No. **645**

An Economic Analysis of the Transferable Development Rights

by

Noboru Sakashita

October 1995

An Economic Analysis of the Transferable Development Rights

Noboru Sakashita*
University of Tsukuba

1. Introduction

Down zoning regulation policy is liked by urban planners as a powerful tool of the urban growth control. This policy takes the form of constrained floor-area ratio (FAR) which is set below the FAR determined by a free market, and by doing so it aims to suppress the construction of office buildings within a specific city. For economists it is almost self-evident that such physical regulation produces inefficiency of resource allocation by distorting free ground-rent formation in the land market.

Since the down zoning regulation is justified on the basis of external diseconomy in the land market, economists must give a quantitative evaluation of the inefficiency which arises from this physical regulation in order to persuade urban planners not to adopt such a policy because of uncertain existence of stated external diseconomy.

On the other hand, in between the down zoning regulation and the free market, there is the scheme of the transferable development rights (TDR). This scheme of TDR permits the transaction of development rights although it regulates the total amount of development within a specific city. It is also self-evident that the degree of inefficiency caused by a TDR policy between that of the down zoning regulation and the free market. We need, however, to know the degree of inefficiency related to TDR exactly.

As previous examples of economic analysis of TDR, there are Mills (1980) and Carpenter et.al. (1982). The model analysed in Mills (1980) is a spaceless one so that it is not so interesting as an urban analysis. On the other hand, the regulation dealt with in Carpenter et.al. (1982) is concerning the "lower" limit of development not that concerning the upper limit so that the aim of their analysis is quite different from that to be dealt with in this paper.

The purpose of the present paper is to quantitatively evaluate the amount of total ground rents which emerge under the conditions of free market, down zoning regulation, and TDR policy by a model with specified functional forms, and to measure the degree of inefficiency related to the latter two

* The author is thankful to Professors Kiyohiko Nishimura, Miki Seko, and (M.) Yoshino for their valuable comments to the earlier version of this paper. Needless to say, all remaining errors are of the author alone.

polices.

The model to be used in this paper is a simplified version of the business firm model developed in Chapter 1, Section 2 of Henderson (1985). In my case, the product of business firms is the office space itself. In Section 2, the case of free development is analysed. Down zoning regulation is dealt with in Section 3, and TDR policy in Section 4. Three policies are compared in Section 5 and concluding remarks are given in Section 6.

2. Market Equilibrium under the Free Development

The city in question is assumed to be of a circular form and its boundary is fixed at the distance of u_0 from the center. For usual urban models, the boundary is to be endogenously determined. In this paper, however, it is assumed to be a fixed one in order to concentrate attention to the business activities. This assumption should be relaxed to accommodate the analysis of residential activities in the future.

Developer-firms produce the homogeneous office space x using land l and capital k as inputs by a Cobb-Douglas production function of :

$$x(u) = C'\{l(u)\}^{\gamma}\{k(u)\}^{1-\gamma} \qquad (1)$$

in which u is the location of a specific firm. $\left(u \in [0,\ u_0]\right)$
If it is assumed that the labour input in office space production (guard man, janitor etc.) are proportional to the output, the cost of labour can be treated implicitly by reducing p_x, the rental price of x, accordingly.

In addition, let us assume that the producer's price of x, $\tilde{p}_x(u)$ undergoes a decay depending on the distance from the centre, u, it is expressed as :

$$\tilde{p}_x(u) = p_x - \theta u \qquad (2)$$

in which p_x is the producer's price at the centre, and θ is the unit rate of distance decay. θ can be interpreted as information cost, transaction cost, and so on.

Let $p_l(u)$ be the (ground) rent at u and p_k be the price of capital service which does not depend on the firm's location. If it is assumed that each firm minimizes the unit production cost and that the profit of any firm becomes zero by the free entry of potential competitors, there will be demand functions for land and capital by single firm as follows :

$$l(u) = \gamma\, \tilde{p}_x x(u)/p_l(u)\ ,\ k(u) = (1-\gamma)\tilde{p}_x x(u)/p_k \qquad (3).$$

Putting Equations (3) into Equation (1), we obtain a factorprice-frontier as :

$$\tilde{p}_x(u)=(C')^{-1}\left\{\gamma^{\gamma}(1-\gamma)^{1-\gamma}\right\}^{-1}\left\{p_l(u)\right\}^{\gamma}p_k^{1-\gamma} \qquad (4),$$

and the rent profile within the city is finally given as :

$$\left.\begin{aligned} p_l(u)&=Ap_k^{-\frac{1-\gamma}{\gamma}}(p_x-\theta u)^{\frac{1}{\gamma}},\quad 0\le u\le u_0\\ A&=\left[C'\left\{\gamma^{\gamma}(1-\gamma)^{1-\gamma}\right\}\right]^{\frac{1}{\gamma}}\end{aligned}\right\} \qquad (5).$$

Equation (5), the rent profile will play a fundamental role in the analyses which follow.

The total rents within this city, R will be expressed by the following integral form :

$$R=\int_0^{u_0}Ap_k^{-\frac{1-\gamma}{\gamma}}(p_x-\theta u)^{\frac{1}{\gamma}}2\pi u\,du \qquad (6).$$

Computation of RHS of Equation (6) will be done in Appendix A. The final result is :

$$\left.\begin{aligned} R&=B'(p_k)\left[\frac{\gamma^2}{\theta^2(1+\gamma)(1+2\gamma)}\left\{p_x^{\frac{1+2\gamma}{\gamma}}-(p_x-\theta u_0)^{\frac{1+2\gamma}{\gamma}}\right\}-\frac{1}{\theta}\frac{\gamma}{1+\gamma}(p_x-\theta u_0)^{\frac{1+\gamma}{\gamma}}u_0\right],\\ B'(p_k)&=2\pi\left[C'\left\{\gamma^{\gamma}(1-\gamma)^{1-\gamma}\right\}\right]^{\frac{1}{\gamma}}p_k^{-\frac{1-\gamma}{\gamma}}\end{aligned}\right\} \qquad (7).$$

On the other hand, Equation (3) gives us the following FAR profile :

$$\rho(u)=\frac{k(u)}{l(u)}=\frac{1-\gamma}{\gamma}\frac{p_l(u)}{p_k} \qquad (8).$$

Therefore, we see that the FAR profile is perfectly similar to the rent profile given by Equation (5). From Equation (8) we can calculate the total capital stock in this city as :

$$K=\frac{1-\gamma}{\gamma}\frac{1}{p_k}\int_0^{u_0}p_l(u)l(u)\,du=\frac{1-\gamma}{\gamma}\frac{1}{p_k}R \qquad (9).$$

The total amount of office development in this city becomes :

$$\left.\begin{aligned} X&=D\left[\frac{\gamma^2}{\theta^2(1+\gamma)}\left\{p_x^{\frac{1+\gamma}{\gamma}}-(p_x-\theta u_0)^{\frac{1+\gamma}{\gamma}}\right\}-\frac{\gamma}{\theta}(p_x-\theta u_0)^{\frac{1}{\gamma}}u_0\right],\\ D&=2\pi C'^{\frac{1}{\gamma}}(1-\gamma)^{\frac{1-\gamma}{\gamma}}p_k^{-\frac{1-\gamma}{\gamma}}\end{aligned}\right\} \qquad (10)$$

because at particular point u, $x(u)$ is calculated as :

$$x(u)=D(p_x-\theta u)^{\frac{1-\gamma}{\gamma}}u \qquad (11).$$

Utilizing the above equations of (7), (9), and (10), we can calculate the values of R, K and X for specific values of parameters as the outcomes of free development in a particular city.

3. Market Equilibrium under the Down Zoning Regulation

An FAR regulation takes the form of exogenously given upper limit to the capital-land ratio at each point within a city. As shown in Equations (8) with (5), the capital-land ratio is a convex function of distance variable u. However, any FAR regulation must be designed before the urban development is practiced so that it must be worked out without knowing the form of the capital-land ratio profile under the free market.

In order to make the content of regulation simple and transparent to everyone, a possibility is to make it a linear form such that :

$$\hat{\rho}(u)=\lambda-\mu u\,,\ \lambda=\rho(0)=\frac{1-\gamma}{\gamma}Ap_k^{-\frac{1}{\gamma}}p_x^{\frac{1}{\gamma}} \qquad (12)$$

with a condition of $\mu=\lambda/u_0$ to make the development at the city border null. (See Figure 1)

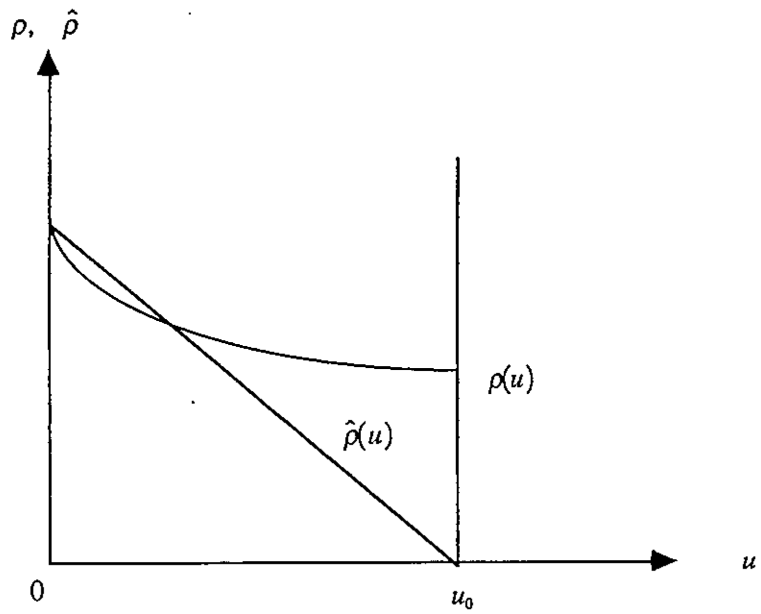

Figure 1. Profiles of capital-land ratio

$\rho(u)$: Market Equilibrium without regulation
$\hat{\rho}(u)$: FAR regulation

Putting Equation (12) into Equation (1), the output at u under the FAR regulation is given as :

$$\begin{aligned}\hat{x}(u) &= C'\{l(u)\}^{\gamma}\{(\lambda-\mu u)\, l(u)\}^{1-\gamma} \\ &= C'(\lambda-\mu u)^{1-\gamma} l(u)\end{aligned} \qquad (13).$$

The rent profile in this case is derived as follows by the condition of zero profit :

$$\hat{p}_l(u) = (p_x - \theta u) C'(\lambda-\mu u)^{1-\gamma} - p_k(\lambda-\mu u) \qquad (14).$$

The total rents in this case, $\hat{R}$ is expressed as :

$$\hat{R} = \int_0^{u_0} \left[C'(p_x-\theta u)(\lambda-\mu u)^{1-\gamma} - p_k(\lambda-\mu u)\right] 2\pi u\, du \qquad (15).$$

The complicated computation of RHS of Equation (15) is done in Appendix B, and the final result is :

$$\begin{aligned}\hat{R} = 2\pi C' \Bigg[& \frac{1}{\mu(2-\gamma)}\left\{-(\lambda-\mu u_0)^{2-\gamma}(p_x-\theta u_0)u_0\right\} \\ & + \frac{\theta}{\mu^2(2-\gamma)(3-\gamma)}\left\{(\lambda-\mu u_0)^{3-\gamma} u_0\right\} \\ & - \frac{1}{\mu^3(2-\gamma)(3-\gamma)}\left\{(\lambda-\mu u_0)^{3-\gamma}(p_x-\theta u_0) - \lambda^{3-\gamma} p_x\right\} \\ & + \frac{2\theta}{\mu^3(2-\gamma)(3-\gamma)(4-\gamma)}\left\{(\lambda-\mu u_0)^{4-\gamma} - \lambda^{4-\gamma}\right\}\Bigg] \\ & - 2\pi p_k\left[-\frac{(\lambda-\mu u_0)^2 u_0}{2\mu} - \frac{1}{6\mu^2}\left\{(\lambda-\mu u_0)^3 - \lambda^3\right\}\right] \qquad (16)\end{aligned}$$

The total capital, $\hat{K}$ under this scheme is computed using Equation (B-8) in Appendix B as :

$$\begin{aligned}\hat{K} &= \int_0^{u_0} (\lambda-\mu u) 2\pi u\, du \\ &= 2\pi\left[-\frac{(\lambda-\mu u_0)^2 u_0}{2\mu} - \frac{1}{6\mu^2}\left\{(\lambda-\mu u_0)^3 - \lambda^3\right\}\right] \qquad (17).\end{aligned}$$

Finally, the total amount of development, $\hat{X}$ under this scheme is :

$$\begin{aligned}\hat{X} &= \int_0^{u_0} C'(\lambda-\mu u)^{1-\gamma} 2\pi u\, du \\ &= 2\pi C'\left[-\frac{1}{\mu(2-\gamma)}(\lambda-\mu u_0)^{2-\gamma} u_0 - \frac{1}{\mu^2(2-\gamma)(3-\gamma)}\left\{(\lambda-\mu u_0)^{3-\gamma} - \lambda^{3-\gamma}\right\}\right] \qquad (18).\end{aligned}$$

Not like as the case of free development, there is no similarity between $\hat{p}_l(u)$ and $\hat{\rho}(u)$ so that there is no simple relationship between $\hat{R}$ and $\hat{K}$ contrary to Equation (9). A stronger FAR regulation implies smaller λ and/or bigger μ which reduce $\hat{R}$ as well as $\hat{K}$. The difference between R (Equation (7)) and $\hat{R}$ (Equation (16)) shows the degree of inefficiency caused by an FAR regulation.

4. Market Equilibrium under the Transferable Development Rights

The target of city planning may be the total amount of development within a city of total land area $L = \pi u_0^2$ equal to $\hat{X}$ given by Equation (18). Alternative method to accomplish this target is to set up a market of development rights instead of a physical constraint in the form of Equation (12) and to realize $\hat{X}$ as a result of the intervened market equilibrium.[1]

With a transfer system of development rights, an individual firm takes the following behaviour :

$$\underset{\langle l(u),k(u)\rangle}{\text{Maximize}} \left[(p_x - \theta u - r)C'\{l(u)\}^{\gamma}\{k(u)\}^{1-\gamma} - p_l(u)l(u) - p_k k(u)\right] \qquad (19).$$

r is the market price of a development right.[2]
From Formula (19), a city-wide demand function for development rights, $X(r)$ is derived and an equilibrium condition of

$$X(r) = \hat{X} \qquad (20)$$

determines the equilibrium value of r. (See Figure 2)

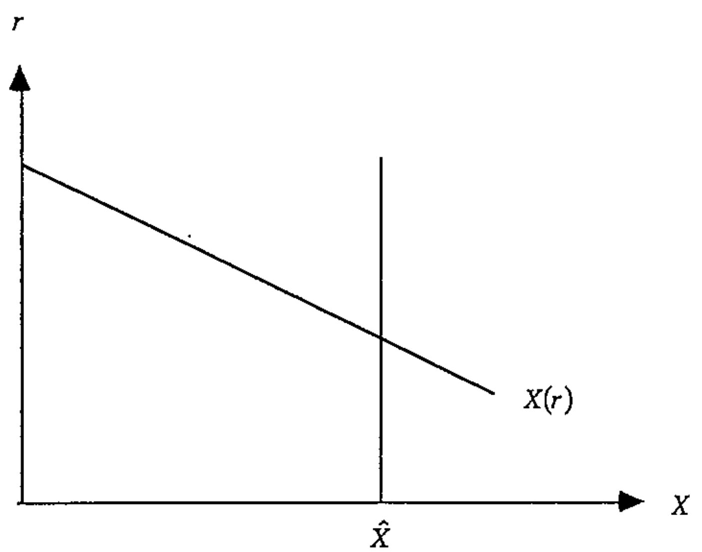

Figure 2. Equilibrium of Development Rights Market

Demand for DR : $X(r)$
Supply of DR : $\hat{X}$

The rent profile under this scheme is similar to Equation (5), and given as :

$$p_l^*(u) = A\, p_k^{-\frac{1-\gamma}{\gamma}} \{(p_x - r) - \theta u\}^{\frac{1}{\gamma}} \qquad (21).$$

The total rents $R^*(r)$ is derived by changing p_x in Equation (7) into $(p_x - r)$, and the total capital $K^*(r)$ is derived by a similar change in Equation (9).

The rent profile of the city will be shifted downward when p_x is lowered to $(p_x - r)$ giving smaller amount of total rents with compensation of revenues in the DR market. The comparison among the three schemes should be done by comparing R, $\hat{R}$, and $R^*(r)$ with r which equalizes $X^*(r)$ (derived by Equation (10) changing p_x into $(p_x - r)$) to $\hat{X}$. Since all of the relevant equations are fairly complicated, we have to rely on numerical simulations to make the above comparison. Such simulations will be made in the next Section.

5. Numerical Simulations

(i) Setting the values of parameters and equilibrium under free development

Parameters in the basic model of Section 2 will be given the following values :

$$C' = 1, \quad \gamma = 0.5, \quad u_0 = 100, \quad p_x = 10, \quad \theta = 0.05, \quad p_k = 5 \qquad (22)$$

With these values, the rent profile of Equation (5) becomes :

$$p_l(u) = 0.05(10 - 0.05u)^2 \qquad (23),$$

and the total rents of Equation (7) becomes :

$$R(p_x) = 0.1\pi\left[33.3333\left\{p_x^4 - (p_x - 5)^4\right\} - 666.6666(p_x - 5)^2\right] \qquad (24).$$

Therefore, when $p_x = 10$, $R(10)$ is given as :

$$R(10) \cong 22{,}916.7\pi \qquad (25).$$

The total development $X(p_x)$ of Equation (10) is :

$$X(p_x) = 0.2\pi\left[66.6666\left\{p_x^3 - (p_x - 5)^3\right\} - 1{,}000(p_x - 5)^2\right] \qquad (26),$$

so that $X(10)$ will be :

$$X(10) \cong 6666.66\pi \quad (27).$$

On the other hand, the realized capital-land ratio profile is given as :

$$\rho(u) = 0.01(10 - 0.05u)^2 \quad (28),$$

and the total capital stock is calculated as :

$$K(10) = 4{,}583.33\pi \quad (29).$$

(ii) Equilibrium under the FAR regulation

The regulation of FAR by the form of Equation (12) with $\lambda = \rho(0) = 1$ and $\mu = 0.01$ gives a value of $\hat{R}$ by Equation (16) as :

$$\hat{R} = 21{,}428.0\pi \quad (30).$$

Corresponding $\hat{K}$ and $\hat{X}$ are given as :

$$\hat{K} = 3{,}333.3\pi, \quad \hat{X} = 5{,}333.2\pi \quad (31)$$

by Equations (17) and (18).

(iii) Equilibrium with TDR

In this case, p_x in Equation (24) is adjusted to give the value of $\hat{X}$ ((31)) using Equation (26). More concretely solving $X(p_x - r) = 5{,}333.33\pi$, we obtain

$$p_x - r = 8.66666, \quad r = 1.33334 \quad (32)$$

and the corresponding total rents is calculated as :

$$R^* \cong R(8.67) = 14{,}916.7\pi \quad (33)$$

(iv) Comparison of the three schemes

Calculations in the above are summarized in the following table :

	Total Land Area	Total Capital	Total Development	Total Rents	Development Rights Revenue
Free Development	10,000	4,583.3 (100)	6,666.7 (100)	22,916.7 (100)	
FAR regulation	10,000	3,333.3 (72.7)	5,333.2 (80.0)	21,428.0 (93.5)	
TDR	10,000	2,983.3 (65.1)	5,333.3 (80.0)	14,916.3 (65.1)	7,111.0 (31.0)

Table 1. Comparison of Different Equilibria
(The multiplier π is omitted from all figures.)

In this table we see that the total rents under the FAR regulation, $\hat{R}$ is reduced to 93.5% of the same under free development, R, and that the same under TDR, R^*, becomes 65.1% of R. If we include the revenue from the sale of development rights in the social surplus, the latter becomes 96.1% of the same under free development.

6. Concluding remarks

In this paper, the development of office space in a city with a fixed border was considered under the three different schemes: (i) free development without any regulation, (ii) market with an arbitrary FAR regulation, and (iii) market with TDR.
For each scheme, the degree of efficiency is evaluated by the amount of the total rents emerged.

The results of numerical simulation are summarized in Table 1, and they clearly indicate the strong inefficiency of an arbitrary FAR regulation. One can achieve the same level of development suppression with much less sacrifice of efficiency by introducing a TDR market.

The method of FAR regulation considered in this paper seems to be too mechanical and it can be made more flexible by introducing some semi-optimizing technique, say, finding optimizing values of λ and μ, or optimizing the function $\hat{\rho}(u)$ itself. Alternatively, the price of capital service p_k can be increased by a form of congestion charge instead of lowering p_x by a TDR purchase. The author's future task is to explore these possibilities.

Appendix A. Calculation of Equation (6)

Except for a constant multiplier of $\left(2\pi A p_k^{-\frac{1-\gamma}{\gamma}}\right)$, it is needed to calculate an integral of :

$$I=\int_0^{u_0}(p_x-\theta u)^{\frac{1}{\gamma}}u\,du \qquad \text{(A-1)}.$$

Since

$$\int (p_x-\theta u)^{\frac{1}{\gamma}}du=-\frac{1}{\theta}\frac{\gamma}{1+\gamma}(p_x-\theta u)^{\frac{1+\gamma}{\gamma}} \qquad \text{(A-2)},$$

using the theorem of partial integral we have :

$$I=\left[\left\{-\frac{1}{\theta}\frac{\gamma}{1+\gamma}(p_x-\theta u)^{\frac{1+\gamma}{\gamma}}\right\}u\right]_0^{u_0}-\int_0^{u_0}\left\{-\frac{1}{\theta}\frac{\gamma}{1+\gamma}(p_x-\theta u)\right\}^{\frac{1+\gamma}{\gamma}}du$$

$$=\left[\left\{-\frac{1}{\theta}\frac{\gamma}{1+\gamma}(p_x-\theta u)^{\frac{1+\gamma}{\gamma}}\right\}u-\frac{1}{\theta^2}\frac{\gamma}{1+\gamma}\frac{(p_x-\theta u)^{\frac{1+2\gamma}{\gamma}}}{\frac{1+2\gamma}{\gamma}}\right]_0^{u_0}$$

$$=\frac{\gamma^2}{\theta^2(1+\gamma)(1+2\gamma)}\left\{p_x^{\frac{1+2\gamma}{\gamma}}-(p_x-\theta u_0)^{\frac{1+2\gamma}{\gamma}}\right\}-\frac{1}{\theta}\frac{\gamma}{1+\gamma}(p_x-\theta u_0)^{\frac{1+\gamma}{\gamma}}u_0 \qquad \text{(A-3)}.$$

From Equation (A-3), Equation (7) in Section 2 is finally derived.

Appendix B. Calculation of Equation (15)

First we need to calculate :

$$I_1=\int_0^{u_0}(p_x-\theta u)(\lambda-\mu u)^{1-\gamma}u\,du \qquad \text{(B-1)}.$$

Again by the theorem of partial integral, it is expressed that :

$$I_1=\left[\int(p_x-\theta u)(\lambda-\mu u)^{1-\gamma}du\right]u-\int\left[\int(p_x-\theta u)(\lambda-\mu u)^{1-\gamma}du\right]du \qquad \text{(B-2)}.$$

Let [ ] in the second term of RHS of the above equation be I_2,

$$
\begin{aligned}
I_2 &= \int (p_x - \theta u)(\lambda - \mu u)^{1-\gamma} du \\
&= \left[\int (\lambda - \mu u)^{1-\gamma} du\right](p_x - \theta u) - \int \left[\int (\lambda - \mu u)^{1-\gamma} du\right](-\theta) du \\
&= -\frac{(\lambda - \mu u)^{2-\gamma}}{\mu(2-\gamma)}(p_x - \theta u) - \frac{\theta}{\mu(2-\gamma)} \int (\lambda - \mu u)^{2-\gamma} du \\
&= -\frac{(\lambda - \mu u)^{2-\gamma}}{\mu(2-\gamma)}(p_x - \theta u) + \frac{\theta}{\mu^2(2-\gamma)(3-\gamma)}(\lambda - \mu u)^{3-\gamma} \qquad \text{(B-3)}.
\end{aligned}
$$

Therefore, the second term of RHS of Equation (B-2) (call it J) will become :

$$
\begin{aligned}
J &= \int I_2(u)\, du \\
&= -\frac{1}{\mu(2-\gamma)}\left[-\frac{(\lambda - \mu u)^{3-\gamma}}{\mu(3-\gamma)}(p_x - \theta u) + \frac{\theta}{\mu(3-\gamma)}\left(\frac{1}{\mu}\frac{(\lambda - \mu u)^{4-\gamma}}{(4-\gamma)}\right)\right] \\
&\quad - \frac{\theta}{\mu^3(2-\gamma)(3-\gamma)(4-\gamma)}(\lambda - \mu u)^{4-\gamma} \\
&= \frac{(\lambda - \mu u)^{3-\gamma}}{\mu^2(2-\gamma)(3-\gamma)}(p_x - \theta u) - \frac{2\theta}{\mu^3(2-\gamma)(3-\gamma)(4-\gamma)}(\lambda - \mu u)^{4-\gamma} \qquad \text{(B-4)}.
\end{aligned}
$$

Using the above results, I_1 is finally expressed as :

$$
\begin{aligned}
[I_1]_0^{u_0} &= \left[\left\{-\frac{(\lambda - \mu u)^{2-\gamma}}{\mu(2-\gamma)}(p_x - \theta u) + \frac{\theta}{\mu^2(2-\gamma)(3-\gamma)}(\lambda - \mu u)^{3-\gamma}\right\}u \right.\\
&\quad \left. - \frac{(\lambda - \mu u)^{3-\gamma}}{\mu^2(2-\gamma)(3-\gamma)}(p_x - \theta u) + \frac{2\theta}{\mu^3(2-\gamma)(3-\gamma)(4-\gamma)}(\lambda - \mu u)^{4-\gamma}\right]_0^{u_0} \\
&= \frac{1}{\mu(2-\gamma)}\left\{-(\lambda - \mu u_0)^{2-\gamma}(p_x - \theta u_0)u_0\right\} + \frac{\theta}{\mu^2(2-\gamma)(3-\gamma)}\left\{(\lambda - \mu u_0)^{3-\gamma} u_0\right\} \\
&\quad - \frac{1}{\mu^2(2-\gamma)(3-\gamma)}\left\{(\lambda - \mu u_0)^{3-\gamma}(p_x - \theta u_0) - \lambda^{3-\gamma} p_x\right\} \\
&\quad + \frac{2\theta}{\mu^3(2-\gamma)(3-\gamma)(4-\gamma)}\left\{(\lambda - \mu u_0)^{4-\gamma} - \lambda^{4-\gamma}\right\} \qquad \text{(B-5)}.
\end{aligned}
$$

Second we calculate

$$
I_3 = \int (\lambda - \mu u) u\, du \qquad \text{(B-6)}.
$$

I_3 simply becomes :

$$
\begin{aligned}
I_3 &= \left[\int (\lambda - \mu u)\, du\right]u - \int \left[\int (\lambda - \mu u) du\right] du \\
&= -\frac{(\lambda - \mu u)^2 u}{2\mu} - \frac{(\lambda - \mu u)^3}{6\mu^2} \qquad \text{(B-7)},
\end{aligned}
$$

therefore we have :

$$[I_3]_0^{u_0} = -\frac{(\lambda - \mu u_0)^2 u_0}{2\mu} - \frac{1}{6\mu^2}\left\{(\lambda - \mu u_0)^3 - \lambda^3\right\} \qquad \text{(B-8)}.$$

Combining Equation (B-5) with Equation (B-8), Equation (16) is finally derived.

Notes

<1> Depending on the choice of setting target by $\hat{K}$ (total capital) or by $\hat{X}$ (total development), there will be some difference in the evaluation of inefficiency. This difference, however, may be small. Anyway the FAR regulation aims a physical constraint on the capital-land ratio. On the other hand, the target of regulation in the TDR scheme is the total amount of development. We must, therefore, be careful about delicate difference between the two schemes.

<2> As a concrete procedure, we can describe the scheme as follows: The planning authority sells development rights in amount of $\hat{X}$ expressed by Equation (18). Developers at each distance u buy DR as they wish at the price r, and the aggregate market of DR determines the equilibrium price of r, say $\bar{r}$. However, at distance u each developer originally has the right of development up to $\hat{x}(u)$ of Equation (13) so that the planning authority will reimburse the money in amount of $\{\bar{r}\hat{x}(u)\}$ to each developer at u. By doing so all amount of development rights selling is refunded to the group of developers.

References

Carpenter, Bruce E. and Dennis R. Heffley (1982), "Spatial Equilibrium Analysis of Transferable Development Rights", Journal of Urban Economics, Vol.12, 238-261

Henderson, J. Vernon (1985), Economic Theory and the Cites (2nd edition), Academic Press, New York

Mills, David E. (1980), "Transferable Development Rights Markets", Journal of Urban Economics, Vol.7, 63-74

An Economic Analysis of the Transferable Development Rights: Supplement

N. Sakashita

The Optimal FAR Profile

In Section 3 of the main paper, an FAR regulation was introduced in the form of a straight line, $\hat{\rho}(u) = \lambda - \mu u$. It can be said that it was a too mechanical way of regulation. Urban planners are much wiser to invent better form of the FAR profile achieving the same constrained amount of the total development.

In this supplement, I will derive the optimal form of the FAR profile in the sense of maximizing total rents under the constraint of Equation (20) in Section 4 of the main paper. It is interesting to see that the optimal FAR profile thus derived exactly coincides with the FAR profile under the TDR scheme (see Equation (21) with Equation (8) in the main paper). It will be proved, therefore, that the TDR scheme is nothing other than the best, deliberately designed, FAR regulation.

Let an undefined FAR profile be $q(u)$. Then we have:

$$k(u) = q(u)\, l(u)$$

$$x(u) = C'\{l(u)\}^{\gamma}\{q(u)\, l(u)\}^{1-\gamma} = C'\, l(u)\{q(u)\}^{1-\gamma} \quad \text{(S1)}$$

$$\pi(u) = (p_x - \theta u) C'\{q(u)\}^{1-\gamma} l(u) - p_l(u)\, l(u) - p_k q(u)\, l(u) \quad \text{(S2)}$$

$$= [(p_x - \theta u) C'\{q(u)\}^{1-\gamma} - p_l(u) - p_k q(u)]\, l(u) \quad \text{(S3)}$$

in which $\pi(u)$ is the profit (density) function at the location u.

From Equation (S3), the rent profile is derived as :

$$p_l(u) = (p_x - \theta u) C'\{q(u)\}^{1-\gamma} - p_k q(u) \quad \text{(S4).}$$

Now an optimizing problem for a wise planner can be formulated as :

$$\underset{\langle q(u)\rangle}{\text{Maximize}}\ I\{q(u)\} = \int_0^{u_0} [(p_x - \theta u) C'\{q(u)\}^{1-\gamma} - p_k q(u)]\, 2\pi u\, du$$

$$\text{subject to} \quad \int_0^{u_0} C'\{q(u)\}^{1-\gamma}\, 2\pi u\, du = \hat{X} \quad \text{(S5)}$$

To solve the above problem, an artificial integrand can be introduced as follows :

$$L\{u, q(u)\} = [(p_x - \theta u)C'\{q(u)\}^{1-\gamma} - p_k q(u) - \omega C'\{q(u)\}^{1-\gamma}]u \quad \text{(S6)}$$

in which ω is a Lagrangean multiplier. The Euler-Lagrange condition with respect to this problem is shown to be :

$$\frac{\partial L}{\partial q} = (p_x - \theta u)(1-\gamma)C' q^{-\gamma} - p_k - \omega C'(1-\gamma)q^{-\gamma} = 0 \qquad \text{(S7)},$$

(See Miller (1979) p.70).

From Equation (S7), the optimal FAR profile, $q^*(u)$ is derived as :

$$q(u) = (1-\gamma)^{\frac{1}{\gamma}} C'^{\frac{1}{\gamma}} p_k^{-\frac{1}{\gamma}} (p_x - \omega - \theta u)^{\frac{1}{\gamma}} \qquad \text{(S8)}.$$

Equation (S8) exactly coincides with Equation (8) of the main paper if Equation (5) of the same is substituted in the term $p_l(u)$ with a change of p_x into $(p_x - \omega)$. This means that Equation (S8) also exactly coincides with Equation (21) of the main paper if r is changed into ω. The value of ω can be adjusted to attain the constraint part of Equation (S5) since the LHS of the constraint is apparently a decreasing function of ω.

From the above discussion, we can conclude that :

Proposition The TDR scheme gives the optimal FAR profile among all down zoning regulations of the same amount of the total developments.

Reference

Miller, R.E. (1979) Dynamic Optimization and Economic Applications, McGraw-Hill, New York