $\delta_t(x_1,x_2) \le \delta_{t-1}(x_1,x_2)$ holds for any x_1 and any x_2 such as $x_1 < x_2$ where note $w_t(x_1) \le w_t(x_2)$ for all t from Theorem 5.9.

(i) In the case of $x_1 < x_2 \le a$, it follows from Lemma 4.3(a) that $v_t(x_1) = v_t(x_2) = v_t(a)$ and $v_{t-1}(x_1) = v_{t-1}(x_2) = v_{t-1}(a)$, hence, $\delta_t(x_1,x_2) = \delta_{t-1}(x_1,x_2)$.

(ii) In the case of $a \le x_1 < x_2 \le \tilde{x}_t$, from Lemma 5.10(I.a), we can express $\delta_t(x_1,x_2)$ as

$$\delta_t(x_1,x_2) = \int_a^{w_t(x_1)} \beta\delta_{t-1}(x_1,x_2) dF(w) + \int_{w_t(x_1)}^{w_t(x_2)} \{\beta v_{t-1}(x_2)+d-\beta v_{t-1}(w)\} dF(w). \tag{5.22}$$

If $w_t(x_1) \le w$, then from Lemmas 5.10(I.a) and 4.1(a), we have $-s+\beta v_{t-1}(x_1) = -d-s+\beta v_{t-1}(w_t(x_1)) \le -d-s+\beta v_{t-1}(w)$, from which we get $d-\beta v_{t-1}(w) \le -\beta v_{t-1}(x_1)$. Hence,

$$\begin{aligned} \int_{w_t(x_1)}^{w_t(x_2)} \{\beta v_{t-1}(x_2)+d-\beta v_{t-1}(w)\} dF(w) &\le \int_{w_t(x_1)}^{w_t(x_2)} \{\beta v_{t-1}(x_2)-\beta v_{t-1}(x_1)\} dF(w) \\ &= \int_{w_t(x_1)}^{w_t(x_2)} \beta\delta_{t-1}(x_1,x_2) dF(w). \end{aligned} \tag{5.23}$$

From Eqs.(5.22) and (5.23), we have

$$
\begin{aligned}
\delta_t(x_1,x_2) &\le \int_a^{w_t(x_1)} \beta\delta_{t-1}(x_1,x_2)dF(w) + \int_{w_t(x_1)}^{w_t(x_2)} \beta\delta_{t-1}(x_1,x_2)dF(w) \\
&= \int_a^{w_t(x_2)} \beta\delta_{t-1}(x_1,x_2)dF(w) \\
&= \beta\delta_{t-1}(x_1,x_2)F(w_t(x_2)) \\
&\le \delta_{t-1}(x_1,x_2).
\end{aligned} \tag{5.24}
$$

(iii) In the case of $\tilde{x}_t \le x_1 < x_2 \le x^o$, ~~by applying Lemma 5.10(I.b)~~ instead of Lemma 5.10(I.a), we can show $\delta_t(x_1,x_2) \le \delta_{t-1}(x_1,x_2)$ in exactly the same fashion as (ii).

(iv) In the case of $x^o \le x_1 < x_2$, from Lemma 5.10(I.c), we get immediately $v_t(x_1) = v_{t-1}(x_1) = T(x_1)+x_1$ and $v_t(x_2) = v_{t-1}(x_2) = T(x_2)+x_2$, hence $\delta_t(x_1,x_2) = \delta_{t-1}(x_1,x_2)$.

(v) In the case of $x_1 \le a < x_2 \le \tilde{x}_t$, applying the above implies

$$
\delta_t(x_1,x_2) = \delta_t(x_1,a) + \delta_t(a,x_2) \le \delta_{t-1}(x_1,a) + \delta_{t-1}(a,x_2) = \delta_{t-1}(x_1,x_2). \tag{5.25}
$$

Also in the other possible cases, that is, $x_1 < a < \tilde{x}_t < x_2 \le x^o$, $x_1 < a < x^o < x_2$, $a \le x_1 < \tilde{x}_t < x_2 \le x^o$, $a \le x_1 < \tilde{x}_t < x^o < x_2$, and $\tilde{x} \le x_1 < x^o < x_2$, we can similarly prove $\delta_t(x_1,x_2) \le \delta_{t-1}(x_1,x_2)$. ∎

Theorem 5.12 *$w_t(x)$ is nondecreasing in t for any x.*

Proof: As a preliminary of the proof, we show that $z^o_{t+1}(x) - z^o_t(x)$ is nonincreasing in x for any t. Clearly we have $z^o_1(x) - z^o_0(x) = \max\{x,\ -s+\beta v_0(x)\} - x = \max\{0,\ -s+\beta v_0(x)-x\}$, which is nonincreasing in x from Lemma 4.4. For $t \ge 1$, it holds from Lemma 5.3(b) that

$$
\begin{aligned}
z^o_{t+1}(x) - z^o_t(x) &= \max\{x,\ -s+\beta v_t(x)\} - \max\{x, -s+\beta v_{t-1}(x)\} \\
&= \begin{cases} \beta(v_t(x) - v_{t-1}(x)) & \text{if } x \le x^o, \\ 0. & \text{if } x^o \le x. \end{cases}
\end{aligned} \tag{5.26}
$$

Applying Eq.(5.26) and Lemma 5.11, we find that $z^o_{t+1}(x) - z^o_t(x)$ is nonincreasing in x. Thus by induction, $z^o_{t+1}(x) - z^o_t(x)$ is nonincreasing in x for any t.

From the above and Lemma 5.1(c), we have

$$
z^o_{t+1}(w_t(x)) - z^o_t(w_t(x)) \le z^o_{t+1}(x) - z^o_t(x). \tag{5.27}
$$

By noting that Lemma 4.1(b) and $d > 0$, it follows from Lemma 8.3 that, for any x,

$$
\begin{aligned}
z^o_t(x) - z^d_t(x) &= \max\{x, -s+\beta v_{t-1}(x)\} - \max\{x, -d-s+\beta v_{t-1}(x)\} \\
&\le \max\{x, -s+\beta v_t(x)\} - \max\{x, -d-s+\beta v_t(x)\} \\
&= z^o_{t+1}(x) - z^d_{t+1}(x),
\end{aligned} \tag{5.28}
$$

hence, we have

$$
z^d_{t+1}(w_t(x)) - z^d_t(w_t(x)) \le z^o_{t+1}(w_t(x)) - z^o_t(w_t(x)). \tag{5.29}
$$

From Eqs.(5.27) and (5.29), we get $z^d_{t+1}(w_t(x)) - z^d_t(w_t(x)) \le z^o_{t+1}(x) - z^o_t(x)$. Due to Lemma 5.2(c), this inequality is equivalent to

$$
z^d_{t+1}(w_t(x)) \le z^o_{t+1}(x). \tag{5.30}
$$

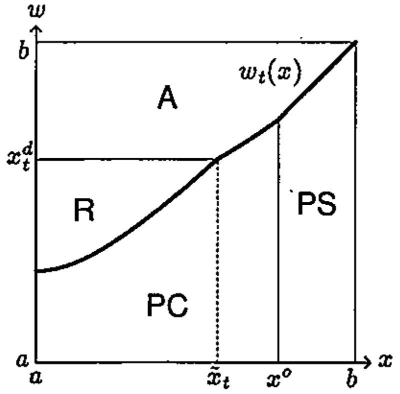

Figure 1. $a < \tilde{x}_t$

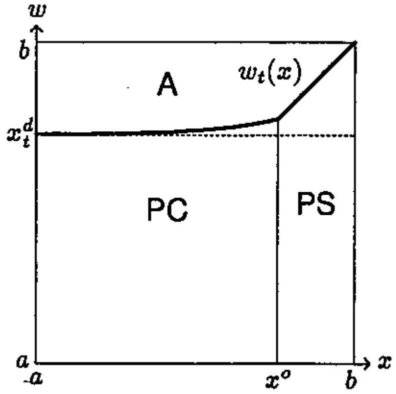

Figure 2. $\tilde{x}_t \le a$

Here, assume $w_{t+1}(x) < w_t(x)$. Then we obtain $z^d_{t+1}(w_{t+1}(x)) \le z^d_{t+1}(w_t(x))$ due to Lemma 4.5(a). Consequently, it follows from Lemma 5.2(c) and Eq.(5.30) that $z^o_{t+1}(x) = z^d_{t+1}(w_{t+1}(x)) \le z^d_{t+1}(w_t(x)) \le z^o_{t+1}(x)$, from which we obtain $z^d_{t+1}(w_t(x)) = z^o_{t+1}(x)$, in other words, $w_t(x)$ satisfies the equation $z^d_{t+1}(w) = z^o_{t+1}(x)$. This contradicts the definition of $w_{t+1}(x)$. Therefore, $w_t(x) \le w_{t+1}(x)$ must hold, that is, $w_t(x)$ is nondecreasing in t. ∎

Theorems 5.9, 5.12, and 5.8 enable us to draw x^o, x^d_t, and $w_t(x)$ on an (x,w)-plane as in Figures 1 and 2 where x represents a leading offer and w a current offer. Meanwhile a and b are, respectively, the lower and upper bounds of the sample space of distribution F, and the curved bold line is $w_t(x)$. Figure 1 is the case of $a < \tilde{x}_t$ and Figure 2 the case of $\tilde{x}_t \le a$. In either case, the optimal decision rule is described as follows. In the instance where the pair (x, w) is in area A, then the current offer w (decision A) should be accepted. The same interpretations are given for the other areas. Here, note that area R does not always exist. This implies that there may be a case where no offer is reserved despite, however low the leading offer may be.

The following theorem provides a necessary and sufficient condition for which there exists an area R at $t = 1$ (see Corollary 5.7).

Theorem 5.13 $a < \tilde{x}_1$ *if and only if* $d < \min\{-s+\beta\mu-a,\ K(-s+\beta\mu)\}$.

Proof: If $a < \tilde{x}_1$, then $z^o_1(a) \le z^o_1(\tilde{x}_1) = x^d_1$ from Lemma 4.5(a). If $z^o_1(a) = x^d_1$, that is, a satisfies the equation $z^o_1(x) = x^d_1$, then this contradicts the definition of $\tilde{x}_1$. Hence it must be $z^o_1(a) < x^d_1$. Conversely, if $z^o_1(a) < x^d_1 = z^o_1(\tilde{x}_1)$, then $a < \tilde{x}_1$ from Lemma 4.5(a). As a result, $a < \tilde{x}_1$ is equivalent to $z^o_1(a) < x^d_1$.

Now, since $z^o_1(a) = \max\{a, -s+\beta\mu\}$ from Eq.(4.14) and $v_0(a) = \mu$, it follows that $z^o_1(a) < x^d_1$ is equivalent to $a < x^d_1$ and $-s+\beta\mu < x^d_1$. Here, $a < x^d_1$ is equivalent to $K(a) > K(x^d_1)$ from Lemma 3.1(a). Noting that $K(a) = -s+\beta\mu-a$ from Lemma 3.1(c) and that $K(x^d_1) = d$ from $0 = g^d_1(x^d_1) = K(x^d_1)-d$, we get that $K(a) > K(x^d_1)$ is equivalent to $-s+\beta\mu-a > d$. Similarly, we can show that $-s+\beta\mu < x^d_1$ is equivalent to $K(-s+\beta\mu) > d$. Therefore, $z^o_1(a) < x^d_1$, or $a < \tilde{x}_1$ is equivalent to the two inequalities, $d < -s+\beta\mu-a$ and $d < K(-s+\beta\mu)$, that is, $d < \min\{-s+\beta\mu-a,\ K(-s+\beta\mu)\}$. ∎

This theorem states that if the reserving cost d satisfies $d \ge \min\{-s+\beta\mu-a,\ K(-s+\beta\mu)\}$, then we should not reserve any offer however low the leading offer may be, and vice versa.

6. Infinite planning horizon

In this section, we examine the case of infinite planning horizon.

Theorem 6.1 *$v_t(x)$, $w_t(x)$, and x_t^d converge to $v(x)$, $w(x)$, and x^d, respectively, where*

$$v(x) = \begin{cases} (x^o+s)/\beta & \text{if } x \le x^o, \\ T(x)+x & \text{if } x^o \le x, \end{cases} \qquad w(x) = \begin{cases} x^o & \text{if } x \le x^o, \\ x & \text{if } x^o \le x, \end{cases} \qquad x^d = x^o - d,$$

and $\tilde{x} = -\infty$.

Proof: First, clearly $v(x) = T(x)+x$ and $w(x) = x$ for $x \ge x^o$ from Lemmas 5.10(I.c) and 5.1(c). Next, $v_t(x)$ and $w_t(x)$ for $x \le x^o$ converge to certain functions $v(x) \le T(x^o)+x^o$ and $w(x) \le x^o$ from Lemma 4.1, Theorem 5.9, and Theorem 5.12. Hence, $z_t^d(x)$, $z_t^o(x)$, and $u_t(x, w)$ converge to certain functions, respectively, $z^d(x)$, $z^o(x)$, and $u(x, w)$. By definitions of $v_t(x)$, $z_t^d(x)$, $z_t^o(x)$, and $w_t(x)$, they can be expressed as

$$v(x) = \int_a^b \max\{z^d(w),\ z^o(x)\}dF(w), \tag{6.1}$$

$$z^d(x) = \max\{x,\ -d-s+\beta v(x)\}, \tag{6.2}$$

$$z^o(x) = \max\{x,\ -s+\beta v(x)\}, \tag{6.3}$$

$$w(x) = \max\{w \mid z^d(w) = z^o(x)\}. \tag{6.4}$$

Now, $x_t^d \le x^o - d$ for any t from Lemma 5.1(b), that is, x_t^d is bounded from above. Because of this and Theorem 5.8(b), x_t^d converges to a certain $x^d \le x^o - d$ as $t \to \infty$.

Here, note that, similarly to Lemma 5.3, we have

$$z^d(x) = \begin{cases} -d-s+\beta v(x) & \text{if } x \le x^d, \\ x & \text{if } x^d \le x, \end{cases} \tag{6.5}$$

$$z^o(x) = \begin{cases} -s+\beta v(x) & \text{if } x \le x^o, \\ x & \text{if } x^o \le x, \end{cases} \tag{6.6}$$

$$u(x, w) = \begin{cases} z^o(x) & \text{if } w \le w(x), \\ z^d(w) & \text{if } w(x) \le w. \end{cases} \tag{6.7}$$

To begin with, we shall show that $v_t(x)$ for $x \le x^o$ converges to $(x^o+s)/\beta$.

(i) Assume $\beta < 1$. Then, it suffices to verify that the function $v(x) = (x^o+s)/\beta$ with $x \le x^o$ is the unique solution of the Eq.(6.1). We shall show this by first revealing that the right hand side of Eq.(6.1) becomes equal to $(x^o+s)/\beta$ if it is rearranged by substituting the function and then confirming that Eq.(6.1) has an unique solution on $x \le x^o$. Let the right hand side be designated by $R(x)$.

First, on substituting $v(x) = (x^o+s)/\beta$ to Eq.(6.2), we obtain $z^d(x) = \max\{x,\ x^o - d\}$, implying that $x^d = x^o - d$. Hence, Eq.(6.5) can be rewritten as follows.

$$z^d(x) = \begin{cases} x^o - d & \text{if } x \le x^d, \\ x & \text{if } x^d \le x. \end{cases} \tag{6.8}$$

Substituting $v(x) = (x^o+s)/\beta$ to Eq.(6.6) yields $z^o(x) = x^o$ for $x \le x^o$. Thus we can rewrite Eq.(6.6) as follows.

$$z^o(x) = \begin{cases} x^o & \text{if } x \le x^o, \\ x & \text{if } x^o \le x. \end{cases} \tag{6.9}$$

By using Eqs.(6.8) and (6.9), we have $z^d(x^o) = x^o = z^o(x)$ for any $x \le x^o$. This means that the equation $z^d(w) = z^o(x)$, given $x \le x^o$, has a solution x^o. Since all $w' > x^o$ satisfy

$z^d(w') = w' > x^o = z^o(x)$ if $x \le x^o$, it follows that $w(x) = x^o$ for any $x \le x^o$. From the above, if $a \le x^o$, hence $a \le w(x)$ for $x \le x^o$, then it follows for $x \le x^o$ that

$$\begin{aligned} R(x) &= \int_a^{w(x)} z^o(x) dF(w) + \int_{w(x)}^b z^d(w) dF(w) \\ &= \int_a^{x^o} z^o(x) dF(w) + \int_{x^o}^b z^d(w) dF(w) \\ &= \int_a^{x^o} x^o \, dF(w) + \int_{x^o}^b w \, dF(w) \\ &= \int_a^b \max\{x^o, w\} dF(w) \\ &= T(x^o) + x^o \\ &= (x^o + s)/\beta \end{aligned} \tag{6.10}$$

where the last equation followed from $T(x^o)+x^o = (x^o+s)/\beta$ from Eq.(3.2) and $K(x^o) = 0$. In the case of $x^o < a$, the same thing as above can also be shown. Hence, it follows that the function $v(x) = (x^o+s)/\beta$ for $x \le x^o$ satisfies the Eq.(6.1) for any $\beta < 1$. Here, of course, this result holds true even for $\beta = 1$.

Next, so as to verify the uniqueness of the solution for $x \le b$ instead of $x \le x^o$ (note that $x^o < b$ from Lemma 5.1(b)). We assume that there exists another finite solution $\bar{v}(x)$ such that $\bar{v}(x) \ne v(x)$ at an $x \le b$ where

$$\bar{v}(x) = \int_a^b \max\{\bar{z}^d(w),\ \bar{z}^o(x)\} dF(w), \tag{6.11}$$

in which $\bar{z}^d(x) = \max\{x,\ -d-s+\beta\bar{v}(x)\}$ and $\bar{z}^o(x) = \max\{x,\ -s+\beta\bar{v}(x)\}$. Now, let $\Delta = \sup_{x \le b} |v(x) - \bar{v}(x)|$. Hence clearly, $0 < \Delta < \infty$. Then using the general formula

$$|\max\{a_1, b_1\} - \max\{a_2, b_2\}| \le \max\{|a_1 - a_2|, |b_1 - b_2|\}, \tag{6.12}$$

we immediately get from Eqs.(6.1) and (6.11),

$$|v(x) - \bar{v}(x)| \le \int_a^b \max\{|z^d(w) - \bar{z}^d(w)|,\ |z^o(x) - \bar{z}^o(x)|\} dF(w). \tag{6.13}$$

Furthermore, by use of Eq.(6.12), we can show $|z^d(w) - \bar{z}^d(w)| \le \beta |v(w) - \bar{v}(w)| \le \beta\Delta$ for $a \le w \le b$ and $|z^o(x) - \bar{z}^o(x)| \le \beta\Delta$ for $x \le b$. Consequently, it follows from Eq.(6.13) that $|v(x) - \bar{v}(x)| \le \beta\Delta$, yielding $\Delta \le \beta\Delta$, so that $1 \le \beta$. This contradicts $\beta < 1$. Eventually, the Eq.(6.1) for $x \le b$ must have the unique solution $v(x)$.

(ii) Assume $\beta = 1$. Here, for convenience, we express $v(x)$ and x^o as, respectively, $v(x;\beta)$ and $x^o(\beta)$. From $T(x^o)+x^o = (x^o+s)/\beta$ and what is stated in the beginning of the proof, we have $v(x;1) \le (x^o(1)+s)/1$ for any $x \le x^o(1)$. Furthermore, $(x^o(\beta)+s)/\beta \le v(x;\beta)$ for any $\beta < 1$ and any x from proof (i). In addition, it can be easily verified by induction that $v(x;\beta) \le v(x;1)$ for any $\beta < 1$ and any x. Hence, we have for $x \le x^o(1)$,

$$(x^o(\beta)+s)/\beta \le v(x;1) \le (x^o(1)+s)/1. \tag{6.14}$$

Noting that $(x^o(\beta)+s)/\beta \to (x^o(1)+s)/1$ as $\beta \to 1$ since $x^o(\beta) \to x^o(1)$ by Lemma 3.1(d), we have $v(x;1) = (x^o(1)+s)/1$ from Eq.(6.14).

Accordingly, the assertion is verified that $v(x) = (x^o+s)/\beta$ for $x \le x^o$ and any $\beta \le 1$. It is shown in the proof (i) that $w(x) = x^o$ for $x \le x^o$ and $x^d = x^o - d$ if the assertion is true.

Finally, since $v(x) \ge (x^o+s)/\beta$, from Eq.(6.3), we have $z^o(x) \le x^o = x^d + d > x^d$. Hence, the equation $z^o(x) = x^d$ has no solution. Thus, by definition, $\tilde{x} = -\infty$. ∎

7. Numerical Examples

This section illustrates the properties of x_t^d, $\tilde{x}_t$, and $w_t(x)$ by some numerical examples where offers are uniformly distributed with $a = 0$ and $b = 1$, thus $\mu = 0.5$ and where s and d are such that $a < -d - s + \beta\mu$ with $\beta = 0.97$. It can be shown from Lemma 5.1 that $a \le x^o$ and $a \le x_t^d$. All calculations are made by stepsizing to 0.0005.

- x_t^d : Figures 3 and 4 depict the relationships of x_t^d with t, d, and s. From the figures, we can confirm that x_t^d is nondecreasing in t and converges to $x^o - d$ (Theorem 5.8(b) and Theorem 6.1). Here, we notice that x_t^d decreases as s or d increases.
- $\tilde{x}_t$: Figures 5 and 6 show the influence of t, d, and s on $\tilde{x}_t$ where $\tilde{x}_t < 0$ is regarded as $\tilde{x}_t = 0$. From the figures, we find that ~~$\tilde{x}_t$ decreases as s or d increases; however,~~ it has not yet been proven whether this property holds generally. Figure 7 shows that $\tilde{x}_t$ is not always decreasing in t where $\beta = 1.0$, $s = 0.005$, and $d = 0.005$.
- $w_t(x)$: Figure 8 illustrates the relationship of $w_t(x)$ (bold line) with t where $s = 0.005$ and $d = 0.04$ (see Figures 1 and 2 for details of the figure). We can confirm that $w_t(x)$ is nondecreasing in x and t and converges to x^o for $x \le x^o$ (Theorems 5.9, 5.12 and 6.1). It was shown from other numerical examples that $w_t(x)$ becomes small as s becomes large or d becomes small.

8. Conclusions

The main results of this paper are summarized as follows.

(a) A leading offer should not be accepted except on the deadline. This result can be explained as follows. To accept a leading offer, its value is required to be at least x_t^o. Since

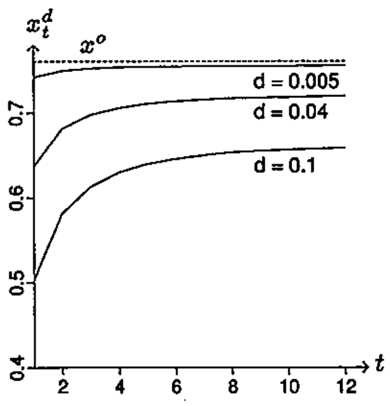

Figure 3. $s = 0.005$

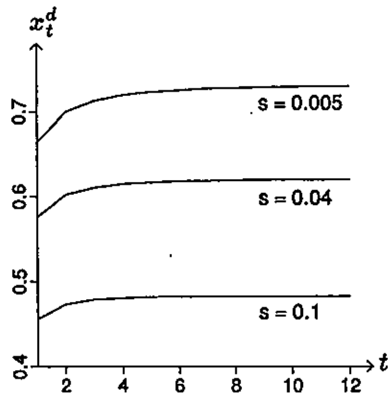

Figure 4. $d = 0.04$

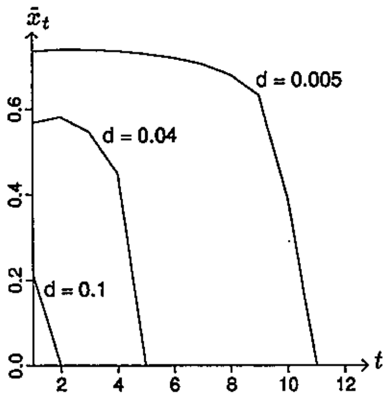

Figure 5. $s = 0.005$

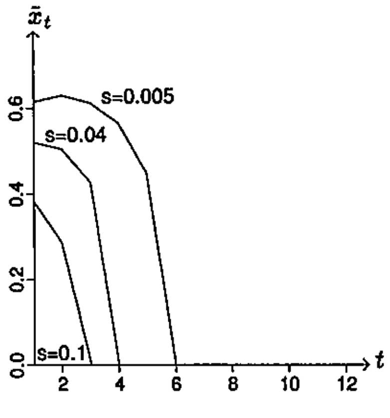

Figure 6. $d = 0.04$

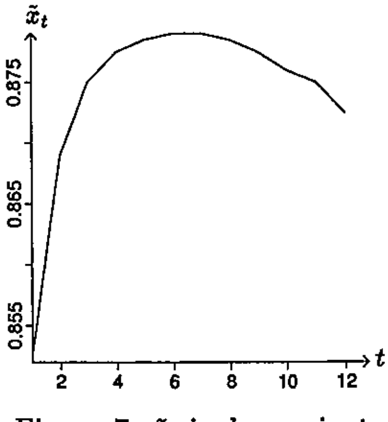

Figure 7. $\tilde{x}_t$ incleases in t

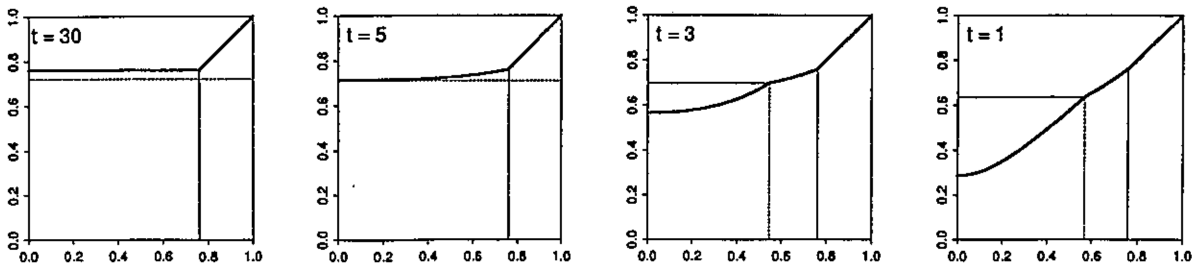

Figure 8. $w_t(x)$ (bold line) with $s = 0.005$ and $d = 0.04$

x_t^o is independent of time t, that is, $x_t^o = x^o$ (Theorem 5.8(a)), the only way to have such a leading offer is reduced to renewing the leading offer by reserving an offer w larger than x^o. However, even if such an offer appears, our decision is not to reserve it but to accept it because of $x_t^d < x^o$. Hence, since we never have an leading offer x larger than x^o up to time 1, no leading offer should be accepted except at time 0 (the deadline). However, from Eq.(4.3), at time 0, if the value of a current offer is less than that of the leading offer, we make a decision to accept it. From the above, it eventually follows that no leading offer should be accepted except on the deadline. This appears counterintuitive to us because it seems not so unreasonable to stop the search by accepting a leading offer before the deadline. We can regard the reserving cost as a sort of insurance against the situation where no desirable offer appears up to the deadline.

(b) Both the lowest value of offer to accept and the offer to reserve become small with time elapse. This result implies that, as the deadline draws near, the searcher inclines to more enthusiastically accept or reserve an offer that appears.

(c) Every time an offer is reserved, both the lowest value of the offer to accept and the offer to reserve become larger. This result is easily explained in the light that the searcher shifts to a favorable situation every time he reserves an offer.

(d) If an offer inferior to the leading offer is obtained, it should be automatically rejected.

(e) Given an infinite planning horizon, we should not reserve any offer over the whole horizon.

(f) At no time should we reserve any offer if the reserving cost is very high. Therefore, it follows that there exists a lower bound d_t^* of reserving cost d for which no offer should be reserved. The lower bound of time 1 is given by $d_1^* = \min\{-s+\beta\mu-a,\ K(-s+\beta\mu)\}$. The evaluation of Theorem 5.13 for $t \geq 2$ is a subject that is yet to be investigated.

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Appendix

Lemma 8.1 *Let f be a continuous function on $(-\infty, b]$ which takes minimum value m and maximum value M where b is a certain given finite number. Then for any $c \in [m, M]$, there exists a maximum element of a set $S(c) = \{w \mid f(w) = c,\ w \leq b\}$.*

Lemma 8.2 *Let f be a continuous and nondecreasing function on $(-\infty, +\infty)$ which takes minimum value m and maximum value M. Then for any $c \in (m, M]$, there exists a minimum element of a set $S(c) = \{w \mid f(w) = c\}$.*

Lemma 8.3 *If $d > 0$ and $f_1(x) \leq f_2(x)$ for any x, then it follows that*

$$\max\{x, f_1(x)\} - \max\{x, -d+f_1(x)\} \leq \max\{x, f_2(x)\} - \max\{x, -d+f_2(x)\}. \tag{8.1}$$

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