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A Note on a Continuation Method for Linear Complementarity Problems

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Abstract. This short report aims to find some properties which can be seen commonly in so-called *noninterior continuation methods* and *interior-point methods* for solving linear complementarity problems. We deal with a method of using a smooth approximation to the plus function $\max\{x, 0\}$ as an example of noninterior continuation methods. Theoretically, we show that the approximate solution provided for the method exists even under a condition which is often imposed in interior-point methods. Our computational results indicate that the number of iterations of the noninterior continuation method is almost constant regardless of the number of variables, similarly as in many interior-point methods.

Keywords. Linear complementarity problems, interior-point methods, noninterior-point methods, positive semi-definite matrices.

1 Introduction

The complementarity problem is to find a pair $(x, y) \in \mathbb{R}^{2n}$ which satisfies

$$(x, y) \geq 0,\ x_i y_i = 0 (i = 1, 2, \ldots, n),\ y = f(x)$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$. Recently, many noninterior continuation methods has been proposed for complementarity problems and/or variational inequality problems ([4, 5, 1, 14], etc.). Among others, the method proposed by Chen and Mangasarian [1] is attractive in the sense that the method is based on the idea of using the sigmoid function

$$s(x, \alpha) = \frac{1}{1 + e^{-\alpha x}}$$

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where $\alpha > 0$ and $x \in R$. The sigmoid function approximates the step function

$$\sigma(x) = \begin{cases} 1, & \text{if } x > 0, \\ 0, & \text{if } x \leq 0 \end{cases}$$

effectively and commonly used in neural networks (see [11], for example).

The complementarity problem can be reformulated using the plus function given by

$$(x)_+ = \max\{x, 0\}$$

for $x \in R$. In fact, we can easily see that the problem to find a $(x, y) \in \mathbb{R}^{2n}$ which satisfies

$$x_i - (x_i - y_i)_+ = 0 \ (i = 1, 2, \ldots, n), \ y = f(x) \tag{1}$$

is equivalent to the complementarity problem. In consideration of the fact that

$$(x)_+ = \int_{-\infty}^{x} \sigma(y) dy,$$

Chen and Mangasarian [1] utilized the integral of the sigmoid function

$$p(x, \alpha) = \int_{-\infty}^{x} s(y) dy = x + \frac{1}{\alpha} \log(1 + e^{-\alpha x})$$

as an approximation to the plus function $(x)_+$, and defined the smooth system

$$x_i - p(x_i - y_i, \alpha) = 0 \ (i = 1, 2, \ldots, n), \ y = f(x) \tag{2}$$

as an approximation to the system (1). As stated in Lemma 1.1 of [1], the function $p(\cdot, \alpha)$ is strictly convex and strictly increasing on $\mathbb{R}$, and $\lim_{\alpha \to \infty} p(x, \alpha) = (x)_+$ for every $x \in \mathbb{R}$ and $\alpha > 0$. Let us assume that the mapping f is linear, i.e., f is given by $f(x) = Mx + q$ for some $n \times n$ matrix M and n-dimensional vector q. In order to ensure the existence of a solution of the system (2), Chen and Mangasarian imposed the following condition on the matrix M:

Condition 1.1. **(i)** *The matrix M is a P_0-matrix, i.e., a matrix with nonnegative principal minors.*

(ii) *The matrix M is an R_0-matrix, i.e., the linear complementarity problem with $f(x) = Mx$ has the only solution $(x, y) = (0, 0)$.*

On the other hand, in the field of interior point methods, it is well-known that the following is a sufficient condition for the existence of the path of centers and/or the polynomial complexity bound of the algorithms (see [7, 9, 6], etc.):

Condition 1.2. **(i)** *There exists an feasible-interior point $(\bar{x}, \bar{y})$ such that $(\bar{x}, \bar{y}) > 0$ and $\bar{y} = M\bar{x} + q$.*

(ii) *The matrix M is positive semidefinite.*

The positive semidefinite matrix M is a P_0-matrix, and Condition 1.1 implies that (i) of Condition 1.2 holds (see Section 3 of [5]). Thus, both of the above conditions commonly require that (i) of Condition 1.1 and (i) of Condition 1.2 hold. However, the 2-dimensional linear complementarity problem with

$$M = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad q = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

satisfies Condition 1.2 but not Condition 1.1 since M is not an R_0-matrix. Therefore, Condition 1.2 does not ensure that Condition 1.1 holds. This short report is motivated by the above difference between the two conditions. In the next section, we will show that the solution of the system (2) has a solution even if Condition 1.2 holds. Section 3 is devoted to a comparison between Chen and Mangasarian's continuation method and an (polynomial) interior-point algorithm proposed in [12]. After a brief description of these methods, some computational results are presented. These preliminary results sugget that the required iteration number of the continuation method is almost constant regardless of number of variables, which is analogous to the case of the interior-point method.

2 The existence of the approximate solution

In this section, we show that the approximate system (2) has a solution under Condition 1.2. First we state the following lemma which is easily checked:

Lemma 2.1 (Lemma 4.1 of [1]) : *The following are equivalent.*

(i) $x = p(x - y, \alpha)$.

(ii) $e^{-\alpha x} + 1e^{-\alpha y} = 1$.

(iii) $x = -\frac{1}{\alpha}\log(1 - e^{-\alpha y})$.

In view of the above lemma, we can see that if the minimization problem

$$\begin{array}{ll} \text{Minimize} & \phi(x, y, \alpha) = \frac{1}{2}\|e^{-\alpha x} + e^{-\alpha y} - 1\|_2^2 \\ \text{subject to} & y = f(x), \end{array} \tag{3}$$

has an optimal solution $(x(\alpha), y(\alpha))$ with $\phi(x(\alpha), y(\alpha)) = 0$ then the point $(x(\alpha), y(\alpha))$ is a solution of the system (2).

In the remainder of this section, we assume that the mapping f is given by $f(x) = Mx + q$ and that Condition 1.2 holds. Under (i) of the condition, we obtain the following result concerning the problem (3) and the approximate system (2).

Theorem 2.2 (Theorem 4.3 of [1]) : *Suppose that (i) of Condition 1.2 holds. Then for every $\alpha > 0$, the stationary point $x(\alpha)$ of the problem (3) is a solution of (2).*

In order to show the existence of a solution of (3), we use the following lemma.

Lemma 2.3. *Suppose that $x \in \mathbb{R}$ and $y \in \mathbb{R}$ satisfy the inequality*

$$0 < 1 - \lambda_{\min} \leq e^{-\alpha x} + e^{-\alpha y} \leq 1 + \lambda_{\max}$$

for some $\lambda_{\min}$ and $\lambda_{\max}$ and $\alpha > 0$. Then we have

$$-\frac{1}{\alpha}\log(1+\lambda_{\max}) \leq x \leq -\frac{1}{\alpha}\log(1-\lambda_{\min}-e^{-\alpha y})$$
$$-\frac{1}{\alpha}\log(1+\lambda_{\max}) \leq y \leq -\frac{1}{\alpha}\log(1-\lambda_{\min}-e^{-\alpha x})$$

Proof: The proof is straightforward by using $e^{-\alpha x} > 0$ and $e^{-\alpha y} > 0$. ∎

The next theorem is the intention of this section, which is analogous to Theorem 4.4 of [1].

Theorem 2.4. *Suppose that Condition 1.2 holds. Then the system (2) has a solution.*

Proof: We show that the level set

$$L(\lambda) = \{(x,y) : \phi(x,y,\alpha) \leq \lambda,\ y = Mx + q\}$$

of the problem (3) is compact for a sufficiently small $\lambda > 0$. It follows from the continuity of the function ϕ that the set $L(\lambda)$ is closed. Hence it suffices to show that the $L(\lambda)$ is bounded. Let $\bar{\beta}$ be the minimum value of the components of the feasible-interior-point $(\bar{x}, \bar{y})$ whose existence is guaranteed by (ii) of Condition 1.2:

$$0 < \bar{\beta} = \min\{\bar{x}_i, \bar{y}_i\ (i = 1, 2, \ldots, n)\}.$$

Let us take

$$0 < \bar{\lambda} < 1 - e^{-\alpha\bar{\beta}} < 1, \tag{4}$$

and define

$$\lambda = \frac{1}{2}\bar{\lambda}^2. \tag{5}$$

From the definition of ϕ and (5), we can see that

$$-\bar{\lambda} \leq e^{-\alpha x_i} + e^{-\alpha y_i} - 1 \leq \bar{\lambda}\ \ (i = 1, 2, \ldots, n) \tag{6}$$

for every $(x, y) \in L(\lambda)$. In view of Lemma 2.3, the above inequalities give us the following bounds for $(x, y) \in L(\lambda)$:

$$-\frac{1}{\alpha}\log(1+\bar{\lambda}) \le x_i \le -\frac{1}{\alpha}\log(1-\bar{\lambda}-e^{-\alpha y_i}) \quad (i = 1, 2, \ldots, n) \tag{7}$$

$$-\frac{1}{\alpha}\log(1+\bar{\lambda}) \le y_i \le -\frac{1}{\alpha}\log(1-\bar{\lambda}-e^{-\alpha x_i}) \quad (i = 1, 2, \ldots, n) \tag{8}$$

(9)

Let us assume that there exists an unbounded sequence $\{(x^k, y^k)\} \subset \mathbb{R}^{2n}$ such that $(x^k, y^k) \in L(\lambda)$ $(k = 1, 2, \ldots)$. Since the inequalities (7) and (8) imply that x_i and y_i are bounded from below for every i, there exists an index i and a subsequence $\{x_i^k\}$ (or $\{y_i^k\}$) such that $x_i^k \to \infty$ (or $y_i^k \to \infty$). Here the requirement on $\bar{\lambda}$ implies that

$$-\frac{1}{\alpha}\log(1-\bar{\lambda}) < \bar{\beta} = \min\{\bar{x}_i, \bar{y}_i \ (i = 1, 2, \ldots, n)\}.$$

Thus, from the inequality (8) (or (7)), there exists a K such that $y_i^k - \bar{y}_i < 0$ (or $x_i^k - \bar{x}_i < 0$) for every $k > K$. This implies that $(x_i^k - \bar{x}_i)(y_i^k - \bar{y}_i) \to -\infty$. However, since every unbounded subsequence possesses this property, we finally see that there exists $\{(x^{k_l}, y^{k_l})\} \subset L(\lambda)$ such that

$$(x^{k_l} - \bar{x})^T(y^{k_l} - \bar{y}) = (x^{k_l} - \bar{x})^T M (x^{k_l} - \bar{x})$$

diverges to $-\infty$ which contradicts to (ii) of Condition 1.2. Thus the level set $L(\lambda)$ is compact and as in the proof of Theorem 4.4 of [1], the problem (3) has a solution which satisfies (2). ∎

3 Numerical results

The aim of this section is to give some numerical results of Chen and Mangasarian's continuation method and to compare them with the results of an interior-point method proposed in [12]. The problem setting follows the paper [12] according to this aim. We briefly describe some elements used in the experimentation.

We considered the two types of the linear complementarity problems problems which are handled in [12] and whose matrix M and vector q are given as follows:

(P1):

$$M = \begin{pmatrix} 1 & 2 & 2 & \cdots & 2 \\ 0 & 1 & 2 & \cdots & 2 \\ 0 & 0 & 1 & \cdots & 2 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \text{ and } q = \begin{pmatrix} -1 \\ -1 \\ -1 \\ \vdots \\ -1 \end{pmatrix}.$$

(P2):

$$M = \begin{pmatrix} 1 & 2 & 2 & \cdots & 2 \\ 2 & 5 & 6 & \cdots & 6 \\ 2 & 6 & 9 & \cdots & 10 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 2 & 6 & 10 & \cdots & 4(n-1)+1 \end{pmatrix} \text{ and } q = \begin{pmatrix} -1 \\ -1 \\ -1 \\ \vdots \\ -1 \end{pmatrix}.$$

It is known that some popular pivoting methods ([2] [10]) require exponential number of steps to solve these problems (see [13, 3], etc.).

It should be noted that Chen and Mangasarian's method is a so-called *non-interior* method and it allows us to start from every non-interior-point. However, in order to prepare the same initial environment in [12], we adopted the method of making an artificial problem described in Section 3 of [12] for each problem, and used a feasible-interior-point of the artificial problem as the initial point. The created artificial problem is equivalent to the original problem (see [8]) and we can easily see that the matrix of the artificial problem problem is a P-matrix, i.e., a matrix with positive principal minors, which ensures that the problem satisfies both of Condition 1.1 and Condition 1.2.

We used an algorithm which is based on Newton's method for solving the approximate system (2). After a number of experiments, we obtained that the following strategy of choosing parameters achieves computational efficiency and numerical stability under the problem setting described above:

(i) The step size: Define the positive orthant in (x, y)-space as a safeguard, and go to 99% of the line segment between the current point and a point on the boundary of the positive orthant along Newton's direction.

(ii) The update of the parameter $\alpha > 0$: Starting from $\alpha = 100$, increase α by a fixed factor of 2.15 to 2.35.

Concerning the stopping criterion, we followed Chen and Mangasarian's rule in [1].

The continuation algorithm described above was implemented in C and run on a Sun OS 4.1.2. Table 1 and Table 2 show the number of iterations required to solve the problems of the types P1 and P2, respectively. CA denotes the continuation method and IPM denotes Algorithm B' of [12], which is a polynomial-time feasible-interior point algorithm with an implementation for improving the computational efficiency. The results for Algorithm B' are quoted from [12]. The hyphen in the tables means that there is no result for the corresponding size of the problem.

Note that both of the algorithms require $O(n^3)$ arithmetic operations at each iteration, which are mainly due to the calculation of the solution $(\Delta x, \Delta y)$ satisfying

$$\begin{pmatrix} D_x & D_y \\ -M & I \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix}$$

where D_x and D_y are positive diagonal matrices. Thus the number of iterations plays an crucial role in the comparison of the two methods. For each of the methods, the number of iterations is almost constant regardless of the number of variables. This implies that the potential for the polynomial-time convergence rate of the continuation method, which is still an open problem.

Table 1: The number of iterations for (P1).

	n=8	n=16	n=32	n=64	n=128	n=256	n=512
CA #	9	10	9	9	8	9	9
IPM #	28	29	30	32	33	—	—

Table 2: The number of iterations for (P2).

	n=8	n=16	n=32	n=64	n=128	n=256	n=512
CA #	6	6	7	7	7	6	7
IPM #	32	34	36	38	39	—	—

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