No. **646**

Symbolic Higher-Order Moments of the Waiting Time in an M/G/1 Queue with Random Order of Service

by

Hideaki Takagi and Seiya Kudoh

October 1995

Symbolic Higher-Order Moments of the Waiting Time in an M/G/1 Queue with Random Order of Service

Hideaki Takagi
Institute of Socio-Economic Planning, University of Tsukuba
1-1-1 Tennoudai, Tsukuba-shi, Ibaraki 305, Japan
and
Seiya Kudoh
Doctoral Program in Socio-Economic Planning, University of Tsukuba
1-1-1 Tennoudai, Tsukuba-shi, Ibaraki 305, Japan

The explicit expressions for the mean and the second moment of the waiting time distribution for an M/G/1 queue with random order of service can be found in the literature. However, higher-order moments have been unavailable, because their calculation is extremely laborious, albeit straightforward in principle. This paper presents the third through sixth order moments of the waiting time in symbolic form, which is obtained by a *Mathematica* program developed for formula manipulation. The results provide more information about the characteristic of the waiting time distribution. In particular, they are used to compare the skewness and kurtosis of the waiting time with those for the first-come first-served and last-come first-served systems.

Key Words: Queue, M/G/1, waiting time, higher-order moments, random order of service, symbolic calculation, *Mathematica.*

1. Introduction

In M/G/1 queueing systems, the mean waiting time of a customer does not depend on the order of service, such as first-come first-served (FCFS), last-come first-served (LCFS), and random order of service (ROS) [8, sec. 3.4]. However, the distribution of the waiting time *does* depend on the order of service, so do the second and higher-order moments. While the symbolic calculation of these moments are relatively easy for FCFS and LCFS systems, it is not the case for an ROS system due to an extremely involved calculation. This is why only the second moment of the waiting time is available in the literature. Higher-order moments would be useful to characterize the waiting time distribution more clearly, for example, in terms of skewness and kurtosis.

This paper presents the explicit symbolic expressions for third to sixth order moments of the waiting time in an ROS M/G/1 queue. They are obtained by a *Mathematica* [12] program developed for symbolic formula manipulation. The program implements a procedure described by Conolly [4, sec. 5.3.5], who remarks that "This is *extremely* laborious, and though straightforward in principle, it is almost inevitable that errors are repeatedly committed." Conolly's procedure is based on the Laplace-Stieltjes transform of the distribution function for the waiting time originally derived by Kingman [7]. See also Cohen [3, sec. III.3.3], Takács [9], and Takagi [10, sec. 1.3] for the description of Kingman's formula. Alternatively, Burke [1] and Carter and Cooper [2] (see also Cooper [5, sec. 5.12]) show approaches which do not use transform methods. Fuhrmann [6] shows interesting relations among the waiting time moments for FCFS, LCFS, and ROS systems. But his arguments apply only to up to the second moment. The results in this paper can be compared with the corresponding results for FCFS and LCFS systems recently obtained up to the 10th order by Takagi and Sakamaki [11].

The rest of this paper is organized as follows. In Section 2, we define our model and notation, and then show Kingman's formulation. Section 3 provides a *Mathematica* program that calculates the moments of the waiting time, together with annotation on its algorithm. The complete symbolic expressions for the third to sixth order moments of the waiting time are presented in Section 4. Section 5 displays the squared coefficient of variation, the skewness, and the kurtosis of the waiting time distribution in FCFS, LCFS, and ROS systems. The program in this paper was written and executed using *Mathematica* Version 2.2 for SPARC from Wolfram Research, Inc. [12].

2. Model and Formulation

Let us introduce our model and associated notation specifically. We consider an M/G/1 queueing system with an infinite capacity. The rate of a Poisson arrival process is denoted by λ. The Laplace-Stieltjes transform (LST) of the distribution function (DF), the mean, and the nth moment of generally distributed service times are denoted by $B^*(s)$, b, and $b^{(n)}$ $(n = 2, 3, \ldots)$, respectively. The service discipline is random-order-of-service (ROS), that is, every customer present in the queue at the end of each service can be selected for the next service with equal probability. The traffic intensity is given by $\rho := \lambda b$, which is assumed to be less than unity for the stability of the system.

The waiting time W of an arbitrary customer is defined as the time interval from its arrival to the service start. The mean and the second moment of the waiting time for W is already available:

$$E[W] = \frac{\lambda b^{(2)}}{2(1-\rho)} \tag{2.1}$$

$$E[W^2] = \frac{2\lambda b^{(3)}}{3(1-\rho)(2-\rho)} + \frac{[\lambda b^{(2)}]^2}{(1-\rho)^2(2-\rho)} \tag{2.2}$$

According to Kingman [7] (also in [3, 9, 10]), the LST $W^*(s)$ of the DF for the waiting time W is given by

$$W^*(s) = 1 - \rho + \frac{\lambda(1-\rho)}{s} \int_{\Theta^*(s)}^{1} \frac{(1-z)[B^*(\lambda - \lambda z) - B^*(s + \lambda - \lambda z)]}{B^*(\lambda - \lambda z) - z} dK(z, s) \tag{2.3}$$

where

$$K(z, s) := \exp\left[-\int_z^1 \frac{du}{u - B^*(s + \lambda - \lambda u)}\right] \tag{2.4}$$

and $\Theta^*(s)$ is the LST of the DF for the length of a busy period, which is given as the unique solution to the equation

$$\Theta^*(s) = B^*[s + \lambda - \lambda\Theta^*(s)] \tag{2.5}$$

The procedure to calculate the moments $E[W^n]$ $(n = 1, 2, \ldots)$ is simple in principle; they can be obtained as the coefficients of the expansion of $W^*(s)$ in (2.3) in powers of s.

This procedure is described by Conolly [4, sec. 5.3.5], but he was able to yield only up to the second moment as given in (2.2). In the next section, we will show a *Mathematica* program that implements his procedure. An alternative way to calculate the moments is provided by

Takács [9], who uses recursive relations for a set of LSTs of the DF for the waiting time of an arbitrary customer conditioned on the number of other customers present in the system at the epoch of service start. This method is more tractable for manual calculation, but seems less amenable to computer program.

3. Mathematica Program for Conolly's Procedure

A *Mathematica* program shown in Figure 1 calculates the moments of the waiting time from the expansion of $W^*(s)$ in (2.3) in powers of s, basically following Conolly [4, sec. 5.3.3].

For this purpose, let us rewrite the integration part in (2.3) as

$$\begin{aligned}\Psi^*(s) &= \int_{\Theta^*(s)}^{1} \frac{(1-z)[B^*(\lambda-\lambda z) - B^*(s+\lambda-\lambda z)]}{B^*(\lambda-\lambda z)-z} dK(z,s) \\ &= \int_{\Theta^*(s)}^{1} K(z,s)\left[1+\frac{1-z}{B^*(\lambda-\lambda z)-z}\right] dz \qquad (3.1)\end{aligned}$$

The first paragraph of the program provides preliminaries. We obtain the nth moment of the busy period length Θ in `busy[n]`, using the method by Takács [9] (see [11] for explicit expressions for $\{E[\Theta^n]; n = 1\text{–}10\}$).

The second part of the program deals with the expression in the brackets in (3.1), which is expanded in powers of $1-z$ as

$$1+\frac{1-z}{B^*(\lambda-\lambda z)-z} = \sum_{n=0}^{\infty} r_n(1-z)^n \qquad (3.2)$$

The coefficients $\{r_n; n = 0, 1, 2, \ldots\}$ are obtained in `g3[n]`. We now change the variable in (3.1) from z to y defined by

$$y = \frac{z-\Theta^*(s)}{1-\Theta^*(s)} \qquad (3.3)$$

Noting

$$\begin{aligned}z &= (1-\Theta^*(s))y + \Theta^*(s) \\ 1-z &= (1-\Theta^*(s))(1-y) \\ dz &= (1-\Theta^*(s))dy \qquad (3.4)\end{aligned}$$

we get

$$\Psi^*(s) = \int_0^1 K\left[(1-\Theta^*(s))y+\Theta^*(s), s\right]\left[\sum_{n=0}^{\infty} r_n(1-\Theta^*(s))^n Y^n\right] dy \qquad (3.5)$$

where $Y := 1-y$. Thus we can obtain the power series expansion of $1+\dfrac{1-z}{B^*(\lambda-\lambda z)-z}$ in s. The coefficients are polynomials in Y stored in `g6[n]`.

The third paragraph is concerned with the power series expansion of $K\left[(1-\Theta^*(s))\,y+\Theta^*(s),s\right]$ around $s=0$. In the equation

$$-\log K\left[(1-\Theta^*(s))\,y+\Theta^*(s),s\right] \;=\; \int_{\Theta(s)+(1-\Theta^*(s))y}^{1} \frac{du}{u-B^*(s+\lambda-\lambda u)} \tag{3.6}$$

we transform the variable from u to v by

$$v=\frac{u-\Theta^*(s)-y\,(1-\Theta^*(s))}{(1-\Theta^*(s))\,(1-y)} \tag{3.7}$$

If V denotes $1-v$, then

$$\begin{aligned} u &= \Theta^*(s)+y\,(1-\Theta^*(s))+(1-\Theta^*(s))\,Yv=1-YV\,(1-\Theta^*(s)) \\ \lambda-\lambda u &= \lambda\,(1-\Theta^*(s))\,YV \\ du &= (1-\Theta^*(s))\,Y dv \end{aligned} \tag{3.8}$$

Thus we get

$$-\log K\left[(1-\Theta^*(s))\,y+\Theta^*(s),s\right] \;=\; Y\,(1-\Theta^*(s))\int_0^1 \frac{dv}{\Delta} \tag{3.9}$$

where

$$\Delta := 1-YV\,(1-\Theta^*(s))-B^*[s+\lambda YV\,(1-\Theta^*(s))] \tag{3.10}$$

We now expand Δ around $s=0$. However, when $s\to 0$, $1-\Theta^*(s)\to 0$. So we use $w=YV\,(1-\Theta^*(s))$. Since $B^*(s+\lambda w)=\sum_{n=0}^{\infty}\frac{B^{*(n)}(0)}{n!}(s+\lambda w)^n$, we have

$$\Delta=-w+\sum_{n=1}^{\infty}\frac{B^{*(n)}(0)}{n!}(s+\lambda w)^n \tag{3.11}$$

as in `f2[n]`. Further we expand $(s+\lambda w)^n$ in `f3[n]`. Hence we obtain Δ as a power series in s whose coefficients are given in `f6[n]`. Note that Δ^{-1} is also a polynomial of s. The coefficients are given in `f8[n]`.

We proceed to calculate $K\left[(1-\Theta^*(s))\,y+\Theta^*(s),s\right]$ as

$$\begin{aligned} K\left[(1-\Theta^*(s))\,y+\Theta^*(s),s\right] &= \exp\left[-(1-\Theta^*(s))\,Y\int_0^1\frac{dv}{\Delta}\right] \\ &= \exp\left[s\sum_{m=1}^{\infty}\frac{E\left[\Theta_m\right]}{m!}s^m\left(\frac{-1}{bs}\right)\left\{\log[1-y]+\sum_{n=1}^{\infty}\Delta_n s^n\right\}\right] \\ &= (1-Y)^{E[\Theta_1]/b}\sum_{n=0}^{\infty}k_n(Y)s^n \end{aligned} \tag{3.12}$$

where the coefficients $\{\Delta_n; n=1,2,3\ldots\}$ are obtained in `f14[n]`, and the coefficient $\{k_n(Y); n=0,1,2\ldots\}$ are given in `f16[n]`. Note that we use notation y, v, and `g4[n]` for Y, V, and $1-\Theta^*(s)$, respectively, in the program.

We are now in a position to calculate the power series expansion of $\Psi^*(s)$ as

$$\begin{aligned} \Psi^*(s) &= \int_0^1 (1-Y)^{E[\Theta_1]/b} \left[\sum_{n=0}^{\infty} k_n(Y)s^n\right] \left[\sum_{n=0}^{\infty} r_n(1-\Theta^*(s))^n Y^n\right] dy \\ &= \sum_{n=1}^{\infty} \psi_n s^n \end{aligned} \tag{3.13}$$

The coefficients $\{\psi_n; n = 1, 2, \ldots\}$ are obtained `h1[n]` in the fifth paragraph.

However, *Mathematica* cannot evaluate the improper integrals in the above. So we unprotect the `Integrate` command, and define a new command, `integrate`. For example,

$$\int_0^1 (1-Y)^{E[\Theta_1]/b} y^m (\log[1-y])^n dy, \tag{3.14}$$

is given by `integrate[y^m (Log[1 - y])^n,y]`. Other integrals are calculated in the same way. This is shown in the fourth paragraph. Finally we can obtain $E[W^n]$ by

$$W^*(s) = (1-\rho)\left\{1 + \frac{\lambda}{s}\Psi^*(s)\right\} \tag{3.15}$$

The moments are stored in `h2[n]`, and to transformed into the LaTeX form in `h3[n]`.

4. Third through Sixth Order Moments of the Waiting Time

We now present the third through sixth order moments of the waiting time distribution in the ROS M/G/1 system obtained by the *Mathematica* program in Figure 1.

$$E[W^3] = \frac{\lambda b^{(4)}(6-\rho+\rho^2)}{4(1-\rho)(2-\rho)(3-2\rho)} + \frac{3[\lambda b^{(2)}]^3(18-15\rho+4\rho^2-\rho^3)}{4(1-\rho)^3(2-\rho)^2(3-2\rho)} + \frac{\lambda^2 b^{(2)} b^{(3)}(30-23\rho+7\rho^2-2\rho^3)}{2(1-\rho)^2(2-\rho)^2(3-2\rho)} \tag{4.1}$$

$$\begin{aligned} E[W^4] &= \frac{2\lambda b^{(5)}(12-5\rho+5\rho^2)}{5(2-\rho)(1-\rho)(3-2\rho)(4-3\rho)} + \frac{3[\lambda b^{(2)}]^4(90-135\rho+76\rho^2-23\rho^3+4\rho^4)}{(1-\rho)^4(2-\rho)^3(3-2\rho)^2} \\ &+ \frac{2\lambda^3 {b^{(2)}}^2 b^{(3)}(864-1896\rho+1668\rho^2-799\rho^3+233\rho^4-34\rho^5)}{(1-\rho)^3(2-\rho)^3(3-2\rho)^2(4-3\rho)} + \frac{4[\lambda b^{(3)}]^2(12-13\rho+7\rho^2-2\rho^3)}{(1-\rho)^2(2-\rho)^2(3-2\rho)(4-3\rho)} \\ &+ \frac{\lambda^2 b^{(2)} b^{(4)}(216-342\rho+219\rho^2-89\rho^3+20\rho^4)}{(1-\rho)^2(2-\rho)^2(3-2\rho)^2(4-3\rho)} \end{aligned} \tag{4.2}$$

$$\begin{aligned} E[W^5] &= \frac{\lambda b^{(6)}(120-86\rho+97\rho^2-19\rho^3+8\rho^4)}{6(1-\rho)(2-\rho)(3-2\rho)(4-3\rho)(5-4\rho)} \\ &+ \frac{15[\lambda b^{(2)}]^5(4032-11688\rho+14302\rho^2-9945\rho^3+4545\rho^4-1475\rho^5+333\rho^6-48\rho^7+4\rho^8)}{2(1-\rho)^5(2-\rho)^4(3-2\rho)^3(4-3\rho)} \\ &+ \frac{5\lambda^4 {b^{(2)}}^3 b^{(3)}(498240-2183712\rho+4208116\rho^2-4744636\rho^3+3531819\rho^4-1860877\rho^5+719917\rho^6-204359\rho^7+41008\rho^8-5420\rho^9+384\rho^{10})}{2(1-\rho)^4(2-\rho)^4(3-2\rho)^3(4-3\rho)^2(5-4\rho)} \\ &+ \frac{5\lambda^3 b^{(2)} {b^{(3)}}^2(44640-142752\rho+197582\rho^2-160485\rho^3+87173\rho^4-33021\rho^5+8537\rho^6-1458\rho^7+144\rho^8)}{3(1-\rho)^3(2-\rho)^3(3-2\rho)^2(4-3\rho)^2(5-4\rho)} \\ &+ \frac{\lambda^3 {b^{(2)}}^2 b^{(4)}(69120-261216\rho+430896\rho^2-418578\rho^3+274679\rho^4-130517\rho^5+45028\rho^6-10768\rho^7+1680\rho^8-144\rho^9)}{2(1-\rho)^3(2-\rho)^3(3-2\rho)^3(4-3\rho)^2(5-4\rho)} \\ &+ \frac{5\lambda^2 b^{(3)} b^{(4)}(2520-4926\rho+4453\rho^2-2540\rho^3+903\rho^4-202\rho^5+32\rho^6)}{6(1-\rho)^2(2-\rho)^2(3-2\rho)^2(4-3\rho)(5-4\rho)} \\ &+ \frac{\lambda^2 b^{(2)} b^{(5)}(10080-25344\rho+28414\rho^2-20727\rho^3+10773\rho^4-3600\rho^5+740\rho^6-96\rho^7)}{2(1-\rho)^2(2-\rho)^2(3-2\rho)^2(4-3\rho)^2(-5+4\rho)} \end{aligned} \tag{4.3}$$

$$\tag{4.4}$$

$$
\begin{aligned}
E[W^6] &= \frac{2\lambda b^{(7)}(360-378\rho+497\rho^2-210\rho^3+91\rho^4)}{7(1-\rho)(2-\rho)(3-2\rho)(4-3\rho)(5-4\rho)(6-5\rho)} \\
&+ \frac{90[\lambda b^{(2)}]^6(48672-206688\rho+389114\rho^2-433067\rho^3+322236\rho^4-172099\rho^5+68371\rho^6-20094\rho^7+4157\rho^8-548\rho^9+36\rho^{10})}{(1-\rho)^6(2-\rho)^5(3-2\rho)^4(4-3\rho)^2} \\
&+ \frac{15\lambda^5 {b^{(2)}}^4 b^{(3)}}{(1-\rho)^5(2-\rho)^5(3-2\rho)^4(4-3\rho)^3(5-4\rho)^2(6-5\rho)} \cdot [440294400-3240829440\rho+10966193856\rho^2-22675140256\rho^3 \\
&+ 32153957512\rho^4-33293853196\rho^5+26178876994\rho^6-16029385309\rho^7+7746765097\rho^8 \\
&- 2957836689\rho^9+880179597\rho^{10}-197896410\rho^{11}+31786788\rho^{12}-3275592\rho^{13}+164448\rho^{14}] \\
&+ \frac{10[\lambda^2 b^{(2)} b^{(3)}]^2}{(1-\rho)^4(2-\rho)^4(3-2\rho)^3(4-3\rho)^3(5-4\rho)^2(6-5\rho)} \cdot [62035200-382230720\rho+1072369968\rho^2-1825112436\rho^3 \\
&+ 2121168028\rho^4-1796703705\rho^5+1151947527\rho^6-568365051\rho^7+214854225\rho^8-60624996\rho^9+12111584\rho^{10}-1542792\rho^{11}+95328\rho^{12}] \\
&+ \frac{20\lambda^3 {b^{(3)}}^3(18720-65376\rho+103610\rho^2-101827\rho^3+69850\rho^4-34106\rho^5+11370\rho^6-2355\rho^7+234\rho^8)}{(1-\rho)^3(2-\rho)^3(3-2\rho)^2(4-3\rho)^2(5-4\rho)(6-5\rho)} \\
&+ \frac{15\lambda^4 {b^{(2)}}^3 b^{(4)}}{(1-\rho)^4(2-\rho)^4(3-2\rho)^4(4-3\rho)^3(5-4\rho)^2(6-5\rho)} \cdot [64281600-433848960\rho+1340086464\rho^2-2522671896\rho^3+3256829928\rho^4 \\
&- 3079739910\rho^5+2222252366\rho^6-1252167821\rho^7+554087948\rho^8-190178753\rho^9+49086518\rho^{10}-9016676\rho^{11}+1060680\rho^{12}-60768\rho^{13}] \\
&+ \frac{20\lambda^3 b^{(2)} b^{(3)} b^{(4)}}{(1-\rho)^3(2-\rho)^3(3-2\rho)^3(4-3\rho)^2(5-4\rho)^2(6-5\rho)}[1296000-6307200\rho+13885200\rho^2-18571044\rho^3 \\
&+ 17094456\rho^4-11521391\rho^5+5795769\rho^6-2142557\rho^7+555537\rho^8-91550\rho^9+7320\rho^{10}] \\
&+ \frac{5[\lambda b^{(4)}]^2}{(1-\rho)^6(2-\rho)^5(3-2\rho)^4(4-3\rho)^3(5-4\rho)^2(6-5\rho)} \cdot [12441600-141212160\rho+755097984\rho^2-2532909024\rho^3+5987900016\rho^4 \\
&+ 14711812352\rho^6-16306235654\rho^7+14710950431\rho^8-10911297764\rho^9+6681799715\rho^{10}-3373602580\rho^{11}+1394122825\rho^{12}-464850868\rho^{13} \\
&+ 122162113\rho^{14}-24373942\rho^{15}+3469828\rho^{16}-313800\rho^{17}+13536\rho^{18}] \\
&+ \frac{3\lambda^3 {b^{(2)}}^2 b^{(5)}}{(1-\rho)^3(2-\rho)^3(3-2\rho)^3(4-3\rho)^3(5-4\rho)^2(6-5\rho)} \cdot [10713600-58527360\rho+145138464\rho^2-219478904\rho^3+230037628\rho^4 \\
&- 179175516\rho^5+106843020\rho^6-48557035\rho^7+16327775\rho^8-3860220\rho^9+582684\rho^{10}-43056\rho^{11}] \\
&+ \frac{2\lambda^2 b^{(3)} b^{(5)}(34560-101088\rho+140592\rho^2-128262\rho^3+81991\rho^4-35319\rho^5+9494\rho^6-1248\rho^7)}{(1-\rho)^2(2-\rho)^2(3-2\rho)^2(4-3\rho)^2(5-4\rho)(6-5\rho)} \\
&+ \frac{2\lambda^2 b^{(2)} b^{(6)}(86400-303840\rho+496056\rho^2-524114\rho^3+397787\rho^4-213717\rho^5+77818\rho^6-18214\rho^7+2184\rho^8)}{[(1-\rho)(2-\rho)(3-2\rho)(4-3\rho)(5-4\rho)]^2(6-5\rho)} \qquad (4.5)
\end{aligned}
$$

5. Comparison with FCFS and LCFS Systems

Using the above results, we compare the squared coefficient of variation (s.c.v.), the skewness, and the kurtosis of waiting time distribution in the FCFS, LCFS, and ROS systems. These quantities characterize the distribution.

According to the previous work [9,11], the second to the fourth moments of the waiting time in the FCFS and LCFS systems are given by

$$E[W^2]_{\mathrm{FCFS}} = \frac{[\lambda b^{(2)}]^2}{2(1-\rho)^2} + \frac{\lambda b^{(3)}}{3(1-\rho)} \qquad (5.1)$$

$$E[W^3]_{\mathrm{FCFS}} = \frac{3[\lambda b^{(2)}]^3}{4(1-\rho)^3} + \frac{\lambda^2 b^{(2)} b^{(3)}}{(1-\rho)^2} + \frac{\lambda b^{(4)}}{4(1-\rho)} \qquad (5.2)$$

$$E[W^4]_{\mathrm{FCFS}} = \frac{3[\lambda b^{(2)}]^4}{2(1-\rho)^4} + \frac{3\lambda^3 {b^{(2)}}^2 b^{(3)}}{(1-\rho)^3} + \frac{2[\lambda b^{(3)}]^2}{3(1-\rho)^2} + \frac{\lambda^2 b^{(2)} b^{(4)}}{(1-\rho)^2} + \frac{\lambda b^{(5)}}{5(1-\rho)} \qquad (5.3)$$

$$E[W^2]_{\mathrm{LCFS}} = \frac{[\lambda b^{(2)}]^2}{2(1-\rho)^3} + \frac{\lambda b^{(3)}}{3(1-\rho)^2} \qquad (5.4)$$

$$E[W^3]_{\mathrm{LCFS}} = \frac{3[\lambda b^{(2)}]^3}{2(1-\rho)^5} + \frac{3\lambda^2 b^{(2)} b^{(3)}}{2(1-\rho)^4} + \frac{\lambda b^{(4)}}{4(1-\rho)^3} \qquad (5.5)$$

$$E[W^4]_{\mathrm{LCFS}} = \frac{15[\lambda b^{(2)}]^4}{2(1-\rho)^7} + \frac{10\lambda^3 {b^{(2)}}^2 b^{(3)}}{(1-\rho)^6} + \frac{4[\lambda b^{(3)}]^2}{3(1-\rho)^5} + \frac{2\lambda^2 b^{(2)} b^{(4)}}{(1-\rho)^5} + \frac{\lambda b^{(5)}}{5(1-\rho)^4} \quad (5.6)$$

We can thus evaluate

$$\text{s.c.v.} = \frac{\mathrm{Var}[W]}{(E[W])^2}$$

$$\text{skewness} = \frac{E\left[(W - E[W])^3\right]}{\{\mathrm{Var[W]}\}^{\frac{3}{2}}}$$

$$\text{kurtosis} = \frac{E\left[(W - E[W])^4\right]}{\{\mathrm{Var[W]}\}^2}$$

where $\mathrm{Var}[W] = E\left[(W - E[W])^2\right] = E[W^2] - (E[W])^2$. Assuming a constant time of unit duration for service times, $B^*(s) = e^{-s}$, we have calculated the s.c.v., skewness and kurtosis of the waiting time in FCFS, LCFS, and ROS systems, and plotted them against ρ in Figures 2(a), 3(a), and 4(a), respectively. Similar values for the case of exponentially distributed service times with unit mean are plotted in Figures 2(b), 3(b), and 4(b).

In these figures, we observe the following:

- In the three systems, the values of s.c.v., skewness, and kurtosis diverge as λ approaches zero.
- In the FCFS and ROS systems, those values converge when λ approaches one.
- In the LCFS system, those values diverge when λ approaches one.
- The curves for the ROS system lie between those for the FCFS system and for the LCFS system.

In the limit $\lambda \to 0$, the order of service makes little difference because the system is almost always empty when a customer arrives. In fact, we see that

$$E[W^n] = \frac{\lambda b^{(n+1)}}{n+1} + O(\lambda^2) \qquad n = 1, 2, \ldots \quad (5.7)$$

for the three systems. Thus, for the constant service time we have $E[W^n] \approx \lambda/(n+1)$ so that

$$\text{s.c.v.} = \frac{4}{3\lambda} + O(1) \quad (5.8)$$

$$\text{skewness} = \frac{3\sqrt{3}}{4\sqrt{\lambda}} + O(\sqrt{\lambda}) \quad (5.9)$$

$$\text{kurtosis} = \frac{9}{5\lambda} + O(1) \quad (5.10)$$

For the exponentially distributed service time we have $E[W^n] \approx \lambda n!$ so that

$$\text{s.c.v.} = \frac{2}{\lambda} + O(1) \quad (5.11)$$

$$\text{skewness} = \frac{3}{\sqrt{2\lambda}} + O(\sqrt{\lambda}) \quad (5.12)$$

$$\text{kurtosis} = \frac{6}{\lambda} + O(1) \quad (5.13)$$

These explain the divergence as λ approaches zero in Figures 2–4.

When ρ approaches one, the dominant terms in the waiting time moments for the FCFS system are the following:

$$E[W] = \frac{\lambda b^{(2)}}{2(1-\rho)} \tag{5.14}$$

$$E[W^2]_{\text{FCFS}} \approx \frac{[\lambda b^{(2)}]^2}{2(1-\rho)^2} \tag{5.15}$$

$$E[W^3]_{\text{FCFS}} \approx \frac{3[\lambda b^{(2)}]^3}{4(1-\rho)^3} \tag{5.16}$$

$$E[W^4]_{\text{FCFS}} \approx \frac{3[\lambda b^{(2)}]^4}{2(1-\rho)^4} \tag{5.17}$$

Therefore, we get

$$\text{s.c.v.} \approx \frac{\frac{[\lambda b^{(2)}]^2}{2(1-\rho)^2} - \left(\frac{\lambda b^{(2)}}{2(1-\rho)}\right)^2}{\left(\frac{\lambda b^{(2)}}{2(1-\rho)}\right)^2} = 1 \tag{5.18}$$

This implies that the s.c.v. does not depend on the distribution of service time. Similarly, we have

$$\text{skewness} = 2 \tag{5.19}$$

$$\text{kurtosis} = 9 \tag{5.20}$$

For the LCFS system, the dominant terms in the waiting time moments are as follows.

$$E[W^2]_{\text{LCFS}} \approx \frac{[\lambda b^{(2)}]^2}{2(1-\rho)^3} \tag{5.21}$$

$$E[W^3]_{\text{LCFS}} \approx \frac{3[\lambda b^{(2)}]^3}{2(1-\rho)^5} \tag{5.22}$$

$$E[W^4]_{\text{LCFS}} \approx \frac{15[\lambda b^{(2)}]^4}{2(1-\rho)^7} \tag{5.23}$$

Thus we have divergence

$$\text{s.c.v.} \approx \frac{2}{1-\rho} \tag{5.24}$$

$$\text{skewness} \approx \frac{3\sqrt{2}}{(1-\rho)^{1/2}} \tag{5.25}$$

$$\text{kurtosis} \approx \frac{30}{1-\rho} \tag{5.26}$$

as shown in the figures. Finally for the ROS system, the dominant terms as $\rho \to 1$ are given by

$$E[W^2]_{\text{ROS}} \approx \frac{[\lambda b^{(2)}]^2}{(1-\rho)^2} \tag{5.27}$$

$$E[W^3]_{\text{ROS}} \approx \frac{9[\lambda b^{(2)}]^3}{2(1-\rho)^3} \tag{5.28}$$

$$E[W^4]_{\text{ROS}} \approx \frac{36[\lambda b^{(2)}]^4}{(1-\rho)^4} \tag{5.29}$$

which yield

$$\begin{aligned} \text{s.c.v.} &= 3 && (5.30)\\ \text{skewness} &= \frac{26}{3\sqrt{3}} && (5.31)\\ \text{kurtosis} &= \frac{151}{3} && (5.32) \end{aligned}$$

We note also for the ROS system that these coefficients do not depend on the service time as ρ approaches one.

Another interesting observation is that the dominant terms in the waiting time moments as $\rho \to 1$ have the forms

$$\begin{aligned} E[W^n] &\propto \frac{[\lambda b^{(2)}]^n}{(1-\rho)^n} \quad \text{for the FCFS and ROS systems} && (5.33)\\ E[W^n] &\propto \frac{[\lambda b^{(2)}]^n}{(1-\rho)^{2n-1}} \quad \text{for the LCFS systems} && (5.34) \end{aligned}$$

which depend only on the first and second moments of the service time.

Acknowledgment

This work is supported in part by the Telecommunications Advancement Foundation for the fiscal year 1995.

References

[1] Burke, P. J. Equilibrium delay distribution for one channel with constant holding time, Poisson input and random service. *The Bell System Technical Journal*, Vol.38, pp.1021–1031, July 1959.

[2] Carter, G. M., and Cooper, R. B. Queues with service in random order. *Operations Research*, Vol.20, No.2, pp.389–405, March–April, 1972.

[3] Cohen, J. W. *The Single Server Queue.* Revised edition, North-Holland Publishing Company, Amsterdam, 1982.

[4] Conolly, B *Lecture Notes on Queueing Systems.* Ellis Horwood Limited, Sussex, England, 1975.

[5] Cooper, R. B. *Introduction to Queueing Theory.* Second edition, North-Holland, New York, 1981.

[6] Fuhrmann, S. W. Second moment relationships for waiting times in queueing systems with Poisson input. *Queueing Systems,* Vol.8, No.4, pp.397–406, June 1991.

[7] Kingman, J. F. C. On queues in which customers are served in random order. *Proceedings of the Cambridge Philosophical Society,* Vol.58, Part 1, pp.79–91, January 1962.

[8] Kleinrock, L *Queueing Systems, Volume 2: Computer Applications.* John Wiley and Sons, New York, 1976.

[9] Takács, L. Delay distributions for one line with Poisson input, general holding times, and various orders of service. *The Bell System Technical Journal,* Vol.42, No.2, pp.487–503, March 1963.

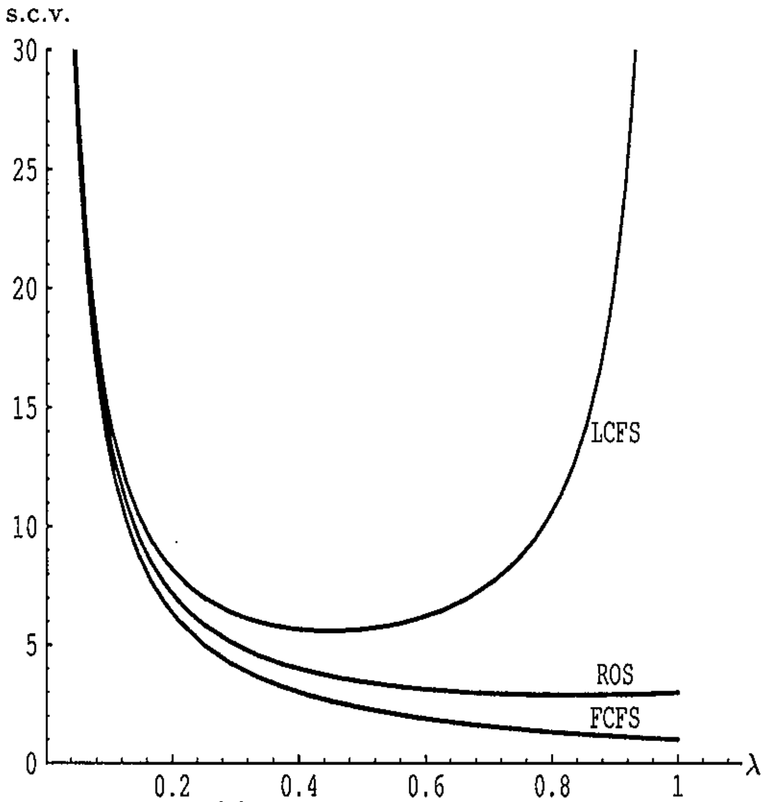
[10] Takagi, H. *Queueing Analysis, Volume 1: Vacation and Priority Systems.* Elsevier, Amsterdam, 1991.

[11] Takagi, H., and Sakamaki, K. 1994. Symbolic Moment Calculation for an M/G/1 Queue, Discussion Paper No.596, Institute of Socio-Economic Planning, University of Tsukuba, September 1994.

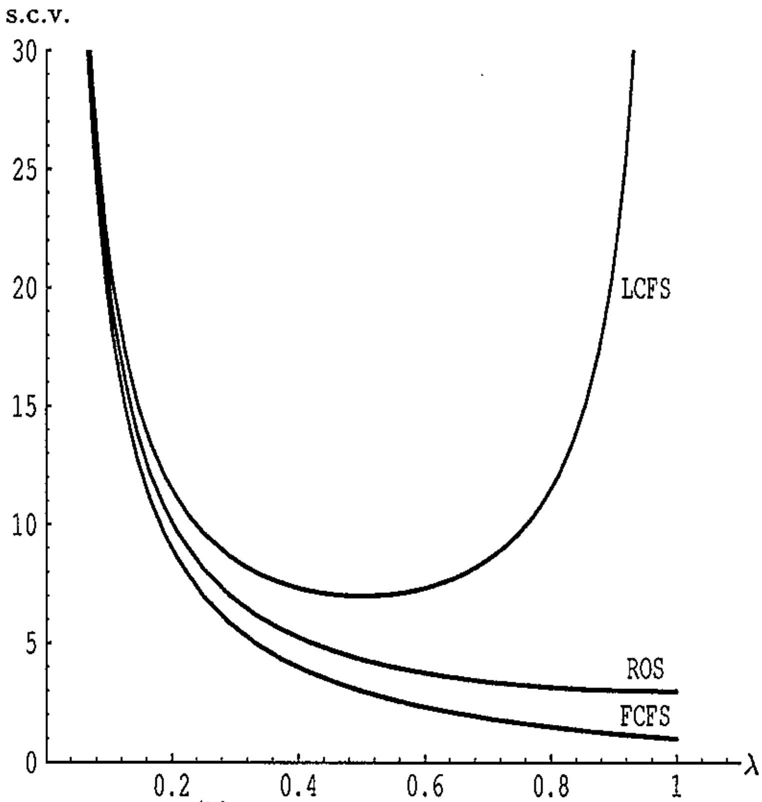
[12] Wolfram, S. *Mathematica: A System for Doing Mathematics by Computer*, Second edition, Addison-Wesley, Reading, Massachusetts, 1991.

```
$RecursionLimit = 16384
e =.
w =.
e = 5  (*dimension of n th moment*)
B[0] = 1
B'[0] = -b
Derivative[n_][B][0]=Derivative[n][b] * (-1)^n
f1[n_] = (s/(s -lambda +lambda B[s]))^n
busy[1] = b/(1-lambda b)
busy[n_]:=Expand[(-1)^(n-1)/lambda * (n-1)! * Coefficient[Series[f1[n],{s,0,n-1}],s^(n-1)]]

g1[z] = Series[B[lambda - lambda z],{z,1,e+1}]
g2[z] = 1 + (1 - z) / (g1[z] - z)
g3[n_] := Apart[Coefficient[Normal[Series[(-1)^n * g2[z],{z,1,e+1}]],(-1+z),n],b]
g4[n_] := Sum[s^i * (-1)^(i+1) * busy[i] /i!,{i,1,n}]
g5[n_] := Normal[Series[(g4[e+1])^n,{s,0,e+1}]]
g6[n_] := Cancel[Collect[Expand[Sum[g5[i] * g3[i-1] * y^(i-1),{i,1,n}]],s]]

f2[n_] := Normal[Series[(s+lambda w)^n,{s,0,e+1}]]
w= g4[e+1] * v * y
f3[n_] := Collect[Expand[f2[n]],s]
f4[s] = Collect[Sum[Derivative[i][B][0] * f3[i] /i!,{i,0,e+1}],s]
f5[s,v] = 1- w - f4[s]
f6[n_] := Cancel[Factor[Coefficient[Cancel[f5[s,v]/ (b s)],s,n]] /(1- y v)]
f7[s,v] = Sum[f6[i] * (-1)^(i+1) * s^i,{i,0,e+1}]
f8[n_] := Factor[Coefficient[Normal[Series[1/f7[s,v],{s,0,e}]],s,n]]
f9[s,v] = Collect[Sum[f8[i] * (-1)^i * s^i,{i,0,e}],{v,y}]
f10[s] = Collect[- Integrate[f9[s,v]/(1- v y),{v,0,1}],s]
f11[n_] := Cancel[Factor[Coefficient[f10[s],s,n]]]
f12[s] = Sum[f11[i] * s^i,{i,0,e}]
f13[s] = Collect[Cancel[Expand[f12[s] * y * g4[e+1]/(b s) ]],s]
f14[n_] := Cancel[Factor[Coefficient[f13[s],s,n]]]
f15[n_] := Sum[f14[i] * s^i ,{i,1,n}]
f16[n_] := Cancel[Factor[Coefficient[Normal[Series[Exp[f15[e]],{s,0,e}]],s,n]]]
f17[s] = Sum[f16[i] * s^i * (-1)^i ,{i,0,e}]

Unprotect[Integrate]
integrate[u_ + w_ ,y_] := integrate[u,y] + integrate[w,y]
integrate[c_ u_ ,y_] := c integrate[u,y] /; FreeQ[c,y]
integrate[y_^m_. (Log[1-y_])^n_.,y_] :=  Sum[Binomial[m,k] * (-1)^k *
                                         (-(1+a+k))^(-n) * n! *(1+a+k)^(-1),{k,0,m}]
integrate[(Log[1-y_])^n_.,y_] := (-1)^n * (1+a)^(-n-1) * n!
integrate[y_^m_.,y_] := m! * Product[(a+i)^(-1),{i,1,m+1}]
integrate[c_,y_] := c * (1+a)^(-1) /; FreeQ[c,y]
a = 1/(1 - b lambda)

h1[n_] := Cancel[Factor[integrate[Cancel[Factor[Coefficient[f17[s] * g6[e],s,n]]],y]]]
h2[n_] := Apart[Cancel[(-1)^n * n! * (1- lambda b) * lambda h1[n+1])]]
h3[n_] := Do[TeXForm[h2[[i]] >>> conc.tex,{i,0,n}]
h4[n_] := Do[h2[[i] >>> conc,{i,0,n}]
Integrate =.
Protect[Integrate]
```

Figure 1. Symbolic calculation of the moments of the waiting time for an ROS M/G/1 system.

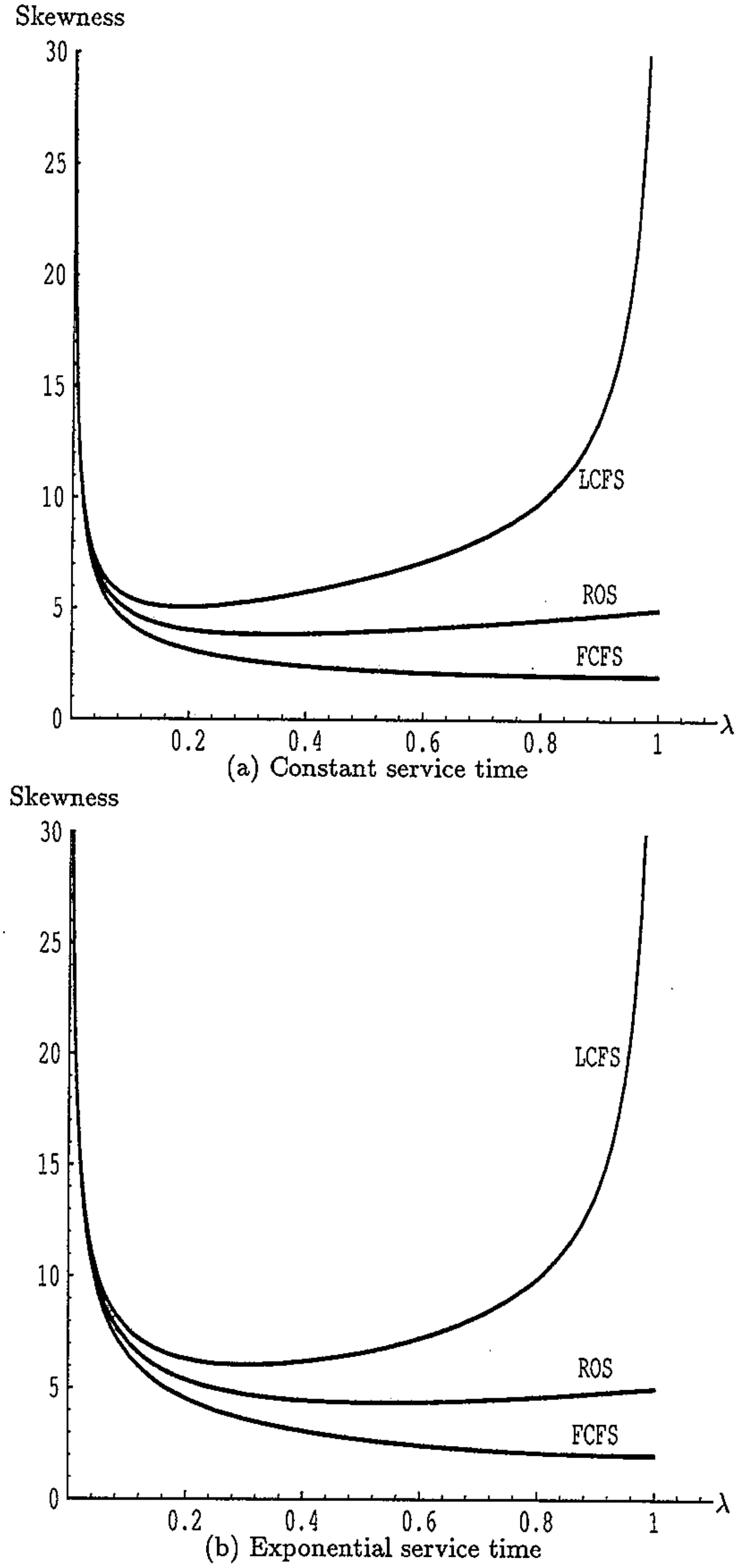


(a) Constant service time



(b) Exponential service time

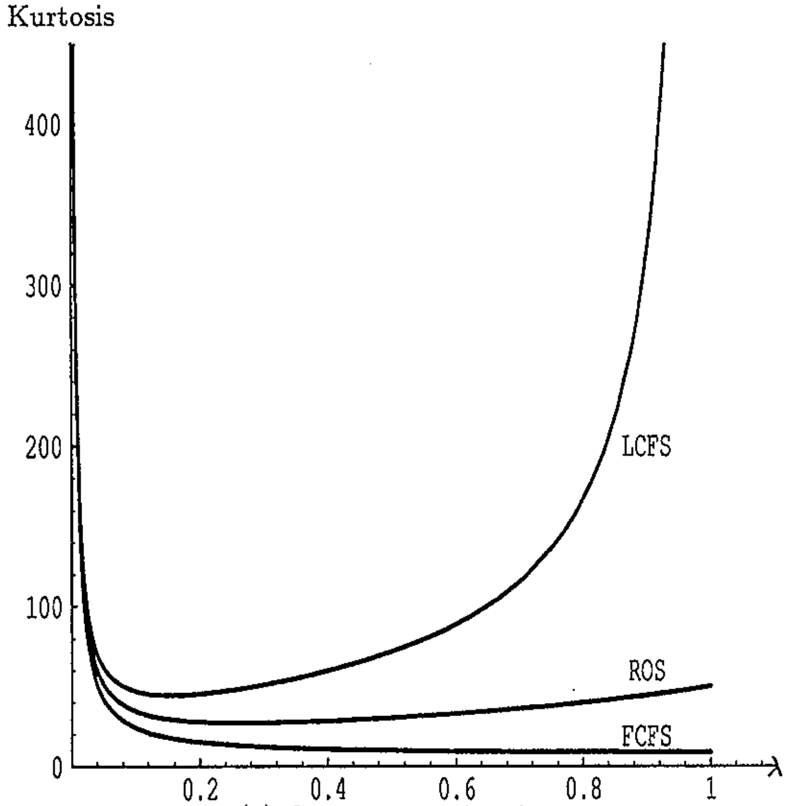
Figure 2. s.c.v. of the waiting time in FCFS, LCFS, and ROS systems.



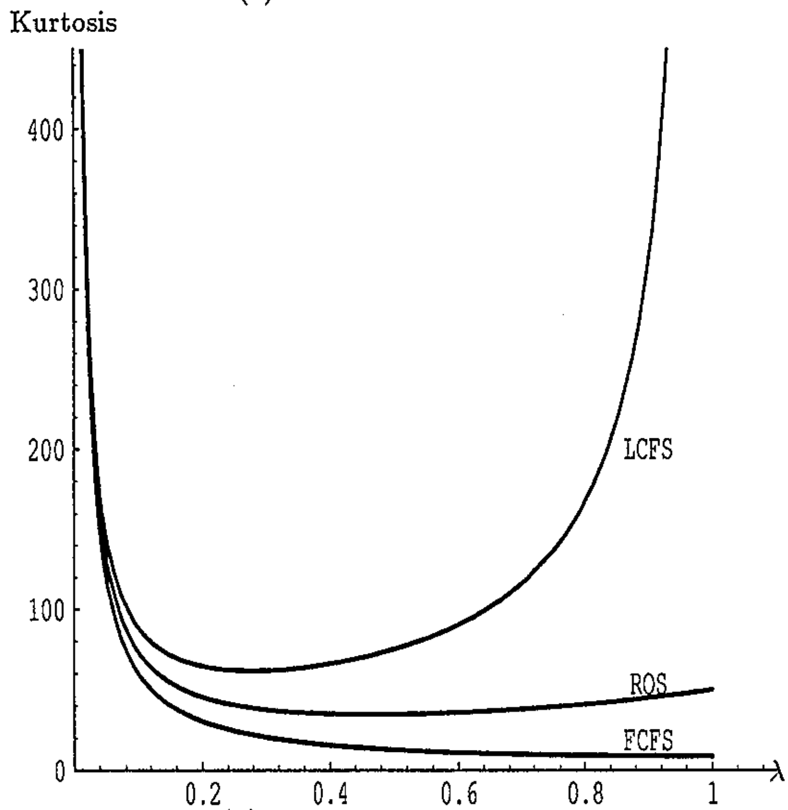
(a) Constant service time

(b) Exponential service time

Figure 3. Skewness of the waiting time in FCFS, LCFS, and ROS systems.



(a) Constant service time



(b) Exponential service time

Figure 4. Kurtosis of the waiting time in FCFS, LCFS, and ROS systems.