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A SLS TYPE OF ALLOCATION PROBLEM WITH RETIREMENT POLICY

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In this paper, we consider the following discrete-time sequential allocation problem with a finite horizon t. Suppose we have i units of material to manufacture certain products which are sold to successively appearing buyers. In producing a product, we can complete it with probability q where it is impossible to recycle unsuccessful units. A distribution of the price offered by each appearing buyer and the probability of successful production are known. For producing, we take a strategy of *shoot-look-shoot* scheme, implying that if a production is unsuccessful, then we must decide whether or not to consume an additional one. At the end of each period, we must further decide whether or not to retire the production activity. In case of retirement, we immediately receive a terminal reward depending on the remaining periods and the remaining units of material. The objective here is to examine the properties of the optimal decision rules which maximize the total expected reward, the expected total sales plus the expected terminal reward. We find two interesting features in the optimal decision rules: (1) the optimal decision rule for production is not always monotone in the number of units of material in hand, (2) the optimal decision rule for retirement may become possibly the following form; if there remain few or too many units of material, then it is optimal for us to retire, or else to continue.

1. Introduction

In this paper, we discuss the following sequential allocation problem with a finite planning horizon. Suppose we have i units of material to manufacture certain products where a unit of material is enough to make a product. In consuming a unit, the manufacturing is successful with probability q and unsuccessful with $1 - q$ where it is impossible to recycle unsuccessful units. Our planning horizon is t periods. At the beginning of each period, we may find a buyer, assumed that every buyer wants to buy only one product. Then, he immediately offers a buying price w, which is a random sample from a known distribution and independent of prices offered by buyers so far. If his offering price is attractive, we try to make a product immediately. Completing the product, we sell it for the price w, and of course an additional production is unnecessary. If we fail, the buyer may disappear with probability r. We can try to make a product over and over while the buyer stands still or the material remains. Repeated productions for the same buyer are assumed to waste no time. Such a way of decision that it is decided whether or not to make an attempt every time we confirm the result of the previous trial is called a *shoot-look-shoot* (for short, SLS) scheme. After each failure, we can also refuse the proposal of the present buyer.

If we sell one unit, lose the present buyer or refuse him, then we have to make an additional decision as to whether or not to retire the production activity. In case of retirement, we immediately receive a terminal reward depending on the remaining periods and the remaining units of material. If the production activity will be continued, then a period proceeds and we will search for another buyer. In this way, we will continue the production activity until arriving at the deadline or deciding to retire. Assume it is impossible to make a product before a buyer offers a price. Our purpose is to maximize the total expected reward, the expected total sales plus the expected terminal reward.

Many authors so far study sequential allocation problems, which can be classified into two types, shooting problems and economic ones. In the former, decision makers who have, for example, i torpedoes must decide how many to allocate to the present target of value w, a random variable. There exist the following two types of policies as to the allocation of torpedoes: SLS policy and *volley*. Here, by volley policy, we means that if the decision maker decides to shoot j torpedoes, then he shoots them in salvo.

Mastran and Thomas [4] treat a military problem in which the computational method to obtain the optimal decision rules for both policies are showed. Kisi [3] considers a model of SLS policy and examines the relation between the approximate solution and the exact. Sakaguchi [9] investigates the continuous-time version of Mastran and Thomas [4]. Namekata et. al. [5] deal with a model of volley policy where there exist two kinds of targets in a sense that the necessary number of torpedoes to sink them are different. Namekata et al. [7] also examine a problem of volley policy with unknown number of periods. The author [10] discuss a problem of SLS policy, in which the search cost must be payed to find a target, and it is derived that a critical value, at which firing or not become indifferent in the optimal decision, is not always decreasing† in the number of remaining bullets. Furthermore, the author [11] examine a problem in which it is possible to replenish some bullets by paying a certain cost.

On the other hand, Derman et al. [2] deal with a problem of volley policy as an economic investment problem. In their model, all investment opportunities have a common profit function depending on the amount of resources allocated. Namekata et al. [6] also discuss an economic problem of volley policy. They assume that a decision maker sells some of his goods to acquire a reward which depends on both the number of goods sold and a class of appearing customer, provided that unsold goods may possibly perish at the beginning of the next period. Prastacos [8] also considers an investment problem of volley policy in which a profit function depends on both

†In this paper, the words "increasing" and "decreasing" mean "nondecreasing" and "nonincreasing", respectively.

the quality of the present opportunity and the quantity of the resources invested.

In models such as stated above, the resources (or torpedoes, etc.) which remain at deadline are assumed to be valueless. In addition, except the author [11], if a decision maker consumes all of resources before the deadline, he only wastes his remaining planning horizon. However, he may get some profit in exchange for remaining resources, and as the proverb "time is money" says, he may be able to use the remaining periods for another purpose. For this reason, in this paper, we employ a concept of terminal reward explained previously.

By the way, there exists a sequential assignment problem by Derman et. al. [1]. Though it is closely related to a sequential allocation problem of SLS policy, what differs from the other are the following points. In a sequential allocation problem of SLS policy, resources are homogeneous and it is possible to invest an additional unit just after observing the result of the previous investment. In a sequential assignment problem, each resource has a certain kind of different value and an additional investment is impossible.

In the next section, we define variables and parameters used in this model, give its optimality equations and reveal fundamental properties of optimal decision rules. In Sections 3 and 4, two special cases are discussed. Conclusions obtained are summarized in Section 5.

2. Optimality Equations and Optimal Decision Rules

Let us define the following:

i	:	number of remaining units of material,	$i \geq 0$,
t	:	remaining planning horizon (point of time),	$t \geq 0$,
w	:	price offered by the present buyer,	$w \in [0, 1]$,
q	:	probability of successful production,	$q \in (0, 1]$,
r	:	probability of losing the present buyer,	$r \in [0, 1]$,
p	:	$p = (1-q)(1-r)$,	$p \in [0, 1)$,
β	:	discount factor,	$\beta \in (0, 1]$.

Furthermore, let

$u_t(i, w)$: maximum of the total expected reward with t periods and i units of material remaining when the present buyer offers w,

$v_t(i)$: expectation of $u_t(i, w)$ in terms of w,

$z_t(i)$: maximum of the total expected reward starting from time t with i units of material remaining when it is decided not to try to produce the product for the present buyer,

$R_t(i)$: terminal reward when t periods remain and i units of material are available, increasing in both t and i and concave in i.

We shall number points of time backward from the horizon point as 0, 1, and so on; the interval between time t and $t-1$ is called period t, as depicted in Figure 1.

A buyer appears with probability $\theta \in (0, 1]$, assumed that more than one buyer does not appear at the same point of time. The price he will offer is a random variable having a known probability distribution function $F_1(w)$, continuous or discrete, where $F_1(w) = 0$ for $w < 0$, $F_1(w) < 1$ for $w < 1$ and $F_1(w) = 1$ for $w \geq 1$. The distribution does not concentrate on only a point, that is, $\Pr(w) < 1$ for any w. The prices offered at successive points of time are assumed to be stochastically independent. Now let $F_0(w)$ be a distribution function where $\Pr(w) = 0$ for any $w \neq 0$. Then, using θ, we can combined $F_1(w)$ and $F_0(w)$ into the following distribution function:

$$F(w) = (1-\theta)F_0(w) + \theta F_1(w). \tag{2.1}$$

Then, we have the following recursive relations:

$$\begin{aligned} u_t(i, w) &= \max\{z_t(i),\ q(w + z_t(i-1)) + (1-q)(rz_t(i-1) + (1-r)u_t(i-1, w))\} \\ &= \max\{z_t(i),\ pu_t(i-1, w) + qw + (1-p)z_t(i-1)\}, \quad t \geq 0,\ i \geq 1, \end{aligned} \tag{2.2}$$

where

$$z_t(i) = \max\{R_t(i),\ \beta v_{t-1}(i)\}, \quad t \geq 1,\ i \geq 0, \tag{2.3}$$

$$v_t(i) = \int_0^1 u_t(i, \xi) dF(\xi), \quad t \geq 0,\ i \geq 1, \tag{2.4}$$

$$z_0(i) = R_0(i), \quad i \geq 0, \tag{2.5}$$

$$u_t(0, w) = v_t(0) = z_t(0), \quad t \geq 0. \tag{2.6}$$

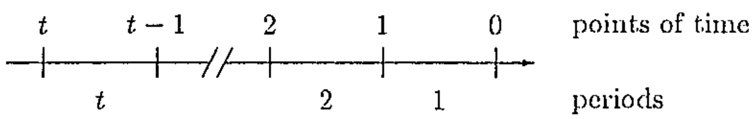

Figure 1 Points of time and periods

Here, assume $R_0(0) = 0$ and

$$R_t(i+1) - R_t(i) < q, \quad t \geq 0, i \geq 0, \tag{2.7}$$

which implies that, provided that a buyer offering the maximum price $w = 1$ appears fortunately and we accept it, the increasing rate of terminal reward as to i is always less than the expected reward from the buyer, that is, $q = 1 \times q + 0 \times (1-q)$. From (2.3), (2.5) and (2.6), immediately we have for $t \geq 1$

$$\begin{aligned} z_t(0) &= \max\{R_t(0),\ \beta v_{t-1}(0)\} \\ &= \max\{R_t(0),\ \beta \max\{R_{t-1}(0),\ \beta v_{t-2}(0)\}\} \\ &\ \vdots \\ &= \max_{0 \leq l \leq t} \beta^{t-l} R_l(0) = R_t(0). \end{aligned} \tag{2.8}$$

Hence, we also get $u_t(0, w) = v_t(0) = R_t(0)$ from (2.6). Now we will show some properties of the optimality equations.

Lemma 1.

(a) $u_t(i,w)$, $v_t(i)$ *and* $z_t(i)$ *are increasing in* t *for any* i *and* w.

(b) $u_t(i,w)$, $v_t(i)$ *and* $z_t(i)$ *are increasing in* i *for any* t *and* w.

(c) $u_t(i+1,w) - u_t(i,w) \leq q$ *for any* t, i *and* w *where the equal sign holds only when* $i = 0$ *and* $w = 1$. *In addition,* $v_t(i+1) - v_t(i) < q$ *and* $z_t(i+1) - z_t(i) < q$ *also hold for any* t *and* i.

(d) $u_t(i,w)$ *is increasing in* w *for any* t *and* i.

Proof: (a) From (2.3) and the definition of $R_t(i)$, we get $z_1(i) \geq R_1(i) \geq R_0(i) = z_0(i)$. Now assume $z_t(i) \geq z_{t-1}(i)$ for any i. Then, easily we get $u_t(0,w) = z_t(0) \geq z_{t-1}(0) = u_{t-1}(0,w)$. Furthermore, supposing $u_t(i-1,w) \geq u_{t-1}(i-1,w)$ as the second inductive assumption, we obtain $u_t(i,w) \geq u_{t-1}(i,w)$ from (2.2). Hence it follows that $u_t(i,w) \geq u_{t-1}(i,w)$ for any i and w. Accordingly we have $v_t(i) \geq v_{t-1}(i)$ for any i, which yields

$$\begin{aligned} z_{t+1}(i) &= \max\{R_{t+1}(i),\ \beta v_t(i)\} \\ &\geq \max\{R_t(i),\ \beta v_{t-1}(i)\} = z_t(i) \end{aligned} \tag{2.9}$$

for any i. By double induction, the statement is proven.

(b) By definition, $R_0(i)$, hence $z_0(i)$ is increasing in i. Now let $z_t(i+1) \geq z_t(i)$ for any i. It is obvious that $u_t(1,w) \geq z_t(1) \geq z_t(0) = u_t(0,w)$. In addition, assume $u_t(i,w) \geq u_t(i-1,w)$ as the second inductive assumption. Then we have

$$\begin{aligned} u_t(i+1,w) &= \max\{z_t(i+1),\ pu_t(i,w) + qw + (1-p)z_t(i)\} \\ &\geq \max\{z_t(i),\ pu_t(i-1,w) + qw + (1-p)z_t(i-1)\} = u_t(i,w). \end{aligned} \tag{2.10}$$

From above, it follows that $u_t(i,w)$ is increasing in i for any w. Therefore, we get $v_t(i+1) \geq v_t(i)$ for any i and w, which yields

$$\begin{aligned} z_{t+1}(i+1) &= \max\{R_{t+1}(i+1),\ \beta v_t(i+1)\} \\ &\geq \max\{R_{t+1}(i),\ \beta v_t(i)\} = z_{t+1}(i) \end{aligned} \tag{2.11}$$

for any i. By double induction, we obtain the statement.

(c) From the definition, we have for any i

$$z_0(i+1) - z_0(i) = R_0(i+1) - R_0(i) < q. \tag{2.12}$$

Now suppose $z_t(i+1) - z_t(i) < q$ for any i. Then , immediately we get

$$\begin{aligned} u_t(1,w) - u_t(0,w) &= \max\{z_t(1),\ qw + z_t(0)\} - z_t(0) \\ &= \max\{z_t(1) - z_t(0),\ qw\} \leq q \end{aligned} \tag{2.13}$$

where the inequality sign $\leq$ can be replaced with $<$ when $w < 1$. Further, let $u_t(i,w) - u_t(i-1,w) \leq q$. Then, using the *general formula* $\max_i a_i - \max_i b_i \leq \max_i(a_i - b_i)$ we have

$$\begin{aligned} u_t(i+1,w) - u_t(i,w) \leq \max\{&z_t(i+1) - z_t(i), \\ &p(u_t(i,w) - u_t(i-1,w)) + (1-p)(z_t(i) - z_t(i-1))\} < q. \end{aligned} \tag{2.14}$$

Hence it follows that $u_t(i+1,w) - u_t(i,w) < q$ for any i and w except $i = 0$ and $w = 1$. Accordingly we get $v_t(i+1) - v_t(i) < q$ and

$$z_{t+1}(i+1) - z_{t+1}(i) \leq \max\{R_{t+1}(i+1) - R_{t+1}(i),\ \beta(v_t(i+1) - v_t(i))\} < q \tag{2.15}$$

for all i. Thus by double induction, the proof is complete.

(d) It is easily proven by induction. ∎

Now, let $\bar{v}_t(\bar{\imath})$ be the function of real number $\bar{\imath} \in [0, \infty)$ defined by successively connecting two points $(i, v_t(i))$ and $(i+1, v_t(i+1))$, $i = 0, 1, 2, \ldots$, with a straight line, and so also be $\bar{R}_t(\bar{\imath})$. Furthermore, we shall define the following functions.

$$g_t(i, w) = pu_t(i-1, w) + qw + (1-p)z_t(i-1) - z_t(i), \quad t \geq 0,\ i \geq 1, \tag{2.16}$$

$$\bar{\psi}_t(\bar{\imath}) = \beta \bar{v}_{t-1}(\bar{\imath}) - \bar{R}_t(\bar{\imath}), \quad t \geq 1,\ \bar{\imath} \geq 0. \tag{2.17}$$

Then, by using $g_t(i, w)$ and $\bar{\psi}_t(\bar{\imath})$, optimal decision rules can be expressed as follows:

(a) If $g_t(i, w) \geq 0$, then product one unit, or else don't product.

(b) If $\bar{\psi}_t(i) \geq 0$, then continue the production activity, or else retire.

An w satisfying $g_t(i, w) = 0$, if it exists, is called a *production critical value* $h_t(i)$ for given t and i, and an $\bar{\imath}$ satisfying $\bar{\psi}_t(\bar{\imath}) = 0$, also if it exists, is called a *retirement critical value* ρ_t for a given t. The above functions have the following properties.

Lemma 2.

(a) *For $t \geq 0$ and $i \geq 1$, $g_t(i, w) = 0$ has a unique solution $h_t(i) \in [0, 1)$.*

(b) *$\bar{\psi}_t(0) \leq 0$ for $t \geq 1$.*

Proof: (a) It can be easily proven by induction that $u_t(i, 0) = z_t(0)$, hence we get

$$\begin{aligned} g_t(i, 0) &= pu_t(i-1, 0) + (1-p)z_t(i-1) - z_t(i) \\ &= z_t(i-1) - z_t(i) \leq 0. \end{aligned} \tag{2.18}$$

On the other hand, it follows from Lemma 1(c) that

$$\begin{aligned} g_t(i, 1) &= pu_t(i-1, 1) + q + (1-p)z_t(i-1) - z_t(i) \\ &\geq pz_t(i-1) + q + (1-p)z_t(i-1) - z_t(i) \\ &\geq q - (z_t(i) - z_t(i-1)) > 0. \end{aligned} \tag{2.19}$$

Furthermore, since $g_t(i, w)$ is a continuous function of w on $[0, 1]$, it follows that $g_t(i, w) = 0$ has a solution $w \in [0, 1)$. It is clear from (2.16) that $g_t(i, w)$ is strictly increasing in w for $t \geq 0$ and $i \geq 1$, hence the solution is unique.

(b) Since $v_t(0) = R_t(0)$, so $\bar{v}_t(0) = \bar{R}_t(0)$, we get $\bar{\psi}_t(0) = \beta \bar{R}_{t-1}(0) - \bar{R}_t(0) \leq 0$. ∎

Because $h_t(i)$ is unique and $h_t(i) \in [0, 1)$ for given t and i, we get the following optimal decision rule: if $w \geq h_t(i)$, then make one unit, or else don't make.

Now suppose $t = 0$. Then, immediately we get from (2.5) and (2.16)

$$0 = g_0(1, h_0(1)) = qh_0(1) + R_0(0) - R_0(1), \tag{2.20}$$

hence

$$h_0(1) = (R_0(1) - R_0(0))/q. \tag{2.21}$$

If $h_0(i) = (R_0(i) - R_0(i-1))/q$, then it follows from the concavity of $R_t(i)$ in i that

$$\begin{aligned} g_0(i+1, h_0(i)) &= pu_0(i, h_0(i)) + qh_0(i) + (1-p)z_0(i) - z_0(i+1) \\ &= qh_0(i) + z_0(i) - z_0(i+1) \\ &= (R_0(i) - R_0(i-1)) - (R_0(i+1) - R_0(i)) \geq 0. \end{aligned} \tag{2.22}$$

Therefore we obtain $h_0(i) \geq h_0(i+1)$, which yields

$$0 = g_0(i+1, h_0(i+1)) = pz_0(i) + qh_0(i+1) + (1-p)z_0(i) - z_0(i+1), \tag{2.23}$$

hence

$$h_0(i+1) = (R_0(i+1) - R_0(i))/q. \tag{2.24}$$

By induction, we have $h_0(i) = (R_0(i) - R_0(i-1))/q \leq h_0(i-1)$ for all $i \geq 1$. Furthermore, the following holds true in general:

$$\begin{aligned} 0 = g_t(i, h_t(i)) &= pu_t(i-1, h_t(i)) + qh_t(i) + (1-p)z_t(i-1) - z_t(i) \\ &\geq qh_t(i) + z_t(i-1) - z_t(i), \end{aligned} \tag{2.25}$$

which is rewritten

$$h_t(i) \leq (z_t(i) - z_t(i-1))/q, \quad t \geq 0,\ i \geq 1. \tag{2.26}$$

The lemma below tells us some more detailed properties of $h_t(i)$.

Lemma 3.

(a) *If $p > 0$, then for $t \geq 1$ and $i \geq 1$,*

$$h_t(i) \geq (<)\ h_t(i+1) \iff h_t(i+1) = (<)\ (z_t(i+1) - z_t(i))/q.$$

When $p = 0$, it always holds true for $t \geq 1$ and $i \geq 1$ that $h_t(i+1) = (z_t(i+1) - z_t(i))/q$.

(b) *For $t \geq 1$ and $i \geq 1$, if $h_t(i) \geq h_t(i+1)$, then $2z_t(i) - z_t(i-1) - z_t(i+1) \geq 0$.*

(c) *For $t \geq 1$ and $i \geq 1$, if $2z_t(i) - z_t(i-1) - z_t(i+1) < 0$, then $h_t(i) < h_t(i+1)$.*

(d) *Assume $h_t(i) = (z_t(i) - z_t(i-1))/q$ for $t \geq 1$ and $i \geq 1$. Then*

$$2z_t(i) - z_t(i-1) - z_t(i+1) \geq 0 \implies h_t(i) \geq h_t(i+1).$$

(e) *For a given t and a given positive integer I, the critical value $h_t(i)$ is decreasing in $i \leq I+1$, if and only if*

$$2z_t(i) - z_t(i-1) - z_t(i+1) \geq 0, \quad 1 \leq i \leq I.$$

(f) *For a given t and a given positive integer I, if $h_t(i)$ is decreasing in $i \leq I+1$, then*

$$2v_t(i) - v_t(i-1) - v_t(i+1) \geq 0, \quad 1 \leq i \leq I.$$

Proof: (a) Suppose $p > 0$ and $h_t(i) \geq (<)\, h_t(i+1)$ for $t \geq 0$ and $i \geq 0$. Then, since $g_t(i, w)$ is strictly increasing in w, we get

$$\begin{aligned} 0 = g_t(i+1, h_t(i+1)) &= pu_t(i, h_t(i+1)) + qh_t(i+1) + (1-p)z_t(i) - z_t(i+1) \\ &=(<)\, qh_t(i+1) + z_t(i) - z_t(i+1), \end{aligned} \tag{2.27}$$

hence

$$h_t(i+1) =(<)\, (z_t(i+1) - z_t(i))/q. \tag{2.28}$$

To go the other way, if $p > 0$ and $h_t(i+1) = (<)\, (z_t(i+1) - z_t(i))/q$ for given t and i, then it follows that

$$\begin{aligned} 0 = g_t(i+1, h_t(i+1)) &= pu_t(i, h_t(i+1)) + qh_t(i+1) + (1-p)z_t(i) - z_t(i+1) \\ &=(<)\, p(u_t(i, h_t(i+1)) - z_t(i)), \end{aligned} \tag{2.29}$$

hence

$$z_t(i) =(<)\, u_t(i, h_t(i+1)) \tag{2.30}$$

implying that

$$h_t(i) \geq (<)\, h_t(i+1). \tag{2.31}$$

When $p = 0$, the statement is immediate from the fact that $g_t(i+1, w) = qw + z_t(i) - z_t(i+1)$.

(b) From the assumption and (2.26), we have

$$0 \leq g_t(i+1, h_t(i)) = qh_t(i) + z_t(i) - z_t(i+1) \leq 2z_t(i) - z_t(i-1) - z_t(i+1). \tag{2.32}$$

(c) The statement is the contraposition of (b).

(d) It follows from (2.26) that

$$\begin{aligned} 0 &\leq 2z_t(i) - z_t(i+1) - z_t(i-1) \\ &= qh_t(i) + z_t(i) - z_t(i+1) \\ &\leq q(h_t(i) - h_t(i+1)), \quad t \geq 1\, i \geq 1. \end{aligned} \tag{2.33}$$

(e) First, we have for all t

$$0 = g_t(1, h_t(1)) = pu_t(0, h_t(1)) + qh_t(1) + (1-p)z_t(0) - z_t(1)$$
$$= qh_t(1) + z_t(0) - z_t(1). \tag{2.34}$$

If $h_t(i) \geq h_t(i+1)$ for $i \leq I$, then from Lemma 3(a) and (2.34), we have

$$z_t(i) - z_t(i-1) \geq z_t(i+1) - z_t(i), \quad 1 \leq i \leq I. \tag{2.35}$$

Conversely, assume $2z_t(i) - z_t(i-1) - z_t(i+1) \geq 0$ for $1 \leq i \leq I$. It is clear $h_t(1) = (z_t(1) - z_t(0))/q$. Supposing $h_t(i) = (z_t(i) - z_t(i-1))/q$, we get

$$h_t(i) \geq h_t(i+1) \tag{2.36}$$

from Lemma 3(d), which yields

$$h_t(i+1) = (z_t(i+1) - z_t(i))/q \tag{2.37}$$

owing to Lemma 3(a). By induction, we get $h_t(i) = (z_t(i) - z_t(i-1))/q$ and $h_t(i-1) \geq h_t(i)$ for $1 \leq i \leq I+1$.

(f) Let $X_t(i, \xi)$ be such that

$$\begin{aligned} X_t(i,\xi) = {} & 2\max\{z_t(i)\ pu_t(i-1,\xi) + q\xi + (1-p)z_t(i-1) + z_t(i)\} \\ & - \max\{z_t(i-1)\ pu_t(i-2,\xi) + q\xi + (1-p)z_t(i-2) + z_t(i-1)\} \\ & - \max\{z_t(i+1)\ pu_t(i,\xi) + q\xi + (1-p)z_t(i) + z_t(i+1)\}, \quad i \geq 1. \end{aligned} \tag{2.38}$$

Using (2.38), we obtain the following equation:

$$2v_t(i) - v_t(i-1) - v_t(i+1) = \int_0^1 X_t(i,\xi) dF(\xi), \quad i \geq 1. \tag{2.39}$$

From the assumption, it is true that $h_t(i+1) \leq h_t(i)$ for $i \leq I$. Then, we get

$$X_t(i,\xi) = 2z_t(i) - z_t(i-1) - z_t(i+1) \geq 0 \tag{2.40}$$

for $0 \leq \xi \leq h_t(i+1)$ from Lemma 3(c),

$$\begin{aligned} X_t(i,\xi) &= 2z_t(i) - z_t(i-1) - (pz_t(i) + q\xi + (1-p)z_t(i)) \\ &= z_t(i) - z_t(i-1) - q\xi = q(h_t(i) - \xi) = 0 \end{aligned} \tag{2.41}$$

for $h_t(i+1) \leq \xi \leq h_t(i)$ from Lemma 3(a), and

$$X_t(i,\xi) = 2u_t(i,\xi) - u_t(i-1,\xi) - (pu_t(i,\xi) + q\xi + (1-p)z_t(i))$$
$$= (2-p)(pu_t(i-1,\xi) + q\xi + (1-p)z_t(i-1)) - u_t(i-1,\xi) - q\xi - (1-p)z_t(i)$$
$$= -(1-p)^2 u_t(i-1,\xi) + (1-p)q\xi - (1-p)z_t(i) + (2-p)(1-p)z_t(i-1) \quad (2.42)$$

for $h_t(i) \leq \xi \leq 1$.

Below, the proof of $X_t(i,\xi) \geq 0$ for $h_t(i) \leq \xi \leq 1$ and $1 \leq i \leq I$ is made by induction. First, for $i = 1$ and $h_t(1) \leq \xi \leq 1$, we obtain

$$X_t(1,\xi) = -(1-p)^2 u_t(0,\xi) + (1-p)q\xi - (1-p)z_t(1) + (2-p)(1-p)z_t(0)$$
$$= (1-p)q\xi - (1-p)(z_t(1) - z_t(0)) = q(1-p)(\xi - h_t(1)) \geq 0. \quad (2.43)$$

Assuming $X_t(i-1,\xi) \geq 0$ for $h_t(i-1) \leq \xi \leq 1$, we have for $h_t(i-1) \leq \xi \leq 1$

$$X_t(i,\xi) = -(1-p)^2(pu_t(i-2,\xi) + q\xi + (1-p)z_t(i-2))$$
$$+(1-p)q\xi - (1-p)z_t(i) + (2-p)(1-p)z_t(i-1)$$
$$= p(-(1-p)^2(u_t(i-2,\xi) + (1-p)q\xi - (1-p)z_t(i-1) + (2-p)(1-p)z_t(i-2))$$
$$+(1-p)(2z_t(i-1) - z_t(i-2) - z_t(i))$$
$$= pX_t(i-1,\xi) + (1-p)(2z_t(i-1) - z_t(i-2) - z_t(i)) \geq 0. \quad (2.44)$$

Further, for $h_t(i) \leq \xi \leq h_t(i-1)$, we get

$$X_t(i,\xi) = -(1-p)^2 z_t(i-1) + (1-p)q\xi - (1-p)z_t(i) + (2-p)(1-p)z_t(i-1)$$
$$= (1-p)(q\xi - z_t(i) + z_t(i-1)) = q(1-p)(\xi - h_t(i)) \geq 0. \quad (2.45)$$

Therefore, it follows that

$$X_t(i,\xi) \geq 0, \quad h_t(i) \leq \xi \leq 1,\ 1 \leq i \leq I. \quad (2.46)$$

Thus, it follows from (2.40), (2.41) and (2.46) that the statement holds true. ∎

In the following two sections, we examine two cases where the terminal reward $R_t(i)$ takes concrete forms.

3. Case 1: $R_t(i) = bi$

In this section, let us assume

$$R_t(i) = bi, \quad 0 \leq b < q,\ t \geq 0,\ i \geq 0, \quad (3.1)$$

where b is a unit price of material, which is constant. Then, the following theorem holds.

Theorem 1. *Suppose $\beta = 1$. Then*

(a) *it is optimal to continue the production activity over the entire planning horizon, that is, $\bar{\psi}_t(i) \geq 0$ for $t \geq 1$ and $i \geq 0$,*

(b) *$h_t(i)$ is decreasing in i for any t,*

(c) *$h_t(i)$ is increasing in t for any i.*

Proof: (a) Because $\beta = 1$, we get for $t \geq 1$ and $i \geq 0$

$$v_t(i) = \int_0^1 u_t(i, \xi) dF(\xi) \geq \int_0^1 z_t(i) dF(\xi) = z_t(i) = \max\{bi, v_{t-1}(i)\} \geq bi. \tag{3.2}$$

In addition, $v_0(i) = \int u_0(i, \xi) dF(\xi) \geq \int R_0(i) dF(\xi) = bi$. Therefore, we obtain

$$\bar{\psi}_t(i) = v_{t-1}(i) - bi \geq 0, \quad t \geq 1, \ i \geq 0. \tag{3.3}$$

(b) We have $h_0(i) = b/q$ from (2.24), so $h_0(i)$ is decreasing in i. Accordingly, it follows from Lemma 3(f) and Theorem 1(a) that

$$2z_1(i) - z_1(i-1) - z_1(i+1) = 2v_0(i) - v_0(i-1) - v_0(i+1) \geq 0, \quad i \geq 1 \tag{3.4}$$

implying from Lemma 3(e) that $h_1(i)$ is decreasing in i. Repeating the similar procedure for $t = 1, 2, \ldots$, we get the conclusion that $h_t(i)$ is decreasing in i for any t. Here note that, in the above proof, we also obtain

$$2z_t(i) - z_t(i-1) - z_t(i+1) \geq 0, \quad t \geq 0, \ i \geq 1. \tag{3.5}$$

This will be used in the proof of (c).

(c) Because $u_t(0, w) = z_t(0) = R_t(0) = 0$, we have

$$u_t(1, w) - u_t(0, w) = u_t(1, w) \geq z_t(1) = z_t(1) - z_t(0). \tag{3.6}$$

Suppose $u_t(i, w) - u_t(i-1, w) \geq z_t(i) - z_t(i-1)$. Then, we obtain for $0 \leq w \leq h_t(i+1)$

$$u_t(i+1, w) - u_t(i, w) = z_t(i+1) - z_t(i), \tag{3.7}$$

for $h_t(i+1) \leq w \leq h_t(i)$

$$u_t(i+1, w) - u_t(i, w) = u_t(i+1, w) - z_t(i) \geq z_t(i+1) - z_t(i), \tag{3.8}$$

and for $h_t(i) \leq w \leq 1$

$$u_t(i+1,w) - u_t(i,w) = p(u_t(i,w) - u_t(i-1,w)) + (1-p)(z_t(i) - z_t(i-1))$$
$$\geq z_t(i) - z_t(i-1) \geq z_t(i+1) - z_t(i) \qquad (3.9)$$

from (3.5). Then, it follows that

$$v_t(i+1) - v_t(i) = \int_0^1 (u_t(i+1,\xi) - u_t(i,\xi))dF(\xi) \geq z_t(i+1) - z_t(i), \qquad (3.10)$$

which yields

$$z_{t+1}(i+1) - z_{t+1}(i) = v_t(i+1) - v_t(i) \geq z_t(i+1) - z_t(i). \qquad (3.11)$$

Therefore, we obtain $h_{t+1}(i) \geq h_t(i)$ from Lemma 3(a,c). By induction, the statement is verified. ∎

In case for $\beta = 1$, we arrive at the intuitive conclusion that $h_t(i)$ is increasing in t and decreasing in i. However, the properties described in Theorem 1 are not always true if $\beta < 1$ and $b > 0$.

Since $h_0(i) = (R_0(i) - R_0(i-1))/q = b/q$, we immediately get the following equation by induction.

$$u_0(i,w) = bi + \frac{1-p^i}{1-p}\max\{qw - b,\ 0\}, \quad i \geq 0. \qquad (3.12)$$

Hence we have

$$v_0(i) = bi + \frac{1-p^i}{1-p}\int_{b/q}^1 (q\xi - b)dF(\xi) = bi + \alpha(1-p^i), \quad i \geq 0, \qquad (3.13)$$

where α is a positive number. Therefore, if $\beta < 1$ and $b > 0$, then clearly

$$\bar{\psi}_1(i) = \beta\alpha(1-p^i) - (1-\beta)bi \to -\infty \ (i \to \infty), \qquad (3.14)$$

that is, it becomes optimal to retire for sufficiently large i.

Next, we shall show an example in which $h_t(i)$ is not decreasing in i, that is, $h_t(i') < h_t(i'+1)$ for at least one $i' \geq 1$. Suppose w follows a continuous uniform distribution on $[0,1]$. Then, (3.13) can be rewritten

$$v_0(i) = bi + \frac{1-p^i}{1-p}\int_{b/q}^1 (q\xi - b)d\xi = bi + \frac{1-p^i}{1-p}\frac{(q-b)^2}{2q}. \qquad (3.15)$$

Furthermore, assuming $p = q = 1/2$ $(r = 0)$, $b = 2/5$ $(< q)$ and $\beta = 160/163$, we get

$$\beta v_0(1) = 328/815 > 2/5 = b, \qquad (3.16)$$

$$\beta v_0(2) = 4/5 = 2b, \qquad (3.17)$$

$$\beta v_0(3) = 974/815 < 6/5 = 3b. \qquad (3.18)$$

Accordingly, it follows that

$$2z_1(2) - z_1(1) - z_1(3) = 4b - \beta v_0(1) - 3b = b - \beta v_0(1) < 0. \tag{3.19}$$

Thus from Lemma 3(c), we obtain $h_1(2) < h_1(3)$. However, it is possible that $h_t(i)$ is decreasing in i on a certain range of i even if $\beta < 1$ and $b > 0$. The next theorem states a sufficient condition for $h_t(i)$ to be decreasing in i.

Theorem 2. *Assume $\beta < 1$ and $0 \leq b \leq \beta q$. Then, the equation $\bar{\psi}_1(\bar{\imath}) = 0$ has only two different solutions $\rho_1 = 0$ and $\rho_1 = \rho_1'(> 1)$. Furthermore, for $t \geq 0$ and $i \leq \rho_1'$, $\bar{\psi}_t(i) \geq 0$ and the production critical value $h_t(i)$ is decreasing in i.*

Proof: From (3.13), $v_0(i)$ is strictly concave in i, hence the difference $\bar{v}_0(\bar{\imath}+1) - \bar{v}_0(\bar{\imath})$ is strictly decreasing in $\bar{\imath}$. Therefore, the difference $\bar{\psi}_1(\bar{\imath}+1) - \bar{\psi}_1(\bar{\imath})$ is also strictly decreasing in $\bar{\imath}$. In addition, we have $\bar{\psi}_1(0) = 0$, $\bar{\psi}_1(1) = \beta v_0(1) - b > \beta q - b \geq 0$ from Lemma 1(c) and $\bar{\psi}_1(\bar{\imath}) \to -\infty$ $(\bar{\imath} \to \infty)$ owing to $\beta < 1$. Accordingly, we get $\rho_1 = 0$ and $\rho_1' \in (1, \infty)$, which leads us to $\bar{\psi}_1(i) \geq 0$ for $i \leq \rho_1'$. Thus, we obtain for $t \geq 1$ and $i \leq \rho_1'$

$$\bar{\psi}_t(i) = \beta \bar{v}_{t-1}(i) - bi \geq \beta \bar{v}_0(i) - bi = \bar{\psi}_1(i) \geq 0 \tag{3.20}$$

from Lemma 1(b). Below, let us prove that $h_t(i)$ is decreasing in $i \in [0, \rho_1']$ for any t. It is obvious that $h_0(i)$ $(= b/q)$ is decreasing in i. Assume $h_t(i)$ is decreasing in $i \leq \rho_1'$. Then, from Lemma 3(f), we obtain

$$2v_t(i) - v_t(i-1) - v_t(i+1) \geq 0, \quad 1 \leq i \leq \rho_1' - 1. \tag{3.21}$$

From (3.20) and (3.21), we have for $1 \leq i \leq \rho_1' - 1$

$$2z_{t+1}(i) - z_{t+1}(i-1) - z_{t+1}(i+1) = \beta(2v_t(i) - v_t(i-1) - v_t(i+1)) \geq 0. \tag{3.22}$$

Thus, eventually it follows from Lemma 3(e) that $h_{t+1}(i)$ is decreasing in $i \leq \rho_1'$. By induction, it is proven that $h_t(i)$ is decreasing in $i \leq \rho_1'$ for all t. ∎

Now we shall state a practical implication of the optimal decision rule characterized by $h_t(i)$ when we take the SLS scheme. Assume $h_t(i)$ is such as in Figure 2.

First, let $w = w_a$. If we have more than twelve units, then we should continue to make a product until at least one of the following three events occurs; we sell a complete product, the remaining number of units of material becomes less than thirteen or the buyer disappears.

It is mentioned in Section 1 that following SLS policy, we must make a decision every time a production is unsuccessful. However, eventually it follows from the above explanation that we need not make a decision every after an unsuccessful result, but it suffices for us to decide up to how many units to consume only when a buyer offers a price. On the other hand, starting with less than thirteen units, we should reject the present buyer and search for the next.

Second, let $w = w_b$. In this case, we should continue to make a product until we complete a product, loses the present buyer or consume all units.

Last, let $w = w_c$. If we have more than eight units, then it is optimal to try to make a product until we complete one, the buyer disappears or the remaining units become less than nine. If we have more than four and less than nine, then we should refuse the price. Furthermore, if we have less than five units, then we should try manufacturing until a product is sold, the buyer disappears or all of units are spent. It goes without saying that such a case never occurs if $h_t(i)$ is decreasing in i.

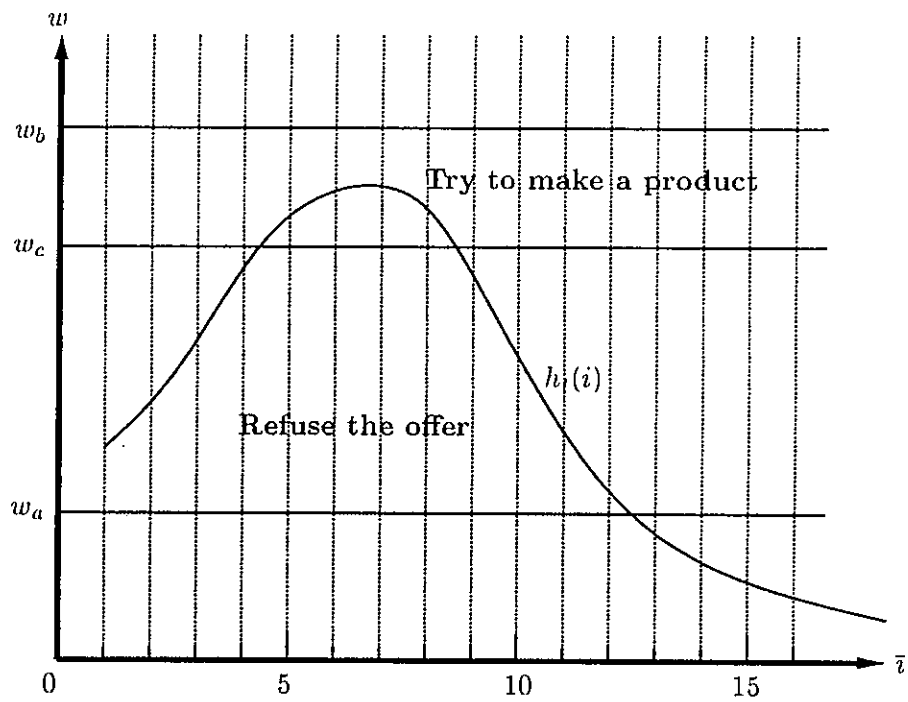

Figure 2 The Sequence of $h_t(i)$ for a given t and certain parameters

4. Case 2: $R_t(i) = bi + k_t$

Next, suppose

$$R_t(i) = bi + k_t, \quad t \geq 0, \, i \geq 0. \tag{4.1}$$

Here, b is a unit price of material and k_t is a reward expected over the remaining periods t where k_t is increasing in t with $k_0 = 0$.

Now we consider the following example.

$$k_t = \begin{cases} tc, & \beta = 1, \\ (1-\beta)c/(1-\beta^t), & \beta < 1, \end{cases} \tag{4.2}$$

where $c\,(\geq 0)$ is a cost required to search for a buyer each period. That is, in this example, k_t can be looked upon as the total search budget under an assumption that we continue the production activity over the whole planning horizon. If we decide to retire at time $t \geq 1$, then we can receive, as a terminal reward, the search budget appropriated for the remaining periods. Therefore, this example is equivalent to a sequential allocation problem in which a search cost is payed each period to find a new buyer. Hence this model can be regarded as a more general form of the author [10].

In the above example, k_t is given from the view point of the search budget. In addition, it may be also possible to consider the situation that we can gain our reward by engaging on another work for the remaining periods. Anyhow, Case 2 is more intricate than Case 1.

As one of the reflexes of the complexity, an optimal decision rule for retirement can become the following form. Now assume $q = 1/2$, $r = 0$ (hence $p = 1/2$), $\beta = 5/6$, $b = 1/10$, $k_1 = 2/15$ and $F(w)$ is a continuous uniform distribution function on $[0, 1]$. Then, it follows that

$$\beta v_0(1) = 13/60 < 7/30 = b + k_1, \tag{4.3}$$

$$\beta v_0(2) = 11/30 > 1/3 = 2b + k_1. \tag{4.4}$$

Furthermore, from (3.14), we get for sufficiently large I

$$\beta v_0(I) < bI + k_1. \tag{4.5}$$

The above example means that if we appropriately take parameters q, r, β, b and k_1, then there may exist $\rho_1' > 1$ and $\rho_1'' > \rho_1' + 1$ that are solutions of $\bar{\psi}_1(\bar{\imath}) = \beta \bar{v}_0(\bar{\imath}) - (b\bar{\imath} + k_1) = 0$, as drawn in Figure 3. Then, the optimal decision rule for retirement becomes the following manner; if $\rho_1' \leq i \leq \rho_1''$, then continue the production activity, or else retire. This implies that if there remain few or too many units, then it is optimal to retire, or else, that is, if moderate number of units remain, then it is

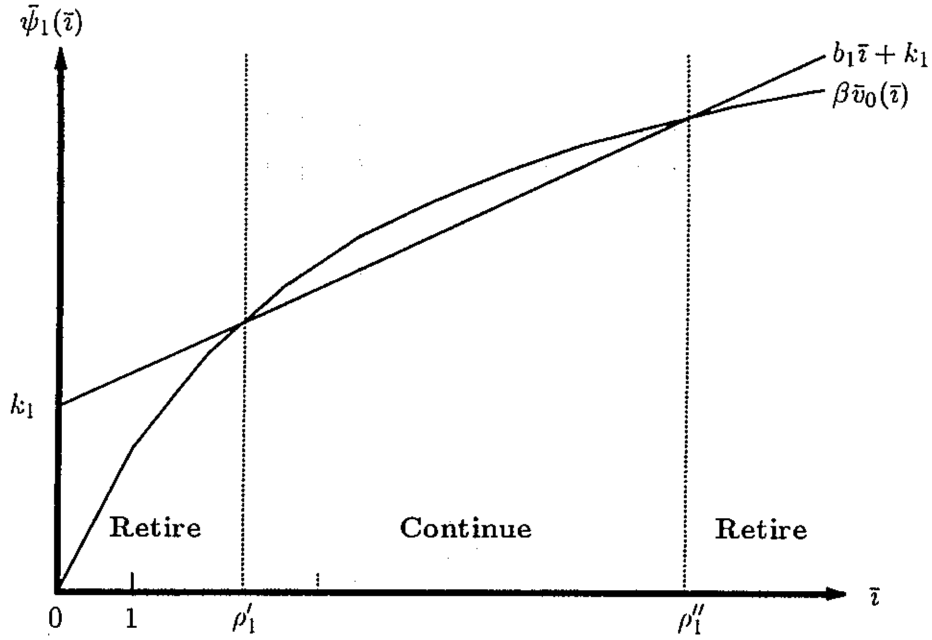

Figure 3 The Existence of ρ_1' and ρ_1''

optimal to continue working. We cannot always assert that such phenomena are counterintuitive. It may be possible that a more complex human behavior is described by a certain kind of definition of $R_t(i)$.

5. Conclusions

We considered a discrete-time finite horizon sequential allocation problem with a terminal reward in which a SLS scheme is adopted. Introducing a concept of terminal reward, we can give certain kinds of value to remaining periods and materials. This is an important concept in applying the framework of this model to realistic problems. In Section 2, we describe some fundamental properties of the optimal decision rules. In Sections 3 and 4, we examine two special cases in terms of $R_t(i)$ and find two characteristic rules: the production critical value $h_t(i)$ is not always decreasing in the number of units of material and the retirement critical value ρ_t is not always unique for a given t.

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