No.658

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January 1996

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Abstract: In this paper, we present two approaches for solving an $\mathcal{NP}$-complete problem: minimum maximal flow problem, i.e., $\min\{\| \xi \| \mid \xi$ is a maximal flow$\}$. We introduce lower bounds on flow, and consider the problem as a minimization of a concave function over a convex set. We solve the problem by a global optimization method. As an application, we consider the minimum maximal matching problem.

Key words: maximal flows, cutting plane, global optimization, bipartite graph, matching

1 Introduction

Throughout this paper, $G(V, E, \partial^+, \partial^-)$ is a directed graph consisting of the node set $V = \{1, 2, ..., n\}$, the edge set $E \subseteq V \times V$ with $|E| = m$, and two incident functions $\partial^+ : E \to V$ and $\partial^- : E \to V$. A path, cycle, etc, and their directed versions are defined as usual. A network $N(G, c)$ (abbreviated by N) is a graph G with a capacity function on the edge set E, say $c : E \to R^m_+$. $c(e)$ is called the capacity of edge $e \in E$. A function ξ: $E \to R^m_+$ is called a flow in the network N if it satisfies the following conservation law:

$$\forall v \in V : \sum_{\partial^+ e = v} \xi(e) = \sum_{\partial^- e = v} \xi(e). \tag{1.1}$$

A flow ξ is called to be feasible if $0 \leq \xi \leq c$. A *two-terminal* network $N(G, s, t, c)$ is a network with two special vertices source s and sink t. An s-t flow ξ is a flow satisfying (1.1) for all nodes of $V \setminus \{s, t\}$. The flow value of an s-t flow ξ is defined by

$$\| \xi \| = \sum_{\partial^+ e = s} \xi(e) - \sum_{\partial^- e = s} \xi(e)$$

or

$$\| \xi \| = \sum_{\partial^- e = t} \xi(e) - \sum_{\partial^+ e = t} \xi(e).$$

A feasible s-t flow ξ is called to be maximal on network N if there does not exist a feasible s-t flow ξ' of N such that

$$\xi' \geq \xi, \quad \xi' \neq \xi.$$

We consider the problem P of finding the minimum value γ^* of maximal s-t flow for network N. That is

$$P: \qquad \gamma^* = \min\{\| \xi \| \mid \xi \text{ is a maximal } s\text{-}t \text{ flow in } N\}.$$

For the sake of notational simplicity, we abbreviate s-t flow by flow.

Very few algorithms successfully address Problem P. In this paper, we show that Problem P is $\mathcal{NP}$-complete and present a global optimization method for this problem.

2 Lower Bound Method for the Problem

To solve Problem P, we introduce a lower bound for flow ξ and consider the relationship between the corresponding maximum flow problem with the lower bound and the minimum maximal flow problem.

Let l be a lower bound for flow ξ with $0 \leq l \leq c$. We abbreviate $N(G, s, t, c)$ with lower bound l by $N(l)$. Then we have

Lemma 2.1 *If ξ is a maximal flow in N, then there exists a lower bound l such that ξ is a maximum flow of $N(l)$.*

Proof: Let $l(e) = \xi(e)$ for every $e \in E$. Suppose that ξ is not a maximum flow of $N(l)$. Then there exists an augmenting path such that

1. $\xi(e) < c(e)$ for all forward edges e in the augmenting path,

2. $\xi(e) > l(e)$ for all backward edges e in the augmenting path.

Since $l(e) = \xi(e)$ for every $e \in E$, the augmenting path does not include any backward edges. Increasing the flow along the path, we see ξ is not a maximum flow of $N(l)$. □

We assume in this paper the following assumption.

Assumption 2.2 *Graph G is connected and acyclic.*

Lemma 2.3 *Under Assumption 2.2, if ξ is a maximum flow of $N(l)$ for some l, then ξ is a maximal flow of N.*

Proof: Let ξ be a maximum flow of $N(l)$ for some l. Suppose that ξ is not maximal. Then there exists a feasible flow η of N such that

$$\eta \geq \xi, \eta \neq \xi.$$

Therefore $\eta(e_i) > \xi(e_i)$ holds for some $e_i \in E$. This indicates that there exist two edges e_{i-1} and e_{i+1} such that

$$\partial^- e_{i-1} = \partial^+ e_i,\ \eta(e_{i-1}) > \xi(e_{i-1}), \text{ and } \partial^+ e_{i+1} = \partial^- e_i,\ \eta(e_{i+1}) > \xi(e_{i+1}).$$

Repeating this procedure with e_i replaced by e_{i+1} and e_{i-1}, we will reach s and t after finitely many steps since G is acyclic. Therefore we obtain a forward directed path such that

$$\xi(e_i) < \eta(e_i) \leq c(e_i) \quad \text{for } e_i \text{ in the forward directed path .}$$

It indicates that ξ is not a maximum flow of $N(l)$. □

Let

(2.1) $$L = \{l \mid 0 \leq l \leq c, \text{ there exists a feasible flow of } N(l)\}$$

and consider the problem

$$P(l): \qquad \max\{\| \xi \| \mid \xi \text{ is a feasible flow of } N(l)\}.$$

We denote by $\gamma(l)$ the optimal value of Problem $P(l)$, i.e.,

$$\gamma(l) = \max\{\| \xi \| \mid \xi \text{ is a feasible flow of } N(l)\}.$$

Theorem 2.4 $\gamma^* = \min\{\gamma(l) \mid l \in L\}$.

Proof: From Lemma 2.1, we have

$$\gamma^* \geq \min\{\gamma(l) \mid l \in L\}.$$

Lemma 2.3 claims that the converse inequality holds. □

By Theorem 2.4, Problem P is cast into the minimization of $\gamma(l)$ over L.

Let A be the incidence matrix of graph G. Then, for a fixed $l \in L$, Problem $P(l)$ can be rewritten as

$$P(l): \qquad \left| \begin{array}{ll} \max & z \\ \text{s.t.} & A\xi = \begin{bmatrix} z \\ 0 \\ -z \end{bmatrix}, \\ & l \leq \xi \leq c. \end{array} \right.$$

The dual of $P(l)$ is

$$D(l): \qquad \left| \begin{array}{ll} \min & qc - rl \\ \text{s.t.} & pA + qI - rI = 0, \\ & -p(s) + p(t) = 1, \\ & q \geq 0, r \geq 0, \end{array} \right.$$

where I stands for an $m \times m$ unit matrix, $p(s)$ and $p(t)$ are the first and the last component of the n-dimensional vector p, respectively. Denote by $\triangle F$ the set of directions of unbounded rays of the feasible region of $D(l)$, i.e.,

$$\triangle F = \left\{ (\triangle p, \triangle q, \triangle r) \;\middle|\; \begin{array}{l} \triangle pA + \triangle qI - \triangle rI = 0, \\ -\triangle p(s) + \triangle p(t) = 0, \\ \triangle q \geq 0, \triangle r \geq 0 \end{array} \right\}.$$

Then we have

Lemma 2.5 $L = \{l \mid 0 \leq l \leq c, \triangle qc - \triangle rl \geq 0 \text{ for all } (\triangle p, \triangle q, \triangle r) \in \triangle F\}$.

Proof: Since $D(l)$ always has a feasible solution, we see from the duality theorem of linear programming that $P(l)$ is feasible if and only if $D(l)$ has a bounded solution. Namely, $l \in L$ if and only if

$$\triangle qc - \triangle rl \geq 0$$

for all $(\triangle p, \triangle q, \triangle r) \in \triangle F$. This is the required result. □

Lemma 2.6 *$\gamma(l)$ is a piecewise linear concave function on L.*

Proof: It follows from the duality theorem of linear programming and the property of parametric linear programming (see, e.g., [2]). □

Lemma 2.7 *If $0 \leq l^2 \leq l^1 \in L$, then $l^2 \in L$ and $\gamma(l^2) \geq \gamma(l^1)$.*

Proof: Directly follows from the definitions L and $\gamma(l)$. □

An element $l \in L$ is said to be a maximal element of L if there does not exist $l' \in L$ such that $l' \geq l$ and $l' \neq l$. Denote by $L_{\max}$ the set of maximal elements of L, that is,

$$L_{\max} = \{l \in L \mid \text{ there is no } l' \in L \text{ such that } l' \geq l, l' \neq l\}.$$

Denote by Ξ the set of feasible flow of N, that is,

$$\Xi = \{\xi \mid \xi \text{ is a feasible flow of } N\}.$$

Also, we denote $\Xi_{\max}$ the set of maximal elements of Ξ. By Lemma 2.7 we have

Corollary 2.8 *$\gamma(\cdot)$ attains the minimum on $L_{\max}$.*

Let $R^m_- = \{l \in R^m \mid l \leq 0\}$ and $R^m_+ = \{l \in R^m \mid l \geq 0\}$. Then

Lemma 2.9 *(i) $\Xi \subseteq L$, (ii) $L = (\Xi + R^m_-) \cap R^m_+$, (iii) $\Xi_{\max} = L_{\max}$.*

Proof: (i) Trivial.
(ii) "$\subseteq$" Suppose $l \in L$. By the definition of L, there exists $\xi \in \Xi$ such that $0 \leq l \leq \xi$. Therefore $l = \xi + \xi_-$ for some $\xi_- \in R^m_-$. It means $l \in (\Xi + R^m_-) \cap R^m_+$.

"$\supseteq$" Suppose $l \in (\Xi + R^m_-) \cap R^m_+$. Then $l \geq 0$ can be written as $l = \xi + \xi_-$ for some $\xi \in \Xi$ and $\xi_- \in R^m_-$. Therefore $0 \leq l \leq \xi \in \Xi$. It implies that $l \in L$ by the definition of L.
(iii) "$\subseteq$" Let $\xi_{\max} \in \Xi_{\max}$ and suppose that $\xi_{\max} \notin L_{\max}$, i.e., there exists an $l \in L$ such that $\xi_{\max} \leq l$ and $\xi_{\max} \neq l$. Since $l \in L$, we see that $l \leq \xi$ for some $\xi \in \Xi$. It follows that $\xi_{\max} \leq \xi$ and $\xi_{\max} \neq \xi$. This is a contradiction.

"$\supseteq$" Let $l_{\max} \in L_{\max}$ and suppose that $l_{\max} \notin \Xi_{\max}$, then there is a flow $\xi \in \Xi$ such that $l_{\max} \leq \xi$ and $l_{\max} \neq \xi$. Since $\Xi \subseteq L$, we have $\xi \in L$. This contradicts that $l_{\max} \in L_{\max}$.

The desired result follows this relation and (i). □

By Lemma 2.7, Corollary 2.8 and Lemma 2.9, we obtain

Theorem 2.10

$$\min\{\gamma(l)\,|\,l\in L\}=\min\{\gamma(\xi)\,|\,\xi\in\Xi\}$$
$$=\min\{\gamma(\xi)\,|\,\xi\in\Xi_{\max}\}=\min\{\gamma(l)\,|\,l\in L_{\max}\}.$$

Proof:

$$\begin{aligned}\min\{\gamma(l)\,|\,l\in L\} &\le \min\{\gamma(\xi)\,|\,\xi\in\Xi\} && (\text{by } \Xi\subseteq L)\\ &\le \min\{\gamma(\xi)\,|\,\xi\in\Xi_{\max}\} && (\text{by } \Xi_{\max}\subseteq\Xi)\\ &= \min\{\gamma(l)\,|\,l\in L_{\max}\} && (\text{by } \Xi_{\max}=L_{\max})\\ &= \min\{\gamma(l)\,|\,l\in L\}. && (\text{by Corollary 2.8})\end{aligned}$$

□

Theorem 2.10 provides four possible domains over which we consider Problem P.

2.1 Cutting Plane Algorithm over Ξ

Now we consider the minimization of $\gamma(\xi)$ over Ξ, i.e.,

$$Q:\quad \begin{array}{ll}\gamma^*=\min & \max\{\|\eta\|\,|\,\eta\in\Xi,\xi\le\eta\}.\\ \text{s.t.} & \xi\in\Xi\end{array}$$

Let B be a submatrix of A with the rows corresponding to s and t deleted, that is,

$$A=\begin{pmatrix}\cdots\\ B\\ \cdots\end{pmatrix}.\quad\begin{array}{l}\leftarrow \text{node } s\\ \\ \leftarrow \text{node } t\end{array}$$

Then Problem Q can be rewritten as

$$Q\quad\left|\begin{array}{ll}\min & \gamma(\xi)\\ \text{s.t.} & B\xi=0,\\ & 0\le\xi\le c.\end{array}\right.$$

Let us denote the rows of B by $b^1,\dots,b^{n-2}$. We use the convention that $\gamma(l)=-\infty$ when $l\notin L$. Based on the above discussion we propose Algorithm I for Problem Q using the cutting plane method [6, 8]. Here we denote the set of vertices of a polyhedral set S by $W(S)$.

The outline of Algorithm I is as follows: At start, the algorithm constructs a hypercube $C_0=\{\xi\,|\,0\le\xi\le c\}$ containing Ξ. Without loss of generality, we can assume that $c\notin\Xi$

since otherwise c would be the optimal solution of P. Therefore c does not meet the conservation law at some node and we can choose $b \in \{b^1, ..., b^{n-2}\}$ such that $cb \neq 0$. Add the cutting plane $\{\xi \mid b\xi = 0\}$ to C_0 and make a smaller polytope C_1. In general step, we have a polytope C_{k-1} and its vertex set $W(C_{k-1})$. We evaluate $\gamma(\cdot)$ at the vertices and choose one with the smallest value. If it belongs to Ξ, we have solved the problem. Otherwise we add an equality constraint, say $b\xi = 0$, of $B\xi = 0$ which is violated by the chosen vertex, and set $C_k := C_{k-1} \cap \{\xi \mid b\xi = 0\}$. By the efficient algorithm proposed by Horst et al. [7], the vertices $W(C_k)$ of C_k is generated.

We state Algorithm I formally as follows:

Algorithm I

Step 0: Let $B_0 := \{b^1, ..., b^{n-2}\}$; $C_0 := \{\xi \mid 0 \leq \xi \leq c\}$;
Choose $b \in B_0$ such that $bc \neq 0$; $k := 1$;

Step 1: $B_k := B_{k-1} \setminus \{b\}$; $C_k := C_{k-1} \cap \{\xi \mid b\xi = 0\}$;
Find $W(C_k)$;

Step 2: $\gamma_k := \min\{\gamma(w) \mid w \in W(C_k)\}$; $\Gamma_k := \{w \in W(C_k) \mid \gamma(w) = \gamma_k\}$;
if $\Gamma_k \cap \Xi \neq \emptyset$
then $\overline{\gamma} := \gamma_k$; Let $\overline{\xi}$ be an arbitrary element of $\Gamma_k \cap \Xi$; Stop
else Choose $w \in \Gamma_k$ and $b \in B_k$ such that $bw \neq 0$;
$k := k + 1$; go to Step 1.

Theorem 2.11 *Algorithm I finds an optimal solution of Problem P within finitely many iterations.*

Proof: Note that polytope C_k contains Ξ and $\gamma(\cdot)$ is a concave function. Therefore in general

$$\gamma_k = \min\{\gamma(w) \mid w \in W(C_k)\} = \min\{\gamma(w) \mid w \in C_k\} \leq \min\{\gamma(\xi) \mid \xi \in \Xi\} = \gamma^*.$$

When Algorithm I terminates at Step 2, $\Gamma_k \cap \Xi \neq \emptyset$, and for any $w \in \Gamma_k \cap \Xi$, we have

$$\gamma_k = \gamma(w) \geq \min\{\gamma(\xi) \mid \xi \in \Xi\} = \gamma^*.$$

Therefore $\overline{\gamma}$ is the optimal value and $\overline{\xi}$ is an optimal solution of Problem Q.
Since Ξ has only $n-2$ equality constraints, after at most $n-2$ iterations C_k is coincide with Ξ and the algorithm terminates. □

2.2 Outer Approximation Algorithm over L

Consider the dual $D(l)$ of $P(l)$ again,

$$D(l): \quad \left| \begin{array}{ll} \min & qc - rl \\ \text{s.t.} & pA + qI - rI = 0, \\ & -p(s) + p(t) = 1, \\ & q \geq 0, r \geq 0. \end{array} \right.$$

Let $N'(l)$ be the network $N(l)$ with a directed edge (t,s) added such that $l(t,s) = 0$ and $c(t,s) = +\infty$. The following lemma provides a well-known necessary and sufficient condition for l to be in L.

Lemma 2.12 (Theorem 6.11 in [1]) *A nonnegative l is in L if and only if*

$$c(X,\overline{X}) \geq l(\overline{X},X)$$

holds for every cut $(X,\overline{X})$ of $N'(l)$, where $c(X,\overline{X}) = \sum_{e\in(X,\overline{X})} c(e)$ and $l(\overline{X},X) = \sum_{e\in(\overline{X},X)} l(e)$.

Therefore, if $l \notin L$, there is a cut $(X,\overline{X})$ such that

$$c(X,\overline{X}) < l(\overline{X},X). \tag{2.2}$$

We see that this cut $(X,\overline{X})$ provides one of the inequalities defining L. Let $(\Delta p, \Delta q, \Delta r)$ be

$$\begin{aligned} \Delta p(v) &= \begin{cases} 1 \text{ for } v \in \overline{X}, \\ 0 \text{ for } v \in X, \end{cases} \\ \Delta q(e) &= \begin{cases} 1 \text{ for } e \in (X,\overline{X}), \\ 0 \text{ otherwise }, \end{cases} \\ \Delta r(e) &= \begin{cases} 1 \text{ for } e \in (\overline{X},X), \\ 0 \text{ otherwise }. \end{cases} \end{aligned} \tag{2.3}$$

Lemma 2.13 *When (2.2) holds, then $(\Delta p, \Delta q, \Delta r)$ defined by (2.3) is in ΔF and $\Delta qc - \Delta rl < 0$.*

Proof: Note first that $(t,s) \notin (X,\overline{X})$ because $c(t,s)=+\infty$. When $(t,s)\in(X,\overline{X})$, define

$$X' = X \cup \{t\}.$$

It is readily seen that $c(X',\overline{X'}) \le c(X,\overline{X})$ and $l(\overline{X'},X') \ge l(\overline{X},X)$ and hence

$$c(X',\overline{X'}) < l(\overline{X'},X').$$

Therefore we can assume without loss of generality that either $s,t\in X$ or $s,t\in\overline{X}$. Then clearly

$$-\triangle p(t) + \triangle p(s) = 0.$$

We also obtain

$$\triangle qc - \triangle rl = c(X,\overline{X}) - l(\overline{X},X) < 0.$$

Furthermore

$$\begin{aligned} & \triangle p(\partial^+ e) - \triangle p(\partial^- e) + \triangle q(e) - \triangle r(e) \\ = & \begin{cases} 0-1+1-0 \text{ for } e\in(X,\overline{X}) \\ 1-0+0-1 \text{ for } e\in(\overline{X},X) \\ d-d+0+0 \text{ otherwise} \end{cases} \\ = & \ 0, \end{aligned}$$

where $d=0$ or $d=1$, which implies $\triangle pA + \triangle qI - \triangle rI = 0$. Clearly $\triangle q \ge 0$ and $\triangle r \ge 0$. Therefore $(\triangle p, \triangle q, \triangle r) \in \triangle F$. □

Lemma 2.14 *If the subgraph consisting of the node set $\overline{X}$ and edges e with $\partial^+ e$, $\partial^- e \in \overline{X}$ is connected, then $(\triangle p,\triangle q,\triangle r)$ defined by (2.3) is an extreme direction of $\triangle F$.*

Proof: Suppose that

$$(\triangle p,\triangle q,\triangle r) = \lambda_1(\triangle p^1,\triangle q^1,\triangle r^1) + \lambda_2(\triangle p^2,\triangle q^2,\triangle r^2)$$

for some $(\triangle p^1,\triangle q^1,\triangle r^1)$, $(\triangle p^2,\triangle q^2,\triangle r^2)\in\triangle F$ and some λ_1, $\lambda_2>0$. Since $\triangle p(e)=0$ for $v\in X$, $\triangle q(e)=0$ for $e\notin(X,\overline{X})$ and $\triangle r(e)=0$ for $e\notin(\overline{X},X)$, we readily see that

$$\begin{aligned} &\triangle p^1(v) = \triangle p^2(v) = 0 \text{ for } v\in X, \\ &\triangle q^1(e) = \triangle q^2(e) = 0 \text{ for } e\notin(X,\overline{X}), \\ &\triangle r^1(e) = \triangle r^2(e) = 0 \text{ for } e\notin(\overline{X},X). \end{aligned} \tag{2.4}$$

Note that for each edge $e \in E$,

$$\begin{aligned}&\Delta p^j(\partial^+ e) - \Delta p^j(\partial^- e) + \Delta q^j(e) - \Delta r^j(e) = 0 \text{ for } j = 1,2,\\ &\Delta p(\partial^+ e) - p(\partial^- e) + \Delta q(e) - \Delta r(e) = 0.\end{aligned} \tag{2.5}$$

Form (2.4) and (2.5), we obtain

$$\begin{aligned}&\Delta p^j(\partial^- e) = \Delta q^j(e) \text{ for } e \in (X, \overline{X}),\ j = 1,2,\\ &\Delta p^j(\partial^+ e) = \Delta r^j(e) \text{ for } e \in (\overline{X}, X),\ j = 1,2\end{aligned} \tag{2.6}$$

and

$$\begin{aligned}&\Delta p(\partial^- e) = \Delta q(e) \text{ for } e \in (X, \overline{X}),\\ &\Delta p(\partial^+ e) = \Delta r(e) \text{ for } e \in (\overline{X}, X).\end{aligned}$$

Since the subgraph of node set $\overline{X}$ and edges e with $\partial^+ e, \partial^- e \in \overline{X}$ is connected, from (2.6) we obtain

$$\Delta p^j(v) = d_j \text{ for } v \in \overline{X}, j = 1,2.$$

Therefore for $j = 1$ and 2, $(\Delta p^j, \Delta q^j, \Delta r^j)$ is of the same direction as $(\Delta p, \Delta q, \Delta r)$, which implies that $(\Delta p, \Delta q, \Delta r)$ is the direction of an extreme ray of ΔF. □

If $\overline{X}$ in the cut $(X, \overline{X})$ satisfying (2.2) does not satisfy the assumption of Lemma 2.14, one can constructe a new cut $(X', \overline{X'})$ which holds (2.2) and the assumption of Lemma 2.14. Suppose that the assumption is not satisfied, i.e., the subgraph of node set $\overline{X}$ and edges e with $\partial^+ e, \partial^- e \in \overline{X}$ is not connected. Without loss of generality, we assume that $\overline{X}$ can be partitioned into two separate node set $\overline{X}^1$ and $\overline{X}^2$ such that the subgraph of node set $\overline{X}^j$ $(j = 1,2)$ and edges e with $\partial^+ e, \partial^- e \in \overline{X}^1$ is connected. From $c(X, \overline{X}) < l(\overline{X}, X)$, we see that

$$c(X, \overline{X}^1) + c(X, \overline{X}^2) < l(\overline{X}^1, X) + l(\overline{X}^2, X).$$

Souppse that $c(X, \overline{X}^1) < l(\overline{X}^1, X)$ holds. Let us set

$$X' = X \cup \overline{X}^2,\ \overline{X'} = \overline{X}^1.$$

Then we see that

$$c(X', \overline{X'}) = c(X, \overline{X'}) < l(\overline{X'}, X) = l(\overline{X'}, X').$$

That is, the assumption of Lemma 2.14 holds without loss of generality.

Based on the above discussion, we can design Algorithm II to solve Problem P. Algorithm II starts with a hypercube $L^0 = \{l \,|\, 0 \le l \le c\}$ containing L. For $l^k \notin L^k$ ($k = 0, 1, \ldots$), by Lemma 2.12 and 2.13, we find $(\triangle p_k, \triangle q_k, \triangle r_k) \in \triangle F$ satisfying $\triangle q_k c - \triangle r_k l^k < 0$. Adding the cutting plane $\{l \,|\, \triangle q_k c - \triangle r_k l \ge 0\}$ to L^k yields the polytope L^{k+1}. Then we find all vertices $W(L^{k+1})$ of polytope L^{k+1}. Checking the values of $\gamma(\cdot)$ on $W(L^{k+1})$, we obtain the minimum value γ_k of $\gamma(\cdot)$ on L^{k+1}. If a vertex of $W(L^{k+1})$ with the minimum value of $\gamma(\cdot)$ belongs to L, then it is an optimal solution of Problem P. Otherwise $l^{k+1} \notin L$ will be found and the procedure is repeated.

Algorithm II

Step 0: Let $L^0 := \{l \,|\, 0 \le l \le c\}$; $l^0 := c$; $k := 0$;

Step 1: Construct a cut $(X, \overline{X})$ satisfying (2.2) and the assumption of Lemma 2.14 for l^k; Find $(\triangle p_k, \triangle q_k, \triangle r_k)$ defined by (2.3);
Let $L^{k+1} := L^k \cap \{l \,|\, \triangle q_k c - \triangle r_k l \ge 0\}$; Find $W(L^{k+1})$;

Step 2: $\gamma_k := \min\{\gamma(w) \,|\, w \in W(L^{k+1})\}$; $W_k := \{w \in W(L^{k+1}) \,|\, \gamma(w) = \gamma_k\}$;
if $\gamma_k \neq -\infty$
then $\overline{\gamma} := \gamma_k$; Let $\bar{l}$ be an arbitrary element of $W_k \cap L$; Stop
else Choose $l \in W_k$; $l^{k+1} := l$;
$k := k + 1$; go to Step 1.

Theorem 2.15 *Algorithm II finds an optimal solution of Problem P within finitely many iterations.*

Proof: It is easy to see that $\overline{\gamma} = \gamma^*$ and $\bar{l}$ is an optimal solution of Problem P when the algorithm terminates. Since cut $(X, \overline{X})$ constructed in Step 1 satisfies (2.2) and the assumption of Lemma 2.14, $(\triangle p_k, \triangle q_k, \triangle r_k)$ is the extreme directions of $\triangle F$. From $l^k \in L^k$ and $l^k \notin L^{k+i}$ for $i \ge 1$, we see that the same $(\triangle p_k, \triangle q_k, \triangle r_k)$ will not appear more than once in Algorithm II. Note that there are only a finite number of extreme directions, then Algorithm II terminates in finitely many iterations. □

To find the cut satisfying (2.2), we consider a new network based on G. The new network is constructed by

(i) adding two nodes S (super source) and T (super sink) to G;

(ii) linking S to v and v to T for every node v of G to make n edges (S, v) and n edges (v, T);

(iii) linking t to s to make new edge (t, s), and setting $l(t, s) = 0, c(t, s) = +\infty$.

For a fixed l, the capacity c' of the new network is determined by

$$\begin{array}{ll} & 0 \le c'(e) = c(e) - l(e) \text{ for all } e \in E, \\ (2.7) & 0 \le c'(S, v) = \sum_{\partial^- e = v} l(e) \text{ for all } v \in V, \\ & 0 \le c'(v, T) = \sum_{\partial^+ e = v} l(e) \text{ for all } v \in V. \end{array}$$

For a given $l > 0$, we can determine a maximum flow of the new network. Meanwhile, we find a minimum cut $(X, \overline{X})$. If the value of the maximum flow is equal to $\sum_{v \in V} l(e)$, we see that $l \in L$. Otherwise, i.e., the value of the maximum flow is less that $\sum_{v \in V} l(e)$, then there is a cut $(X, \overline{X})$ satisfying (2.2). In this case, by Lemma 2.13, we can obtain an extreme ray $(\triangle p, \triangle q, \triangle r)$ satisfying $\triangle qc - \triangle rl < 0$.

3 Minimum Maximal Matching Problem

In this section, we consider a special case of Problem P: minimum maximal matching problem (see, e.g., [3]). This problem can be stated as follows:

> *Instance:* Graph $G = (V, E)$, positive integer $K \le |E|$.
> *Question:* Is there a subset $E' \subseteq E$ with $|E'| \le K$ such that E' is a maximal matching, i.e., no two edges in E' share a common endpoint and every edge in $E \setminus E'$ shares a common endpoint with some edge in E' ?

Even for a bipartite graph, this problem is $\mathcal{NP}$-complete (see, e.g., [3]). Throughout this section, we assume G is a bipartite graph. That is, node set V is partitioned into two subsets V_1 and V_2 such that for each edge $e \in E$ its two endpoints belong to the distinct set V_1 and V_2, respectively. To transform the minimum maximal matching problem to minimum maximal flow problem, we first make a directed version of the underlying graph G by designating all edges as pointing from the nodes in V_1 to the nodes in V_2. Then, we add a source node s and a sink node t, with edges connecting s to each node in V_1 and edges connecting each node in V_2 to t. Denote by $\overline{V}$ the enlarged node set with s and t, by $\overline{E}$ the enlarged edge set. For each edge in the network, we set the capacity c to 1. Denote by $\overline{N} = ((\overline{V}, \overline{E}), 1)$ the transformed network. Note that in $\overline{N}$, every node in V_1 has one incoming edge and every node in V_2 has one outgoing edge. Therefore, a matching with cardinality K has a one-to-one correspondence to an integral flow of value K in $\overline{N}$.

Clearly, $\overline{N}$ satisfies Assumption (2.2). Now we focus on the relationship between maximal matching problem and maximal flow problem.

Lemma 3.1 *If matching M is maximal of G, then ξ defined by*

$$\xi(e) = \begin{cases} 1 & \text{if } e \in M \\ 1 & \text{if } e = (s, \partial^+ f) \text{ for some } f \in M \\ 1 & \text{if } e = (\partial^- f, t) \text{ for some } f \in M \\ 0 & \text{otherwise} \end{cases}$$

is a maximal flow of $\overline{N}$.

Proof: It is clear that ξ is a flow. Suppose ξ is not maximal. Then there exists a feasible flow η such that $\eta \geq \xi, \eta \neq \xi$. Let

$$\begin{aligned} E_1 &= \{e \mid e = (s, \partial^+ f) \text{ for some } f \in M\}, \\ E_2 &= \{e \mid e \in M\}, \\ E_3 &= \{e \mid e = (\partial^- f, t) \text{ for some } f \in M\}. \end{aligned}$$

From $\eta \geq \xi, \eta \neq \xi$, we see that there exists an edge e^0 such that $\eta(e^0) > \xi(e^0)$. Suppose that $e^0 \in E_2$. Then there exist $e^1 \in E_1$ and $e^3 \in E_3$ such that $\eta(e^1) > \xi(e^1), \eta(e^3) > \xi(e^3)$. Therefore $\xi(e^1) = \xi(e^0) = \xi(e^3) = 0$. Since (e^1, e^0, e^3) is a s-t path, $M' = M \cup \{e^0\}$ is still a matching of G. This is a contradiction. For the cases of $e^0 \in E_1$ and $e^0 \in E_3$, the proof will be the same as above. □

Then we have

Lemma 3.2 *If ξ is a maximal integral flow of $\overline{N}$, then $\{e \mid e \in E, \xi(e) = 1\}$ is a maximal matching of G.*

Proof: It is easy to see that $M = \{e \mid e \in E, \xi(e) = 1\}$ is a matching of G. Suppose that M is not maximal. Then there exists an edge $f \in E \setminus M$ such that $M \cup \{f\}$ is still a matching. Let

$$\eta(e) = \begin{cases} 1 & \text{if } e = f, \text{ or } e = (s, \partial^+ f), \text{ or } e = (\partial^- f, t), \\ \xi(e) & \text{otherwise.} \end{cases}$$

Then η is a feasible flow. This contradicts the maximality of ξ. □

From Lemma 3.1 and 3.2, we see that minimum maximal matching problem is equivalent to the following problem:

$$R: \qquad \gamma^*_{\overline{N}} = \min\{\| \xi \| \mid \xi \text{ is a maximal integral flow of } \overline{N}\}.$$

Denote by $\overline{N}(l)$ the the network $\overline{N}$ with lower bound $l \leq 1$.

Lemma 3.3 *If ξ is a maximal integral flow of $\overline{N}$, then there exists a lower bound l such that ξ is a maximum flow of $\overline{N}(l)$.*

Proof: Similar to the proof of Lemma 2.1. □

Lemma 3.4 *If ξ is a maximum integral flow of $\overline{N}(l)$ for some l, then ξ is a maximal flow of $\overline{N}$.*

Proof: Note that $\overline{N}$ satisfies Assumption 2.2. The lemma follows from Lemma 2.3. □

Denote

$$L_Z = \{l \mid 0 \leq l \leq 1, \text{ there exists an integral feasible flow of } \overline{N}(l)\}.$$

Moreover, let $\gamma_Z(l) = \max\{\| \xi \| \mid \xi \text{ is an integral flow of } \overline{N}(l)\}$. Similar to Theorem 2.4, we have

Lemma 3.5 $\gamma^*_{\overline{N}} = \min\{\gamma_Z(l) \mid l \in L_Z\}$.

Proof: From Lemma 3.3 and 3.4. □

Let us denote

$$L_{\overline{N}} = \{l \mid 0 \leq l \leq 1, \text{ there exists a feasible flow } \xi \text{ of } \overline{N}(l)\},$$
$$\Xi_{\overline{N}} = \{\xi \mid \xi \text{ is a feasible flow of } \overline{N}(l)\}.$$

Lemma 3.6 *If $l \in L_Z$ is integral, then $\gamma_Z(l) = \gamma(l)$.*

Proof: By the definitions of $\gamma_Z(l)$ and $\gamma(l)$, we see that $\gamma_Z(l) \leq \gamma(l)$. Since l is integral, by Integrality Theorem (Theorem 6.5 in [1]) there is an integral solution ξ of $\gamma(l)$, i.e., $\gamma(l) = \| \xi \|$. This implies that $\gamma_Z(l) \geq \| \xi \| = \gamma(l)$. Therefore $\gamma_Z(l) = \gamma(l)$. □

Theorem 3.7 $\min\{\gamma_Z(l) \,|\, l \in L_Z\} = \min\{\gamma(l) \,|\, l \in L_{\overline{N}}\}$.

Proof: Let $\bar{l} = \arg\min\{\gamma_Z(l) \,|\, l \in L_Z\}$ and $\bar{\xi}$ be a solution of $\gamma_Z(\bar{l})$. Then $\bar{\xi}$ is integral and $\bar{\xi} \in L_Z$. From Lemma (3.6), $\gamma_Z(\bar{\xi}) = \gamma(\bar{\xi})$. From $\bar{l} \leq \bar{\xi}$ and the definition of $\gamma_Z(\cdot)$, we see that $\gamma_Z(\bar{l}) \geq \gamma_Z(\bar{\xi})$. Therefore

$$\min\{\gamma_Z(l) \,|\, l \in L_Z\} = \gamma_Z(\bar{l}) \geq \gamma_Z(\bar{\xi}) = \gamma(\bar{\xi}) \geq \min\{\gamma(l) \,|\, l \in L_{\overline{N}}\}.$$

Now we prove the converse inequality. From Theorem 2.10, we have

$$\min\{\gamma(l) \,|\, l \in L_{\overline{N}}\} = \min\{\gamma(\xi) \,|\, \xi \in \Xi_{\overline{N}}\}.$$

From Lemma 2.6, we see that

$$\gamma(\cdot) \text{ attains the minimum at a vertex of } \Xi_{\overline{N}}.$$

Note that every vertex of $\Xi_{\overline{N}}$ is integral. Then there exists an integral l^* such that

$$\gamma(l^*) = \min\{\gamma(l) \,|\, l \in L_{\overline{N}}\}.$$

By Integrality Theorem in [1], $\gamma(l^*)$ has is an integral solution. It means $l^* \in L_Z$. By Lemma (3.6), $\gamma(l^*) = \gamma_Z(l^*)$. Therefore

$$\min\{\gamma_Z(l) \,|\, l \in L_Z\} \leq \gamma_Z(l^*) = \gamma(l^*) = \min\{\gamma(l) \,|\, l \in L_{\overline{N}}\}.$$

Thus, the converse inequality holds. □

By the above theorem, we can use Algorithm I and II to solve minimum maximal matching problem over $L_{\overline{N}}$.

Theorem 3.8 *Problem P is $\mathcal{NP}$-complete.*

Proof: Note that minimum maximal matching problem is $\mathcal{NP}$-complete. Therefore Problem R is an $\mathcal{NP}$-complete problem. From Lemma 3.5 and Theorem 3.7, we see that $\min\{\gamma(l)\,|\,l \in L_{\overline{N}}\}$ is $\mathcal{NP}$-complete. Note that $\min\{\gamma(l)\,|\,l \in L_{\overline{N}}\}$ is a special case of $\min\{\gamma(l)\,|\,l \in L\}$. Hence $\min\{\gamma(l)\,|\,l \in L\}$ is $\mathcal{NP}$-complete. It means that P is an $\mathcal{NP}$-complete problem by Theorem 2.4. □

4 Conclusion

In this paper, we set up a connection between the global optimization and an $\mathcal{NP}$-complete problem: minimum maximal flow problem. This connection can be exploited to solve the *u-flow* (*uncontrollable flow*) problem raised by Iri [4]:

> For an s-t network N with capacity c, an s-t flow ξ is called a u-flow if ξ is represented as a positive combination of elementary s-t paths. A u-flow ξ is said maximal if there does not exist a u-flow η in N such that $\eta \geq \xi$ and $c \geq \eta \neq \xi$. How to solve $\min\{\|\xi\|\,|\,\xi$ is a maximal u-flow of $N\}$?

This problem is called *minimum maximal u-flow* problem. As shown in [4], problem

$$\min\{\|\xi\|\,|\,\xi \text{ is maximal u-flow of } N\}$$

is $\mathcal{NP}$-complete. Under Assumption 2.2, one can see that

$$\begin{aligned} & \min\{\|\xi\|\,|\,\xi \text{ is feasible maximal u-flow of } N\} \\ = \; & \min\{\|\xi\|\,|\,\xi \text{ is feasible maximal flow of } N\}. \end{aligned}$$

It means that the approaches in this paper can be exploited to solve minimum maximal u-flow problem. Due to $\mathcal{NP}$-hardness of the problems, it is difficult to estimate the order of complexity of the proposed algorithms. Of course, to investigate the efficiency, computer experiments of the algorithm are necessary. We leave the computer experiments as further works and report the experimental results in next paper.

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