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Abstract
The greater part of demand data to be used in locational analysis lies on the surface of the earth. However, most are not spherical data but instead are planar data obtained from a map produced by some types of map projection. Errors occur as a result of using these planar location data. The objective of this paper is to analyze how map projection affects location error defined as the difference between the optimal location using spherical data and the location using planar data in the Weber model. Specifically, we examine the location error on the maps projected by equirectangular projections when all the demand points lie on Hemispheres.

Key words
map projection; Weber model; location error

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1. Introduction

In his widely known paper, Weber(1909) examined the model of finding a point on a plane where the sum of the weighted distances to given demand points on the plane is minimized. Since Weber's work, a considerable number of operations researchers, economists, geographers and mathematicians have explored this model of planar location: for a review see Hansen, Labbe, Peeters and Thisse(1987). As economical, industrial and transportation systems, as well as environmental destruction become progressively more global, locational analysis on a sphere becomes more important. In answer to this issue, several researchers have formulated the Weber model on a sphere; Drezner and Wesolowsky (1978), Litwhiler and Aly(1979,1980), Katz and Cooper(1980), Drezner (1985,1989). In their formulations, demand points and the facility to be located are assumed to lie on a sphere instead of lying on a plane. Transportation costs are assumed to be proportional to the great circle distances instead of Euclidean distances. Because no map can correctly represent the distances and the directions between all the points on a sphere, the spherical Weber model is essentially different from the planar one; for a review see Wesolowsky(1982).

Although most demand data used in locational analysis lies on the surface of the earth, most of them are not spherical data but planar data obtained from a map produced by some type of map projection. In the extreme case, such planar data are the only data available to analyze location models. As the earth is round and maps are flat, it is impossible to flatten the surface of the earth without stretching or tearing. In other words, the geometrical relationships on the surface of the earth cannot be perfectly duplicated, regardless of map projection. As a result, error is incurred if such planar location data are used. Accordingly, the conclusions drawn using planar location data are of questionable validity. In this paper, such an error is termed a *projection error*. Of course, the projection error may not be great as long as the area to be mapped is very small, irrespective of map projection. For example, if a map shows only a very small portion of the district such as one block, the projection error will be negliable for pratical purposes. But as the area to be mapped increases, so does the projection error and its' seriousness. Hence, the map projection may cause serious error in locational analysis. Although examining projection error is of importance to decision-makers, little work has been done to theoretically clarify the characteristics of projection error.

In the last decade, there has been progress in investigating the distance measurement error in geographical data from the facility location model perspective. For examples, Hillsman and Rhoda (1978) examined the magunitudes of the location and cost errors caused by data aggregation in some facility location models; see also Goodchild (1979), Casillas (1987), Current and Schilling(1987). Ohsawa, Koshizuka and Kurita (1991) analyzed the characteristics of the location and cost errors due to rounding in minisum and minimax models in one-dimensional continuous space.

The aim of this paper is to analyze how map projection affect the *location error*, defined as the difference between the optimal location using spherical data and that using planar data, as in the Weber model. Specifically, we examine the projection error of the maps projected by equirectangular projection when all the demand points lie on the Northern Hemisphere. This projection, which is classified as cylindrical projections, is very simple to construct and to interpret. In addition, this projection, sometimes called the cylindrical equidistant projection with two standard parallels, is composed of an evenly spaced network of horizontal parallels and vertical meridians, and has the added benefit that the scale is true on all meridians and on two parallels. This projection is good for city maps or estate maps; see Raisz(1962). The simplicity of construction and some elements of scale preservation gave this type of projection a role as an outline map, appearing on some computer screens of the 1980s as a quickly drawn base for insertion of other data: see Synder(1993). The U.S. Geological Survey has used it for index maps to show the status of mapping of topographic quadrangles and the like.

This paper is organized as follows: Section 2 describes the formulations and the solution methods of the planar and spherical Weber models. The transformation of equirectangular projection from a sphere to a plane is also given. In Section 3, the location errors obtained by three computational examples are reported and interpreted. Section 4 provides the theoretical results to explain the tendencies of location error. Finally, Section 5 contains our conclusions.

In this paper we will use the traditional latitude-longitude system with north and east being the positive direction in order to specify the locations of points on a sphere, and the Cartesian coordinate to express the ones in a plane. For clarity, throughout the paper, $(\cdot,\cdot)$ should be understood to mean the Cartesian coordinate. $\{\cdot,\cdot\}$ should be understood to specify the traditional latitude-longitude coordinate.

2. WEBER MODELS AND MAP PROJECTIONS

Planar Weber model

The planer Weber model iş formulated as follows:

$$\min_{x,y} f(x,y) = \sum_{i\in I} \omega_i D_i(x,y),$$

where, I is the set of the indices such that each demand point is indexed by i, $i \in I = \{1,\ldots,n\}$, (x,y) is the Cartesian coordinate of the facility to be located, (x_i, y_i) is the Cartesian coordinate of the i-th demand point, $\omega_i(> 0)$ is the weight of the i-th demand, $D_i(x,y)$ is the Euclidean distance between (x,y) and (x_i,y_i), that is, $D_i(x,y) = \sqrt{(x-x_i)^2 + (y-y_i)^2}$.

It is well known that the optimal point, referred as *planar Weber point*, cannot be expressed analytically. Therefore, a numerical calculation is necessary to find the planar Weber point. The most common method to solve this model is carried out in two steps.

(p1) Check whether the minimum is reached at the demand points. To determine this, the following conditions are used. The function $f(x,y)$ is minimized at the k-th demand point if and only if

$$\left(\sum_{i\in I-\{k\}} \omega_i \frac{x_k - x_i}{D_i(x_k,y_k)}\right)^2 + \left(\sum_{i\in I-\{k\}} \omega_i \frac{y_k - y_i}{D_i(x_k,y_k)}\right)^2 \leq {\omega_k}^2. \qquad (1)$$

(p2) The partial derivatives of $f(x,y)$ with respect to x and y are:

$$\frac{\partial f(x,y)}{\partial x} = \sum_{i\in I} \omega_i \frac{x - x_i}{D_i(x,y)}, \qquad (2)$$

$$\frac{\partial f(x,y)}{\partial y} = \sum_{i\in I} \omega_i \frac{y - y_i}{D_i(x,y)}. \qquad (3)$$

Solve the following simultaneous equations for x and y that are obtained by setting the derivatives (2) and (3) equal to zero:

$$\sum_{i\in I} \omega_i \frac{x - x_i}{D_i(x,y)} = 0, \qquad (4)$$

$$\sum_{i\in I} \omega_i \frac{y - y_i}{D_i(x,y)} = 0. \qquad (5)$$

The condition (1) was given in Francis and White(1974). The iterative procedure to solve the simultaneous equations (4) and (5) was proposed by Weiszfeld(1937). It should be noted that the condition (4) means graphically that the sum of the cosines of the angles between the horizontal line and the set of lines connecting the demand points with the planar Weber point must equal zero. Similarly, the condition (5) means the sum of the sines of the above-mentioned angles be zero. These interpretations will be of use in our further work.

Spherical Weber model

For convenience, we may assume that the radius of the sphere is equal to one. This model is formulated as follows:

$$\min_{\phi,\lambda} g(\phi,\lambda) = \sum_{i\in I} \omega_i A_i(\phi,\lambda), \tag{6}$$

where, $\{\phi,\lambda\}$ is the latitude and longitude coordinates of the facility to be located, $\{\phi_i,\lambda_i\}$ is the latitude and longitude coordinates of the i-th demand point, $A_i(\phi,\lambda)$ is the great circle distance between $\{\phi,\lambda\}$ and $\{\phi_i,\lambda_i\}$, that is,

$$A_i(\phi,\lambda) = \arccos\left(\cos\phi\cos\phi_i\cos(\lambda-\lambda_i)+\sin\phi\sin\phi_i\right).$$

Let it now be assumed that $-\frac{\pi}{2} \le \phi_i \le \frac{\pi}{2}$ and $-\pi < \lambda_i \le \pi$. Accordingly, every point on the sphere, except the North and South poles, has a unique pair of coordinates.

As is the case of the planar Weber point, the local optimal points, referred to as *spherical Weber points*, cannot be expressed analytically. Generally, the procedure to solve this model is carried out in two steps as follows:

(s1) Check for local minima at demand points using the following the conditions. The function $g(\phi,\lambda)$ has a local minimum at the k-th demand point if and only if

$$\left(\sum_{i\in I-\{k\}} \omega_i \frac{\sin\phi_k\cos\phi_i[1-\cos(\lambda_i-\lambda_k)]+\sin(\phi_i-\phi_k)}{\sin A_i(\phi_k,\lambda_k)}\right)^2 + \left(\sum_{i\in I-\{k\}} \omega_i \frac{\cos\phi_i\sin(\lambda_i-\lambda_k)}{\sin A_i(\phi_k,\lambda_k)}\right)^2 \le {\omega_k}^2. \tag{7}$$

(s2) The partial derivatives of $g(\phi,\lambda)$ with respect to ϕ and λ are:

$$\frac{\partial g(\phi,\lambda)}{\partial\phi} = -\sum_{i\in I}\omega_i \frac{\sin\phi\cos\phi_i[1-\cos(\lambda_i-\lambda)]+\sin(\phi_i-\phi)}{\sin A_i(\phi,\lambda)}, \tag{8}$$

$$\frac{\partial g(\phi,\lambda)}{\partial\lambda} = -\cos\phi\sum_{i\in I}\omega_i \frac{\cos\phi_i\sin(\lambda_i-\lambda)}{\sin A_i(\phi,\lambda)}. \tag{9}$$

Solve the following simultaneous equations for ϕ and λ that are obtained by setting the derivatives (8) and (9) equal to zero:

$$\sum_{i\in I}\omega_i \frac{\sin\phi\cos\phi_i[1-\cos(\lambda_i-\lambda)]+\sin(\phi_i-\phi)}{\sin A_i(\phi,\lambda)} = 0, \tag{10}$$

$$\sum_{i\in I}\omega_i \frac{\cos\phi_i\sin(\lambda_i-\lambda)}{\sin A_i(\phi,\lambda)} = 0. \tag{11}$$

Choose the local minimizer from the solution of these equations and its antipode.

The condition (7) was given by Katz and Cooper(1980). To solve the simultaneous equations (10) and (11), the iterative scheme, which is analogous to that of Weiszfeld, was proposed by Drezner and Wesolowsky (1978). The spherical Weber model differs significantly from the planer Weber model in several respects with regard to the objective function. Whereas the condition (1) can characterize a global minimum at a demand point, the condition (7) characterizes only a local minimum due to the nature of the non-convexity of (6). However, it was proven by Drezner and Wesolowsky (1978) that if all the demand points are covered by a spherical disk with radius $\frac{\pi}{4}$, then a local minimizer is also a global minimizer. In this paper, we shall refer the local (global) minimizers at demand points as *local (global) E-minimizers*, and the other local (global) minimizers as *local (global) I-minimizers.*

Equirectangular Projection and Plane Coordinate

Standard lines are defined as lines which do not change length when projected from a generating sphere to a map. When the θ-th parallels ($0 \leq \theta < \frac{\pi}{2}$) are standard and the Greenwich meridian is centered, the mapping from $\{\phi, \lambda\}$ on a sphere into (x, y) on a plane can be represented mathematically as follows; see McDonnell(1979):

$$\begin{aligned} x &= \lambda \cos \theta, \\ y &= \phi. \end{aligned}$$

Tokyo is located at $\{35°42'N, 139°46'E\}$. This projection with standard parallels at $35°42'$ and with a central line at $139°42'$ of east longitude is illustrated in Figure 1. In the Northern Hemisphere, the northern parallels are too long and the southern parallels are too short. It is evident that if the standard parallel is the equator, we have a grid made up of even squares. The map is not conformal or an equal area.

3. COMPUTATIONAL RESULTS

Numerical Examples

We take up the following three data sets to investigate numerically the tendency of projection errors:

(d1) national data in the European Communities;

(d2) state data in the United States;

(d3) prefectural data in Japan.

In the first data set, we use the locations of the capitals of 12 nations as the demand points, and their national populations as their weights. Similarly, in the second (third) data, we use the locations of the capitals of 51 states (47 prefectures) and the population of their states (prefectures). It should be noted that in each data set all local minimizers become global minimizers. This is because all the demand points on each data set are covered by a spherical disk with radius $\frac{\pi}{4}$ multiplied by the radius of the earth. For each data set, first, we have derived the spherical Weber points. The the spherical Weber point for the first data set is located at {49°20′N,3°48′E}. The one for the second and third data sets are located at {39°16′N,86°04′W} and {35°28′N,137°17′E}, respectively. Second, we have calculated planar Weber points on the maps expressed by the equirectangular projection with standard parallels ranging from 0° to 85° at 5° intervals. Third, we have calculated the great circle distances between the spherical Weber point and the planar Weber points. Computational results on the planar Weber points and the location errors are provided in Tables 1, 2 and 3. In each table, the entries in the first column are the latitudes of the standard parallels. Entries in the second and fourth columns are the locations in the latitude-longitude system of the planar Weber points obtained by the projection. The ones in the third and fifth columns are the location errors, that is, the great circle distances between the spherical Weber point and the planar Weber points. An asterisk(*) indicates that the planar Weber point is located on the south of the spherical Weber point.

Interpreting the Results of Errors

Based on the numerical experiences, the following four natures have been drawn:

(n1) In both projections, most planar Weber points are located on the south of the spherical Weber point;

(n2) In the United States, data set where the demand points are distributed most widely of the three data set, this tendency frequently holds;

(n3) In particular, if we select standard parallels near the center of the area to be considered, this tendency necessarily holds.

These natures will be graphically interpreted and explained theoretically in the following sections.

4. INTERPRETATIONS

Orthographic Projection

In this projection the surface of a sphere is projected with parallel rays on a perpendicular plane. The eye point of this projection is at infinity. This projection preserves distance and bearing from the polar point to all points on a sphere. When $\{\phi^*, \lambda^*\}$ is centered, $\{\phi_i, \lambda_i\}$ on a sphere is transformed into (x_i, y_i) on a plane as follows; see Robinson and Sale (1964):

$$x_i = \cos\phi_i \sin(\lambda_i - \lambda^*), \tag{12}$$

$$y_i = \sin\phi^* \cos\phi_i[1 - \cos(\lambda_i - \lambda^*)] + \sin(\phi_i - \phi^*). \tag{13}$$

Note that

$$x_i^2 + y_i^2 = \sin^2 A_i(\phi^*, \lambda^*). \tag{14}$$

As is evident from the definition, $\{\phi^*, \lambda^*\}$ is transformed into (0,0), i.e., the origin of the map. Figure 2 presents this projection as centered on Tokyo.

Property 1. *$\{\phi^*, \lambda^*\}$ is a local minimizer to the spherical Weber model if and only if (0,0) is the optimal point to the planar model obtained by the orthographic projection centered on $\{\phi^*, \lambda^*\}$.*

Litwhiler and Aly (1979) demonstrated the necessary part of this property. The proof of Property 1 is given in the Appendix. This Property will be of use in our further work. This is because the spherical Weber model can be interpreted in terms of the planar Weber model. Figure 3 expresses the locations of the twelve capitals in the European Community on the orthographic projection map centered on their spherical Weber point. This Property states that the minimizer to the planar Weber model on the map shown in Figure 3 becomes the center of the map.

We shall refer to $(x_i(\theta), y_i(\theta))$ and (x^*, y^*) as the Cartesian coordinate of $\{\phi_i, \lambda_i\}$ and $\{\phi^*, \lambda^*\}$ projected by the equirectangular projection with standard parallels at θ, respectively. Moreover, we shall refer to $(\tilde{x}_i, \tilde{y}_i)$ as the Cartesian coordinate of $\{\phi_i, \lambda_i\}$ projected by the orthographic projection centered on $\{\phi^*, \lambda^*\}$.

In the following subsections, we consider only the case where any spherical Weber point would never be the same as the location of the demand points. Therefore, the conditions (4) and (5) would give the necessary and sufficient conditions for the planar Weber point. Following Property 1, on the the orthographic projection map centered on $\{\phi^*, \lambda^*\}$, the sum of the sines of the angles between the horizontal line and the lines connecting $(\tilde{x}_i, \tilde{y}_i)$'s with the origin must equal zero. As a result, if this sum is greater than the sum of the sines of the angles between the horizontal line and the lines connecting $(x_i(\theta), y_i(\theta))$ with (x^*, y^*)

on the equirectangular projection map, the southern direction at (x^*, y^*) becomes a descent direction in terms of the planar Weber model. Accordingly, we may conclude that on the equirectangular projection map with standard paralleles at θ, the southern neighborhood of (x^*, y^*) contains points which are better than (x^*, y^*) in terms of the planar Weber model. That is to say, mathematically, there exists $\epsilon(>0)$ such that:

$$\sum_{i \in I} \omega_i \sqrt{(x^* - x_i)^2 + (y^* - \epsilon - y_i)^2} < \sum_{i \in I} \omega_i \sqrt{(x^* - x_i)^2 + (y^* - y_i)^2}.$$

Moreover, we limit the discussion to the case where all the demand points lie on the Northern Hemisphere and where on the equirectangular projection map the central meridian is fixed at the λ-th meridian.

Interpretation of the nature(n1)

We call the demand data set $\{\phi_i, \lambda_i\}, (i \in I)$ *symmetrical on* $\{\phi^*, \lambda^*\}$ if $|I|$ is even and for any $i \in \{1, \cdots, \frac{|I|}{2}\}$ $\omega_i = \omega_{\frac{|I|}{2}+i}$, and the shorter great circle arc connecting the i-th demand point and the $\frac{|I|}{2}+i$-th demand point also passes $\{\phi^*, \lambda^*\}$. As is evident from the definition, $\{\phi^*, \lambda^*\}$ becomes a local I-minimizer to the spherical Weber model.

Moreover, on the orthographic projection map, any great circle arc through the center appears as a straight line segment; see Snyder(1993). In contrast to this, on the equirectangular projection map, any great circle arc through the center on the Northern Hemisphere becomes a concave curve, irrespective of selecting standard parallels; see Figures 1. This is because the equirectangular projection is classified as cylindrical projections.

Property 2. *On the equirectangular projection map, the locus of the greatest circle arc through the center on the Northern Hemisphere appears as a concave curve, irrespective of selecting standard parallels.*

From Property 2, on the equirectangular projection map, the locus of greatest circle arc connecting the i-th and the $\frac{|I|}{2}+i$-th demand points, and passing through $\{\phi^*, \lambda^*\}$ become a concave curve. Because of this concavity, if the data set is symmetrical it follows that the sum of the sines on the orthographic projection map are greater than that of the equirectangular projection. Therefore, this together with Property 1 yields the following proposition:

Proposition 1. *For the data $\{\phi_i, \lambda_i\}, (i \in I)$ which are symmetrical on $\{\phi^*, \lambda^*\}$, then the southern direction at (x^*, y^*) becomes a decent direction in terms of the planar Weber model obtained by the equirectangular projection with parallels at θ, irrespective of $(0 <)\theta(< \frac{\pi}{2})$.*

Hence, we can explain theoretically the nature (n1) for the case of the equirectangular projection. In addition, directly from Proposition 1, we see that for four demand points with equal weights which are on the vertices of a quadrilateral, the southern direction at (x^*, y^*) bècomes a decent direction in terms of the planar Weber model obtained by the equirectangular projection with parallels at θ, irrespective of $(0 < \theta < \frac{\pi}{2})$.

Of course, the actual data are never truly symmetrical. But this assumption is reasonable when the area to be mapped is homogeneous. In addition, it is important to examine a case for demand distribution which we can imagine easily.

Interpretation of the nature(n2)

In Figure 1, as the point on the great circle arc is located further from Tokyo, the angle between the horizontal line and the line connecting the point with Tokyo decreases. Obviously, this result was shown theoretically by Property 2. In contrast, in Figure 2 the above-mentioned angle on the orthographic projection map is constant. Accordingly, it follows that as the demand points are located further to the center, keeping the same angles between the horizontal line and the lines connecting the demand points and the spherical Weber point on the orthographic projection map, the sum of the sines of these angles may be greater than that of the equirectangular projection, irrespective of selecting the standard parallels.

Interpretation of the nature(n3)

Figure 4 presents the locations of the twelve European capitals on the equirectangular projection map with standard parallels at the latitude of their spherical Weber points. Figure 5 shows the two types of locations for these capitals obtained by merging Figures 3 and 4, such that the direction and the projection of the spherical Weber point on Figure 3 coincide with those on Figure 4. In the latter Figure, the capitals on the orthographic projection, and the equirectangular projection are connected with the spherical Weber point by solid lines, and dotted lines, respectively. We see that in each city the solid lines always lie above the dotted lines. Accordingly, we would expect that if the standard parallels coincide with the latitude of the spherical Weber point, the sum of the sines on the orthographic projection map is greater than that of the equirectangular projection.

Property 3. *For the locus of the greatest circle arc connecting any two points on the Northern Hemisphere, its projection on the orthographic projection map lies above its projection on the equirectangular projection map with the standard paralleles at the latitude of one point, when the direction and the projection of the point on the former map coincide with the direction and the projection of the point on the latter map.*

The mathematical proof of this Property is given in the Appendix. Therefore, Property 3 together with Property 1 gives the following proposition.

Proposition 2. *In terms of the planar model obtained by the equirectangular projection with the standard paralleles at θ^*, then the southern direction at (x^*, y^*) becomes decent direction in terms of the planar Weber model obtained by the equirectangular projection with parallels at θ, irrespective of $(0 < \theta < \frac{\pi}{2})$.*

5. CONCLUSIONS

We have discussed how the equirectangular and equal-area projections affect location errors in the Weber model. Most models in location theory have assumed that the demand points lie on a plane. As the earth is not flat, it is important to examine the error due to the map projection. In brief, this paper allows us to make two conclusions about projection error. First, we can see that most planar Weber points are located on the south of the spherical Weber point. In particular, this tendency tends to hold as the demand point distribution is symmetrical (as the demand points are distributed widely), irrespective of selecting standard parallels. Secondly, if we select standard parallels near the center of the area to be considered, this tendency necessarily holds.

We have assumed in this paper that all of the demand points lie on the Northern Hemisphere. It is clear, however, from symmetry that the same results hold for the case where all demand points lie on the Southern Hemisphere. In addition, it is clear that Proposition 1 holds for any cylindrical projection such as Mercator projection, cylindrical equal-area projection, Miller's cylindrical projection, Gall's stereographic projection and so on.

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APPENDIX

Proof of Property 1

There are two cases, corresponding as to whether the spherical Weber point is or is not at a demand point: case 1:$\{\phi^*, \lambda^*\}$ is a local I-minimizer; case 2:$\{\phi^*, \lambda^*\}$ is a local E-minimizer.

For the first case, on referring to (8) and (9), we have

$$\begin{aligned} \frac{\partial g(\phi,\lambda)}{\partial \phi}\bigg|_{\phi=\phi^*,\lambda=\lambda^*} &= -\sum_{i\in I}\omega_i \frac{\sin\phi^*\cos\phi_i[1-\cos(\lambda_i-\lambda^*)]+\sin(\phi_i-\phi^*)}{\sin A_i(\phi^*,\lambda^*)}, \\ \frac{\partial g(\phi,\lambda)}{\partial \lambda}\bigg|_{\phi=\phi^*,\lambda=\lambda^*} &= -\cos\phi^*\sum_{i\in I}\omega_i \frac{\cos\phi_i\sin(\lambda_i-\lambda^*)}{\sin A_i(\phi^*,\lambda^*)}. \end{aligned}$$

On the other hand, using (2), (3), (12), (13) and (14), with $r(x,y)$ defined by the objective function of the planar Weber model obtained by the orthographic projection centered on $\{\phi^*, \lambda^*\}$, we can get

$$\begin{aligned} \frac{\partial r(x,y)}{\partial y}|_{x=0,y=0} &= -\sum_{i\in I}\omega_i \frac{\sin\phi^*\cos\phi_i[1-\cos(\lambda_i-\lambda^*)]+\sin(\phi_i-\phi^*)}{\sin A_i(\phi^*,\lambda^*)}, \\ \frac{\partial r(x,y)}{\partial x}|_{x=0,y=0} &= -\sum_{i\in I}\omega_i \frac{\cos\phi_i\sin(\lambda_i-\lambda^*)}{\sin A_i(\phi^*,\lambda^*)}. \end{aligned}$$

Accordingly, we have

$$\begin{aligned} \frac{\partial g(\phi,\lambda)}{\partial \phi}\bigg|_{\phi=\phi^*,\lambda=\lambda^*} &= \frac{\partial r(x,y)}{\partial y}\bigg|_{x=0,y=0}, \\ \frac{\partial g(\phi,\lambda)}{\partial \lambda}\bigg|_{\phi=\phi^*,\lambda=\lambda^*} &= \cos\phi^*\frac{\partial r(x,y)}{\partial x}\bigg|_{x=0,y=0}. \end{aligned}$$

Noting that Cartesian coordinate $(0,0)$ on this plane corresponds to the latitude-longitude coordinate $\{\phi^*, \lambda^*\}$ on the sphere, we see that $\{\phi^*, \lambda^*\}$ is a local I-minimizer if and only if $(0,0)$ is the optimal point to the planar model obtained by the orthographic projection centered on $\{\phi^*, \lambda^*\}$.

In the second case, $\{\phi_k, \lambda_k)$ is a local E-minimizer if and only if there exists k such that (7) holds. On referring to equations (1), (12), (13) and (14), (7) holds if and only if $(0,0)$

is optimal to the planar Weber model obtained by the orthographic projection centered on $\{\phi^*, \lambda^*\}$. □

Proof of Property 3

Consider the planar Weber model obtained by projecting the demand points onto the plane using the equirectangular projection with standard parallels at θ and with a central line at the Greenwich meridian.

For $0 < \phi^* < \frac{\pi}{2}$, define

$$T_E(\phi, \lambda) = \frac{\phi - \phi^*}{\lambda \cos \theta}, \tag{15}$$

$$T_O(\phi, \lambda) = \frac{\sin \phi^* \cos \phi [1 - \cos \lambda] + \sin(\phi - \phi^*)}{\sin \lambda \cos \phi}. \tag{16}$$

Geometrically, $T_E(\phi, \lambda)$ and $T_O(\phi, \lambda)$ represent the tangent of the angles between the horizontal line and the line connecting (x, y) with (x^*, y^*) on the equirectangular projection map, and the tangent of the angles between the horizontal line and the line connecting $(\tilde{x}, \tilde{y})$ with the origin on the orthographic projection map, respectively.

We will show that given $-\infty < \beta < \infty$, for $0 \leq \phi < \frac{\pi}{2}$, $0 < \lambda < \pi$ such that $T_O(\phi, \lambda) = \beta$,

$$\lim_{\lambda \to 0} T_E(\phi, \lambda) = \beta \frac{\cos \phi^*}{\cos \theta}. \tag{17}$$

As is evident from the definition of $T_E(\phi, \lambda)$, the equation (17) with Property 2 means that when $\theta = \phi^*$, for $\lambda > 0$ such that $T_O(\phi, \lambda) = \beta$,

$$T_E(\phi, \lambda) < \beta.$$

This proves Property 3.

The substitution of (16) into $T_O(\phi, \lambda) = \beta$ yields

$$\frac{\sin(\phi - \phi^*)}{\cos \phi} = \beta \sin \lambda - \sin \phi^* (1 - \cos \lambda)$$

$$\Leftrightarrow \quad \frac{\sin(\phi - \phi^*)}{\cos(\phi - \phi^*) \cos \phi^* - \sin(\phi - \phi^*) \sin \phi^*} = \beta \sin \lambda - \sin \phi^* (1 - \cos \lambda).$$

That is to say,

$$\tan(\phi - \phi^*) = \frac{\cos \phi^* [\beta \sin \lambda - \sin \phi^* (1 - \cos \lambda)]}{1 + \sin \phi^* [\beta \sin \lambda - \sin \phi^* (1 - \cos \lambda)]}.$$

This together with (15) means that for $\lambda > 0$ such that $T_O(\phi, \lambda) = \beta$,

$$T_E(\phi, \lambda) = \frac{1}{\lambda \cos \theta} \arctan\{\frac{\cos \phi^* [\beta \sin \lambda - \sin \phi^* (1 - \cos \lambda)]}{1 + \sin \phi^* [\beta \sin \lambda - \sin \phi^* (1 - \cos \lambda)]}\}.$$

Therefore, using the formula of l'Hospital, we have the equation (17).

Table 1. **Spherical Weber Points and Location Error Using European Data**

$*$ *indicates that the planar Weber point is located on the south of the spherical Weber point*

standard parallels	coordinate of planar Weber point	location error(km)
0°	$\{49°05'N, 3°46'E\}$*	27.7
5°	$\{49°05'N, 3°46'E\}$*	28.0
10°	$\{49°05'N, 3°45'E\}$*	27.9
15°	$\{49°05'N, 3°45'E\}$*	27.8
20°	$\{49°05'N, 3°45'E\}$*	27.8
25°	$\{49°05'N, 3°45'E\}$*	27.6
30°	$\{49°05'N, 3°44'E\}$*	27.6
35°	$\{49°05'N, 3°44'E\}$*	27.5
40°	$\{49°05'N, 3°43'E\}$*	27.4
45°	$\{49°06'N, 3°43'E\}$*	26.8
50°	$\{49°06'N, 3°42'E\}$*	26.6
55°	$\{49°07'N, 3°40'E\}$*	25.9
60°	$\{49°07'N, 3°39'E\}$*	26.0
65°	$\{49°08'N, 3°37'E\}$*	26.3
70°	$\{49°08'N, 3°34'E\}$*	27.8
75°	$\{49°07'N, 3°29'E\}$*	32.8
80°	$\{49°04'N, 3°16'E\}$*	48.1
85°	$\{48°53'N, 2°29'E\}$*	108.1

Table 2. **Spherical Weber Points and Location Error Using United State Data**

** indicates that the planar Weber point is located on the south of the spherical Weber point*

standard parallels	coordinate of planar Weber point	location error(km)
0°	{38°17′N, 86°08′W}*	109.5
5°	{38°17′N, 86°08′W}*	109.2
10°	{38°18′N, 86°08′W}*	108.4
15°	{38°19′N, 86°07′W}*	106.9
20°	{38°20′N, 86°07′W}*	105.0
25°	{38°21′N, 86°07′W}*	102.5
30°	{38°23′N, 86°07′W}*	99.2
35°	{38°25′N, 86°06′W}*	95.3
40°	{38°27′N, 86°05′W}*	90.7
45°	{38°30′N, 86°05′W}*	85.3
50°	{38°34′N, 86°04′W}*	79.0
55°	{38°38′N, 86°04′W}*	71.4
60°	{38°43′N, 86°05′W}*	62.2
65°	{38°49′N, 86°07′W}*	50.9
70°	{38°56′N, 86°11′W}*	38.3
75°	{39°05′N, 86°20′W}*	30.7
80°	{39°15′N, 86°40′W}*	52.6
85°	{39°22′N, 87°49′W}	151.0

Table 3. **Spherical Weber Points and Location Error Using Japanese Data**

$*$ *indicates that the planar Weber point is located on the south of the spherical Weber point*

standard parallels	coordinate of planar Weber point	location error(km)
0°	$\{35°29'N, 137°23'E\}$	9.6
5°	$\{35°29'N, 137°23'E\}$	8.8
10°	$\{35°29'N, 137°22'E\}$	9.1
15°	$\{35°29'N, 137°22'E\}$	8.8
20°	$\{35°29'N, 137°22'E\}$	7.6
25°	$\{35°28'N, 137°21'E\}*$	6.6
30°	$\{35°27'N, 137°20'E\}*$	5.8
35°	$\{35°27'N, 137°20'E\}*$	6.2
40°	$\{35°26'N, 137°19'E\}*$	6.2
45°	$\{35°25'N, 137°18'E\}*$	7.0
50°	$\{35°24'N, 137°18'E\}*$	8.5
55°	$\{35°23'N, 137°17'E\}*$	9.3
60°	$\{35°22'N, 137°18'E\}*$	11.0
65°	$\{35°22'N, 137°19'E\}*$	12.2
70°	$\{35°21'N, 137°21'E\}*$	14.3
75°	$\{35°22'N, 137°27'E\}*$	19.2
80°	$\{35°24'N, 137°39'E\}*$	34.8
80°	$\{35°30'N, 138°19'E\}*$	93.4

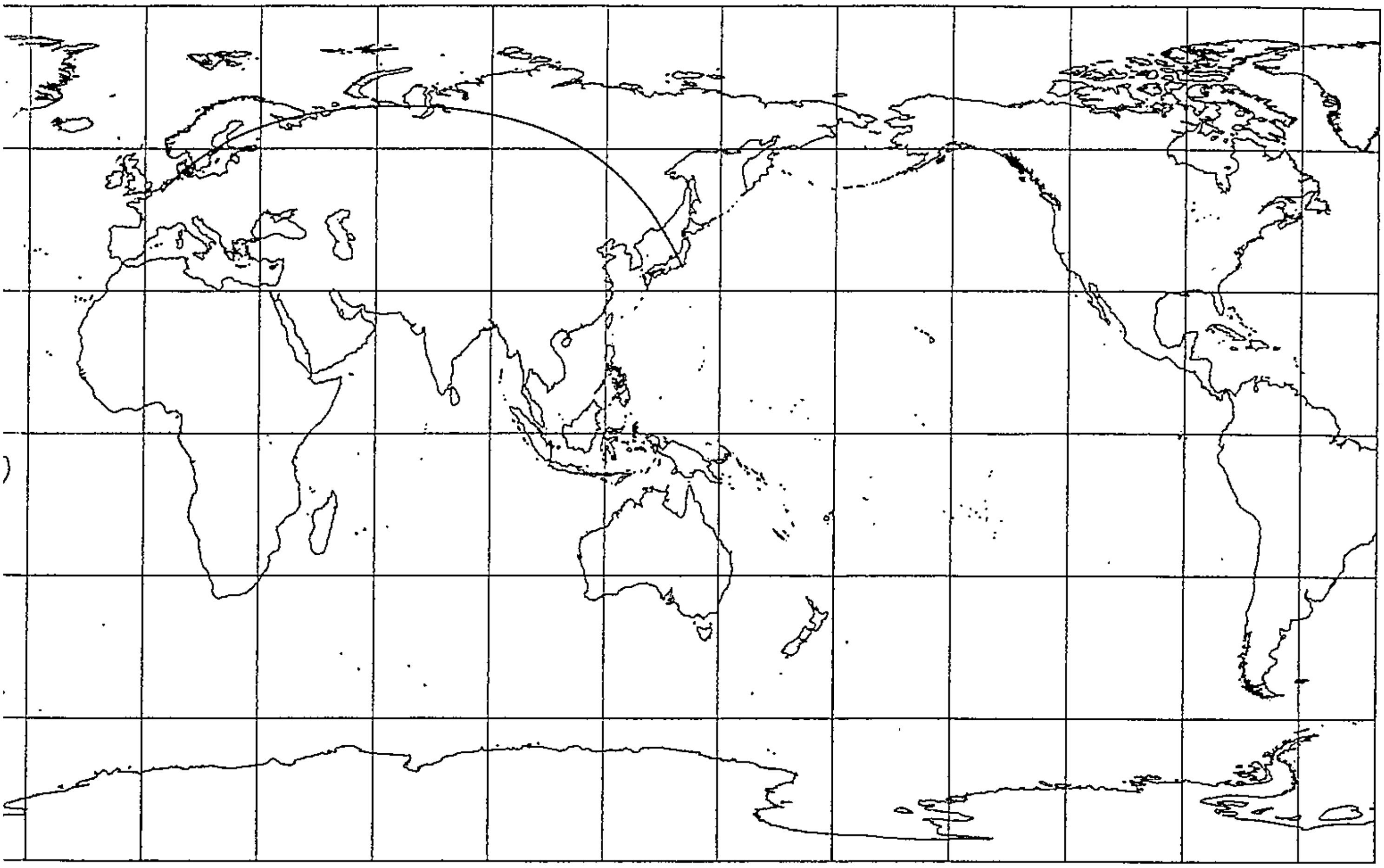

Figure 1. World Map on an Equirectangular Projection

The parallels at $35°42'$ are standard in the case shown. The great circle arc connecting Tokyo with Brussels is also shown.

Figure 2. Hemisphere Map on an Orthographic Projection

Tokyo is centered on. The great circle arc connecting Tokyo with Brussels is also shown.

Figure 3. European Map on an Orthographic Projection

The spherical Weber point is centered on. o represents the location of the spherical Weber point. • represents the location of each capital

Figure 4. European Map on an Equirectangular Projection

The spherical Weber point is centered on. o represents the location of the spherical Weber point. • represents the location of each capital

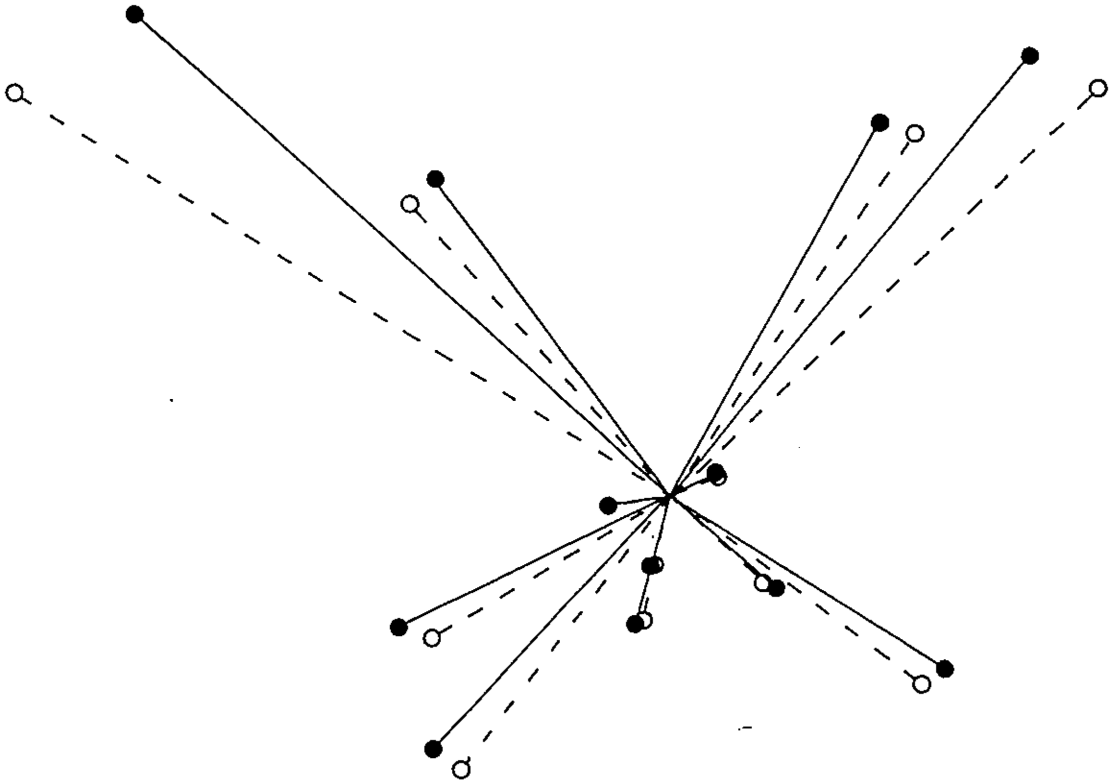

Figure 5. Two Types of the Locations of European Capitals

o represents the location of them on the orthographic projection map shown in Figure 3. • represents the location of them on the equirectangular projection map shown in Figure 4. Each location is connected with the projection of the spherical Weber point by a straight line.