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Airline Seat Management with Multiple Flight-legs

by

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Abstract *It is common for airline companies to classify a pool of identical seats on the same flight into several booking classes through the application of restrictions or service on tickets. In previous studies, other researchers have tried to develop models to determine whether to accept or deny a customer's request for a certain booking class with the assumption that once the customer is accepted, he/she must buy the ticket at the price associated with his/hers booking request. Yet, in reality, a lot of customers will request for price quotations before they purchase their tickets. Therefore, this paper will try to develop a dynamic programming model which brings about selection rule and pricing policy for flights of multiple flight-legs with multiple booking classes and illustrate them with numerical examples. It is common to think that the price of the seat for a trip is nonincreasing with respect to the seats available in each individual flight-leg of the trip, and nondecreasing with respect to the seats available in all other flight-legs. However, this holds true only if the number of flight-legs is less than or equal to two.*

1 Instruction

Airline market is a highly competitive market. In such a competitive market, competition has been focused on quality of service, pricing policy, route structure and so on. In order to attract customers with different purchasing power, it is common for airline companies to classify a pool of identical seats on the same flight into several booking classes through the application of restrictions or service on tickets.

A number of papers have focused on the problem of single flight-leg with multiple fare classes. Among them are ROBINSON, GALLEGO and RYZIN, LEE and HERSH, Brumelle and McGill, Wollmer, Belobaba, Alstrup, Littlewood. WONG, KOPPELMAN and DASKIN developed a model for multiple flight-legs with single fare class by using the flexible assignment approach, which led to develop rules for trip seat allocation. DROR and TRUDEAU presented a mathematical approach (maximal flow) allocating seats to different categories of customers.

This paper develops a discrete-time dynamic programming model which brings about a decision rules (selection of customers and pricing policy) for the problem associated with multiple flight-legs with multiple booking classes. From this model, we will try to develop a selection rule which determines whether to accept or deny an arriving customer who requests for a certain booking class of a trip and a pricing policy which determines the appropriate price to be offered to a customer once the airline company decides to accept the customer.

The objective of this paper is to maximize the expected total present discounted selling profit, which is the expected total present discounted selling price minus the expected total present cost.

- *Key assumptions in previous research*

The problem of seat booking management is highly affected by customers' arriving patterns and permissible purchasing powers. The assumptions of some previous researches are that customers of low fares come before customers of high fares, and all customers must purchase their tickets once they are accepted by the airline. Yet, in reality, it may always be the case, because customers of higher fares may arrive before customers of lower fares. Lee and Hersh (1993) have considered an uncertain arrival pattern in their model under the assumption that all accepted customers must purchase tickets. However, in most cases, customers will request for price quotations before they buy their tickets and may switch to other companies if they decide that the prices offered by the airline are too high. Therefore, it is not likely that all accepted customers will purchase their tickets.

2 Model

Consider the following discrete time sequential stochastic decision process with a finite planning horizon. First, for convenience, let points in time be numbered backward from the final point in time of the planning horizon as t, $t-1$,... and so on, where the interval between two successive points in time, say time t and time $t-1$, is period t. Here, the period is small enough that no more than one request (from any booking class in any trip) arrives and no more than one flight departs during that period. In this paper, let us denote the following notations:

Notations

0	starting point (airport) of a route.
N	destination of a route.
$d_{j,j+1}$	a flight-leg from airport j to the next airport $j+1$.
$i_{j,j+1}$	seats available for flight-leg $d_{j,j+1}$.
f_{jk}	a trip from airport j to k which is composed of flight-legs $\{d_{j,j+1},\cdots,d_{k-1,k}\}$.
s_{jk}	seats available for trip f_{jk}. ($s_{jk}=\min\{i_{m,m+1} \mid m=j,\cdots,k-1\}$)
i	a set with elements that are the number of seats available for all flight-legs, that is $i=\{i_{01},i_{12},\ldots,i_{N-1,N}\}$. Let $0=i$ with $i_{j,j+1}=0, j=0,1,\cdots,N-1$.
$S(i)$	the set of all trips in which the remaining seats available is no less than 1. $S(i)=\{f_{jk} \mid s_{jk}\geq 1, 0\leq j<k\leq N\}$.
L_{jk}	the number of booking classes of a trip f_{jk}.
t_j	departure time from airport j where $0=t_{N-1}<t_{N-2}<\ldots<t_1<t_0$.
$h(t)$	the present or the next airport to depart from at time t, given by $h(t)=\min\{j \mid t_j\leq t, j=,1,\cdots,N-1\}$
$\lambda_t(f_{jk},l)$	the probability that a customer of class l in trip f_{jk} arrives at time t.
$\lambda_t(0)$	the probability that no customer arrives at time t, given by $\lambda_t(0)=1-\sum_{j=h(t)}^{N-1}\sum_{k=j+1}^{N}\sum_{l=1}^{L_{jk}}\lambda_t(f_{jk},l)$
c_{jkl}	cost for carrying a passenger of class l of f_{jk}.
β	per-period discount factor, $\beta\in(0,1]$.

Furthermore, let $p_{jkl}(x)$ be the probability that a customer of class l of f_{jk} buys a ticket if the offered selling price is x. Here, for given a_{jkl} and b_{jkl} such that $0\leq a_{jkl}<b_{jkl}$, let $p_{jkl}(x)=1$ for $x\leq a_{jkl}$, $p_{jkl}(x)=0$ for $b_{jkl}\leq x$, and $p_{jkl}(x)$ be strictly decreasing in x for $a_{jkl}\leq x\leq b_{jkl}$. Let $P_{jkl}(\theta)$ be the probability distribution function of the maximum permissible purchasing price θ that a customer of class l of f_{jk} has in mind for the ticket. That is, if the offered price is smaller than θ, then the customer will decide to buy. In this case $p_{jkl}(x)$ can be given by

$$p_{jkl}(x)=\int_x^{\infty} dP_{jkl}(\theta). \tag{2.1}$$

- *Basic concepts*

Consider a route (Figure 1) from airport 0 to airport N via airports 1, 2, ..., $N-1$, with departure time from airports 0, 1, ..., $N-1$ be $t_0, t_1, \ldots, t_{N-1} = 0$, respectively. Assume one or more aircrafts are used in this route and that no more than one aircraft is used in the same flight-leg. A flight-leg is a nonstop component of the route. Furthermore, assume that all the trips on this route are connected by these flight-legs, in which case, the number of types of trips is the binomial coefficient $N(N+1)/2$, (i.e. $f_{01}, f_{02}, \cdots, f_{0N}, f_{12}, \cdots, f_{N-1,N}$).

It is impossible for a customer to make a reservation after the departure time. Therefore, the trips that are open for reservation are those whose departure time are not larger than the present time. For example, assume the present time is t and $0 = t_{N-1} < t_{N-2} < \ldots \leq t_j < t < t_{j-1}$, then the trips that open for reservation are $f_{j,j+1}, f_{j,j+2}, \cdots, f_{N-1,N}$.

Since a trip is composed of one or more flight-legs, seats available for any trips are the minimum number of seats available for those flight-legs. When a seat for a certain trip is reserved, then the seats available for those flight-legs consisting the trip should be reduced by one.

In order to illustrate the above assumption, let us consider a route from airport 0 to airport 2 via airport 1. In this case, assume that Aircraft A departs from airport 0 at time $t_0 = 10$ and arrives in airport 1 at time t_0^* before time t_1 and that Aircraft B leaves airport 1 at time $t_1 = 0$ and arrives in airport 2 at time t_1^*. Now, assume the present time is 20, then, since $20 > 10$, the trips that open for reservation are f_{01}, f_{12} and f_{02}. Furthermore, assume the seats available for flight-legs d_{01} and d_{12} are $i_{01} = 35$ and $i_{12} = 30$, respectively. Then, the seats available for f_{01}, f_{12} and f_{02} are $s_{01} = 35$, $s_{12} = 30$ and $s_{02} = \min\{35, 30\} = 30$, respectively.

As a valuation, let us consider the case of a single aircraft with multiple flight-legs and that no customers get off at immediate airport (Figure 2). In this case, seats available for any trip are the same since they used the common flight-leg $d_{N-1,N}$.

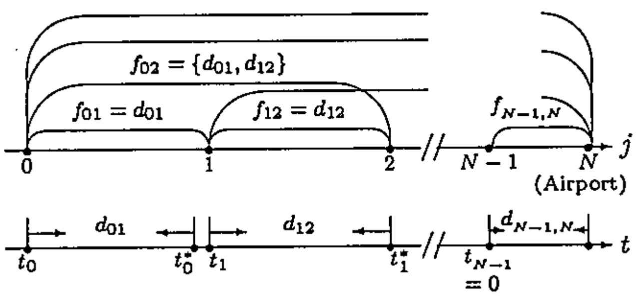

Figure 1

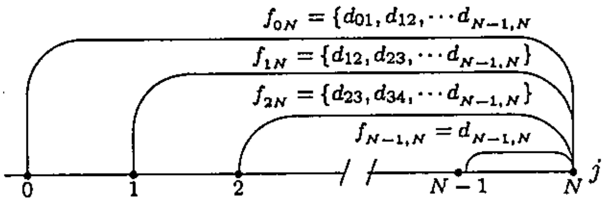

Figure 2

- *Decision structure of the model*

The decision structure of the airline seat booking problems is depicted in Figure 1. It shows the airline's decision making process on whether to accept a customer and how to offer a price once customer is accepted and the customer's decision making process on whether to accept the price being offered.

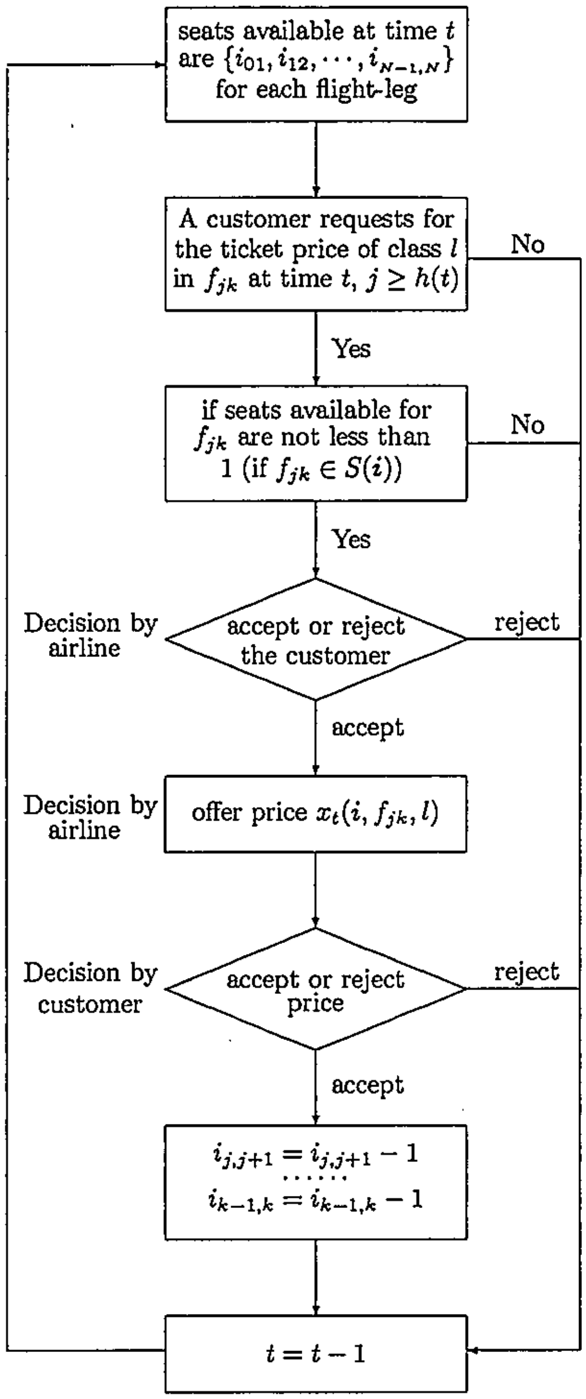

Figure 3
Structure of Decisions

3 Preliminaries: K-function

For given a and b such that $0 \le a < b$, let $p(x)$ be a strictly decreasing function of x for $a \le x \le b$ with $p(x) = 1$ for $x \le a$ and $p(x) = 0$ for $b \le x$. And, for any real number ν, define

$$K(\nu) \;=\; \max_x p(x)(x-\nu). \tag{3.1}$$

Let $x(\nu)$ denote the smallest x attaining the maximum of the right hand side of (3.1) if it exists.

Lemma 3.1

(a) *$K(\nu)$ is nonincreasing in ν and $K(\nu)+\nu$ is nondecreasing in ν.*

(b) *$K(\nu) \ge 0$ for all ν and $K(\nu) > 0$ for $\nu < b$.*

(c) *$x(\nu)$ is nondecreasing in ν.*

(d) *$a \le x(\nu) \le b$ for all ν, $a \le x(\nu) < b$ for $\nu < b$, and $x(\nu) = b$ for $\nu \ge b$.*

(e) *If $\nu_1 \le \nu_2$, then $K(\nu_1) - K(\nu_2) \le \nu_2 - \nu_1$.*

(f) *If $\nu_1 \ge \nu_2$, then $K(\nu_1) - K(\nu_2) \ge \nu_2 - \nu_1$.*

(g) *If $\nu_1 \le \min\{\nu_2,\nu_3\}$ and $\max\{\nu_2,\nu_3\} \le \nu_4$, then*
$K(\nu_1) - K(\nu_2) + K(\nu_4) - K(\nu_3) \ge p(x(\nu_2))(\nu_2 - \nu_1 + \nu_3 - \nu_4)$.

(h) *If $\nu_2 \le \min\{\nu_1,\nu_4\}$ and $\max\{\nu_1,\nu_4\} \le \nu_3$, then*
$K(\nu_1) - K(\nu_2) + K(\nu_4) - K(\nu_3) \le p(x(\nu_1))(\nu_2 - \nu_1 + \nu_3 - \nu_4)$

Proof: See Appendix A.

4 Formulation

Let $v_t(i)$ denote the maximum total expected present discounted profit starting from time t with i seats available for reservation. Then, clearly, we have

$$\begin{aligned} v_t(0) &= 0, \qquad\qquad t \ge 0, & (4.1)\\ v_t(i) &= \lambda_t(0)\beta v_{t-1}(i) \\ &\quad + \sum_{j=h(t)}^{N-1} \sum_{\substack{k=j+1 \\ f_{jk}\notin S(i)}}^{N} \sum_{l=1}^{L_{jk}} \lambda_t(f_{jk},l)\beta v_{t-1}(i) \\ &\quad + \sum_{j=h(t)}^{N-1} \sum_{\substack{k=j+1 \\ f_{jk}\in S(i)}}^{N} \sum_{l=1}^{L_{jk}} \lambda_t(f_{jk},l)\max\{g_t(i,f_{jk},l), \beta v_{t-1}(i)\} \qquad t \ge 1, & (4.2)\end{aligned}$$

where $g_t(i, f_{jk}, l)$ is the maximum total expected present discounted profit starting from time $t \ge t_j$ with i seats remaining, provided that the airline accepts the customer of class l of trip f_{jk} who has just arrived and decides to offer a price to the customer. Then $g_t(i, f_{jk}, l)$ can be expressed as

$$g_t(i,f_{jk},l) = \max_x\{p_{jkl}(x)(x - c_{jkl} + \beta v_{t-1}(i_{jk})) + (1 - p_{jkl}(x))\beta v_{t-1}(i)\}, \tag{4.3}$$

where

$$e_{jk} = \{\underset{1}{0}, \underset{2}{0}, \cdots \underset{j+1}{1}, \underset{j+2}{1}, \cdots \underset{k}{1}, \underset{k+1}{0}, \underset{k+2}{0}, \cdots \underset{N}{0}\}, \tag{4.4}$$

$$i_{jk} = i - e_{jk} = \{i_{01}, i_{12}, \cdots, i_{j,j+1} - 1, i_{j+1,j+2} - 1, \cdots, i_{k-1,k} - 1, i_{k,k+1}, \ldots, i_{N-1,N}\}. \tag{4.5}$$

The final condition is given as follows:

$$v_0(i) = \sum_{\substack{l=1 \\ f_{N-1,N} \in S(i)}}^{L_{N-1,N}} \lambda_0(f_{N-1,N}, l) \max\{g_0(i, f_{N-1,N}, l), 0\} \tag{4.6}$$

where

$$g_0(i, f_{\mathrm{N-1,N}}, l) = \max_x p_{N-1,Nl}(x)(x - c_{N-1,Nl}). \tag{4.7}$$

Here, note that $v_0(i)$ can be expressed by (4.2) with $t = 0$ if setting $v_{-1}(i) = 0$ for all i.

Let

$$\Delta_{jk} v_t(i) = v_t(i) - v_t(i_{jk}), \qquad f_{jk} \in S(i_{jk}), t \geq 0, \tag{4.8}$$

$$z_t(i, f_{jk}, l) = c_{jkl} + \beta \Delta_{jk} v_t(i), \qquad f_{jk} \in S(i), t \geq 0. \tag{4.9}$$

Furthermore, for convenience, let

$$z_{-1}(i, f_{jk}, l) = c_{jkl}. \tag{4.10}$$

Then by using (3.1), $g_t(i, f_{jk}, l)$ can be rewritten as follows.

$$\begin{aligned} g_t(i, f_{jk}, l) &= \max_x \{p_{jkl}(x)(x - c_{jkl} + \beta v_{t-1}(i_{jk})) + (1 - p_{jkl}(x))\beta v_{t-1}(i)\} \\ &= \beta v_{t-1}(i) + \max_x p_{jkl}(x)(x - c_{jkl} + \beta \Delta_{jk} v_{t-1}(i)) \\ &= \beta v_{t-1}(i) + K_{jkl}(z_{t-1}(i, f_{jk}, l)). \end{aligned} \tag{4.11}$$

Theorem 4.1 $g_t(i, f_{jk}, l) \geq \beta v_t(i)$.

Proof: Clearly from the fact $K_{jkl}(\nu) \geq 0$ for all i, k, l, ν from Lemma 3.1(b). ∎

Hence, $v_t(i)$ can be rewritten as follows.

$$v_t(i) = \beta v_{t-1}(i) + \sum_{\substack{j=h(t) \\ f_{jk} \in S(i)}}^{N-1} \sum_{k=j+1}^{N} \sum_{l=1}^{L_{jk}} \lambda_t(f_{jk}, l) K_{jkl}(z_{t-1}(i, f_{jk}, l)) \tag{4.12}$$

where we have

$$v_0(i) = \sum_{\substack{l=1 \\ f_{N-1,N} \in S(i)}}^{L_{N-1,N}} \lambda_0(f_{N-1,N}, l) K_{N-1,Nl}(c_{\mathrm{N-1,N}l}). \tag{4.13}$$

By $x_t(i, f_{jk}, l)$ we denote the smallest x attaining $K_{jkl}(z_{t-1}(i, f_{jk}, l))$. Then, it is of couse that the optimal price $x_t(i, f_{jk}, l)$ is given by x attaining $K_{jkl}(z_{t-1}(i, f_{jk}, l))$, however, if more then one such x's exist, then we let the smallest x of then to be $x_t(i, f_{jk}, l)$.

- **Proprieties of selection rules**

From theorem 4.1 and (4.2), we find that it is always optimal to accept every arriving customer, However, it is possible that the customer decides to reject the offered price after being accepted. This would imply that the price being offered by the airline has been set too high. Theoretically, this occurs when $b_{jkl} \leq z_t(i, f_{jk}, l)$; in this case $x_t(i, f_{jk}, l) = b_{jkl}$ from Lemma 3.1(d). Under this condition, even if a customer is accepted, the deal will not go through. Therefore, when $b_{jkl} \leq z_t(i, f_{jk}, l)$, the airline can decide to reject a customer of class l arbitrarily before making a price offer.

• **Proprieties of the optimal price**

Depending on the Rule of Supply and Demand, we presuppose the following properties of the optimal price.

(1) The optimal price $x_t(i, f_{jk}, l)$ is nonincreasing in $i_{m,m+1} \in \{i_{j,j+1}, \dots, i_{k-1,k}\}$ and nondecreasing in $i_{m,m+1} \notin \{i_{j,j+1}, i_{j+1,j+2}, \dots, i_{k-1,k}\}$ for all t.

(2) The optimal price $x_t(i, f_{jk}, l)$ is nondecreasing in t.

However, in reality, they do not always hold true. For (1), it always hold true for $N \leq 2$, but not for $N \geq 3$. For (2), if $\beta = 1$, then it holds true for $N = 1$, and for $x_t(i_{01}, i_{12}, f_{02}, l)$. The properties of the optimal decision can be described by the following theorems. the proof of the theorems will be given by Appendix A

Lemma 4.1

(a) *$v_t(i)$ is nondecreasing in $i_{m,m+1}$ for all $m = 0, 1, \cdots, N-1$.*

(b) *If there exist j^*, k^*, l^* such that $\lambda_t(f_{j^*k^*}, l^*) > 0$, then $v_t(i) > 0$.*

(c) *$v_t(i)$ is not always nondecreasing in t.*

(d) *If $\beta = 1$, then $v_t(i)$ is nondecreasing in t.*

Proof: See Appendix A.

Lemma 4.2 $\Delta_{jk} v_t(i)$, hence $z_t(i_{jk}, f_{jk}, l)$ is not always nondecreasing in t where $0 \leq j < k \leq N$.

Proof: See Appendix A.

Lemma 4.3

(a) *If $N \geq 3$, $\Delta_{jk} v_t(i)$ is not always nondecreasing in $i_{m,m+1}$ if $m \notin \{j, j+1, \cdots, k-1\}$.*

(b) *If $N \geq 3$, $\Delta_{jk} v_t(i)$ is not always nonincreasing in $i_{m,m+1}$ if $m \in \{j, j+1, \cdots, k-1\}$.. .*

Proof: See section 7 (Examples 4-7).

5 Two flight-legs – $N = 2$ –

In the case of $N = 2$, by definition we have $t_1 = 0, t_0 > 0$ and $i = \{i_{01}, i_{12}\}$. Let $I(\cdot)$ denote an indicator function where $I(A) = 1$ if A is true, or else $I(A)$=0. Then from (4.12) and (4.13) we have

$$\begin{aligned} v_t(0,0) &= 0, \qquad\qquad t \geq 0, \\ v_t(i_{01}, i_{12}) &= \beta v_{t-1}(i_{01}, i_{12}) \\ &+ \sum_{l=1}^{L_{01}} \lambda_t(f_{01}, l) K_{01l}(z_{t-1}(i_{01}, i_{12}, f_{01}, l)) I(i_{01} \geq 1) I(t \geq t_0) \\ &+ \sum_{l=1}^{L_{12}} \lambda_t(f_{12}, l) K_{12l}(z_{t-1}(i_{01}, i_{12}, f_{12}, l)) I(i_{12} \geq 1) \\ &+ \sum_{l=1}^{L_{02}} \lambda_t(f_{02}, l) K_{02l}(z_{t-1}(i_{01}, i_{12}, f_{02}, l)) I(i_{02} \geq 1) I(t \geq t_0), \qquad t \geq 1, \end{aligned} \tag{5.1}$$

where we have

$$v_0(i_{01}, i_{12}) = \sum_{l=1}^{L_{12}} \lambda_0(f_{12}, l) K_{12l}(c_{12l}) I(i_{12} \geq 1). \tag{5.2}$$

Lemma 5.1

(a) $\Delta_{02}v_t(i_{01}, i_{12})$, *hence* $z_t(i_{01}, i_{12}, f_{02}, l)$ *is nonincreasing in* i_{01} *and* i_{12} *for* $t \geq 0$.

(b) $\Delta_{12}v_t(i_{01}, i_{12})$, *hence* $z_t(i_{01}, i_{12}, f_{12}, l)$ *is nondecreasing in* i_{01} *for* $t \geq 0$.

(c) $\Delta_{01}v_t(i_{01}, i_{12})$, *hence* $z_t(i_{01}, i_{12}, f_{01}, l)$ *is nondecreasing in* i_{12} *for* $t \geq 0$.

(d) $\Delta_{01}v_{t-1}(i_{01}, i_{12}) \leq \Delta_{01}v_{t-1}(i_{01}-1, i_{12}-1)$, *hence* $z_{t-1}(i_{01}, i_{12}, f_{01}, l) \leq z_{t-1}(i_{01}-1, i_{12}-1, f_{01}, l)$ *for* $i_{01} \geq 2, i_{12} \geq 1$ *and* $t \geq 1$.

(e) $\Delta_{12}v_{t-1}(i_{01}, i_{12}) \leq \Delta_{12}v_{t-1}(i_{01}-1, i_{12}-1)$, *hence* $z_{t-1}(i_{01}, i_{12}, f_{12}, l) \leq z_{t-1}(i_{01}-1, i_{12}-1, f_{12}, l)$ *for* $i_{01} \geq 1, i_{12} \geq 2$ *and* $t \geq 1$.

Proof : See Appendix A.

Lemma 5.2

(a) $\Delta_{01}v_t(i_{01}, i_{12})$, *hence* $z_t(i_{01}, i_{12}, f_{01}, l)$ *is nonincreasing in* i_{01} *for any* $t \geq 0$.

(b) $\Delta_{12}v_t(i_{01}, i_{12})$, *hence* $z_t(i_{01}, i_{12}, f_{12}, l)$ *is nonincreasing in* i_{12} *for any* $t \geq 0$.

Proof : See Appendix A.

Lemma 5.3 If $\beta = 1$, then $\Delta_{02}v_t(i_{01}, i_{12})$, hence $z_t(i_{01}, i_{12}, f_{02}, l)$ is nondecreasing in t for $i_{01} \geq 1$ and $i_{12} \geq 1$.

Proof : See Appendix A.

Theorem 5.1

(a) $x_t(i_{01}, i_{12}, f_{01}, l)$ *is nonincreasing in* i_{01} *and nondecreasing in* i_{12} *for any* $t \geq 0$.

(b) $x_t(i_{01}, i_{12}, f_{12}, l)$ *is nonincreasing in* i_{12} *and nondecreasing in* i_{01} *for any* $t \geq 0$.

(c) $x_t(i_{01}, i_{12}, f_{02}, l)$ *is nonincreasing in* i_{01} *and* i_{12} *for any* $t \geq 0$.

(d) *If* $\beta = 1$, *then* $x_t(i_{01}, i_{12}, f_{02}, l)$ *is nondecreasing in* t.

Proof : (a) The former part is immediate from Lemma 3.1(c) and Lemma 5.2(a). The latter part is immediate from Lemma 3.1(c) and Lemma 5.1(c). (b) The former part is immediate from Lemma 3.1(c) and Lemma 5.2(b). The latter part is immediate from Lemma 3.1(c) and Lemma 5.1(b).

(c) Immediate from Lemma 3.1(c) and Lemma 5.1(a).

(d) Immediate from Lemma 3.1(c) and Lemma 5.3. ■

6 Single aircraft without customers' getting off at immediate airports

In this case, all customers reach the same destination, airport N by the same aircraft regardless of their departing airports. Since the seats available for all flight-legs are the same initially and that all trips include the flight-leg $d_{N-1,N}$, the seats available for any trips that open for reservation at any booking period are the same. That is, $\mathbf{i}$ becomes i. Then, clearly, we have

$$\begin{aligned} v_t(0) &= 0, & t \geq 0, \\ v_t(i) &= \beta\lambda_t(0)v_{t-1}(i) + \sum_{j=h(t)}^{N-1}\sum_{l=1}^{L_{jN}} \lambda_t(f_{jN}, l)\max\{g_t(i, f_{jN}, l), \beta v_{t-1}(i)\} & t \geq 1. \end{aligned} \tag{6.1}$$

And the final condition is given as follows:

$$v_0(i) = \lambda_0(f_{N-1,N}, l)K_{N-1,Nl}(c_{N-1,Nl}), i \geq 1. \tag{6.2}$$

Let

• Example 1 (One flight-leg) Figure 4(a) shows that $x_{80}(i_{01}, f_{01}, 2)$ is nonincreasing in i_{01} (Theorem 6.1(a)). Note that if $i = 1$, then the optimal price equals to $950(= b_{012})$, implying that when $i = 1$, airline company can either reject the customer before offering the price b_{012}, or decide to offer the price to the customer and let the customer reject the offered price. If $i \geq 6$, then the optimal price equals to $800(= a_{012})$, implying that when $i \geq 6$, airline companies can offer the price $800(= a_{012})$ to attract the customer with probability of 1. Figure 4(b) shows that $x_t(2, f_{01}, 2)$ is nondecreasing in t (Theorem 6.1(b)). Note that if $t \geq 92$, then the optimal price equals to $950(= b_{012})$, implying that when $t \geq 92$, airline companies can either reject the customer before offering the price $950(= b_{012})$, or offer the price to the customer and let the customer reject the offered price with probability of 1. If $t \leq 3$, the optimal price equals to $800(= a_{012})$, and the logic mentioned above applies.

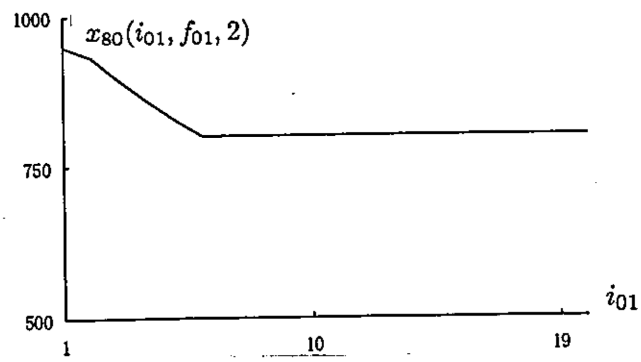

Figure 4(a)

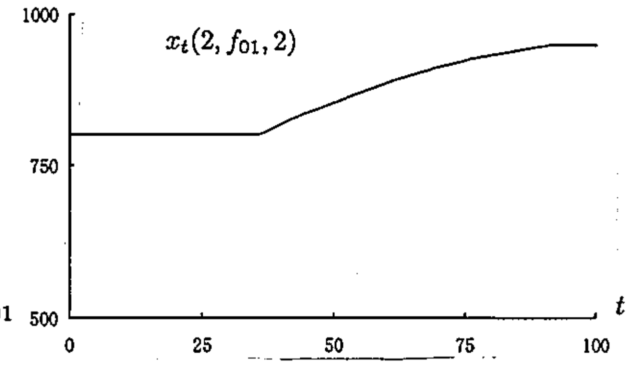

Figure 4(b)

• Example 2 (Two flight-legs) Figure 5(a) shows that $x_{80}(3, i_{12}, f_{12}, 2)$ is nonincreasing in i_{12} (Theorem 5.1(b)). Figure 5(b) shows that $x_{80}(i_{01}, 7, f_{12}, 2)$ is nondecreasing in i_{01} (Theorem 5.1(b)). Figure 5(c) shows that $x_t(3, 7, f_{12}, 2)$ is nondecreasing in t ($x_t(i_{01}, i_{12}, f_{j,j+1}, l)$ is nondecreasing in t. A counterargument is shown in Example 4). Figure 5(d) shows that $x_t(3, 7, f_{02}, 2)$ is nondecreasing in t (Theorem 5.1(d)).

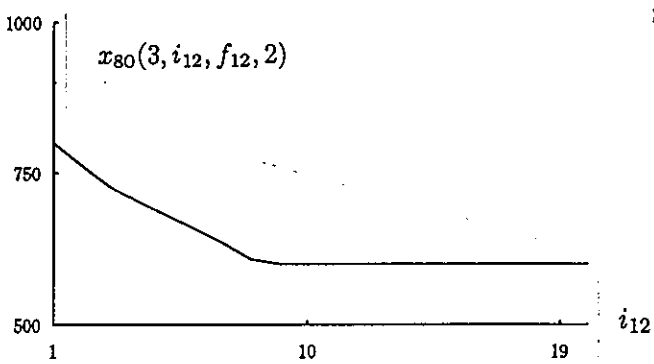

Figure 5(a)

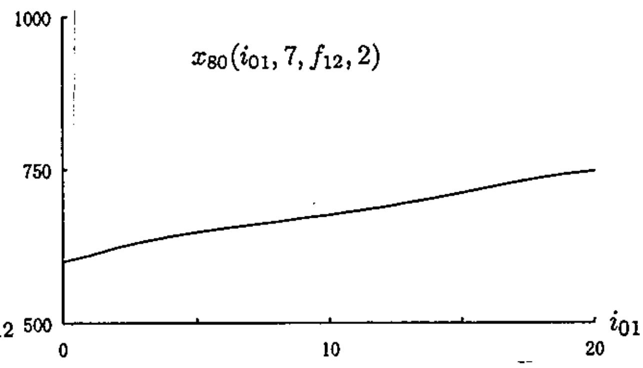

Figure 5(b)

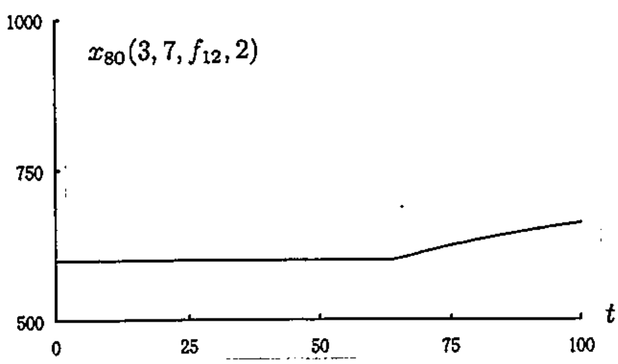

Figure 5(c)

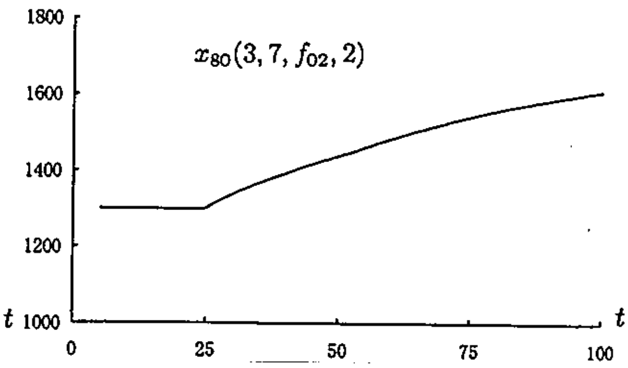

Figure 5(d)

• Example 3 (Three flight-legs) Figure 6(a) shows that $x_{50}(2, 6, i_{23}, f_{01}, 2)$ is nondecreasing in i_{23} ($x_t(i_{01}, i_{12}, i_{23}, f_{jk}, l)$ is nondecreasing in $i_{m,m+1}$ $m \notin \{j, \cdots, k-1\}$. A counterargument is shown in Example 5). Figure 6(b) shows that $x_{50}(2, i_{12}, 10, f_{03}, 2)$ is nonincreasing in i_{12} ($x_t(i_{01}, i_{12}, i_{23}, f_{jk}, l)$ is nonincreasing in $i_{m,m+1}$ $m \in \{j, \cdots, k-1\}$. A counterargument is shown in Example 6). Figure 6(c) shows that $x_t(3, 6, 10, f_{03}, 2)$ is nondecreasing in t ($x_t(i_{01}, i_{12}, i_{23}, f_{jk}, l)$ is nondecreasing in t. A counterargument case is shown in Example 7).

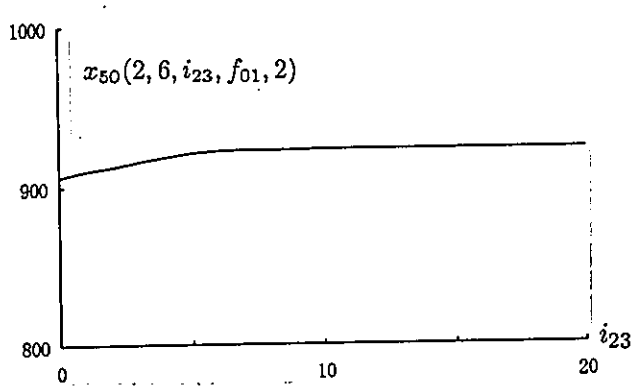

Figure 6(a)

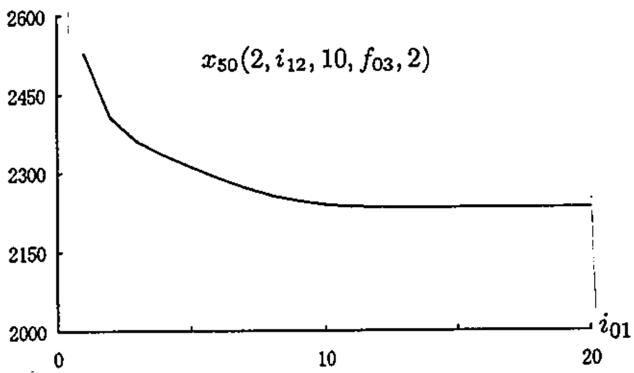

Figure 6(b)

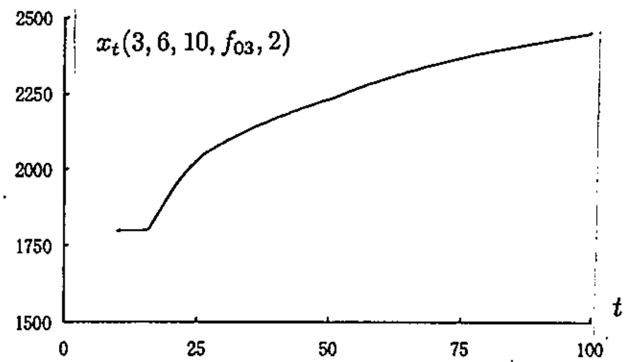

Figure 6(c)

• Example 4 (Two flight-legs) Here, we show one case that when $\beta = 1$, $x_t(i_{01}, i_{12}, f_{12}, l)$ is not nondecreasing in t. From (4.9) and (5.2) we have

$$\begin{aligned} z_7(1, 1, f_{12}, 2) &= v_7(1, 1) - v_7(1, 0) + 140 \\ &= 0.6K_{022}(z_6(1, 1, f_{02}, 2)) + 140 \\ &= 0.6K_{022}(300) + 140 \\ &= 740. \end{aligned}$$

$$\begin{aligned} z_8(1,1,f_{12},2) &= v_8(1,1) - v_8(1,0) + 140 \\ &= z_7(1,1,f_{12},2) + 0.5K_{012}(z_7(1,1,f_{01},2)) - 0.5K_{012}(z_7(1,0,f_{01},2)) \\ &= 740 + 0.5K_{012}(v_7(1,1) - v_7(0,1) + 180) - 0.5K_{012}(v_7(1,0) + 180) \\ &= 740 + 0.5K_{012}(0.6K_{022}(300) + 180) - 0.5K_{012}(180) \\ &= 740 + 0.5(K_{012}(780) - K_{012}(180)) \\ &= 740 + 0.2 - 120 \\ &= 454.085. \end{aligned}$$

Hence, we have $z_8(1,1,f_{12},2) < z_7(1,1,f_{12},2)$. Furthermore, we obtain $x_9(1,1,f_{12},2) = 647.0425 < x_8(1,1,f_{12},2) = 770$.

- Example 5 (Three flight-legs) Here, we show one case that $x_t(i_{01}, i_{12}, i_{23}, f_{01}, l)$ is not nondecreasing in i_{23}. From (4.9) and (4.12) we have

$$\begin{aligned} & z_8(1,1,1,f_{01},2) = v_8(1,1,1) - v_8(0,1,1) + 180 \\ &= v_7(1,1,1) + 0.6K_{022}(z_7(1,1,1,f_{02},2)) - v_7(0,1,1) + 180 \\ &= 0.6K_{022}(v_7(1,1,1) - v_7(0,0,1) + 300) + 180 \\ &= 0.6K_{022}(K_{132}(400) + 300) + 180 \\ &= 0.6K_{022}(1400) + 180 \\ &= 213.75. \end{aligned}$$

$$\begin{aligned} & z_8(1,1,0,f_{01},1) = v_8(1,1,0) - v_8(0,1,0) + 180 \\ &= v_7(1,1,0) + 0.6K_{022}(z_7(1,1,0,f_{02},2)) - v_7(0,1,0) + 180 \\ &= 0.6K_{022}(v_7(1,1,0) + 300) + 180 \\ &= 0.6K_{022}(300) + 180 \\ &= 780. \end{aligned}$$

Hence, we have $z_8(1,1,1,f_{01},2) < z_8(1,1,0,f_{01},2)$. Furthermore, we obtain $x_9(1,1,1,f_{01},2)$ =800< $x_9(1,1,0,f_{01},2)$ =865.

- Example 6 (Three flight-legs) Here, we will show one case that $x_t(i_{01}, i_{12}, i_{23}, f_{03}, l)$ is not nonincreasing in i_{12}. From (4.9) and (4.12) we have

$$\begin{aligned} & z_8(1,2,1,f_{03},2) = v_8(1,2,1) - v_8(0,1,0) + 550 \\ &= v_7(1,2,1) + 0.6K_{022}(z_7(1,2,1,f_{02},2)) - v_7(0,1,0) + 550 \\ &= K_{132}(400) + 550 + 0.6K_{022}(v_7(1,2,1) - v_7(0,1,1) + 300) \\ &= 1650 + 0.6K_{022}(300) \\ &= 2250. \end{aligned}$$

$$\begin{aligned} & z_8(1,1,1,f_{03},2) = v_8(1,1,1) + 550 \\ &= v_7(1,1,1) + 0.6K_{022}(z_7(1,1,1,f_{02},2)) + 550 \\ &= K_{132}(400) + 550 + 0.6K_{021}(v_7(1,1,1) - v_7(0,0,1) + 300) \\ &= 1650 + 0.6K_{021}(K_{132}(400) + 300) \\ &= 1650 + 0.6K_{021}(1400) \\ &= 1683.75. \end{aligned}$$

Hence, we have $z_8(1,2,1,f_{03},2) > z_8(1,1,1,f_{03},2)$. Furthermore, we obtain $x_9(1,1,1,f_{03},2) = 2291.875 < x_9(1,2,1,f_{03},2) = 2575$.

• Example 7 (Three flight-legs) Here, we show one case that $x_t(i_{01},i_{12},i_{23},f_{03},l)$ is not nondecreasing in t. From (4.9) and (4.12) we have

$$\begin{aligned}
z_7(1,2,1,f_{03},2) &= v_7(1,2,1) - v_7(0,1,0) + 550 \\
&= v_6(1,2,1) + 0.5K_{022}(z_6(1,2,1,f_{12},2)) - v_6(0,1,0) + 550 \\
&= K_{132}(400) + 0.5K_{022}(140) + 550 \\
&= 1100 + 580 + 550 \\
&= 2230.
\end{aligned}$$

$$\begin{aligned}
z_7(1,2,1,f_{12},2) &= v_7(1,2,1) - v_7(1,1,1) + 140 \\
&= v_6(1,2,1) + 0.5K_{022}(z_6(1,2,1,f_{02},2)) - v_6(1,1,1) - 0.5K_{022}(z_6(1,1,1,f_{02},2)) + 140 \\
&= 0.5K_{022}(v_6(1,2,1) - v_6(0,1,1) + 300) - 0.5K_{022}(v_6(1,1,1) - v_6(0,0,1) + 300) + 140 \\
&= 0.5(K_{022}(300) - K_{022}(K_{132}(400) + 300)) + 140 \\
&= 0.5(1000 - K_{022}(1400)) + 140 \\
&= 611.875.
\end{aligned}$$

$$\begin{aligned}
z_8(1,2,1,f_{03},2) &= v_8(1,2,1) - v_8(0,1,0) + 550 \\
&= v_7(1,2,1) + 0.5K_{122}(z_7(1,2,1,f_{12},2)) - v_7(0,1,0) - 0.5K_{122}(z_7(0,1,0,f_{12},2)) + 550 \\
&= v_7(1,2,1) + 0.5K_{122}(611.875) - v_7(0,1,0) - 0.5K_{122}(140) + 550 \\
&= z_7(1,2,1,f_{03},2) + 0.5(K_{122}(611.875) - K_{122}(140)) \\
&= z_7(1,2,1,f_{03},2) + 0.5(44.23877 - 460) \\
&= 2022.1194.
\end{aligned}$$

Hence, we have $z_8(1,2,1,f_{03},2) < z_7(1,2,1,f_{03},2)$. Furthermore, we obtain $x_9(1,2,0,f_{03},2) = 2461.06 < x_8(1,2,1,f_{03},2 = 2565$.

8 Conclusion

In this paper, the optimal price is defined by $x_t(i,f_{jk},l)$, which stands for the price at the time when there are still t periods and i seats remaining for the customer of class l in leg f_{jk}. The conclusions obtained in this paper are summarized as follows:

1. It is always optimal to accept every arriving customer (Theorem 4.1).
2. It is possible, although a customer has being accepted, he/she may decide to reject the offered price. This would imply that the price being offered by the airline has being set too high. Theoretically, this occurs when $b_{jkl} \le z_t(i,f_{jk},l)$; in this case $x_t(i,f_{jk},l) = b_{jkl}$ from Lemma 4.1. Under this condition, even if a customer is accepted, the deal will not go through. Therefore, when $b_{jkl} \le z_t(i,f_{jk},l)$, the airline can decide to reject a customer of class l arbitrarily before making a price offer.
3. The optimal price $x_t(i,f_{jk},l)$ when there is no passenger getting off at immediate airports are summarized as follows.

   (1) If $\beta < 1$, then $x_t(i,f_{jN},l)$ is not always monotone in t for all j and i.

   (2) If $\beta = 1$, then $x_t(i,f_{jN},l)$ is nondecreasing in t.

(3) $x_t(i, f_{jN}, l)$ is nonincreasing in i.

4. The optimal price $x_t(i, f_{jk}, l)$ in the case of one flight-legs ($N = 1$) is summarized as 3.

5. The optimal price $x_t(i, f_{jk}, l)$ in the case of two flight-legs ($N = 2$) are summarized as follows.

   (1) $x_t(i_{01}, f_{01}, f_{01}, l)$ and $x_t(i_{01}, f_{01}, f_{12}, l)$ is not always nondecreasing in t.

   (2) If $\beta = 1$, then $x_t(i_{01}, f_{01}, f_{02}, l)$ is nondecreasing in t.

   (3) $x_t(i_{01}, i_{12}, f_{01}, l)$ is nondecreasing in i_{12} and nonincreasing in i_{01}.

   (4) $x_t(i_{01}, i_{12}, f_{12}, l)$ is nondecreasing in i_{01} and nonincreasing in i_{12}.

   (5) $x_t(i_{01}, i_{12}, f_{02}, l)$ is nonincreasing in i_{01} and i_{12}.

6. When the number of flight-legs is equal to or more than 3 ($N \geq 3$), the optimal price $x_t(i, f_{jk}, l)$ are summarized as follows:

   (1) It is not always true that the price of booking seat for a trip is nonincreasing with respect to the seats available in each flight-leg of the trip, and nondecreasing with respect to the seats available of all remaining flight-legs.

   (2) It is not always true that the price of booking seat for a trip is nondecreasing with respect to the remaining time.

Appendix A

Proof of Lemma 3.1

(a) The former part is immediate since $p(x)(x - \nu)$ is nonincreasing in ν for all x. The latter part is clear from the fact that $p(x)(x - \nu) + \nu (= p(x) + (1 - p(x))\nu)$ is nondecreasing in ν for all x.

(b) The former part is clear from the fact that $K(\nu) \geq p(b)(b - \nu) = 0$. The latter can be proved to be true since for $\xi > 0$ such that $\xi + \nu < b$,

$$K(\nu) \geq p(\xi + \nu)(\nu + \xi - \nu) = p(\xi + \nu)\xi > 0.$$

(c) For any $\xi > 0$, we have

$$\begin{aligned}
K(\nu + \xi) &= \max_x p(x)(x - (\nu + \xi)) \\
&= p(x(\nu + \xi))(x(\nu + \xi) - (\nu + \xi)) \\
&= p(x(\nu + \xi))(x(\nu + \xi) - \nu) - p(x(\nu + \xi))\xi \\
&\leq p(x(\nu))(x(\nu) - \nu) - p(x(\nu + \xi))\xi \\
&= p(x(\nu))(x(\nu) - (\nu + \xi)) + \xi(p(x(\nu)) - p(x(\nu + \xi))) \\
&\leq p(x(\nu + \xi))(x(\nu + \xi) - (\nu + \xi)) + \xi(p(x(\nu)) - p(x(\nu + \xi))) \\
&= K(\nu + \xi) + \xi(p(x(\nu)) - p(x(\nu + \xi))).
\end{aligned}$$

Therefore, we have $0 \leq p(x(\nu)) - p(x(\nu + \xi))$, that is, $p(x(\nu)) \geq p(x(\nu + \xi))$, implying that $x(\nu) \leq x(\nu + \xi)$ because $p(x)$ is strictly decreasing in x for $a \leq x \leq b$.

(d) First, assume $b < x(\nu)$. Then

$$K(\nu) = p(x(\nu))(x(\nu) - \nu) = 0 = p(b)(b - \nu),$$

which contradicts the definition of $x(\nu)$. Assume that $x(\nu) < a$, then

$$K(\nu) = p(x(\nu))(x(\nu) - \nu) = x(\nu) - \nu < a - \nu = p(a)(a - \nu) \leq K(\nu),$$

which is also a contradiction. Therefore, it must be that $a \le x(\nu) \le b$. Second, for any $\xi > 0$ such that $\nu+\xi < b$, we have $K(\nu) \ge p(\nu+\xi)(\nu+\xi-\nu) = \xi p(\nu+\xi) > 0$. Hence it follows that $a \le x(\nu) < b$ if $\nu < b$. Finally, for $b \le \nu$, we have $p(x)(x-\nu) < 0$ for all $x < b$ and $p(x)(x-\nu) = 0$ for all $x \ge b$; therefore, we have $b \le x(\nu)$. Combining this with the definition of $x(\nu)$ results in $b = x(\nu)$.

(e) From (a), for $\nu_1 \le \nu_2$ we have

$$K(\nu_1) + \nu_1 \le K(\nu_2) + \nu_2.$$

Hence, it follows that $K(\nu_1) - K(\nu_2) \le \nu_2 - \nu_1$ for $\nu_1 \le \nu_2$.

(f) Same as the proof of (e).

(g) Noting that for any ν_1 and ν_2

$$K(\nu_1) - K(\nu_2) = \max_x p(x)(x-\nu_1) - \max_x p(x)(x-\nu_2) \ge p(x(\nu_2))(\nu_2 - \nu_1).$$

Similarly we have

$$K(\nu_4) - K(\nu_3) = \max_x p(x)(x-\nu_4) - \max_x p(x)(x-\nu_3) \ge p(x(\nu_3))(\nu_3 - \nu_4).$$

For ν_1, $\nu 2$, ν_3 and ν_4 such that $\nu_1 \le \min\{\nu_2, \nu_3\}$ and $\max\{\nu_2, \nu_3\} \le \nu_4$, if $\nu_2 \le \nu_3$ we have

$$\begin{aligned} K(\nu_1) - K(\nu_2) - K(\nu_3) + K(\nu_4) &= K(\nu_1) - K(\nu_2) + K(\nu_4) - K(\nu_3) \\ &\ge p(x(\nu_2))(\nu_2 - \nu_1) + p(x(\nu_3))(\nu_3 - \nu_4) \\ &\ge p(x(\nu_2))(\nu_2 - \nu_1) + p(x(\nu_2))(\nu_3 - \nu_4) \\ &= p(x(\nu_2))(\nu_2 - \nu_1 + \nu_3 - \nu_4). \end{aligned}$$

If $\nu_2 \ge \nu_3$ we have

$$\begin{aligned} K(\nu_1) - K(\nu_2) - K(\nu_3) + K(\nu_4) &= K(\nu_1) - K(\nu_3) + K(\nu_4) - K(\nu_2) \\ &\ge p(x(\nu_3))(\nu_3 - \nu_1) + p(x(\nu_2))(\nu_2 - \nu_4) \\ &\ge p(x(\nu_2))(\nu_3 - \nu_1) + p(x(\nu_2))(\nu_2 - \nu_4) \\ &= p(x(\nu_2))(\nu_2 - \nu_1 + \nu_3 - \nu_4). \end{aligned}$$

(h) Noting that for any ν_1 and ν_2

$$K(\nu_1) - K(\nu_2) = \max_x p(x)(x-\nu_1) - \max_x p(x)(x-\nu_2) \le p(x(\nu_1))(\nu_2 - \nu_1).$$

Similarly, we have

$$K(\nu_4) - K(\nu_3) = \max_x p(x)(x-\nu_4) - \max_x p(x)(x-\nu_3) \le p(x(\nu_4))(\nu_3 - \nu_4).$$

For ν_1, ν_2, ν_3 and ν_4 such that $\nu_2 \le \min\{\nu_1, \nu_4\}$ and $\max\{\nu_1, \nu_4\} \le \nu_3$, if $\nu_1 \le \nu_4$ we have

$$\begin{aligned} K(\nu_1) - K(\nu_2) - K(\nu_3) + K(\nu_4) &= K(\nu_1) - K(\nu_2) + K(\nu_4) - K(\nu_3) \\ &\le p(x(\nu_1))(\nu_2 - \nu_1) + p(x(\nu_4))(\nu_3 - \nu_4) \\ &\le p(x(\nu_1))(\nu_2 - \nu_1) + p(x(\nu_1))(\nu_3 - \nu_4) \\ &= p(x(\nu_1))(\nu_2 - \nu_1 + \nu_3 - \nu_4). \end{aligned}$$

If $\nu_1 \ge \nu_4$ we have

$$\begin{aligned} K(\nu_1) - K(\nu_2) - K(\nu_3) + K(\nu_4) &= K(\nu_1) - K(\nu_3) + K(\nu_4) + K(\nu_2) \\ &\le p(x(\nu_1))(\nu_3 - \nu_1) + p(x(\nu_4))(\nu_2 - \nu_4) \\ &\le p(x(\nu_1))(\nu_3 - \nu_1) + p(x(\nu_1))(\nu_2 - \nu_4) \\ &= p(x(\nu_1))(\nu_2 - \nu_1 + \nu_3 - \nu_4). \quad \blacksquare \end{aligned}$$

Proof of Lemma 4.1

(a) (By induction) For any $i_{m,m+1}$ such that $i_{m,m+1} \geq 1$ and $m \in \{0, 1, \cdots, N-1\}$, from (4.13) we have

$$\begin{aligned}
&\Delta_{m,m+1} v_0(i) \\
&= \sum_{\substack{l=1 \\ f_{N-1,N} \in S(i)}}^{L_{N-1,N}} \lambda_0(f_{N-1,N}, l) K_{N-1,Nl}(c_{N-1,Nl}) - \sum_{\substack{l=1 \\ f_{N-1,N} \in S(i_{m,m+1})}}^{L_{N-1,N}} \lambda_0(f_{N-1,N}, l) K_{N-1,Nl}(c_{N-1,Nl}) \\
&\geq \sum_{\substack{l=1 \\ f_{N-1,N} \in S(i_{m,m+1})}}^{L_{N-1,N}} \lambda_0(f_{N-1,N}, l) K_{N-1,Nl}(c_{N-1,Nl}) - \sum_{\substack{l=1 \\ f_{N-1,N} \in S(i_{m,m+1})}}^{L_{N-1,N}} \lambda_0(f_{N-1,N}, l) K_{N-1,Nl}(c_{N-1,Nl}) \\
&= 0
\end{aligned}$$

due to $K(\nu) \geq 0$ and the set $S(i) \supset S(i_{m,m+1})$. Hence, the assertion holds true for $t = 0$. Assume it holds true for $t-1$. Then, since $v_{t-1}(i)$ is nondecreasing in $i_{m,m+1}$, we have from (4.12)

$$\begin{aligned}
\Delta_{m,m+1} v_t(i) &= \beta \Delta_{m,m+1} v_{t-1}(i) \\
&\quad + \sum_{j=h(t)}^{N-1} \sum_{\substack{k=j+1 \\ f_{jk} \in S(i)}}^{N} \sum_{l=1}^{L_{jk}} \lambda_t(f_{jk}, l) K_{jkl}(z_{t-1}(i, f_{jk}, l)) \\
&\quad - \sum_{j=h(t)}^{N-1} \sum_{\substack{k=j+1 \\ f_{jk} \in S(i_{m,m+1})}}^{N} \sum_{l=1}^{L_{jk}} \lambda_t(f_{jk}, l) K_{jkl}(z_{t-1}(i_{m,m+1}, f_{jk}, l)) \\
&\geq \beta \Delta_{m,m+1} v_{t-1}(i) \\
&\quad + \sum_{j=h(t)}^{N-1} \sum_{\substack{k=j+1 \\ f_{jk} \in S(i_{m,m+1})}}^{N} \sum_{l=1}^{L_{jk}} \lambda_t(f_{jk}, l) (K_{jkl}(z_{t-1}(i, f_{jk}, l)) - K_{jkl}(z_{t-1}(i_{m,m+1}, f_{jk}, l))).
\end{aligned}$$

Here, partition $\{f_{jk} \mid f_{jk} \in S(i_{m,m+1}), h(t) \leq j < k \leq N\}$ into sets $\mathcal{A}$ if $z_{t-1}(i, f_{jk}, l) \leq z_{t-1}(i_{m,m+1}, f_{jk}, l)$ and $\mathcal{B}$ if $z_{t-1}(i, f_{jk}, l) \geq z_{t-1}(i_{m,m+1}, f_{jk}, l)$. Then, on set $\mathcal{A}$ we have from Lemma 3.1(a)

$$K_{jkl}(z_{t-1}(i, f_{jk}, l)) - K_{jkl}(z_{t-1}(i_{m,m+1}, f_{jk}, l)) \geq 0.$$

and on set $\mathcal{B}$ we have from Lemma 3.1(f)

$$\begin{aligned}
K_{jkl}(z_{t-1}(i, f_{jk}, l)) - K_{jkl}(z_{t-1}(i_{m,m+1}, f_{jk}, l)) &\geq z_{t-1}(i_{m,m+1}, f_{jk}, l) - z_{t-1}(i, f_{jk}, l) \\
&= \beta(\Delta_{jk} v_{t-1}(i_{m,m+1}) - \Delta_{jk} v_{t-1}(i)) \\
&= \beta(\Delta_{m,m+1} v_{t-1}(i_{jk}) - \Delta_{m,m+1} v_{t-1}(i)) \\
&\geq -\beta \Delta_{m,m+1} v_{t-1}(i_{m,m+1}).
\end{aligned}$$

Hence, we have

$$\begin{aligned}
\Delta_{m,m+1} v_t(i) &\geq \beta \Delta_{m,m+1} v_{t-1}(i) - \beta \sum_{j=h(t)}^{N-1} \sum_{\substack{k=j+1 \\ f_{jk} \in \mathcal{B}}}^{N} \sum_{l=1}^{L_{jk}} \lambda_t(f_{jk}, l) \Delta_{m,m+1} v_{t-1}(i) \\
&\geq \lambda_t(0) \beta \Delta_{m,m+1} v_{t-1}(i) \geq 0.
\end{aligned}$$

(b) It can be easily shown by induction that $v_t(i) \geq 0$ for all t and i. Then for a given $f_{j^*k^*}$ such that $f_{j^*k^*} \in S(i)$ and $\lambda_t(f_{j^*k^*}, l^*) > 0$, from (4.12) and (4.9) we have

$$v_t(i) \geq \beta v_{t-1}(i) + \lambda_t(f_{j^*k^*}, l^*) K_{j^*k^*l^*}(z_{t-1}(i, f_{j^*k^*}, l^*))$$
$$\geq \lambda_t(f_{j^*k^*}, l^*)(\beta v_{t-1}(i) + K_{j^*k^*l^*}(z_{t-1}(i, f_{j^*k^*}, l^*)))$$
$$= \lambda_t(f_{j^*k^*}, l^*)(c_{j^*k^*l^*} + \beta \Delta_{j^*k^*} v_{t-1}(i) + K_{j^*k^*l^*}(c_{j^*k^*l^*} + \beta \Delta_{j^*k^*} v_{t-1}(i)) + \beta v_{t-1}(i_{j^*k^*}) - c_{j^*k^*l^*}).$$

Here, since $c_{j^*k^*l^*} + \beta \Delta_{j^*k^*} v_{t-1}(i) \geq c_{j^*k^*l^*} - \beta v_{t-1}(i_{j^*k^*l^*})$, we have from Lemma 3.1(a)

$$c_{j^*k^*l^*} + \beta \Delta_{j^*k^*} v_{t-1}(i) + K_{j^*k^*l^*}(c_{j^*k^*l^*} + \beta \Delta_{j^*k^*} v_{t-1}(i))$$
$$\geq c_{j^*k^*l^*} - \beta v_{t-1}(i_{j^*k^*}) + K_{j^*k^*l^*}(c_{j^*k^*l^*} - \beta v_{t-1}(i_{j^*k^*}))$$

Hence,

$$v_t(i) \geq \lambda_t(f_{j^*k^*}, l^*) K_{j^*k^*l^*}(c_{j^*k^*l^*} - \beta v_{t-1}(i_{j^*k^*})).$$

Furthermore, since $c_{j^*k^*l^*} - \beta v_{t-1}(i_{j^*k^*}) < b_{j^*k^*l^*}$, we have $K_{j^*k^*l^*}(c_{j^*k^*l^*} - \beta v_{t-1}(i_{j^*k^*})) > 0$ from Lemma 3.1(b), hence the result follows.

(c) Assume $\lambda_t(f_{jk}, l) = 0$ for all j, k, l and for a certain j^*,k^*,l^* such that $\lambda_{t-1}(f_{j^*k^*}, l) > 0$. Then, for $\beta < 1$ from (4.12) and $v_{t-1}(i) > 0$ from (b) we have

$$v_t(i) = \beta v_{t-1}(i) < v_{t-1}(i).$$

(d) Clearly from (4.12) and $K(\nu) \geq 0$ from Lemma 3.1(b), we have for $\beta = 1$

$$v_t(i) = v_{t-1}(i) + \sum_{j=h(t)}^{N-1} \sum_{\substack{k=j+1 \\ f_{jk} \in S(i)}}^{N} \sum_{l=1}^{L_{jk}} \lambda_t(f_{jk}, l) K_{jkl}(z_{t-1}(i, f_{jk}, l)) \geq v_{t-1}(i). \quad \blacksquare$$

Proof of Lemma 4.2

It can be proven by the example in which for a certain t^* such that $h(t^* - 1) = j$ and $\lambda_t(f_{jk}, l) = 0$ for $t \leq t^*$ except for $\lambda_{t^*-1}(f_{jk^*}, l^*) > 0$, then from (4.12) and Lemma 4.1(b) we have for $\beta < 1$

$$\Delta_{jk^*} v_{t^*}(1, 1, \ldots, 1) - \Delta_{jk^*} v_{t^*-1}(1, 1, \ldots, 1)$$
$$= (\beta - 1)\Delta_{jk^*} v_{t^*-1}(1, 1, \ldots, 1)$$
$$= (\beta - 1)(\Delta_{jk^*} v_{t^*-2}(1, 1, \ldots, 1) + \lambda_{t^*-1}(f_{jk^*}, l^*) K_{jk^*l^*}(z_{t^*-2}(1, 1, \ldots, 1, f_{jk^*}, l^*))) \qquad (*)$$

Here, since $\lambda_t(f_{jk}, l) = 0$ for all $t \leq t^* - 2$, clearly, $\Delta_{jk^*} v_{t^*-2}(1, 1, \ldots, 1) = 0$, accordingly $z_{t^*-2}(1, 1, \cdot \cdots, 1, f_{jk^*}, l^*) = c_{j^*k^*l^*} < b_{j^*k^*l^*}$. Then, since $K_{jk^*l^*}(z_{t^*-2}(1, 1, \ldots, 1, f_{jk^*}, l^*)) > 0$ from Lemma 3.1(b), it follow that $(*) < 0$, hence, the result follows.

Proof of Lemma 5.1

For $i_{01} \geq 1, i_{12} \geq 1$ and $t \geq 0$,

$$\Delta_{12} v_t(i_{01}, i_{12}) - \Delta_{12} v_t(i_{01} - 1, i_{12}) = \Delta_{01} v_t(i_{01}, i_{12}) - \Delta_{01} v_0(i_{01}, i_{12} - 1).$$

Hence, if (b) holds true, so also do (c). Now, by induction, for $i_{01} \geq 2$ and $i_{12} \geq 1$ we have from (5.3)

$$\Delta_{02} v_0(i_{01}, i_{12}) - \Delta_{02} v_0(i_{01} - 1, i_{12})$$
$$= v_0(i_{01}, i_{12}) - v_0(i_{01} - 1, i_{12} - 1) + v_0(i_{01} - 2, i_{12} - 1) - v_0(i_{01} - 1, i_{12})$$
$$= \sum_{l=1}^{L_{12}} \lambda_0(f_{12}, l)(K_{12l}(c_{12l}) - K_{12l}(c_{12l}) I(i_{12} \geq 2) + K_{12l}(c_{12l}) I(i_{12} \geq 2) - K_{12l}(c_{12l})) = 0.$$

For $i_{01} \geq 1$ and $i_{12} \geq 2$ we have from (5.3)

$$\begin{aligned}
&\Delta_{02}v_0(i_{01}, i_{12}) - \Delta_{02}v_0(i_{01}, i_{12}-1) \\
&\quad = v_0(i_{01}, i_{12}) - v_0(i_{01}-1, i_{12}-1) + v_0(i_{01}-1, i_{12}-2) - v_0(i_{01}, i_{12}-1) \\
&\quad = \sum_{l=1}^{L_{12}} \lambda_0(f_{12}, l)(K_{12l}(c_{12l}) - K_{12l}(c_{12l}) + K_{12l}(c_{12l})I(i_{12} \geq 3) - K_{12l}(c_{12l})) \\
&\quad = \sum_{l=1}^{L_{12}} \lambda_0(f_{12}, l)(I(i_{12} \geq 3) - 1)K_{12l}(c_{12l}) \leq 0.
\end{aligned}$$

For $i_{01} \geq 1$ and $i_{12} \geq 1$ we have from (5.3)

$$\begin{aligned}
&\Delta_{12}v_0(i_{01}, i_{12}) - \Delta_{12}v_0(i_{01}-1, i_{12}) \\
&\quad = v_0(i_{01}, i_{12}) - v_0(i_{01}, i_{12}-1) + v_0(i_{01}-1, i_{01}-1) - v_0(i_{01}-1, i_{12}) \\
&\quad = \sum_{l=1}^{L_{12}} \lambda_0(f_{12}, l)(K_{12l}(c_{12l}) - K_{12l}(c_{12l})I(i_{12} \geq 2) + K_{12l}(c_{12l})I(i_{12} \geq 2) - K_{12l}(c_{12l})) \\
&\quad = 0.
\end{aligned}$$

Hence, the assertions (a) (b),hence (c) hold true for $t = 0$. Assume (a) and (b) hold true for $t-1$. Then, (1) $\Delta_{02}v_{t-1}(i_{01}, i_{12})$, hence $z_{t-1}(i_{01}, i_{12}, f_{02}, l)$ is nonincreasing in i_{01} and i_{12}; (2) $\Delta_{12}v_{t-1}(i_{01}, i_{12})$, hence $z_{t-1}(i_{01}, i_{12}, f_{12}, l)$ is nondecreasing in i_{01}, and (3) $\Delta_{01}v_{t-1}(i_{01}, i_{12})$, hence $z_{t-1}(i_{01}, i_{12}, f_{01}, l)$ is nondecreasing in i_{12}. Then, we have for $i_{01} \geq 2$ and $i_{12} \geq 1$,

$$\Delta_{01}v_{t-1}(i_{01}, i_{12}) - \Delta_{01}v_{t-1}(i_{01}-1, i_{12}-1) = \Delta_{02}v_{t-1}(i_{01}, i_{12}) - \Delta_{02}v_{t-1}(i_{01}-1, i_{12}) \leq 0.$$

Furthermore, for $i_{01} \geq 1$ and $i_{12} \geq 2$ we have

$$\Delta_{12}v_{t-1}(i_{01}, i_{12}) - \Delta_{12}v_{t-1}(i_{01}-1, i_{12}-1) = \Delta_{02}v_{t-1}(i_{01}, i_{12}) - \Delta_{02}v_{t-1}(i_{01}, i_{12}-1) \leq 0.$$

Hence, if statements (a) (b) and (c) hold true, the statements (d) and (e) will also hold true. Now, since we have for $i_{01} \geq 1$ and $i_{12} \geq 1$ from (5.2)

$$\begin{aligned}
&\Delta_{12}v_t(i_{01}, i_{12}) - \Delta_{12}v_t(i_{01}-1, i_{12}) \\
&\quad = v_t(i_{01}, i_{12}) - v_t(i_{01}, i_{12}-1) + v_t(i_{01}-1, i_{12}-1) - v_t(i_{01}-1, i_{12}) \\
&\quad = \left.\beta(\Delta_{12}v_{t-1}(i_{01}, i_{12}) - \Delta_{12}v_{t-1}(i_{01}-1, i_{12}))\right\} G_1 \\
&\qquad \left.\begin{aligned}&+ \sum_{l=1}^{L_{01}} \lambda_t(f_{01}, l)\Big(K_{01l}(z_{t-1}(i_{01}, i_{12}, f_{01}, l)) - K_{01l}(z_{t-1}(i_{01}, i_{12}-1, f_{01}, l)) \\
&+ K_{01l}(z_{t-1}(i_{01}-1, i_{12}-1, f_{01}, l))I(i_{01} \geq 2) \\
&- K_{01l}(z_{t-1}(i_{01}-1, i_{12}, f_{01}, l))I(i_{01} \geq 2)\Big)I(t \geq t_0)\end{aligned}\right\} G_2 \\
&\qquad \left.\begin{aligned}&+ \sum_{l=1}^{L_{12}} \lambda_t(f_{12}, l)\Big(K_{12l}(z_{t-1}(i_{01}, i_{12}, f_{12}, l)) - K_{12l}(z_{t-1}(i_{01}, i_{12}-1, f_{12}, l))I(i_{12} \geq 2) \\
&+ K_{12l}(z_{t-1}(i_{01}-1, i_{12}-1, f_{12}, l))I(i_{12} \geq 2) - K_{12l}(z_{t-1}(i_{01}-1, i_{12}, f_{12}, l))\Big)\end{aligned}\right\} G_3 \\
&\qquad \left.\begin{aligned}&+ \sum_{l=1}^{L_{02}} \lambda_t(f_{02}, l)\Big(K_{02l}(z_{t-1}(i_{01}, i_{12}, f_{02}, l)) - K_{02l}(z_{t-1}(i_{01}, i_{12}-1, f_{02}, l))I(i_{12} \geq 2) \\
&+ K_{02l}(z_{t-1}(i_{01}-1, i_{12}-1, f_{02}, l))I(i_{02} \geq 2) \\
&- K_{02l}(z_{t-1}(i_{01}-1, i_{12}, f_{02}, l))I(i_{01} \geq 2)\Big)I(t \geq t_0)\end{aligned}\right\} G_4
\end{aligned}$$

Here, let the first, second, third and forth item of the above equation be denoted by G_1, G_2, G_3 and G_4, respectively. Then, we have $G_1 \geq 0$ from induuction hypothesis. For G_2, if $i_{01} \geq 2$, since $z_{t-1}(i_{01}-1, i_{12}-1, f_{01}, l) \leq z_{t-1}(i_{01}-1, i_{12}, f_{01}, l)$, from Lemma 3.1(a) we have

$$K_{01l}(z_{t-1}(i_{01}-1, i_{12}-1, f_{01}, l)) - K_{01l}(z_{t-1}(i_{01}-1, i_{12}, f_{01}, l)) \geq 0.$$

Hence, for all $i_{01} \geq 1$ we have

$$G_2 \geq \sum_{l=1}^{L_{01}} \lambda_t(f_{01}, l)(K_{01l}(z_{t-1}(i_{01}, i_{12}, f_{01}, l)) - K_{01l}(z_{t-1}(i_{01}, i_{12}-1, f_{01}, l)))I(t \geq t_0).$$

Furthermore, since $z_{t-1}(i_{01}, i_{12}-1, f_{01}, l) \leq z_{t-1}(i_{01}, i_{12}, f_{01}, l)$ for all $i_{01} \geq 1$ and $i_{12} \geq 2$, hence from Lemma 3.1(f) we have

$$\begin{aligned} G_2 &\geq \sum_{l=1}^{L_{01}} \lambda_t(f_{01}, l)(z_{t-1}(i_{01}, i_{12}-1, f_{01}, l) - z_{t-1}(i_{01}, i_{12}, f_{01}, l))I(t \geq t_0) \\ &= \sum_{l=1}^{L_{01}} \lambda_t(f_{01}, l)\beta(\Delta_{01}v_{t-1}(i_{01}, i_{12}-1) - \Delta_{01}v_{t-1}(i_{01}, i_{12}))I(t \geq t_0) \\ &= \sum_{l=1}^{L_{01}} \lambda_t(f_{01}, l)\beta(\Delta_{12}v_{t-1}(i_{01}-1, i_{12}) - \Delta_{12}v_{t-1}(i_{01}, i_{12}))I(t \geq t_0) \\ &= -\sum_{l=1}^{L_{01}} \lambda_t(f_{01}, l)I(t \geq t_0)G_1. \end{aligned}$$

For G_3, if $i_{12} \geq 2$, since $z_{t-1}(i_{01}-1, i_{12}-1, f_{01}, l) \leq z_{t-1}(i_{01}, i_{12}-1, f_{01}, l)$, we have from Lemma 3.1(a)

$$K_{12l}(z_{t-1}(i_{01}-1, i_{12}-1, f_{12}, l)) - K_{12l}(z_{t-1}(i_{01}, i_{12}-1, f_{12}, l)) \geq 0.$$

Hence, for all $i_{12} \geq 1$ we have

$$G_3 \geq \sum_{l=1}^{L_{12}} \lambda_t(f_{12}, l)(K_{12l}(z_{t-1}(i_{01}, i_{12}, f_{12}, l)) - K_{12l}(z_{t-1}(i_{01}-1, i_{12}, f_{12}, l))).$$

Furthermore, since $z_{t-1}(i_{01}-1, i_{12}, f_{12}, l) \leq z_{t-1}(i_{01}, i_{12}, f_{12}, l)$ for all $i_{01} \geq 2$ and $i_{12} \geq 1$, we have from Lemma 3.1(f)

$$\begin{aligned} G_3 &\geq \sum_{l=1}^{L_{12}} \lambda_t(f_{12}, l)(z_{-1}(i_{01}-1, i_{12}, f_{12}, l) - z_{t-1}(i_{01}, i_{12}, f_{12}, l)) \\ &= \sum_{l=1}^{L_{12}} \lambda_t(f_{12}, l)\beta(\Delta_{12}v_{t-1}(i_{01}-1, i_{12}) - \Delta_{12}v_{t-1}(i_{01}, i_{12})) \\ &= -\sum_{l=1}^{L_{12}} \lambda_t(f_{12}, l)G_1. \end{aligned}$$

For G_4, there are four distinct cases that should be considered:

(1) if $i_{01} = 1$ and $i_{12} = 1$, then clearly $G_4 \geq 0$ due to $K(\nu) \geq 0$ from Lemma 3.1(b).

(2) if $i_{01} \geq 2$ and $i_{12} = 1$, since $z_{t-1}(i_{01}, i_{12}, f_{02}, l) \leq z_{t-1}(i_{01}-1, i_{12}, f_{02}, l)$, we have from Lemma 3.1(a) $G_4 \geq 0$.

(3) if $i_{01} = 1$ and $i_{12} \geq 2$, since $z_{t-1}(i_{01}, i_{12}, f_{02}, l) \leq z_{t-1}(i_{01}, i_{12}-1, f_{02}, l)$, we have from Lemma 3.1(a) $G_4 \geq 0$.

(4) if $i_{02} = 2$, since

$$z_{t-1}(i_{01}, i_{12}, f_{02}, l) \leq \min\{z_{t-1}(i_{01}, i_{12}-1, f_{02}, l), z_{t-1}(i_{01}-1, i_{12}, f_{02}, l)\}$$

and

$$z_{t-1}(i_{01}-1, i_{12}-1, f_{02}, l) \geq \max\{z_{t-1}(i_{01}, i_{12}-1, f_{02}, l), z_{t-1}(i_{01}-1, i_{12}, f_{02}, l)\},$$

we have from Lemma 3.1(g)

$$G_4 \geq \sum_{l=1}^{L_{02}} \lambda_t(f_{02}, l)p_{02l}(x_{t-1}(i_{01}, i_{12}-1, f_{02}, l))\Big(z_{t-1}(i_{01}, i_{12}-1, f_{02}, l) - z_{t-1}(i_{01}, i_{12}, f_{02}, l)$$

$$+z_{t-1}(i_{01}-1,i_{12},f_{02},l)-z_{t-1}(i_{01}-1,i_{12}-1,f_{02},l)\Big)I(t\geq t_0)$$

$$= \beta\sum_{l=1}^{L_{02}}\lambda_t(f_{02},l)p_{02l}(x_{t-1}(i_{01},i_{12}-1,f_{02},l))\Big(\varDelta_{02}v_{t-1}(i_{01},i_{12}-1)-\varDelta_{02}v_{t-1}(i_{01},i_{12})$$

$$+\varDelta_{02}v_{t-1}(i_{01}-1,i_{12})-\varDelta_{02}v_{t-1}(i_{01}-1,i_{12}-1)\Big)I(t\geq t_0)$$

$$\geq \beta\sum_{l=1}^{L_{02}}\lambda_t(f_{02},l)p_{02l}(x_{t-1}(i_{01}-1,i_{12},f_{02},l))(\varDelta_{12}v_{t-1}(i_{01}-1,i_{12})-\varDelta_{12}v_{t-1}(i_{01},i_{12}))$$

$$I(t\geq t_0)$$

$$\geq -\sum_{l=1}^{L_{02}}\lambda_t(f_{02},l)I(t\geq t_0)G_1.$$

Applying above inequalities as to G_2, G_3 and G_4, we have

$$\varDelta_{12}v_t(i_{01},i_{12})-\varDelta_{12}v_t(i_{01}-1,i_{12})$$

$$\geq (1-\sum_{l=1}^{L_{01}}\lambda_t(f_{01},l)I(t\geq t_0)-\sum_{l=1}^{L_{12}}\lambda_t(f_{12},l)-\sum_{l=1}^{L_{02}}\lambda_t(f_{02},l)I(t\geq t_0))G_1$$

$$\geq \lambda_t(0)G_1\geq 0.$$

Thus, the proof of (b) for t completes. Now, let us prove the proof of (b) for t. For $i_{01}\geq 2$ and $i_{12}\geq 1$ we have from (5.2)

$$\begin{aligned}
&\varDelta_{02}v_t(i_{01},i_{12})-\varDelta_{02}v_t(i_{01}-1,i_{12})\\
&= v_t(i_{01},i_{12})-v_t(i_{01}-1,i_{12}-1)+v_t(i_{01}-2,i_{12}-1)-v_t(i_{01}-1,i_{12})\\
&= \left.\beta(\varDelta_{02}v_{t-1}(i_{01},i_{12})-\varDelta_{02}v_{t-1}(i_{01}-1,i_{12}))\right\} Q_1\\
&\left.\begin{aligned}&+\sum_{l=1}^{L_{01}}\lambda_t(f_{01},l)\Big(K_{01l}(z_{t-1}(i_{01},i_{12},f_{01},l))-K_{01l}(z_{t-1}(i_{01}-1,i_{12}-1,f_{01},l))\\
&+K_{01l}(z_{t-1}(i_{01}-2,i_{12}-1,f_{01},l))I(i_{01}\geq 3)-K_{01l}(z_{t-1}(i_{01}-1,i_{12},f_{01},l))\Big)I(t\geq t_0)\end{aligned}\right\} Q_2\\
&\left.\begin{aligned}&+\sum_{l=1}^{L_{12}}\lambda_t(f_{12},l)\Big(K_{12l}(z_{t-1}(i_{01},i_{12},f_{12},l))-K_{12l}(z_{t-1}(i_{01}-1,i_{12}-1,f_{12},l))I(i_{12}\geq 2)\\
&+K_{12l}(z_{t-1}(i_{01}-2,i_{12}-1,f_{12},l))I(i_{12}\geq 2)-K_{12l}(z_{t-1}(i_{01}-1,i_{12},f_{12},l))\Big)\end{aligned}\right\} Q_3\\
&\left.\begin{aligned}&+\sum_{l=1}^{L_{02}}\lambda_t(f_{02},l)\Big(K_{02l}(z_{t-1}(i_{01},i_{12},f_{02},l))-K_{02l}(z_{t-1}(i_{01}-1,i_{12}-1,f_{02},l))I(i_{12}\geq 2)\\
&+K_{02l}(z_{t-1}(i_{01}-2,i_{12}-1,f_{02},l))I(i_{01}\geq 3,i_{12}\geq 2)\\
&-K_{02l}(z_{t-1}(i_{01}-1,i_{12},f_{02},l))\Big)I(t\geq t_0)\end{aligned}\right\} Q_4
\end{aligned}$$

For Q_2, if $i_{01}=2$, since $-K(\nu)\leq 0$ from Lemma 3.1(b); and if $i_{01}\geq 3$, since $z_{t-1}(i_{01}-1,i_{12},f_{01},l)\leq z_{t-1}(i_{01}-2,i_{12},f_{01},l)$, we have from Lemma 3.1(a)

$$K_{01l}(z_{t-1}(i_{01}-2,i_{12}-1,f_{01},l)-K_{01l}(z_{t-1}(i_{01}-1,i_{12},f_{01},l))\leq 0.$$

Hence, for all $i_{01}\geq 2$ and $i_{12}\geq 1$, we have

$$Q_2\leq \sum_{l=1}^{L_{01}}\lambda_t(f_{01},l)(K_{01l}(z_{t-1}(i_{01},i_{12},f_{01},l))-K_{01l}(z_{t-1}(i_{01}-1,i_{12}-1,f_{01},l))I(t\geq t_0).$$

Furthermore, since $z_{t-1}(i_{01},i_{12},f_{01},l)\leq z_{t-1}(i_{01}-1,i_{12}-1,f_{01},l)$ for all $i_{01}\geq 2$ and $i_{12}\geq 1$, we have from Lemma 3.1(e)

$$Q_2\leq \sum_{l=1}^{L_{01}}\lambda_t(f_{01},l)(z_{t-1}(i_{01}-1,i_{12}-1,f_{01},l)-z_{t-1}(i_{01},i_{12},f_{01},l))I(t\geq t_0)$$

$$= \sum_{l=1}^{L_{01}}\lambda_t(f_{01},l)\beta(\varDelta_{01}v_{t-1}(i_{01}-1,i_{12}-1)-\varDelta_{01}v_{t-1}(i_{01},i_{12}))I(t\geq t_0)$$

$$= \sum_{l=1}^{L_{01}} \lambda_t(f_{01},l)\beta(\Delta_{02}v_{t-1}(i_{01}-1,i_{12}) - \Delta_{02}v_{t-1}(i_{01},i_{12}))I(t \geq t_0)$$
$$= -\sum_{l=1}^{L_{01}} \lambda_t(f_{01},l)I(t \geq t_0)Q_1.$$

For Q_3, there are two distinct cases that should be considered:

(1) if $i_{12} = 1$, since $z_{t-1}(i_{01}-1,i_{12},f_{01},l) \leq z_{t-1}(i_{01},i_{12}-1,f_{01},l)$ for all $i_{01} \geq 1$, from Lemma 3.1(a) we have $Q_3 \leq 0$.

(2) if $i_{12} = 1$, since

$$z_{t-1}(i_{01}-1,i_{12},f_{12},l) \leq \min\{z_{t-1}(i_{01},i_{12},f_{12},l), z_{t-1}(i_{01}-2,i_{12}-1,f_{12},l)\}$$

and

$$z_{t-1}(i_{01},i_{12},f_{12},l) \geq \max\{z_{t-1}(i_{01},i_{12},f_{12},l), z_{t-1}(i_{01}-2,i_{12}-1,f_{12},l)\},$$

from Lemma 3.1(h) we have

$$\begin{aligned}
Q_3 \leq& \sum_{l=1}^{L_{12}} \lambda_t(f_{12},l)p_{12l}(x_{t-1}(i_{01},i_{12},f_{12},l))\Big(z_{t-1}(i_{01}-1,i_{12}-1,f_{12},l) \\
&-z_{t-1}(i_{01},i_{12},f_{12},l) + z_{t-1}(i_{01}-1,i_{12},f_{12},l)) - z_{t-1}(i_{01}-2,i_{12}-1,f_{12},l)\Big) \\
=& \beta\sum_{l=1}^{L_{12}} \lambda_t(f_{12},l)p_{12l}(x_{t-1}(i_{01},i_{12},f_{12},l))\Big(\Delta_{12}v_{t-1}(i_{01}-1,i_{12}-1) - \Delta_{12}v_{t-1}(i_{01},i_{12}) \\
&+\Delta_{12}v_{t-1}(i_{01}-1,i_{12}) - \Delta_{12}v_{t-1}(i_{01}-2,i_{12}-1)) \\
=& \beta\sum_{l=1}^{L_{12}} \lambda_t(f_{12},l)p_{12l}(x_{t-1}(i_{01},i_{12},f_{12},l))\Big(\Delta_{02}v_{t-1}(i_{01}-1,i_{12}) - \Delta_{02}v_{t-1}(i_{01},i_{12}) \\
&+\Delta_{02}v_{t-1}(i_{01},i_{12}-1) - \Delta_{02}v_{t-1}(i_{01}-1,i_{12}-1)\Big) \\
\leq& \beta\sum_{l=1}^{L_{12}} \lambda_t(f_{12},l)p_{12l}(x_{t-1}(i_{01},i_{12},f_{12},l))(\Delta_{02}v_{t-1}(i_{01}-1,i_{12}) - \Delta_{02}v_{t-1}(i_{01},i_{12})) \\
\leq& -\sum_{l=1}^{L_{12}} \lambda_t(f_{12},l)Q_1
\end{aligned}$$

For Q_4, if $i_{01} = 2$, since $-K(\nu) \leq 0$ from Lemma 3.1(b) and if $i_{01} \geq 3$ and $i_{12} \geq 2$, since $z_{t-1}(i_{01}-1,i_{12}-1,f_{02},l) \leq z_{t-1}(i_{01}-2,i_{12}-1,f_{02},l)$, from Lemma 3.1(a) we have

$$-K_{02l}(z_{t-1}(i_{01}-1,i_{12}-1,f_{02},l)) + K_{02l}(z_{t-1}(i_{01}-2,i_{12}-1,f_{02},l)) \leq 0.$$

Hence, for all $i_{01} \geq 2$ and $i_{12} \geq 1$ we have

$$Q_4 \leq \sum_{l=1}^{L_{02}} \lambda_t(f_{02},l)(K_{02l}(z_{t-1}(i_{01},i_{12},f_{02},l)) - K_{02l}(z_{t-1}(i_{01}-1,i_{12},f_{02},l)))I(t \geq t_0).$$

Furthermore, since $z_{t-1}(i_{01},i_{12},f_{02},l) \leq z_{t-1}(i_{01}-1,i_{12},f_{02},l)$ for all $i_{01} \geq 2$ and $i_{12} \geq 1$, from Lemma 3.1(e) we have

$$\begin{aligned}
Q_4 \leq& \sum_{l=1}^{L_{02}} \lambda_t(f_{02},l)(z_{t-1}(i_{01}-1,i_{12},f_{02},l) - z_{t-1}(i_{01},i_{12},f_{02},l))I(t \geq t_0) \\
=& \sum_{l=1}^{L_{02}} \lambda_t(f_{02},l)\beta(\Delta_{02}v_{t-1}(i_{01}-1,i_{12}) - \Delta_{02}v_{t-1}(i_{01},i_{12}))I(t \geq t_0) \\
=& -\sum_{l=1}^{L_{02}} \lambda_t(f_{02},l)I(t \geq t_0)Q_1.
\end{aligned}$$

Applying the above inequalities as to Q_2, Q_3 and Q_4, we have

$$\Delta_{02}v_t(i_{01},i_{12}) - \Delta_{02}v_t(i_{01}-1,i_{12})$$
$$\leq (1 - \sum_{l=1}^{L_{01}} \lambda_t(f_{01},l) I(t \geq t_0) - \sum_{l=1}^{L_{12}} \lambda_t(f_{02},l) - \sum_{l=1}^{L_{02}} \lambda_t(f_{02},l) I(t \geq t_0)) Q_1 \leq 0.$$

Accordingly, the proof of the monotoncity in i_{01} completes. The proof of monotoncity in i_{12} is similare to the above. Hence, Lemma 5.1 has been proven. ■

Proof of Lemma 5.2

(a) For $i_{01} \geq 1$ and $i_{12} \geq 1$ from (5.2) we have

$$\Delta_{02}v_t(i_{01},i_{12}) = \Delta_{01}v_t(i_{01},i_{12}) + \Delta_{12}v_t(i_{01}-1,i_{12}).$$

Then, since that the left hand side of the above equation, $\Delta_{02}v_t(i_{01},i_{12})$ is nonincreasing in i_{01} from Lemma 5.1(a) and that the right hand side of the above equation, $\Delta_{12}v_t(i_{01}-1,i_{12})$ is nondecreasing in i_{01} from Lemma 5.1(b). Hence, it must follow that $\Delta_{01}v_t(i_{01},i_{12})$ is nonincreasing is i_{01}.

(b) Same as the proof of (a). ■

Proof of Lemma 5.3

For any $t \geq 1$, $i_{01} \geq 1$ and $i_{12} \geq 1$ we have from (5.2)

$$\begin{aligned}
&\Delta_{02}v_t(i_{01},i_{12}) - \Delta_{02}v_{t-1}(i_{01},i_{12}) \\
&= I(t \geq t_0) \sum_{l=1}^{L_{01}} \lambda_t(f_{01},l)(K_{01l}(z_{t-1}(i_{01},i_{12},f_{01},l)) - K_{01l}(z_{t-1}(i_{01}-1,i_{12}-1,f_{01},l)) I(i_{01} \geq 2)) \\
&\quad + \sum_{l=1}^{L_{12}} \lambda_t(f_{12},l)(K_{12l}(z_{t-1}(i_{01},i_{12},f_{12},l)) - K_{12l}(z_{t-1}(i_{01}-1,i_{12}-1,f_{12},l)) I(i_{12} \geq 2)) \\
&\quad + I(t \geq t_0) \sum_{l=1}^{L_{02}} \lambda_t(f_{02},l)(K_{02l}(z_{t-1}(i_{01},i_{12},f_{02},l)) \\
&\qquad - K_{02l}(z_{t-1}(i_{01}-1,i_{12}-1,f_{02},l)) I(i_{01} \geq 2) I(i_{12} \geq 2)).
\end{aligned}$$

Here, since $z_{t-1}(i_{01},i_{12},f_{jk},l) \leq z_{t-1}(i_{01}-1,i_{12}-1,f_{jk},l)$, $0 \leq j < k \leq 1$, then from lemma 5.1 (d), Lemma 5.1(e), Lemma 5.1(a) and Lemma 3.1(a), the result follows. ■

Proof of Lemma 6.1

(a) From (6.9) we have

$$\Delta v_0(2) - \Delta v_0(1) = -\lambda_0(f_{N-1,N},l) K_{jNl}(c_{N-1,Nl}) \leq 0.$$

For $i \geq 3$, we have $\Delta v_0(i) - \Delta v_0(i-1) = 0$. Hence, the assertion holds true for $t = 0$. Assume it holds true for $t-1$. Since $\Delta v_{t-1}(i)$, hence $z_{t-1}(i,f_{jN},l)$ is nonincreasing in i, we have from (6.8) and Lemma 3.1(b)

$$\begin{aligned}
\Delta v_t(2) - \Delta v_t(1) &= \beta(\Delta v_{t-1}(2) - \Delta v_{t-1}(1)) \\
&\quad + \sum_{j=h(t)}^{N-1} \sum_{l=1}^{L_{jN}} \lambda_t(f_{jN},l)(K_{jNl}(z_{t-1}(2,f_{jN},l)) - 2K_{jNl}(z_{t-1}(1,f_{jN},l))) \\
&\leq \beta(\Delta v_{t-1}(2) - \Delta v_{t-1}(1)) \\
&\quad + \sum_{j=h(t)}^{N-1} \sum_{l=1}^{L_{jN}} \lambda_t(f_{jN},l)(K_{jNl}(z_{t-1}(2,f_{jN},l)) - K_{jNl}(z_{t-1}(1,f_{jN},l))).
\end{aligned}$$

Here, since $z_{t-1}(2, f_{jN}, l) \leq z_{t-1}(1, f_{jN}, l)$, from Lemma 3.1(f) we have

$$\begin{aligned}
\Delta v_t(2) - \Delta v_t(1) &\leq \beta(\Delta v_{t-1}(2) - \Delta v_{t-1}(1)) + \sum_{j=h(t)}^{N-1} \sum_{l=1}^{L_{jN}} \lambda_t(f_{jN}, l)(z_{t-1}(1, f_{jN}, l) - z_{t-1}(2, f_{jN}, l)) \\
&= \beta(\Delta v_{t-1}(2) - \Delta v_{t-1}(1)) + \beta \sum_{j=h(t)}^{N-1} \sum_{l=1}^{L_{jN}} \lambda_t(f_{jN}, l)(\Delta v_{t-1}(1) - \Delta v_{t-1}(2)) \\
&= \beta\lambda_t(0)(\Delta v_{t-1}(2) - \Delta v_{t-1}(1)) \leq 0.
\end{aligned}$$

For $i \geq 3$, from (4.12) we have

$$\begin{aligned}
\Delta v_t(i) - \Delta v_t(i-1) &= \beta(\Delta v_{t-1}(i) - \Delta v_{t-1}(i-1)) \\
&+ \sum_{j=h(t)}^{N-1} \sum_{l=1}^{L_{jN}} \lambda_t(f_{jN}, l)(K_{jNl}(z_{t-1}(i, f_{jN}, l)) - K_{jNl}(z_{t-1}(i-1, f_{jN}, l))) \\
&+ \sum_{j=h(t)}^{N-1} \sum_{l=1}^{L_{jN}} \lambda_t(f_{jN}, l)(K_{jNl}(z_{t-1}(i-2, f_{jN}, l)) - K_{jNl}(z_{t-1}(i-1, f_{jN}, l))).
\end{aligned}$$

Here, since $z_{t-1}(i, f_{jN}, l)$ is nonincreasing in i from induction hypothesis, we have from Lemma 3.1(a) and Lemma 3.1(f)

$$\begin{aligned}
\Delta v_t(i) - \Delta v_t(i-1) &\leq \beta(\Delta v_{t-1}(i) - \Delta v_{t-1}(i-1)) \\
&+ \sum_{j=h(t)}^{N-1} \sum_{l=1}^{L_{jN}} \lambda_t(f_{jN}, l)(K_{jNl}(z_{t-1}(i, f_{jN}, l)) - K_{jNl}(z_{t-1}(i-1, f_{jN}, l))) \\
&\leq \beta(\Delta v_{t-1}(i) - \Delta v_{t-1}(i-1)) + \sum_{j=h(t)}^{N-1} \sum_{l=1}^{L_{jN}} \lambda_t(f_{jN}, l)(z_{t-1}(i-1, f_{jN}, l) - z_{t-1}(i, f_{jN}, l)) \\
&= \beta(\Delta v_{t-1}(i) - \Delta v_{t-1}(i-1)) + \beta \sum_{j=h(t)}^{N-1} \sum_{l=1}^{L_{jN}} \lambda_t(f_{jN}, l)(\Delta v_{t-1}(i-1) - \Delta v_{t-1}(i)) \\
&= \beta\lambda_t(0)(\Delta v_{t-1}(i) - \Delta v_{t-1}(i-1) \leq 0.
\end{aligned}$$

(b) For $\beta = 1$, from (6.8) and $K(\nu) \geq 0$ Lemma 3.1(b) we have

$$v_t(i) = v_{t-1}(i) + \sum_{j=h(t)}^{N-1} \sum_{l=1}^{L_{jN}} \lambda_t(f_{jN}, l) K_{jNl}(z_{t-1}(i, f_{jN}, l)) \geq v_{t-1}(i).$$

Hence, we have $v_t(i)$ is nondecreasing in t. Accordingly,

$$\Delta v_t(1) - \Delta v_{t-1}(1) = v_t(1) - v_{t-1}(1) \geq 0.$$

For $i \geq 2$ from (6.8) we have

$$\Delta v_t(i) - \Delta v_{t-1}(i) = \sum_{j=h(t)}^{N-1} \sum_{l=1}^{L_{jN}} \lambda_t(f_{jN}, l)(K_{jNl}(z_{t-1}(i, f_{jN}, l)) - K_{jNl}(z_{t-1}(i-1, f_{jN}, l))).$$

Here, since $z_{t-1}(i, f_{jN}, l) \leq z_{t-1}(i-1, f_{jN}, l)$ from (a), hence, from Lemma 3.1(a) the result follows. ■

Appendix B

Table 1

arriving probabilities of $\lambda_t(f_{01}, l)$

Class	Decision Period			
	t_0 ~25	26 ~ 50	51 ~ 75	76 ~ 100
$l = 1$	0.054	0.034	0.053	0.042
$l = 2$	0.038	0.045	0.047	0.052
$l = 3$	0.052	0.062	0.042	0.041

Table 2

arriving probabilities of $\lambda_t(f_{02}, l)$

Class	Decision Period			
	t_0 ~25	26 ~ 50	51 ~ 75	76 ~ 100
$l = 1$	0.024	0.013	0.028	0.030
$l = 2$	0.040	0.026	0.022	0.027
$l = 3$	0.033	0.030	0.029	0.033

Table 3

arriving probabilities of $\lambda_t(f_{03}, l)$

Class	Decision Period			
	t_0 ~25	26 ~ 50	51 ~ 75	76 ~ 100
$l = 1$	0.009	0.013	0.010	0.020
$l = 2$	0.010	0.002	0.012	0.002
$l = 3$	0.014	0.004	0.011	0.007

Table 4

arriving probabilities of $\lambda_t(f_{12}, l)$

Class	Decision Period			
	t_1 ~25	26 ~ 50	51 ~ 75	76 ~ 100
$l = 1$	0.023	0.040	0.025	0.031
$l = 2$	0.046	0.033	0.035	0.024
$l = 3$	0.022	0.027	0.030	0.029

Table 5

arriving probabilities of $\lambda_t(f_{13}, l)$

Class	Decision Period			
	t_1 ~25	26 ~ 50	51 ~ 75	76 ~ 100
$l = 1$	0.015	0.006	0.010	0.006
$l = 2$	0.010	0.004	0.010	0.007
$l = 3$	0.006	0.003	0.010	0.008

Table 6

arriving probabilities of $\lambda_t(f_{23}, l)$

Class	Decision Period			
	0 ~25	26 ~ 50	51 ~ 75	76 ~ 100
$l = 1$	0.020	0.015	0.010	0.017
$l = 2$	0.011	0.010	0.008	0.005
$l = 3$	0.010	0.009	0.005	0.008

Table 7

the value of a_{jkl}

class	a_{01l}	a_{02l}	a_{03l}	a_{12l}	a_{13l}	a_{23l}
$l = 1$	1100	1400	2200	700	1700	1200
$l = 2$	800	1300	1800	600	1500	1100
$l = 3$	500	900	1350	500	1100	800

Table 8

the value of b_{jkl}

class	b_{01l}	b_{02l}	b_{03l}	b_{12l}	b_{13l}	b_{23l}
$l = 1$	1500	2500	4000	1100	2800	2300
$l = 2$	950	1700	2900	800	2100	1500
$l = 3$	650	1100	2050	570	1400	1000

Table 9

the value of c_{jkl}

(j,k)	$l = 1$	$l = 2$	$l = 3$
(0,1)	300	180	80
(0,2)	500	300	130
(0,3)	900	550	200
(1,2)	250	140	50
(1,3)	700	400	130
(2,3)	500	200	100

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