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Optimal Stopping Problem with Pricing Policy

by

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Abstract The paper deals with a variation of conventional optimal stopping problem where a searcher (price maker) offers a price to price taker, who could be either a seller or a buyer of an item, depending on the situation. Most of the previous studies have dealt with the problem by having the searcher to try to sell individual items at the highest price, or purchase specific items at the lowest price available, with the offers varying continuously. In other words, in the traditional methodology, the so called searcher does not actually have a best price in mind. Yet, in reality, most of searcher do preset various price levels for different price takers, depending on the price taker's demand. In order for the searcher to reach the best decision, two considerations are necessary: (1) Selection of price taker. (2) Pricing policy. This paper will try to develop two dynamic programming models and illustrate them with numerical examples. These models can be applied to solve problems such as assets selling problem, seat booking problem, personnel recruiting problem and so on.

1. Introduction

First, let us consider the conventional optimal stopping problem [1-17] where, within a given planning horizon, an item must be sold for a price offered by buyers, or must be bought for a price offered by sellers. As a valuation of the problem, let us consider that a number of identical items must be sold, or purchased, for a price offered by a searcher. In the former case, the objective of the searcher is profit maximization and that of the latter case is cost minimization. The problems that can be solved by the models include:

- *Assets selling* (Maximization model) [5]

Suppose that a real estate agent must sell a number of identical houses by a certain day in the future. Here, let us assume that a selling price is presented after a buyer (price taker) has arrived. The agent will try to sell his houses at highest possible price to buyers before the deadline. If some houses are not sold out before the deadline, they must be disposed at a price offered by a dealer, regardless of how low it may be. Apparently, it is risky to keep offering at high price, because it is likely that buyers will reject the high price, and ultimately some of the houses will remain unsold by the deadline. On the other hand, if the houses are offered at low price, the total sales will be small even if all the houses are sold out. Therefore, in order to come up with the suitable price, two considerations are necessary: (1) selection rule of buyer, and (2) pricing policy. In this case, it is common that the optimal pricing policy depends on the remaining period before deadline and the remaining unsold houses.

- *Seat booking problem* (Maximization model) [7 – 13]

When a customer (price taker) decides to travel by air, it is necessary to book reservations with airline companies prior to the departure. In most cases, customers will request for price quotations before deciding to buy tickets. Depending on the price being offered, the customers would decide whether or not to make the purchase. Since the demand of service or convenience may be different for different kinds of customers, it is reasonable to classify a pool of identical seats on the same flight into several booking classes through the application of restrictions on tickets. Under this practice, for the purpose of maximizing the total expected revenue in a single flight, the following two decisions must be made over the whole decision period. First, deciding whether to accept or reject a request for a certain booking class and, depending on the seats available at the time and the probability that the customers will buy the tickets, determining the appropriate price that is to be offered to the customers once the airline company decides to accept the specific customer.

- *Personnel recruiting* (Minimization model)

Suppose that a personnel department of a certain company has been assigned the task of finding a number of staffs within a given period. The department then tries to find staffs at lower salary before the deadline. If the department is not capable of finding the required number of staffs within the time limit, then it must accept the salary proposed by personnel recruiting company by the deadline, however high it may be. In order to accomplish the recruiting task at the lowest possible cost, the manager should come up with a selection rule which helps determine whether to accept or reject the interviewee, and payment policy determining how much pay should be offered to the interviewee once he/she is accepted. The job search problem [6], explained by the maximization model, where an unemployed worker tries to find a job within a given period, is similar to the problem discussed above.

- *Decision structure of the models*

The decision structure of the above problems is depicted in Figure 1. It shows the searcher's decision making process of whether to accept a price taker and how to offer a price, and the price taker's decision making process of whether to accept the price being offered.

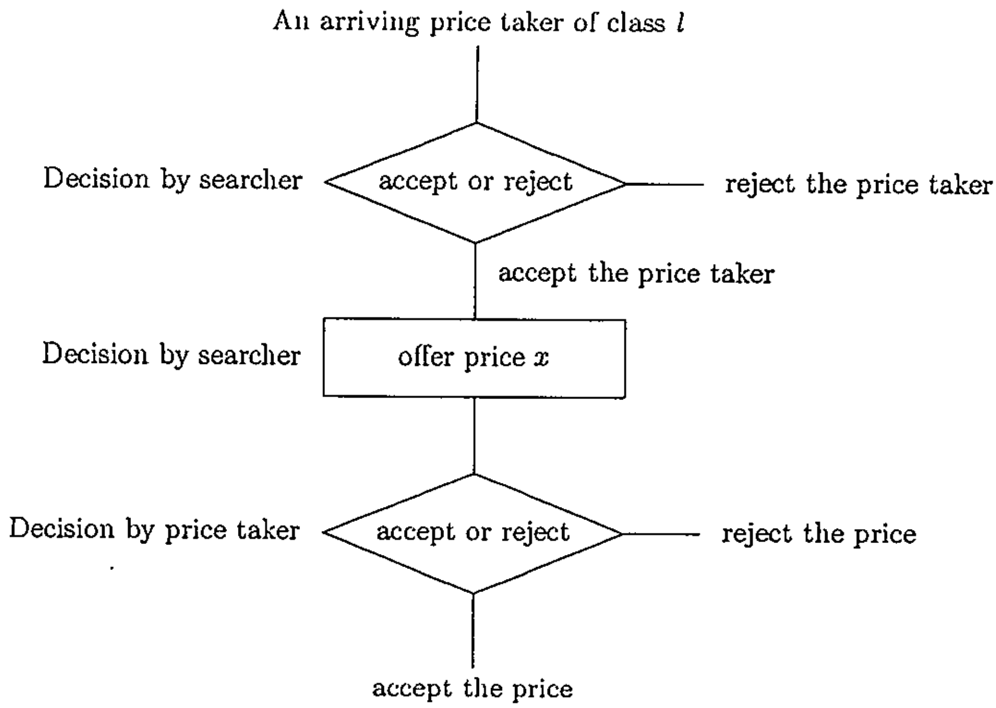

Figure 1
Structure of Decisions

2 Models

2.1 Profit Maximization Model

Consider the following discrete time sequential stochastic decision process with a finite planning horizon. First, for convenience, let points in time be numbered backward from the final point in time of the planning horizon as t, $t-1$,... and so on, where an interval between two successive points in time, say time t and time $t-1$, is called period t.

Assume that there exists a finite number of identical items, say consumer products, lands, houses and so on, that must be sold to buyers (price takers) arriving one by one at a time within the given planning horizon. Further, assume that there are $L \geq 1$ classes of buyers, and let $\lambda_t(l)$ be the probability that a buyer of class l will arrive where $\lambda_t(0) = 1 - \sum_{l=1}^{L} \lambda_t(l)$ is the probability that no buyers will appear.

Let $p_l(x)$ be the probability that a buyer of class l buys an item if the offered selling price is x. Here, for given a_l and b_l such that $0 \leq a_l < b_l$, let $p_l(x) = 1$ for $x \leq a_l$, $p_l(x) = 0$ for $b_l \leq x$, and $p_l(x)$ be strictly decreasing in x for $a_l \leq x \leq b_l$. Let $P_l(\theta)$ be the probability distribution function of the maximum permissible purchasing price θ that a buyer of class l has in mind for the items, that is, if the offered price is smaller than θ, then the buyer will decide to buy, and refuse to make the purchase if otherwise. Then $p_l(x)$ can be given by

$$p_l(x) = \int_x^{\infty} dP_l(\theta).$$

Assume that cost $c_l < b_l$ (service cost, value-added cost and so on) will be incurred if an item is sold to a buyer of class l.

Thoughout the paper, let $\beta \in (0, 1]$ be a per-period discount factor. Finally, assume that if i items still remain at time 0 (deadline), then they must be disposed at price $\alpha(i)$, usually a very small amount.

The objective here is to maximize the total expected present discounted selling profit, the total expected present discounted selling price minus the total expected present discounted cost.

2.2 Cost Minimization Model

As the profit maximization model is applied in selling items, the cost minimization model is applied when trying to buy a finite number of identical items.

Let $\lambda_t(l)$ be the probability that a seller (price taker) of class l will arrive, and let $q_l(x)$ be the probability that a seller of class l will sell the item if the purchasing price is x. Here, for given a_l and b_l such that $0 \leq a_l < b_l$, let $q_l(x) = 1$ for $b_l \leq x$, $q_l(x) = 0$ for $x \leq a_l$, and $q_l(x)$ is strictly increasing in x for $a_l \leq x \leq b_l$. Let $Q_l(\theta)$ be the probability distribution function of the minimum permissible selling price θ that a seller of class l has in mind for the items, that is, if the offered price is larger than θ, then the seller will be willing to sell; otherwise, he does not sell. Then $q_l(x)$ can be given by

$$q_l(x) = \int_0^{x} dQ_l(\theta).$$

Assume that cost $c_l < b_l$ will be incurred when an item is bought from a seller of class l. Further, assume that if i items remain unpurchased by time 0, then those items should be supplied by the total cost of $\gamma(i) \geq 0$.

The objective here is to minimize the total expected present discounted purchasing cost, the total expected present discounted purchasing price plus the total expected present discounted cost.

3 Preliminaries

3.1 K–function

For given a and b such that $0 \leq a < b$, let $p(x)$ be a strictly decreasing function of $x \geq 0$ for $a \leq x \leq b$ with $p(x) = 1$ for $x \leq a$ and $p(x) = 0$ for $b \leq x$. And, for any real number ν, define

$$K(\nu) \quad = \quad \max_x p(x)(x - \nu). \tag{3.1}$$

Let $x(\nu)$ denote the smallest x attaining the maximum of the right hand side of (3.1) if exists.

Lemma 3.1

(a) $K(\nu)$ *is nonincreasing in* ν *and* $K(\nu) + \nu$ *is nondecreasing in* ν.

(b) $K(\nu) \geq 0$ *for all* ν *and* $K(\nu) > 0$ *for* $\nu < b$.

(c) $x(\nu)$ *is nondecreasing in* ν.

(d) $a \leq x(\nu) \leq b$ *for all* ν, $a \leq x(\nu) < b$ *for* $\nu < b$, *and* $x(\nu) = b$ *for* $\nu \geq b$.

(e) *If* $\nu_1 \leq \nu_2$, *then* $K(\nu_1) - K(\nu_2) \leq \nu_2 - \nu_1$.

(f) *If* $\nu_1 \geq \nu_2$, *then* $K(\nu_1) - K(\nu_2) \geq \nu_2 - \nu_1$.

Proof: (a) The former part is immediate since $p(x)(x-\nu)$ is nonincreasing in ν for all x. The latter part is clear from the fact that $p(x)(x - \nu) + \nu(= p(x) + (1 - p(x))\nu)$ is nondecreasing in ν for all x.

(b) The former part is clear from the fact that $K(\nu) \geq p(b)(b - \nu) = 0$. The latter can be proved to be true since for $\xi > 0$ such that $\xi + \nu < b$,

$$K(\nu) \geq p(\xi + \nu)(\nu + \xi - \nu) = p(\xi + \nu)\xi > 0.$$

(c) For any $\xi > 0$, we have

$$\begin{aligned} K(\nu + \xi) &= \max_x p(x)(x - (\nu + \xi)) \\ &= p(x(\nu + \xi))(x(\nu + \xi) - (\nu + \xi)) \\ &= p(x(\nu + \xi))(x(\nu + \xi) - \nu) - p(x(\nu + \xi))\xi \\ &\leq p(x(\nu))(x(\nu) - \nu) - p(x(\nu + \xi))\xi \\ &= p(x(\nu))(x(\nu) - (\nu + \xi)) + \xi(p(x(\nu)) - p(x(\nu + \xi))) \\ &\leq p(x(\nu + \xi))(x(\nu + \xi) - (\nu + \xi)) + \xi(p(x(\nu)) - p(x(\nu + \xi))) \\ &= K(\nu + \xi) + \xi(p(x(\nu)) - p(x(\nu + \xi))). \end{aligned}$$

Therefore, we have $0 \leq p(x(\nu)) - p(x(\nu + \xi))$, that is, $p(x(\nu)) \geq p(x(\nu + \xi))$, implying that $x(\nu) \leq x(\nu + \xi)$ because $p(x)$ is strictly decreasing in x for $a \leq x \leq b$.

(d) First, assume $b < x(\nu)$. Then

$$K(\nu) = p(x(\nu))(x(\nu) - \nu) = 0 = p(b)(b - \nu),$$

which contradicts the definition of $x(\nu)$. Assume that $x(\nu) < a$, then

$$K(\nu) = p(x(\nu))(x(\nu) - \nu) = x(\nu) - \nu < a - \nu = p(a)(a - \nu) \leq K(\nu),$$

which is also a contradiction. Therefore, it must be that $a \leq x(\nu) \leq b$. Second, for any $\xi > 0$ such that $\nu + \xi < b$, we have

$$K(\nu) \geq p(\nu + \xi)(\nu + \xi - \nu) = \xi p(\nu + \xi) > 0.$$

Hence it follows that $a \leq x(\nu) < b$ if $\nu < b$. Finally, for $b \leq \nu$, we have $p(x)(x - \nu) < 0$ for all $x < b$ and $p(x)(x - \nu) = 0$ for all $x \geq b$; therefore, we have $b \leq x(\nu)$. Combining this with the definition of $x(\nu)$ results in $b = x(\nu)$.

(e) From (a), for $\nu_1 \leq \nu_2$ we have

$$K(\nu_1) + \nu_1 \leq K(\nu_2) + \nu_2.$$

Hence, it follows that $K(\nu_1) - K(\nu_2) \leq \nu_2 - \nu_1$ for $\nu_1 \leq \nu_2$.

(f) Same as the proof of (e). ∎

3.2 S–function

For given a and b such that $0 \leq a < b$, let $q(x)$ be a strictly increasing function of $x \geq 0$ for $a \leq x \leq b$ with $q(x) = 0$ for $x \leq a$ and $q(x) = 1$ for $b \leq x$. And, for any real number ν, define

$$S(\nu) \quad = \quad \min_x q(x)(x - \nu). \tag{3.2}$$

By $x(\nu)$ let us denote the largest x attaining the minimum of (3.2) if exists.

Lemma 3.2

(a) *$S(\nu)$ is nonincreasing in ν and $S(\nu) + \nu$ is nondecreasing in ν.*

(b) *$S(\nu) \leq 0$ for all ν and $S(\nu) < 0$ for $\nu > a$.*

(c) *$x(\nu) \leq \nu$ and $S(\nu) + \nu \geq 0$ for $\nu \geq 0$.*

(d) *$x(\nu)$ is nonincreasing in ν.*

(e) *$a \leq x(\nu) \leq b$ for all ν, $a < x(\nu) \leq b$ for $a < \nu$, and $x(\nu) = a$ for $\nu \leq a$.*

(f) *If $\nu_1 \geq \nu_2$, then $S(\nu_1) - S(\nu_2) \geq \nu_2 - \nu_1$*

Proof: (a) Same as the proof of Lemma 3.1(a).

(b) The former part is clear from the fact that $S(\nu) \leq q(a)(a - \nu) = 0$. The latter can be proved as follows, for $\xi > 0$ such that $\nu - \xi > a$,

$$S(\nu) \leq q(\nu - \xi)(\nu - \xi - \nu) = -\xi q(\nu - \xi) < 0.$$

(c) The former part is apparent. The latter is immediate from a fact that for $\nu \geq 0$,

$$S(\nu) + \nu = q(x(\nu))(x(\nu) - \nu) + \nu = q(x(\nu))x(\nu) + (1 - q(x(\nu)))\nu \geq 0.$$

(d) For any $\xi > 0$, we have

$$
\begin{aligned}
S(\nu+\xi) &= \min_x q(x)(x-(\nu+\xi)) \\
&= q(x(\nu+\xi))(x(\nu+\xi)-(\nu+\xi)) \\
&= q(x(\nu+\xi))(x(\nu+\xi)-\nu) - q(x(\nu+\xi))\xi \\
&\geq q(x(\nu))(x(\nu)-\nu) - q(x(\nu+\xi))\xi \\
&= q(x(\nu))(x(\nu)-(\nu+\xi)) + \xi(q(x(\nu)) - q(x(\nu+\xi))) \\
&\geq q(x(\nu+\xi))(x(\nu+\xi)-(\nu+\xi)) + \xi(q(x(\nu)) - q(x(\nu+\xi))) \\
&= S(\nu+\xi) + \xi(q(x(\nu)) - q(x(\nu+\xi))).
\end{aligned}
$$

Therefore, we have $q(x(\nu)) - q(x(\nu+\xi)) \leq 0$, that is, $q(x(\nu)) \leq q(x(\nu+\xi))$, implying that $x(\nu) \leq x(\nu+\xi)$ because $q(x)$ is strictly increasing in x for $a \leq x \leq b$.

(e) First, assume $b < x(\nu)$. Then

$$S(\nu) = q(x(\nu))(x(\nu)-\nu) = x(\nu)-\nu > b-\nu = q(b)(b-\nu),$$

which is a contradiction. Assume $x(\nu) < a$. Then

$$S(\nu) = q(x(\nu))(x(\nu)-\nu) = 0 = q(a)(a-\nu),$$

which contradicts the definition of $x(\nu)$. Therefore, it must be that $a \leq x \leq b$. Second, for $\xi > 0$ such that $a < \nu - \xi$, we have

$$S(\nu) \leq q(\nu-\xi)(\nu-\xi-\nu) = -\xi q(\nu-\xi) < 0,$$

hence, it follows that $a < x(\nu) \leq b$ if $\nu > a$. Finally, for $\nu \leq a$, we have $q(x)(x-\nu) > 0$ for $x > a$ and $q(x)(x-\nu) = 0$ for $x \leq a$; therefore, we have $x(\nu) \leq a$, leading to $a = x(\nu)$ by the definition of $x(\nu)$.

(f) Immediate from the latter part of (a). ■

4 Analysis I – Profit Maximization Model –

Let i denote the number of items that are available at present time and $v_t(i)$ be the maximum total expected present discounted profit starting from time $t \geq 0$ with i items remaining. Then, clearly, we have

$$v_t(0) = 0, \qquad t \geq 0, \qquad (4.1)$$

$$v_t(i) = \lambda_t(0)\beta v_{t-1}(i) + \sum_{l=1}^{L} \lambda_t(l) \max\{g_t(i,l), \beta v_{t-1}(i)\}, \qquad i \geq 1,\ t \geq 1, \qquad (4.2)$$

where $g_t(i,l)$ is the maximum total expected present discounted profit starting from time $t \geq 0$ with i items remaining, provided that the seller (searcher) accepts the buyer (price taker) of class l who has just arrived. Then $g_t(i,l)$ can be expressed as

$$g_t(i,l) = \max_x\{p_l(x)(x - c_l + \beta v_{t-1}(i-1)) + (1-p_l(x))\beta v_{t-1}(i)\}, \quad i \geq 1,\ t \geq 1. \quad (4.3)$$

The final condition is given as follows:

$$v_0(i) = \lambda_0(0)\alpha(i) + \sum_{l=1}^{L} \lambda_0(l) \max\{g_0(i,l), \alpha(i)\} \qquad i \geq 1, \qquad (4.4)$$

where

$$g_0(i,l) = \max_x\{p_l(x)(x - c_l + \alpha(i-1)) + (1 - p_l(x))\alpha(i)\}, \quad i \geq 1. \qquad (4.5)$$

Here, note that $v_0(i)$ can be expressed by (4.2) with $t = 0$ if setting

$$v_{-1}(i) = \alpha(i)/\beta.$$

Let

$$\begin{aligned} \Delta\alpha(i) &= \alpha(i) - \alpha(i-1), & i \geq 1, \\ \Delta v_t(i) &= v_t(i) - v_t(i-1), & i \geq 1,\ t \geq 0, \\ z_t(i,l) &= c_l + \beta\Delta v_t(i). & i \geq 1,\ t \geq 0, \qquad (4.6) \\ x(z_t(i,l)) &= x_t(i,l). \end{aligned}$$

And, for convenience, let

$$z_{-1}(i,l) = c_l + \beta\Delta v_{-1}(i).$$

Then, $g_t(i,l)$ can be rewritten as follows.

$$\begin{aligned} g_t(i,l) &= \max_x\{p_l(x)(x - c_l + \beta v_{t-1}(i-1)) + (1 - p_l(x))\beta v_{t-1}(i)\} \\ &= \beta v_{t-1}(i) + \max_x p_l(x)(x - c_l + \beta v_{t-1}(i-1) - \beta v_{t-1}(i)) \\ &= \beta v_{t-1}(i) + K_l(z_{t-1}(i,l)). \end{aligned}$$

Theorem 4.1 $g_t(i,l) \geq \beta v_{t-1}(i)$

Proof: Clear from the fact that $K_l(\nu) \geq 0$ from Lemma 3.1(b). ∎

From Theorem 4.1, $v_t(i)$ eventually leads to

$$v_t(i) = \beta v_{t-1}(i) + \sum_{l=1}^{L} \lambda_t(l) K_l(z_{t-1}(i,l)), \qquad i \geq 1,\ t \geq 0, \quad (4.7)$$

where we have

$$v_0(i) = \alpha(i) + \sum_{l=1}^{L} \lambda_0(l) K_l(c_l + \Delta\alpha(i)), \qquad i \geq 1. \qquad (4.8)$$

Lemma 4.1 *If* $z_t(i,l) \geq b_l$, *then* $x_t(i,l) = b_l$.

Proof: Immediate from Lemma 3.1(d). ∎

Lemma 4.2 *Assume that* $\Delta\alpha(i)$ *is nonincreasing in* i. *Then*
(a) *If* $\alpha(i) \geq 0$, *then* $v_t(i) \geq 0$ *for all* t *and* i.
(b) *If* $\alpha(i) \geq 0$ *and there exists* l *such that* $\lambda_t(l) > 0$, *then* $v_t(i) > 0$.
(c) $\Delta v_t(i)$, *hence* $z_t(i,l)$ *is nonincreasing in* i *for* $t \geq 0$.
(d) $\Delta v_t(i)$, *hence* $z_t(i,l)$ *is not always nondecreasing in* t *for* $i \geq 1$.

proof : (a) Clear from (4.7), (4.8) and Lemma 3.1(b).

(b) Suppose $\lambda_t(l^*) > 0$ for a certain class l^*. First, from (4.7) and (a) we have

$$\begin{aligned} v_t(i) &= \beta v_{t-1}(i) + \sum_{l=1}^{L} \lambda_t(l) K_l(c_l + \beta \Delta v_{t-1}(i)) \\ &= \sum_{l=0}^{L} \beta v_{t-1}(i) + \sum_{l=1}^{L} \lambda_t(l) K_l(c_l + \beta \Delta v_{t-1}(i)) \\ &\geq \sum_{l=1}^{L} \lambda_t(l)(\beta v_{t-1}(i) + K_l(c_l + \beta \Delta v_{t-1}(i))) \\ &= \sum_{l=1}^{L} \lambda_t(l)(c_l + \beta \Delta v_{t-1}(i) + K_l(c_l + \beta \Delta v_{t-1}(i)) + \beta v_{t-1}(i-1) - c_l). \end{aligned}$$

Here, since $c_l + \beta \Delta v_{t-1}(i) \geq c_l - \beta v_{t-1}(i-1)$ due to $v_t(i) \geq 0$ from (a), we have from Lemma 3.1(a)

$$\begin{aligned} v_t(i) &\geq \sum_{l=1}^{L} \lambda_t(l)(c_l - \beta v_{t-1}(i-1) + K_l(c_l - \beta v_{t-1}(i-1)) + \beta v_{t-1}(i-1) - c_l). \\ &= \sum_{l=1}^{L} \lambda_t(l) K_l(c_l - \beta v_{t-1}(i-1)). \end{aligned}$$

Furthermore, since $c_l - \beta v_{t-1}(i-1) < b_l$ due to $c_l < b_l$ and (a), we have $K_l(c_l - \beta v_{t-1}(i-1)) > 0$ from Lemma 3.1(b); hence, it follows that $v_t(i) > 0$ due to $\lambda_t(l^*) > 0$.

(c) Clearly we have from (4.8) and Lemma 3.1(b).

$$\begin{aligned} \Delta v_0(2) - \Delta v_0(1) &= \Delta\alpha(2) - \Delta\alpha(1) + \sum_{l=1}^{L} \lambda_0(l)(K_l(c_l + \Delta\alpha(2)) - 2K_l(c_l + \Delta\alpha(1))). \\ &\leq \Delta\alpha(2) - \Delta\alpha(1) + \sum_{l=1}^{L} \lambda_0(l)(K_l(c_l + \Delta\alpha(2)) - K_l(c_l + \Delta\alpha(1))). \end{aligned}$$

Here, since $c_l + \Delta\alpha(2) \leq c_l + \Delta\alpha(1)$, from Lemma 3.1(e) we have

$$\begin{aligned} \Delta v_0(2) - \Delta v_0(1) &\leq \Delta\alpha(2) - \Delta\alpha(1) + \sum_{l=1}^{L} \lambda_0(l)(\Delta\alpha(1) - \Delta\alpha(2)) \\ &= \lambda_0(0)(\Delta\alpha(2) - \Delta\alpha(1)) \leq 0. \end{aligned}$$

For $i \geq 3$,

$$\begin{aligned} \Delta v_0(i) - \Delta v_0(i-1) = {} & \Delta\alpha(i) - \Delta\alpha(i-1) \\ & + \sum_{l=1}^{L} \lambda_0(l)(K_l(c_l + \Delta\alpha(i)) - K_l(c_l + \Delta\alpha(i-1))) \\ & + \sum_{l=1}^{L} \lambda_0(l)(K_l(c_l + \Delta\alpha(i-2)) - K_l(c_l + \Delta\alpha(i-1))). \end{aligned}$$

Here, since $\Delta\alpha(i) \leq \Delta\alpha(i-1)$ and $\Delta\alpha(i-1) \leq \Delta\alpha(i-2)$, from Lemma 3.1(a) and Lemma 3.1(e) we have

$$\Delta v_0(i) - \Delta v_0(i-1) \leq \Delta\alpha(i) - \Delta\alpha(i-1) + \sum_{l=1}^{L} \lambda_0(l)(K_l(c_l + \Delta\alpha(i)) - K_l(c_l + \Delta\alpha(i-1)))$$

$$\leq \Delta\alpha(i) - \Delta\alpha(i-1) + \sum_{l=1}^{L} \lambda_0(l)(\Delta\alpha(i-1) - \Delta\alpha(i))$$
$$= \lambda_0(0)(\Delta\alpha(i) - \Delta\alpha(i-1)) \leq 0.$$

Hence, the assertion holds true for $t = 0$. Assume it also holds true for $t-1$. Then, $\Delta v_{t-1}(i)$, hence $z_{t-1}(i,l)$ is nonincreasing in i, from (4.7) and Lemma 3.1(b) we have

$$\begin{aligned}\Delta v_t(2) - \Delta v_t(1) &= \beta(\Delta v_{t-1}(2) - \Delta v_{t-1}(1)) + \sum_{l=1}^{L} \lambda_t(l)(K_l(z_{t-1}(2,l)) - 2K_l(z_{t-1}(1,l))) \\ &\leq \beta(\Delta v_{t-1}(2) - \Delta v_{t-1}(1)) + \sum_{l=1}^{L} \lambda_t(l)(K_l(z_{t-1}(2,l)) - K_l(z_{t-1}(1,l))).\end{aligned}$$

Here, since $z_{t-1}(2,l) \leq z_{t-1}(1,l)$, from Lemma 3.1(e) we have

$$\begin{aligned}\Delta v_t(2) - \Delta v_t(1) &\leq \beta(\Delta v_{t-1}(2) - \Delta v_{t-1}(1)) + \sum_{l=1}^{L} \lambda_t(l)(z_{t-1}(1,l) - z_{t-1}(2,l)) \\ &= \beta(\Delta v_{t-1}(2) - \Delta v_{t-1}(1)) + \beta \sum_{l=1}^{L} \lambda_t(l)(\Delta v_{t-1}(1) - \Delta v_{t-1}(2)) \\ &= \beta\lambda_t(0)(\Delta v_{t-1}(2) - \Delta v_{t-1}(1)) \leq 0.\end{aligned}$$

For $i \geq 3$ we have

$$\begin{aligned}\Delta v_t(i) - \Delta v_t(i-1) = \beta(&\Delta v_{t-1}(i) - \Delta v_{t-1}(i-1)) \\ &+ \sum_{l=1}^{L} \lambda_t(l)(K_l(z_{t-1}(i,l)) - K_l(z_{t-1}(i-1,l))) \\ &+ \sum_{l=1}^{L} \lambda_t(l)(K_l(z_{t-1}(i-2,l)) - K_l(z_{t-1}(i-1,l))).\end{aligned}$$

Here, since $z_{t-1}(i,l) \leq z_{t-1}(i-1,l)$ and $z_{t-1}(i-1,l) \leq z_{t-1}(i-2,l)$, from Lemma 3.1(a) and Lemma 3.1(e) we have

$$\begin{aligned}\Delta v_t(i) - \Delta v_t(i-1) &\leq \beta(\Delta v_{t-1}(i) - \Delta v_{t-1}(i-1)) + \sum_{l=1}^{L} \lambda_t(l)(K_l(z_{t-1}(i)) - K_l(z_{t-1}(i-1)) \\ &\leq \beta(\Delta v_{t-1}(i) - \Delta v_{t-1}(i-1)) + \sum_{l=1}^{L} \lambda_t(l)(z_{t-1}(i-1,l) - z_{t-1}(i,l)) \\ &= \beta(\Delta v_{t-1}(i) - \Delta v_{t-1}(i-1)) + \beta \sum_{l=1}^{L} \lambda_t(l)(\Delta v_{t-1}(i-1) - \Delta v_{t-1}(i)) \\ &= \beta\lambda_t(0)(\Delta v_{t-1}(i) - \Delta v_{t-1}(i-1)) \leq 0.\end{aligned}$$

(d) For example, let $\lambda_1(l) = 0$ for $l = 1, 2, \ldots, L$, $\lambda_0(l) > 0$ for a certain l, and $\alpha(i) \geq 0$. Then, we have $v_1(i) > 0$ from (b) and $v_1(1) = \beta v_0(1)$ from (4.7), hence, for a certain $\beta < 1$ we have

$$\Delta v_1(1) - \Delta v_0(1) = v_1(1) - v_0(1) = (\beta - 1)v_0(1) < 0. \qquad \blacksquare$$

Theorem 4.2 *Assume $\Delta\alpha(i)$ is nonincreasing in i. Then*
(a) *$v_t(i)$ is not always nondecreasing in i for $t \geq 0$.*
(b) *If $\Delta\alpha(i) \geq 0$, then $v_t(i)$ is nondecreasing in i for $t \geq 0$.*

(c) $v_t(i)$ *is not always nondecreasing in* t *for* $i \geq 0$.

(d) *If* $\beta = 1$, *then* $v_t(i)$ *is nondecreasing in* t *for* $i \geq 0$.

(e) *If* $\lambda_t(l)$ *is independent of* t *and* $\alpha(i) = 0$, *then* $v_t(i)$ *is nondecreasing in* t *for* $i \geq 0$.

Proof: (a) If $\alpha(i) < 0$ and $\lambda_0(l) = 0$ for $l = 1, 2, \ldots, L$, then we have from (4.8)

$$v_0(1) = \alpha(1) + \sum_{l=1}^{L} \lambda_0(l) K_l(c_l + \alpha(1)) = \alpha(1) < 0 = v_0(0).$$

(b) First, it is clear that $v_0(1) \geq v_0(0)$ from Lemma 4.2(a). Next, for $i \geq 2$ we have

$$v_0(i) - v_0(i-1) = \Delta\alpha(i) + \sum_{l=1}^{L} \lambda_0(l)(K_l(c_l + \Delta\alpha(i)) - K_l(c_l + \Delta\alpha(i-1))).$$

Here, since $\Delta\alpha(i) \leq \Delta\alpha(i-1)$, from Lemma 3.1(a) we have $K_l(c_l + \Delta\alpha(i)) \geq K_l(c_l + \Delta\alpha(i-1))$. Hence it follows that $v_0(i) - v_0(i-1) \geq \Delta\alpha(i) \geq 0$. Assume the assertion holds true for $t-1$, so $v_{t-1}(i)$ is nondecreasing in i. It is clear that $v_t(1) \geq v_t(0)$ from Lemma 4.2(a). For $i \geq 2$ we have

$$v_t(i) - v_t(i-1) = \beta(v_{t-1}(i) - v_{t-1}(i-1)) + \sum_{l=1}^{L} \lambda_t(l)(K_l(z_{t-1}(i,l)) - K_l(z_{t-1}(i-1,l)))$$

Here, since $z_{t-1}(i,l) \leq z_{t-1}(i-1,l)$ from Lemma 4.2(c), we have from Lemma 3.1(a)

$$v_t(i) - v_t(i-1) \geq \beta(v_{t-1}(i) - v_{t-1}(i-1)) \geq 0.$$

Hence, by induction the statement has been proven.

(c) Let $\lambda_t(l) = 0$ for $l = 1, 2, \ldots, L$, $\lambda_{t-1}(l) > 0$ for a certain l, $\alpha(i) \geq 0$, and $\beta < 1$. Then from (4.7) and Lemma 4.2(b) we have $v_t(i) = \beta v_{t-1}(i) < v_{t-1}(i)$.

(d) If $\beta = 1$, then from (4.7) and Lemma 3.1(b) we have

$$v_t(i) = v_{t-1}(i) + \sum_{l=1}^{L} \lambda_t(l) K_l(z_{t-1}(i,l)) \geq v_{t-1}(i), \qquad i \geq 1, t \geq 0. \tag{4.9}$$

(e) If $\lambda_t(l)$ is independent of t and $\alpha(i) = 0$, then from (4.7) we have

$$v_t(i) = \beta v_{t-1}(i) + \sum_{l=1}^{L} \lambda(l) K_l(z_{t-1}(i,l)), \qquad i \geq 1, t \geq 0, \tag{4.12}$$

where from (4.8)

$$v_0(i) = \sum_{l=1}^{L} \lambda(l) K_l(c_l) \geq 0, \qquad i \geq 1. \tag{4.13}$$

Then, from (4.12) and (4.13), we have

$$\begin{aligned} v_1(1) - v_0(1) &= \beta v_0(1) + \sum_{l=1}^{L} \lambda(l)(K_l(z_0(1,l)) - K_l(c_l)) \\ &= \beta v_0(1) + \sum_{l=1}^{L} \lambda(l)(K_l(c_l + \beta v_0(1)) - K_l(c_l)). \end{aligned}$$

Here, since $c_l + \beta v_0(1) \geq c_l$ due to $v_0(1) \geq 0$ from Lemma 4.2(a), we have from Lemma 3.1(f)

$$v_1(1) - v_0(1) \geq \beta v_0(1) - \sum_{l=1}^{L} \beta\lambda(l) v_0(1) = \beta\lambda(0) v_0(1) \geq 0.$$

For $i \geq 2$, since $z_0(i,l) = c_l + \beta \Delta v_0(i) = c_l$ from (4.13), we have from (4.12) and (4.13)

$$\begin{aligned} v_1(i) - v_0(i) &= \beta v_0(i) + \sum_{l=1}^{L} \lambda(l)(K_l(z_0(i,l)) - K_l(c_l)) \\ &= \beta v_0(i) + \sum_{l=1}^{L} \lambda(l)(K_l(c_l) - K_l(c_l)) \\ &= \beta v_0(i) \geq 0. \end{aligned}$$

Hence, $v_1(i) \geq v_0(i)$. Assume $v_{t-1}(i) \geq v_{t-2}(i)$ for all i. Then, for $t \geq 2$ and $i \geq 1$ we have

$$v_t(i) - v_{t-1}(i) = \beta(v_{t-1}(i) - v_{t-2}(i)) + \sum_{l=1}^{L} \lambda(l)(K_l(z_{t-1}(i,l)) - K_l(z_{t-2}(i,l))).$$

Here, if $z_{t-1}(i) \leq z_{t-2}(i)$, then from Lemma 3.1(a) we have

$$v_t(i) - v_{t-1}(i) \geq \beta(v_{t-1}(i) - v_{t-2}(i)) \geq 0.$$

If $z_{t-1}(i) \geq z_{t-2}(i)$, then from Lemma 3.1(f) we have

$$\begin{aligned} v_t(i) - v_{t-1}(i) &\geq \beta(v_{t-1}(i) - v_{t-2}(i)) + \sum_{l=1}^{L} \lambda(l)(z_{t-2}(i,l) - z_{t-1}(i,l)) \\ &= \beta(v_{t-1}(i) - v_{t-2}(i)) + \beta \sum_{l=1}^{L} \lambda(l)(\Delta v_{t-2}(i) - \Delta v_{t-1}(i)) \\ &= \beta\lambda(0)(v_{t-1}(i) - v_{t-2}(i)) + \beta \sum_{l=1}^{L} \lambda(l)(v_{t-1}(i-1) - v_{t-2}(i-1)) \geq 0. \quad \blacksquare \end{aligned}$$

Lemma 4.3 *Assume $\Delta\alpha(i)$ is nonincreasing in i. Then*

(a) *If $\beta = 1$, then $\Delta v_t(i)$, hence $z_t(i,l)$, is nondecreasing in t for $i \geq 1$.*

(b) *If $\lambda_t(l)$ is independent of t and $\alpha(i) = 0$, then $\Delta v_t(i)$, hence $z_t(i,l)$ is nondecreasing in t for $i \geq 1$.*

Proof: (a) First, clearly from Theorem 4.2(d) we have $\Delta v_t(1) - \Delta v_{t-1}(1) = v_t(1) - v_{t-1}(1) \geq 0$. Second, from (4.9) for $i \geq 2$

$$\Delta v_t(i) = \Delta v_{t-1}(i) + \sum_{l=1}^{L} \lambda_t(l)(K_l(z_{t-1}(i,l)) - K_l(z_{t-1}(i-1,l))).$$

Hence, for $t \geq 1$ and $i \geq 2$ we have

$$\Delta v_t(i) - \Delta v_{t-1}(i) = \sum_{l=1}^{L} \lambda_t(l)(K_l(z_{t-1}(i,l)) - K_l(z_{t-1}(i-1,l))).$$

Here, since $z_{t-1}(i,l) \leq z_{t-1}(i-1,l)$ from Lemma 4.2(c), the statement has been proven by Lemma 3.1(a).

(b) From Theorem 4.2(e), it is clear

$$\Delta v_t(1) - \Delta v_{t-1}(1) = v_t(1) - v_{t-1}(1) \geq 0.$$

For $i \geq 2$, since $\Delta v_0(i) = 0$ from (4.13), we have

$$\Delta v_1(i) - \Delta v_0(i) = \Delta v_1(i) \geq 0$$

Thus, the assertion holds true for $t = 1$. Assume it holds true for $t-1$, So $\Delta v_{t-1}(i) \geq \Delta v_{t-2}(i)$ and $z_{t-1}(i) \geq z_{t-2}(i)$ for all $i \geq 1$. From (4.12) for $i \geq 2$ we have

$$\begin{aligned}
\Delta v_t(i) - \Delta v_{t-1}(i) &= \beta(\Delta v_{t-1}(i) - \Delta v_{t-2}(i)) + \sum_{l=1}^{L} \lambda(l)(K_l(z_{t-1}(i,l)) - K_l(z_{t-1}(i-1,l))) \\
&\quad - \sum_{l=1}^{L} \lambda(l)(K_l(z_{t-2}(i,l)) - K_l(z_{t-2}(i-1,l))) \\
&= \beta(\Delta v_{t-1}(i) - \Delta v_{t-2}(i)) + \sum_{l=1}^{L} \lambda(l)(K_l(z_{t-1}(i,l)) - K_l(z_{t-2}(i,l))) \\
&\quad + \sum_{l=1}^{L} \lambda(l)(K_l(z_{t-2}(i-1,l)) - K_l(z_{t-1}(i-1,l))).
\end{aligned}$$

Here, since $z_{t-2}(i-1,l)) \leq z_{t-1}(i-1,l))$ and $z_{t-1}(i,l) \geq z_{t-2}(i,l)$, from Lemma 3.1(a) and Lemma 3.1(f) we have

$$\begin{aligned}
\Delta v_t(i) - \Delta v_{t-1}(i) &\geq \beta(\Delta v_{t-1}(i) - \Delta v_{t-2}(i)) + \sum_{l=1}^{L} \lambda(l)(K_l(z_{t-1}(i,l)) - K_l(z_{t-2}(i,l))) \\
&\geq \beta(\Delta v_{t-1}(i) - \Delta v_{t-2}(i)) + \sum_{l=1}^{L} \lambda(l)(z_{t-2}(i,l) - z_{t-1}(i,l)) \\
&= \beta(\Delta v_{t-1}(i) - \Delta v_{t-2}(i)) + \beta \sum_{l=1}^{L} \lambda(l)(\Delta v_{t-2}(i) - \Delta v_{t-1}(i)) \\
&\geq \beta\lambda(0)(\Delta v_{t-1}(i) - \Delta v_{t-2}(i)) \geq 0. \qquad \blacksquare
\end{aligned}$$

Theorem 4.3 *Assume $\Delta\alpha(i)$ is nonincreasing in i. Then*
(a) *$x_t(i,l)$ is nonincreasing in i for $t \geq 0$.*
(b) *If $\beta = 1$, then $x_t(i,l)$ is nondecreasing in t for all i.*
(c) *If $\lambda_t(l) = \lambda(l)$ for all l and $\alpha(i) = 0$, then $x_t(i,l)$ is nondecreasing in t for all i.*

proof : (a) Immediate from Lemma 3.1(c) and Lemma 4.2(c).

(b) Clear from Lemma 3.1(c) and Lemma 4.3(a).

(c) Clear from Lemma 3.1(c) and Lemma 4.3(b). ■

5 Analysis II – Cost Minimization Model –

Let i denote the number of items that should be purchased and $v_t(i)$ be the minimum total expected present discounted cost starting from time t with i items to be purchased. Then, clearly, we have

$$v_t(0) = 0, \qquad t \geq 0, \qquad (5.1)$$

$$v_t(i) = \lambda_t(0)\beta v_{t-1}(i) + \sum_{l=1}^{L} \lambda_t(l) \min\{g_t(i,l), \beta v_{t-1}(i)\}, \qquad i \geq 1,\ t \geq 1, \qquad (5.2)$$

where $g_t(i,l)$ is the total expected present discounted cost starting from time $t \geq 0$ with i items to be purchased. Suppose that the buyer (searcher) accepts the seller (price taker) of class l who has just arrived, then $g_t(i,l)$ can be expressed as

$$g_t(i,l) = \min_x\{q_l(x)(x + c_l + \beta v_{t-1}(i-1)) + (1 - q_l(x))\beta v_{t-1}(i)\}, \quad i \geq 1, \; t \geq 1. \quad (5.3)$$

The final conditions is given as follows:

$$v_0(i) = \lambda_0(0)\gamma(i) + \sum_{l=1}^{L}\min\{g_0(i,l), \gamma(i)\} \qquad i \geq 1, \qquad (5.4)$$

where

$$g_0(i,l) = \min_x\{q_l(x)(x + c_l + \gamma(i-1)) + (1 - q_l(x))\gamma(i)\}, \qquad i \geq 1. \qquad (5.5)$$

Now, note that $v_0(i)$ can be expressed by (5.2) with $t = 0$ if setting $v_{-1}(i) = \gamma(i)/\beta$.
Let

$$\begin{aligned} x(z_t(i,l)) &= x_t(i,l) \\ \Delta\gamma(i) &= \gamma(i) - \gamma(i-1), && i \geq 1, \\ \Delta v_t(i) &= v_t(i) - v_t(i-1), && i \geq 1, \; t \geq 0, \\ z_t(i,l) &= \beta\Delta v_t(i) - c_l, && i \geq 1, \; t \geq 0, \qquad (5.6) \end{aligned}$$

and, for convenience, let $z_{-1}(i,l) = \beta\Delta v_{-1}(i) - c_l$. Then, for $i \geq 1$ and $t \geq 0$ we have

$$\begin{aligned} g_t(i,l) &= \min_x\{p_l(x)(x + c_l + \beta v_{t-1}(i-1)) + (1 - p_l(x))\beta v_{t-1}(i)\} \\ &= \beta v_{t-1}(i) + \min_x\{p_l(x)(x + c_l + \beta v_{t-1}(i-1) - \beta v_{t-1}(i))\} \\ &= \beta v_{t-1}(i) + S_l(z_{t-1}(i,l)). \end{aligned}$$

Theorem 5.1 $g_t(i,l) \leq \beta v_{t-1}(i)$

Proof: Clear from the fact that $S_l(\nu) \leq 0$ from Lemma 3.2(b). ■

From the Theorem 5.1, $v_t(i)$ can be rearranged as follows

$$v_t(i) = \beta v_{t-1}(i) + \sum_{l=1}^{L}\lambda_t(l)S_l(z_{t-1}(i,l)), \qquad i \geq 1, \; t \geq 0, \qquad (5.7)$$

where

$$v_0(i) = \gamma(i) + \sum_{l=1}^{L}\lambda_0(l)S_l(\Delta\gamma(i) - c_l), \qquad i \geq 1. \qquad (5.8)$$

Lemma 5.1 *If* $z_t(i,l) \leq a_l$, *then* $x_t(i,l) = a_l$.

Proof: Immediate from Lemma 3.2(e). ■

Theorem 5.2 *If* $\Delta\gamma(i)$ *is nondecreasing in* i. *Then*
(a) $v_t(i) \geq 0$ *for all* t *and* i.
(b) $v_t(i)$ *is nonincreasing in* t *for all* i.

proof : (a) It can be proved by induction starting with

$$\begin{aligned} v_0(i) &= \gamma(i) + \sum_{l=1}^{L} \lambda_0(l) S_l(\Delta\gamma(i) - c_l) \\ &= \sum_{l=0}^{L} \lambda_0(l)\gamma(i) + \sum_{l=1}^{L} \lambda_0(l) S_l(\Delta\gamma(i) - c_l) \\ &\geq \sum_{l=1}^{L} \lambda_0(l)(\gamma(i) + S_l(\Delta\gamma(i) - c_l)). \end{aligned}$$

Here, since $\Delta\gamma(i) - c_l \leq \gamma(i)$, from Lemma 3.2(a) we have

$$v_0(i) \geq \sum_{l=1}^{L} \lambda_0(l)(\gamma(i) + S_l(\gamma(i))).$$

Furthermore, since $\gamma(i) \geq 0$, from Lemma 3.2(c) we have $v_0(i) \geq 0$. Hence, the assertion holds true for $t = 0$. Assume it holds true for $t-1$. Then since $v_{t-1}(i) \geq 0$, from Lemma 3.2(a) and Lemma 3.2(c) we have

$$\begin{aligned} v_t(i) &= \beta v_{t-1}(i) + \sum_{l=1}^{L} \lambda_t(l) S_l(\beta\Delta v_{t-1}(i) - c_l) \\ &\geq \sum_{l=1}^{L} \lambda_t(l)(\beta v_{t-1}(i) + S_l(\beta\Delta v_{t-1}(i) - c_l)) \\ &\geq \sum_{l=1}^{L} \lambda_t(l)(\beta v_{t-1}(i) + S_l(\beta v_{t-1}(i))) \geq 0. \end{aligned}$$

(b) Since $v_t(i) \geq 0$ and $S_l(\nu) \leq 0$, it is immediate that

$$v_t(i) - v_{t-1}(i) = (\beta - 1)v_{t-1}(i) + \sum_{l=1}^{L} \lambda_t(l) S_l(\Delta v_{t-1}(i) - c_l) \leq 0. \quad \blacksquare$$

Lemma 5.2 *If $\Delta\gamma(i)$ is nondecreasing in i, then, $\Delta v_t(i)$, hence $z_t(i,l)$ is nondecreasing in i for $t \geq 0$.*

Proof: Clearly from (5.8) and Lemma 3.2(b) we have

$$\begin{aligned} \Delta v_0(2) - \Delta v_0(1) &= \Delta\gamma(2) - \Delta\gamma(1) + \sum_{l=1}^{L} \lambda_0(l)(S_l(\Delta\gamma(2) - c_l) - 2S_l(\Delta\gamma(1) - c_l)) \\ &\geq \Delta\gamma(2) - \Delta\gamma(1) + \sum_{l=1}^{L} \lambda_0(l)(S_l(\Delta\gamma(2) - c_l) - S_l(\Delta\gamma(1) - c_l)) \end{aligned}$$

Here, since $\Delta\gamma(2) \geq \Delta\gamma(1)$, from Lemma 3.2(f) we have

$$\begin{aligned} \Delta v_0(2) - \Delta v_0(1) &\geq \Delta\gamma(2) - \Delta\gamma(1) + \sum_{l=1}^{L} \lambda_0(l)(\Delta\gamma(1) - \Delta\gamma(2)) \\ &= \lambda_0(0)(\Delta\gamma(2) - \Delta\gamma(1)) \geq 0. \end{aligned}$$

For $i \geq 3$,

$$\begin{aligned} \Delta v_0(i) - \Delta v_0(i-1) = \Delta\gamma(i) &- \Delta\gamma(i-1) \\ &+ \sum_{l=1}^{L} \lambda_0(l)(S_l(\Delta\gamma(i) - c_l) - S_l(\Delta\gamma(i-1) - c_l)) \\ &+ \sum_{l=1}^{L} \lambda_0(l)(S_l(\Delta\gamma(i-2) - c_l) - S_l(\Delta\gamma(i-1) - c_l)). \end{aligned}$$

Here, since $\Delta\gamma(i) \geq \Delta\gamma(i-1)$ and $\Delta\gamma(i-1) \geq \Delta\gamma(i-2)$, from Lemma 3.2(a) and Lemma 3.2(f) we have

$$\begin{aligned}\Delta v_0(i) - \Delta v_0(i-1) &\geq \Delta\gamma(i) - \Delta\gamma(i-1) + \sum_{l=1}^{L}\lambda_0(l)(S_l(\Delta\gamma(i) - c_l) - S_l(\Delta\gamma(i-1) - c_l)) \\ &\geq \Delta\gamma(i) - \Delta\gamma(i-1) + \sum_{l=1}^{L}\lambda_0(l)(\Delta\gamma(i-1) - \Delta\gamma(i)) \\ &= \lambda_0(0)(\Delta\gamma(i) - \Delta\gamma(i-1)) \geq 0.\end{aligned}$$

Hence, the assertion holds true for $t = 0$. Assume it holds true for $t-1$, so $\Delta v_{t-1}(i)$ and $\Delta z_{t-1}(i,l)$ are nondecreasing in i, Then from (5.7) and from Lemma 3.2(b) we have

$$\begin{aligned}\Delta v_t(2) - \Delta v_t(1) &= \beta(\Delta v_{t-1}(2) - \Delta v_{t-1}(1)) + \sum_{l=1}^{L}\lambda_t(l)(S_l(z_{t-1}(2,l)) - 2S_l(z_{t-1}(1,l))). \\ &\geq \beta(\Delta v_{t-1}(2) - \Delta v_{t-1}(1)) + \sum_{l=1}^{L}\lambda_t(l)(S_l(z_{t-1}(2,l)) - S_l(z_{t-1}(1,l))).\end{aligned}$$

Here, since $z_{t-1}(2,l) \geq z_{t-1}(1,l)$, from Lemma 3.2(f) we have

$$\begin{aligned}\Delta v_t(2) - \Delta v_t(1) &\geq \beta(\Delta v_{t-1}(2) - \Delta v_{t-1}(1)) + \sum_{l=1}^{L}\lambda_t(l)(z_{t-1}(1,l) - z_{t-1}(2,l)) \\ &= \beta(\Delta v_{t-1}(2) - \Delta v_{t-1}(1)) + \beta\sum_{l=1}^{L}\lambda_t(l)(\Delta v_{t-1}(1) - \Delta v_{t-1}(2)) \\ &= \beta\lambda_t(0)(\Delta v_{t-1}(2) - \Delta v_{t-1}(1)) \geq 0.\end{aligned}$$

For $i \geq 3$ we have

$$\begin{aligned}\Delta v_t(i) - \Delta v_t(i-1) = \beta(&\Delta v_{t-1}(i) - \Delta v_{t-1}(i-1)) \\ &+ \sum_{l=1}^{L}\lambda_t(l)(S_l(z_{t-1}(i,l)) - S_l(z_{t-1}(i-1,l))) \\ &+ \sum_{l=1}^{L}\lambda_t(l)(S_l(z_{t-1}(i-2,l)) - S_l(z_{t-1}(i-1,l))).\end{aligned}$$

Here since $z_{t-1}(i,l) \geq z_{t-1}(i-1,l)$ and $z_{t-1}(i-1,l) \geq z_{t-1}(i-2,l)$, from Lemma 3.2(a) and Lemma 3.2(f) we have

$$\begin{aligned}\Delta v_t(i) - \Delta v_t(i-1) &\geq \beta(\Delta v_{t-1}(i) - \Delta v_{t-1}(i-1)) + \sum_{l=1}^{L}\lambda_t(l)(S_l(z_{t-1}(i)) - S_l(z_{t-1}(i-1)) \\ &\geq \beta(\Delta v_{t-1}(i) - \Delta v_{t-1}(i-1)) + \sum_{l=1}^{L}\lambda_t(l)(z_{t-1}(i-1,l) - z_{t-1}(i,l)) \\ &= \beta(\Delta v_{t-1}(i) - \Delta v_{t-1}(i-1)) + \beta\sum_{l=1}^{L}\lambda_t(l)(\Delta v_{t-1}(i-1) - \Delta v_{t-1}(i)) \\ &= \beta\lambda_t(0)(\Delta v_{t-1}(i) - \Delta v_{t-1}(i-1)) \geq 0. \qquad \blacksquare\end{aligned}$$

Theorem 5.3 *If $\Delta\gamma(i)$ is nondecreasing in i, then $v_t(i)$ is nondecreasing in i for $t \geq 0$.*

proof : From Theorem 5.2(a), clearly $v_0(1) \geq v_0(0)$. For $i \geq 2$ we have

$$v_0(i) - v_0(i-1) = \gamma(i) - \gamma(i-1) + \sum_{l=1}^{L}\lambda_0(l)(S_l(\Delta\gamma(i) - c_l) - S_l(\Delta\gamma(i-1) - c_l)).$$

Here, since $\Delta\gamma(i)$ is nondecreasing in i and by the definition of $\gamma(0) = 0$, we have $\Delta\gamma(i) \geq 0$. For $i \geq 2$, since $\Delta\gamma(i) - c_l \geq \Delta\gamma(i-1) - c_l$, from Lemma 3.2(f) we have

$$\begin{aligned} v_0(i) - v_0(i-1) &\geq \Delta\gamma(i) + \sum_{l=1}^{L} \lambda_0(l)(\Delta\gamma(i-1) - \Delta\gamma(i)) \\ &\geq \lambda_0(0)\Delta\gamma(i) \geq 0. \end{aligned}$$

Hence, the assertion holds true for $t = 0$. Assume it holds true for $t-1$. Then, $v_{t-1}(i)$ is nondecreasing in i. Since we have $v_t(1) \geq 0 = v_t(0)$ from Theorem 5.2(a). And for $i \geq 2$ we have from (5.7)

$$v_t(i) - v_t(i-1) = \beta(v_{t-1}(i) - v_{t-1}(i-1)) + \sum_{l=1}^{L} \lambda_t(l)(S_l(z_{t-1}(i,l)) - S_l(z_{t-1}(i-1,l))).$$

Here, since $z_{t-1}(i,l) \geq z_{t-1}(i-1,l)$ from Lemma 5.2, we have from Lemma 3.2(f)

$$\begin{aligned} v_t(i) - v_t(i-1) &\geq \beta\Delta v_{t-1}(i) + \sum_{l=1}^{L} \lambda_t(l)(z_{t-1}(i-1,l) - z_{t-1}(i,l)) \\ &= \beta\Delta v_{t-1}(i) + \beta\sum_{l=1}^{L} \lambda_t(l)(\Delta v_{t-1}(i-1) - \Delta_{t-1}(i)) \\ &\geq \beta\lambda_t(0)\Delta v_{t-1}(i) \geq 0. \end{aligned}$$

Hence, the statement has been proven. ∎

Lemma 5.3 *If $\Delta\gamma(i)$ is nondecreasing in i, then $\Delta v_t(i)$, hence $z_t(i,l)$ is nonincreasing in t for all i.*

proof : Clear from Theorem 5.2(b), we have

$$\Delta v_t(1) - \Delta v_{t-1}(1) = v_t(1) - v_{t-1}(1) \leq 0.$$

For $i \geq 2$, we have

$$\Delta v_t(i) - \Delta v_{t-1}(i) = (\beta - 1)\Delta v_{t-1}(i) + \sum_{l=1}^{L} \lambda_t(l)(S_l(z_{t-1}(i,l)) - S_l(z_{t-1}(i-1,l))).$$

Here, since $z_{t-1}(i,l) \geq z_{t-1}(i-1,l)$ from Lemma 5.2, we have from Lemma 3.2(a) and Theorem 5.3

$$\Delta v_t(i) - \Delta v_{t-1}(i) \leq (\beta - 1)\Delta v_{t-1}(i) \leq 0. \quad ∎$$

Theorem 5.4 *If $\Delta\gamma(i)$ is nondecreasing in i, then*
(a) *$x_t(i)$ is nondecreasing in i for all $t \geq 0$.*
(b) *$x_t(i)$ is nonincreasing in t for all i.*

proof : (a) Immediate from Lemma 3.2(d) and Lemma 5.2.
(b) Immediate from Lemma 3.2(d) and Lemma 5.3. ∎

6 Numerical examples

Here, let us demonstrate the properties of optimal pricing policy by six numerical examples where Examples 1 to 4 are for profit maximization model and Examples 5 to 6 are for cost minimization model. In both models, it is assumed that there exist four classes of customers, and $P_l(\theta)$ and $Q_l(\theta)$ are both uniform distribution on $[a_l, b_l]$ shown in Table 3, and costs c_l are as in Table 4.

The values of $\lambda_t(l)$ are assumed to be such shown in Table 1 for Examples 1 to 3 and 5 to 6, and in Table 2 for Example 4. For $\alpha(i)$, let us assume

$\alpha(i) = i(685 + 15i)$ $(\Delta\alpha(i) = 670 + 30i)$ for Example 1,

$\alpha(i) = i(715 - 15i)$ $(\Delta\alpha(i) = 730 - 30i)$ for Example 2,3,4.

For $\gamma(i)$, let us assume

$\gamma(i) = i(2015 - 15i)$ $(\Delta\gamma(i) = 2030 - 30i)$ for Example 5,

$\gamma(i) = i(1985 + 15i)$ $(\Delta\gamma(i) = 1970 + 30i)$ for Example 6. Furthermore, For the discounted factor β, let us assume

$\beta = 1$ for Examples 3, 5,

$\beta = 0.98$ for Examples 1,2,4,6.

Table 1

time	0 ~15	16 ~ 30	31 ~ 45	46 ~ 60	61 ~ 75	76 ~ 90	91 ~ 100
$\lambda_t(1)$	0.20	0.14	0.07	0.15	0.10	0.18	0.22
$\lambda_t(2)$	0.18	0.15	0.12	0.18	0.13	0.17	0.19
$\lambda_t(3)$	0.15	0.12	0.07	0.18	0.12	0.18	0.16
$\lambda_t(4)$	0.16	0.13	0.10	0.20	0.15	0.19	0.17

Table 2

time	1 ~120
$\lambda_t(1)$	0.20
$\lambda_t(2)$	0.18
$\lambda_t(3)$	0.15
$\lambda_t(4)$	0.16

Table 3

class	a_l	b_l
$l = 1$	1200	1700
$l = 2$	900	1450
$l = 3$	800	1150
$l = 4$	750	920

Table 4

class	c_l
$l = 1$	250
$l = 2$	170
$l = 3$	100
$l = 4$	50

The optimal price $x_t(i, l)$ for these examples are depicted as in Figures 2 to 7.

- Example 1 ($\beta = 0.98$, $\Delta\alpha(i)$ is nondecreasing in i). Figure 2(a) shows that $x_t(5, 2)$ is not always nondecreasing in t. Figure 2(b) shows that $x_{20}(i, 2)$ is not always nonincreasing in i.
- Example 2 ($\beta = 0.98$, $\Delta\alpha(i)$ is nonincreasing in i). Figure 3(a) shows that $x_t(5, 2)$ is not always nondecreasing in t. Figure 3(b) shows that $x_{20}(i, 2)$ is nonincreasing in i.
- Example 3 ($\beta = 1$ and $\Delta\alpha(i)$ is nonincreasing in i). Figure 4(a) shows that $x_t(5, 2)$ is nondecreasing in t. Figure 4(b) shows that $x_{20}(i, 2)$ is nonincreasing in i.
- Example 4 ($\beta = 0.98$ and $\alpha(i) = 0$). Figure 5(a) shows that $x_t(5, 2)$ is nondecreasing in t. Figure 5(b) shows that $x_{20}(i, 2)$ is nonincreasing in i.
- Example 5 ($\beta = 1$, $\Delta\gamma(i)$ is nonincreasing in i). Figure 6(a) shows that $x_t(20, 2)$ is not always nonincreasing in t. Figure 6(b) shows that $x_{10}(i, 2)$ is not always nondecreasing in i.
- Example 6 ($\beta = 0.98$ and $\Delta\gamma(i) \geq 0$). Figure 7(a) shows that $x_t(20, 2)$ is nonincreasing in t. Figure 7(b) shows that $x_{10}(i, 2)$ is nondecreasing in i.

7 Conclusion

In the current paper, the optimal price is defined by $x_t(i,l)$, which stands for the price in which there are still t periods and i items remaining for the price taker of class l. The conclusions obtained here are summarized as follows:

1. It is always optimal to accept every arriving price taker (Theorem 4.1, 5.1).
2. The optimal price $x_t(i,l)$ in profit maximization problem can be summarized as in Table 5.

Table 5 Optimal Price $x_t(i,l)$

$\Delta\alpha(i)$ is nondecreasing in i	$\Delta\alpha(i)$ is nonincreasing in i		$\alpha(i)=0$ and $\lambda_t(l)$ is independent of t
	$0<\beta<1$	$\beta=1$	
not always monotone in t not always monotone in i	not always monotone in t nonincreasing in i	nondecreasing in t nonincreasing in i	nondecreasing in t nonincreasing in i

(1) If $\Delta\alpha(i)$ is nondecreasing in i, then $x_t(i,l)$ is not always monotone in t and i (Example 1, Figure 2(a,b)).

(2) If $\Delta\alpha(i)$ is nonincreasing in i and $0<\beta<1$, then $x_t(i,l)$ is not always monotone in t, but always nonincreasing in i (Theorem 4.3(a), Example 2, Figure 3(a,b)).

(3) If $\Delta\alpha(i)$ is nonincreasing in i and $\beta=1$, then $x_t(i,l)$ is nondecreasing in t and nonincreasing in i (Theorem 4.3(a,b), Example 3, Figure 4(a,b)).

(4) If $\alpha(i)=0$ and $\lambda_t(l)=\lambda(l)$, then $x_t(i,l)$ is nondecreasing in t and nonincreasing in i (Theorem 4.3(a,c), Example 4, Figure 5(a,b)).

(5) It may happen that, although a customer has being accepted, the price taker may decide to reject the offered price. This would imply that the price being offered by the searcher has being set too high. Theoretically, this occurs when $b_l \le z_t(i,l)$; in this case $x_t(i,l)=b_l$ from Lemma 4.1. Under this condition, even if a customer is accepted, the deal will not go through. Therefore, when $b_l \le z_t(i,l)$, the searcher can decide to reject a price taker of class l arbitrarily before making a price offer.

3. The optimal price $x_t(i,l)$ in cost minimization model can be summarized as in Table 6.

Table 6 Optimal Price $x_t(i,l)$

$\Delta\alpha(i)$ is nonincreasing in i	$\Delta\alpha(i)$ is nondecreasing in i
not always monotone in t not always monotone in i	nonincreasing in t nondecreasing in i

(1) If $\Delta\gamma(i)$ is nonincreasing in i, then $x_t(i,l)$ is not always monotone in t and i (Example 5, Figure 6(a,b)).

(2) If $\Delta\gamma(i)$ is nondecreasing in i and $\Delta\gamma(i)\ge 0$, then $x_t(i,l)$ is nonincreasing in t and nondecreasing in i (Theorem 5.4, Example 6, Figure 7(a,b)).

(3) It may happen that, although a price taker has being accepted, the price taker may decide to reject the offered price. This would imply that the price being offered by searcher has being set too low. Theoretically, this occurs when $z_t(i,l)\le a_l$; in this case $x_t(i,l)=a_l$ from Lemma 5.1. Under this condition, even if a price taker is accepted, the deal will not go through. Therefore, when $z_t(i,l)\le a_l$, the searcher can decide to reject a price taker of class l arbitrarily before making a price offer.

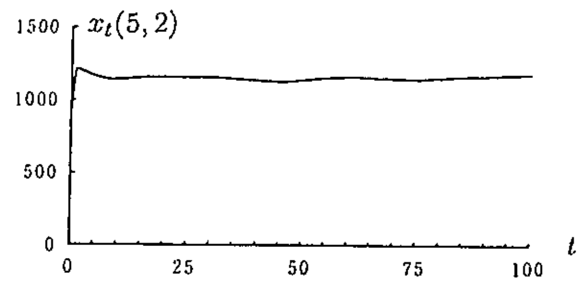

Figure 2(a) $\beta = 0.98$
Optimal price is not always monotone in t

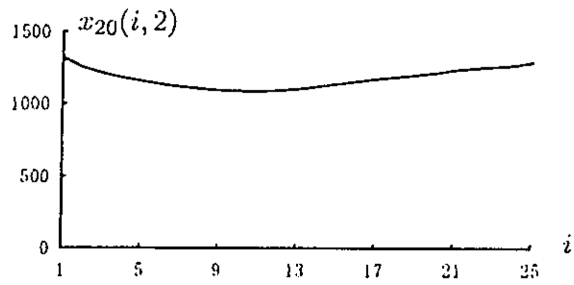

Figure 2(b) $\beta = 0.98$
Optimal price is not always monotone in i

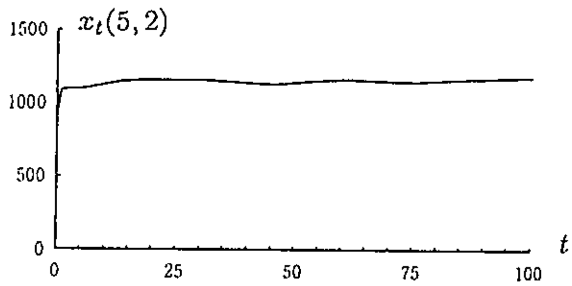

Figure 3(a) $\beta = 0.98$
Optimal price is not always monotone in t

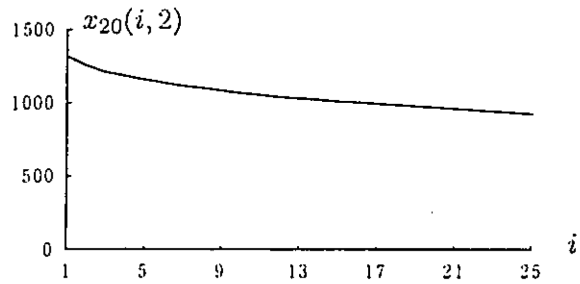

Figure 3(b) $\beta = 0.98$
Optimal price is nonincreasing in i
(Theorem 4.3(a))

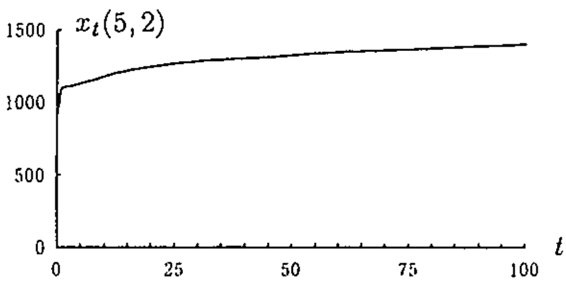

Figure 4(a) $\beta = 1$
Optimal price is nondecreasing in t
(Theorem 4.3(b))

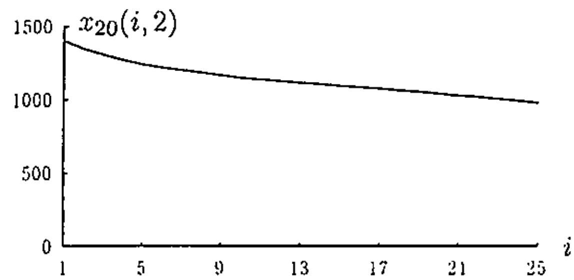

Figure 4(b) $\beta = 1$
Optimal price is nonincreasing in i
(Theorem 4.3(a))

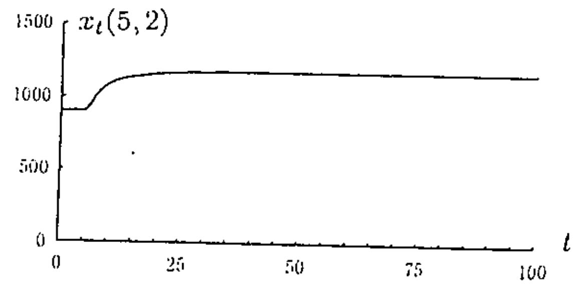

Figure 5(a) $\beta = 0.98$
Optimal price is nondecreasing in t
(Theorem 4.3(c))

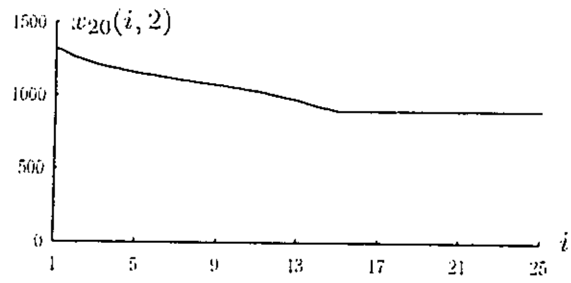

Figure 5(b) $\beta = 0.98$
Optimal price is nonincreasing in i
(Theorem 4.3(a))

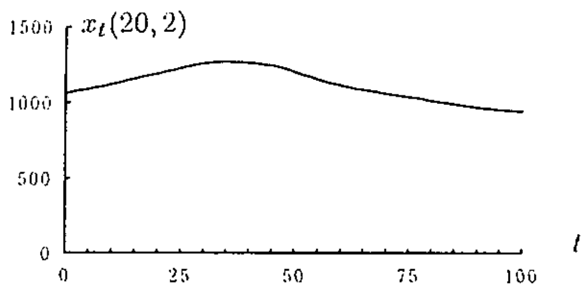

Figure 6(a) $\beta = 1$
Optimal price is not always monotone in t

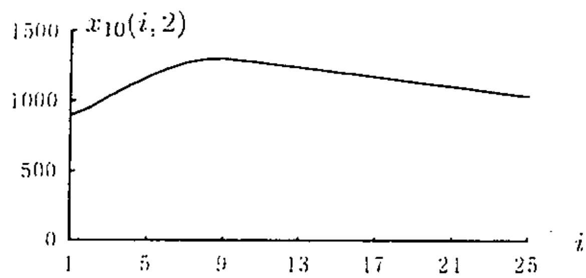

Figure 6(b) $\beta = 1$
Optimal price is not always monotone in i

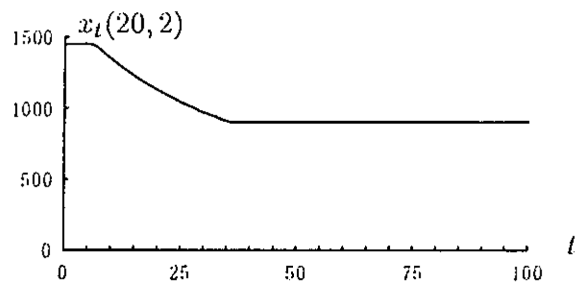

Figure 7(a) $\beta = 0.98$
Optimal price is nonincreasing in t
(Theorem 5.4(b))

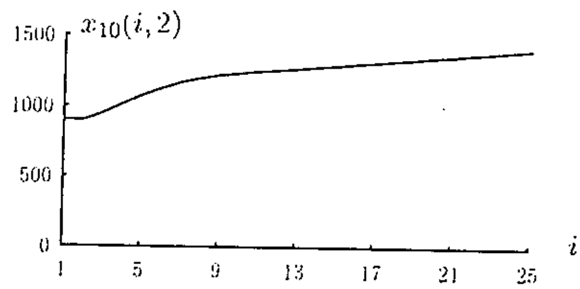

Figure 7(b) $\beta = 0.98$
Optimal price is nondecreasing in i
(Theorem 5.4(a))

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