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Abstract

In this paper, slightly modified versions of the Hotelling spatial competition model on linear markets are formulated. In our model, the governments which divide the whole market segment compete by optimizing its government revenues. There are two types of firms: ordinary firms and transportation firms. The necessary and sufficient conditions for equilibrium and its uniqueness are then derived. The states of price equilibria are also examined. In particular, how the size of contries, and their spatial arrangement, affects taxes and government revenues in equilibrium are examined.

Key words: spatial competition; transportation firms; Nash equilibrium
JEL Classification: R48; R53

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1. Introduction

Hotelling (1929) examined the state of a mill price equilibrium for two firms of a homogeneous commodity when customers of that commodity are uniformly distributed over a line segment. He assumed that each customer purchases one unit of the commodity and bears all transport costs. Subsequent to this work, a number of studies have been undertaken: see Eiselt and Laporte(1989), Gabszewicz and Thisse(1986,1992), Anderson, de Palma and Thisse(1992), Eiselt, Laporte and Thisse (1993) for a survey.

To our knowledge, most existing papers discussing spatial competition model the interaction between firms' decisions. However, governments compete with each other for revenues. Specific examples are the taxes on retail trade imposed by governments. In Europe, with the establishment of the European Community, the European market has become more open. This gives more healthy competition among the European countries as is the case among the states in America. The sales tax in Luxembourg is relatively low compared with Belgium, France and Germany. Accordingly, many people from Belgium, France and Germany go shopping to Luxembourg to seek a good bargain. In addition, most tourists passing through Luxembourg gas up their cars at gas stations there. This is because the gasoline taxes of Luxembourg are low, making the cost of gasoline in Luxembourg the cheapest in Europe. By underselling their competitors, both the government in Luxembourg are able to obtain a lot of revenues and make high profits. As, in the United States, many people go shopping to other states to obtain good bargains because the sales taxes are different between states,. For many states sales taxes are among the principal sources of revenues. As a result, the state governments regard the strategy of price controls such as local taxes as important.

Obviously, a decrease in the tax of a government extends its market area. However, the revenues obtainable from its firms decreases. Thus governments face a trade-off, and attempt to make this trade-off effective. The spatial competition situation analyzed here differs from the standard version of the spatial competition model only in that the governments which divide the whole market segment compete by optimizing its government revenues.

The objective of this paper is to formulate the spatial competition model of imposing a tax between $N(\geq 2)$ governments on a line segment, and to examine the relationships between the revenues in equilibria and the spatial arrangements of the governments and between the revenues in equilibria and the sizes of the governments. More specifically, we shall compute all taxes in equilibria for some sets of geographical and technological parameters.

Two notes are in order on this model. First, in this paper, locations are exogeneously fixed, that is, governments compete on taxes only. Second, the extension of the standard version to more than two firms is quite complex and has received relatively little attention up until now. In this paper we formulate the model for $N(\geq 2)$ governments.

This paper is divided into two parts: *ordinary firms* and *transportation firms*. In the case of ordinary firms, customers have to go to a firm from a demand location in order to be served, then go back to the same demand location, as shown in Figure 1. Shopping stores are typical examples of such ordinary firms. On the other hand, in the case of transportation firms, customers have to travel to a firm from an origin point and continue their journey to a destination point, as shown in Figure 2. Specific examples are gas stations. The service provided by such firms takes place during from one location to another: see Berlin et al.(1976), Hodgson(1981), Ohsawa(1989). It is clear from the definitions that the ordinary firm can be considered as a special case of the transportation firm.

2. The model for ordinary firms

Statement of the model

Consider the line segment along which customers and firms are evenly spread with a unit density. There are $N(\geq 2)$ governments which divide this whole market segment into N line segments. They are indexed by $1, \cdots, N$. There is a single homogeneous commodity that is produced by all firms at zero marginal production cost. The assumption of zero marginal production cost is introduced here so as to simplify the analysis; however, our results do not depend on it. Each government demands a tax, denoted by p_i, from the firms within it. Each customer buys the commodity from the firm offering the lowest *full price*, defined as the mill price plus the transport cost between the firm and the customer, irrespective of its full price. Thus, the demand is perfectly inelastic. Transport costs are linear in distance and equal to γ per unit distance. It should be noted that all firms in the i-th government would charge the same and constant mill prices, p_i. This is because as the firms compete with each other for customers, given the continuum of firms all firms would price at marginal cost.

We define the *revenue* of the i-th government as the sum of the taxes from all firms within it, referred to as $\pi_i(p_1, \ldots, p_N)$. And we define the *total demand* of the i-th government as the sum of the demands of all the firms within it, referred to as $D_i(p_1, \ldots, p_N)$. Then each government maximizes its revenue by changing its tax, assuming that it considers the others' taxes as fixed. In this paper, we define a tax equilibrium by a Nash equilibrium of a non-cooperative N-person game whose players are governments, strategies are taxes and payoffs are revenues. Let the tax and the size for the i-th government be denoted by p_i and $L_i(>0)$ respectively. As usual, a vector $(p^*_1, \ldots, p^*_N)$ is a *tax vector in equilibrium* if and only if $\pi_i(p^*_1, \ldots, p_i, \ldots, p^*_N) \leq \pi_i(p^*_1, \ldots, p^*_i, \ldots, p^*_N)$ for any $p_i \geq 0$ and $i = 1, \cdots, N$. To make our notation simpler, we denote $\pi_i(p^*_1, \cdots, p^*_N)$ by $\pi_i{}^*$ and $D_i(p^*_1, \cdots, p^*_N)$ by $D^*{}_i$.

A government that has only one adjoining neighbour is called a *peripheral*; A government that has exactly two adjoining neighbours is called an *interior*. On this linear market, exactly two governments are peripheral; the rest are interior. We number the governments in such a way that the first government is neighboured only by the second one, and the N-th government is neighboured only by the $(N-1)$-th one, and for $i = 2, \cdots, N-1$, the i-th government is neighboured by both the $(i-1)$-th and $(i+1)$-th governments. This configuration of locations is illustrated in Figure 3.

Because a customer will buy from the firm with the lowest full price, the price to be paid by any customer on the whole market segment is the lower envelope of the full price schedules of all firms. The market areas of the $(i-1)$-th, the i-th and the $(i+1)$-th government are illustrated in Figure 4 and 5. Distances are measured along the horizontal scale and full prices along the vertical. The left boundary of the i-th government is located at point A, and the right boundary is located at point B. In each figure, three dashed line segments express the lower envelopes of full price functions of all firms within each government, respectively. There are now two possibilities, corresponding to whether $D_i(p_1, \cdots, p_N) = 0$ or not. For example, in Figure 4, $D_i(p_1, \cdots, p_N) = 0$ since the i-th government is undercut by the both its adjoining governments. But in this case no Nash equilibrium exists since the i-th government can always secure its local market area by changing a positive tax smaller than the price at which the full prices of both adjoining governments intersect. This states that if an equilibrium exists, $D_i(p_1^*, \cdots, p_N^*) > 0$ for any $i \in \{2, \cdots, N-1\}$. That is, if there exists a tax vector in equilibrium, two corresponding lower envelopes of adjoining neighbours have to intersect and all intersections have to define each market boundary. In this case, the total demands in equilibrium are expressed by

$$D_1^* \quad = \quad L_1 + \frac{1}{\gamma}(p^*{}_2 - p^*{}_1);$$

$$
\begin{aligned}
D_i^* &= L_i + \frac{1}{\gamma}(p^*{}_{i-1} + p^*{}_{i+1} - 2p^*{}_i), \qquad i \in \{2, \cdots, N-1\}; \qquad (1)\\
D_N^* &= L_N + \frac{1}{\gamma}(p^*{}_{N-1} - p^*{}_N).
\end{aligned}
$$

In Figure 5, the market boundaries are defined at a and b. As the customers on bB go shopping to the firms along the boundary in the $(i+1)$-th government, the firms on bB never get demands, and the firms along the boundary on the $(i+1)$-th government get many demands. In fact, there are few gas stations in Belgium near the border between Belgium and Luxembourg. On the other hand, there are many gas stations in Luxembourg close to the border. It is straightforward to see that their revenues in equilibrium are stated formally as:

$$
\begin{aligned}
\pi_1^* &= -\frac{1}{\gamma}\{p^*{}_1{}^2 - (p^*{}_2 + \gamma L_1)p^*{}_1\},\\
\pi_i^* &= -\frac{1}{\gamma}\{2p^*{}_i{}^2 - (p^*{}_{i-1} + p^*{}_{i+1} + \gamma L_i)p^*{}_i\} \qquad i \in \{2, \cdots, N-1\}; \qquad (2)\\
\pi_N^* &= -\frac{1}{\gamma}\{p^*{}_N{}^2 - (p^*{}_{N-1} + \gamma L_N)p^*{}_N\}.
\end{aligned}
$$

Completing the square on $p^*{}_i$ in (2) gives

$$
\begin{aligned}
\pi_1^* &= -\frac{1}{\gamma}\{p^*{}_1 - \frac{1}{2}(p^*{}_2 + \gamma L_1)\}^2 + \frac{1}{4\gamma}(p^*{}_2 + \gamma L_1)^2,\\
\pi_i^* &= -\frac{2}{\gamma}\{p^*{}_i - \frac{1}{4}(p^*{}_{i-1} + p^*{}_{i+1} + \gamma L_i)\}^2 + \frac{1}{8\gamma}(p^*{}_{i-1} + p^*{}_{i+1} + \gamma L_i)^2, i \in \{1, \cdots, N\} \quad (3)\\
\pi_N^* &= -\frac{1}{\gamma}\{p^*{}_N - \frac{1}{2}(p^*{}_{N-1} + \gamma L_N)\}^2 + \frac{1}{4\gamma}(p^*{}_{N-1} + \gamma L_N)^2.
\end{aligned}
$$

Hence, if an equilibrium exists, $p^*{}_i$'s have to meet the following simultaneous system of linear equations:

$$
\begin{aligned}
2p^*{}_1 - p^*{}_2 &= \gamma L_1,\\
-p^*{}_{i-1} + 4p^*{}_i - p^*{}_{i+1} &= \gamma L_i, \qquad i \in \{2, \cdots, N-1\}; \qquad (4)\\
-p^*{}_{N-1} + 2p^*{}_N &= \gamma L_N.
\end{aligned}
$$

It is easy to check that $p_i^* > 0$ for all $i \in \{1, \cdots, N\}$, thus we have $D_i^* > 0$ for all $i \in \{1, \cdots, N\}$. It should be noted that the solution to the system (4) is unique. It can be concluded that if an equilibrium exists, it is given by the solution to the system (4) and it is unique. The analytical solution to the system (4) is given by (33) in Appendix. Substituting (4) into (1) gives the total demand in equilibrium as follows:

$$
D^*{}_1 = \frac{1}{\gamma}p^*{}_1,
$$

$$\begin{aligned} D^*_i &= \frac{2}{\gamma}p^*_i, \qquad i \in \{2, \cdots, N-1\}; \\ D^*_N &= \frac{1}{\gamma}p^*_N, \end{aligned} \tag{5}$$

provided that an equilibrium exists. The insertion of (4) in (2) yields the revenue in equilibrium as follows:

$$\begin{aligned} \pi^*_1 &= \frac{1}{\gamma}{p^*_1}^2, \\ \pi^*_i &= \frac{2}{\gamma}{p^*_i}^2, \qquad i \in \{2, \cdots, N-1\}; \\ \pi^*_N &= \frac{1}{\gamma}{p^*_N}^2, \end{aligned} \tag{6}$$

provided that an equilibrium exists.

Equilibrium for Two Governments

We start with a duopoly analysis.

Proposition 1. *When the number of governments is two, there exists a unique tax vector in equilibrium as follows:*

$$p^*_1 = \frac{\gamma}{3}(2L_1 + L_2), \qquad p^*_2 = \frac{\gamma}{3}(L_1 + 2L_2). \tag{7}$$

Proof. For a given p_2, the first government's demand function is given by

$$D_1(p_1, p_2) = \begin{cases} L_1 + L_2, & 0 \leq p_1 < p_2 - \gamma L_2; \\ L_1 + \frac{1}{\gamma}(p_2 - p_1), & p_2 - \gamma L_2 \leq p_1 < p_2; \\ 0, & p_2 + \gamma L_1 \leq p_1. \end{cases} \tag{8}$$

The derivation of this demand function is illustrated in Figure 6. For $0 \leq p_1 < p_2 - \gamma L_2$, the first government everywhere undercuts the second one in terms of full price. The case of $p_2 + \gamma L_1 \leq p_1$ is the converse to this. For $p_2 - \gamma L_2 < p_1 < p_2$, the marginal customer lies between O and $L_1 + L_2$. Thefore, its revenue function is described by

$$\pi_1(p_1, p_2) = \begin{cases} (L_1 + L_2)p_1, & 0 \leq p_1 < p_2 - \gamma L_2; \\ L_1 + \frac{1}{\gamma}(p_2 - p_1)p_1, & p_2 - \gamma L_2 \leq p_1 < p_2; \\ 0, & p_2 + \gamma L_1 \leq p_1. \end{cases} \tag{9}$$

The revenue function (9) is shown in Figure 7. As seen from the figure, this function is continuous and quasi-concave in p_1 over the domain $0 \leq p_1$ for a given $p_2(\geq 0)$. From the fixed-point theory, this ensures the existence of a Nash equilibrium, whatever the sizes $L_1(> 0)$ and $L_2(> 0)$ may be: see Friedman (1977). Therefore, by solving the system (4), we have (7). □

Substituting (7) into (6) gives

$$\pi^*{}_1 = \frac{\gamma}{9}(2L_1 + L_2)^2, \qquad \pi^*{}_2 = \frac{\gamma}{9}(L_1 + 2L_2)^2. \tag{10}$$

Three major properties can be drawn from the proposition and (10) as follows:

(n1) In the standard version, the profit function is neither discontinuous nor quasi-concave, and no Nash equilibrium existes for some cases: see d'Aspremont, Gabszewicz, Thisse(1979). Contrary this, in our model a Nash equilibrium necessarily and uniquely exists. It is worth noting that the existence property holds for any continuous monotone increasing cost function with respect to distance.

(n2) Let $\delta \equiv \frac{L_1}{L_1+L_2}$, based on (10), we have

$$\frac{\pi^*{}_1}{\pi^*{}_1 + \pi^*{}_2} = \frac{(1+\delta)^2}{(1+\delta)^2 + (2-\delta)^2}. \tag{11}$$

For comparative purposes, we have sketched the revenue ratio $\frac{\pi^*{}_1}{\pi^*{}_1+\pi^*{}_2}$ in Figure 8. The figure will help to clarify the relationship between the size ratios and the revenue ratios. An observation of the figure shows that as a fixed size ratio $\delta = \frac{L_1}{L_1+L_2}$ increases, the revenue ratio $\frac{\pi^*{}_1}{\pi^*{}_1+\pi^*{}_2}$ increases. In addition,

$$\frac{1}{5} < \frac{\pi^*{}_1}{\pi^*{}_1 + \pi^*{}_2} < \frac{4}{5}.$$

Surprisingly perhaps, this first inequality states that the revenue of a government becomes more than one-fifth than that of the other, irrespective of the size ratios.

(n3) We place the dotted line at 45° to each axis in Figure 8. The revenue ratio $\frac{\pi^*{}_1}{\pi^*{}_1+\pi^*{}_2}$ lies above the dotted line for $\delta < \frac{1}{2}$, that is, $L_1 < L_2$. Mathematically, using (11), we have $\frac{\pi^*{}_1}{\pi^*{}_1+\pi^*{}_2} > \frac{L_1}{L_1+L_2}$ for $L_1 < L_2$. This inequality is equivalent to $\frac{\pi^*{}_1}{\pi^*{}_2} > \frac{L_1}{L_2}$. This means that the smaller government is profitable because the revenue of the smaller government is more than the size ratios multiplied by that of the bigger one.

Equilibrium for Three Governments

In this subsection, we specialize to the sizes of governments but consider three governments.

Proposition 2a. *When the number of governments is three and the sizes of the peripheral governments are the same, i.e., $L_1 = L_3 (\equiv L)$, there exists a unique tax vector in equilibrium given by*

$$p^*{}_1 = \frac{\gamma}{6}(4L + L_2), \qquad p^*{}_2 = \frac{\gamma}{6}(2L + 2L_2), \qquad p^*{}_3 = \frac{\gamma}{6}(4L + L_2). \tag{12}$$

Proof. If an equilibrium exists, by solving the system (4), we have (12). In order to ascertain that (p_1^*, p_2^*, p_3^*) is a vector in equilibrium, one must further exclude the possibility that each government can increase its revenue by changing its tax.

(1) For fixed p^*_2 and p^*_3, if $p^*_2 \leq p^*_3$, p^*_3 is so small that the first government cannot undercut the third government for any $p_1 (> 0)$. Accordingly the first government's revenue function is given by

$$\pi_1(p_1, p_2^*, p_3^*) = \begin{cases} \frac{1}{\gamma}(p_3^* + \gamma L - p_1)p_1, & 0 \leq p_1 < p_2^* - \gamma L_2; \\ \frac{1}{\gamma}(p_2^* + \gamma L - p_1)p_1, & p_2^* - \gamma L_2 < p_1 < p_2^* + \gamma L; \\ 0, & p_2^* + \gamma L \leq p_1. \end{cases} \tag{13}$$

The derivation of $\pi_1(p_1, p_2^*, p_3^*)$ is given in Figure 10. In contrast to the case of two governments, the function $\pi_1(p_1, p_2^*, p_3^*)$, which is illustrated in Figure 11, is discontinuous at the point that corresponds to the price at which the first government just undercuts the second one. Note that if $p_2^* > \gamma L_2$

$$p_2^* - \gamma L_2 = \frac{\gamma}{6}(2L - 4L_2) < \frac{\gamma}{6}(4L + L_2) = \frac{p_2^* + \gamma L}{2} \leq \frac{p_3^* + \gamma L}{2}$$

and

$$\begin{aligned} \pi_1^*(p_2^* - \gamma L_2, p_2^*, p_3^*) = & \; \tfrac{\gamma}{36} \;\; (16L^2 - 22LL_2 - 20L_2^2) \\ < & \; \tfrac{\gamma}{36} \;\; (16L^2 + 8LL_2 + L_2^2) = \pi_1^*(\frac{p_2^* + \gamma L}{2}, p_2^*, p_3^*). \end{aligned}$$

Hence, its best reply is $\frac{p_2^* + \gamma L}{2}$. In the case of $p^*_2 > p^*_3$, both p^*_2 and p^*_3 are so small that the first government cannot undercut any other competitors. Obviously, in this case its best reply is $\frac{p_2^* + \gamma L}{2}$. In conclusion, $\frac{p_2^* + \gamma L}{2}$, i.e., p_1^* is optimal for any given p_2^* and p_3^*.

(2) As $p^*_1 = p^*_3$, the second government's revenue function for fixed p^*_1 and p^*_3 is represented by

$$\pi_2(p_1^*, p_2, p_3^*) = \begin{cases} (2L + L_2)p_2, & 0 \leq p_2 < p_1^* - \gamma L; \\ \frac{1}{\gamma}(2p_1^* + \gamma L_2 - 2p_2)p_2, & p_1^* - \gamma L \leq p_2 < \frac{2p_1^* + \gamma L_2}{2}; \\ 0, & \frac{2p_1^* + \gamma L_2}{2} \leq p_2. \end{cases} \tag{14}$$

The derivation of $\pi_2(p_1^*, p_2, p_3^*)$ is given in Figure 9. The function $\pi_2(p_1^*, p_2, p_3^*)$ is continuous in the positive region. In addition, it is unimodal, i.e., it has a single relative maximum. By noting that

$$p_1^* - \gamma L = \frac{\gamma}{6}(-2L + L_2) < \frac{\gamma}{6}(2L + 2L_2) = \frac{2p_1^* + \gamma L_2}{4},$$

its best reply against p_1^* and p_3^* is $\frac{2p_1^* + \gamma L_2}{4}$, i.e., p_2^*.

(3) The symmetric property ensures that the best reply of the third government is p_3^*. □

The insertion of (12) in (6) yields

$$\pi_a{}^*{}_1 = \frac{\gamma}{36}(4L + L_2)^2, \qquad \pi_a{}^*{}_2 = \frac{2\gamma}{9}(L + L_2)^2, \qquad \pi_a{}^*{}_3 = \frac{\gamma}{36}(4L + L_2)^2. \tag{15}$$

Proposition 2b. *When the number of governments is three and the sizes of the adjoining governments are the same, i.e., $L_2 = L_3 (\equiv L)$, there exists a unique tax vector in equilibrium given by (16) if and only if $L_1 \leq \frac{4\sqrt{13}+1}{3} L (\approx 5.141 L_1)$.*

$$p^*{}_1 = \frac{\gamma}{12}(7L_1 + 3L), \qquad p^*{}_2 = \frac{\gamma}{12}(2L_1 + 6L), \qquad p^*{}_3 = \frac{\gamma}{12}(L_1 + 9L). \tag{16}$$

Proof. If an equilibrium exists, by solving the system (4), we have (16).

(1) For fixed $p^*{}_2$ and $p^*{}_3$ such that $p^*{}_2 \leq p^*{}_3$, both $p^*{}_2$ and $p^*{}_3$ are so small that the first government cannot undercut any other competitors. It follows that its best reply is $\frac{p_2^* + \gamma L}{2}$. For fixed $p^*{}_2$ and $p^*{}_3$ such that $p^*{}_2 > p^*{}_3$, $p^*{}_3$ is so small that the first government cannot undercut only the third government for any $p_1 (> 0)$. Consequently, the first government's revenue function is expressed as follows:

$$\pi_1(p_1, p_2^*, p_3^*) = \begin{cases} \frac{1}{\gamma}(p_3^* + \gamma L_1 - p_1)p_1, & 0 \leq p_1 < p_3^* - \gamma L; \\ \frac{1}{2\gamma}(p_3^* + 2\gamma L_1 + \gamma L - p_1)p_1, & p_3^* - \gamma L \leq p_1 < 2p_2^* - p_3^* - \gamma L; \\ \frac{1}{\gamma}(p_2^* + \gamma L_1 - p_1)p_1, & 2p_2^* - p_3^* - \gamma L \leq p_1 < p_2^* + \gamma L_1; \\ 0, & p_2^* + \gamma L_1 \leq p_1. \end{cases} \tag{17}$$

The derivation of $\pi_1(p_1, p_2^*, p_3^*)$ is given in Figure 12. Note that

$$2p_2^* - p_3^* - \gamma L \;\; = \;\; \frac{\gamma}{12}(3L_1 - 9L) < \frac{\gamma}{24}(13L_1 + 9L) = \frac{p_3^* + \gamma L_1}{2}.$$

It follows from $\frac{p_3^* + \gamma L_1}{2} < \frac{p_3^* + 2\gamma L_1 + \gamma L}{2}$ and $\frac{p_3^* + \gamma L_1}{2} < \frac{p_2^* + \gamma L_1}{2}$ that its best reply is $\frac{p_2^* + \gamma L}{2}$. In conclusion, $\frac{p_2^* + \gamma L}{2}$, i.e., p_1^* is optimal for any given p_2^* and p_3^*.

(2) For fixed $p^*{}_1$ and $p^*{}_3$ such that two full prices of peripheral governments intersect within the second government, i.e., $p^*{}_1 - \gamma L \leq p^*{}_3$, $p^*{}_3$ is so small that the second government cannot undercut the third one for any $p_2 (> 0)$. As a result, the second government's revenue function is represented by

$$\pi_2(p_1^*, p_2, p_3^*) = \begin{cases} \frac{1}{\gamma}(p_3^* + \gamma L + \gamma L_1 - p_2)p_2, & 0 \leq p_2 < p_1^* - \gamma L_1; \\ \frac{1}{\gamma}(p_1^* + p_3^* + \gamma L - 2p_2)p_2, & p_1^* - \gamma L_1 \leq p_2 < \frac{p_1^* + p_3^* + \gamma L}{2}; \\ 0, & \frac{p_1^* + p_3^* + \gamma L}{2} \leq p_2. \end{cases} \tag{18}$$

The derivation of $\pi_2(p_1^*, p_2, p_3^*)$ is given in Figure 13. By noting that

$$p_1^* - \gamma L_1 \;\; = \;\; \frac{\gamma}{12}(-5L_1 + 3L) < \frac{\gamma}{12}(2L_1 + 6L) = \frac{p_1^* + p_3^* + \gamma L}{4}$$

and $\frac{p_1^*+p_3^*+\gamma L}{4}, < \frac{p_3^*+\gamma L+\gamma L_1}{2}$, we see that its best reply is $\frac{p_1^*+p_3^*+\gamma L_2}{4}$, i.e., p_2^*. For fixed $p^*{}_1$ and $p^*{}_3$ such that $p^*{}_3+\gamma L \leq p^*{}_1$, $p^*{}_1$ is so small that the second government cannot undercut the first one for any $p_2(>0)$. As a result, the second government's revenue function is expressed as follows:

$$\pi_2(p_1^*, p_2, p_3^*) = \begin{cases} \frac{1}{\gamma}(p_1^* + 2\gamma L - p_2)p_2, & 0 \leq p_2 < p_3^* - \gamma L; \\ \frac{1}{\gamma}(p_1^* + p_3^* + \gamma L - 2p_2)p_2, & p_3^* - \gamma L < p_2 < p_3^* + \gamma L; \\ 0, & p_3^* + \gamma L \leq p_1. \end{cases} \quad (19)$$

The derivation of $\pi_2(p_1^*, p_2, p_3^*)$ is given in Figure 14. The function $\pi_2(p_1^*, p_2, p_3^*)$, which is shown in Figure 15, is discontinuous at the point that corresponds to the price such that the second government elects to price just below the third government's full price within the first government, i.e., the second government is just not undercut. Note that

$$p_3^* + \gamma L = \frac{\gamma}{12}(L_1 + 21L) \leq \frac{\gamma}{12}(2L_1 + 6L) = \frac{p_1^* + p_3^* + \gamma L}{4}$$

is equivalent to $15L \leq L_1$. Accordingly, if $15L \leq L_1$, its best reply is $p_3^* + \gamma L_2$. In this case, the third government would have an incentive to lower its tax in order to recover the market within the first government, thus yielding no equilibrium. Otherwise its best reply is p_2^*.

(3) For fixed $p^*{}_1$ and $p^*{}_2$ such that $p^*{}_2 \leq p^*{}_1$, $p^*{}_1$ is so small that the third government cannot undercut the first government for any $p_3(>0)$. In this case an argument similar to the first part of the proof of Proposition 2a holds. The third government's revenue function is expressed as follows:

$$\pi_3(p_1^*, p_2^*, p_3) = \begin{cases} \frac{1}{\gamma}(p_1^* + \gamma L_1 - p_3)p_3, & 0 \leq p_3 < p_2^* - \gamma L; \\ \frac{1}{\gamma}(p_2^* + \gamma L_1 - p_3)p_3, & p_2^* - \gamma L < p_3 < p_2^* + \gamma L_1; \\ 0, & p_2^* + \gamma L_1 \leq p_3. \end{cases} \quad (20)$$

The function is discontinuous at $p_3 = p_2^* - \gamma L$. Note that

$$p_2^* - \gamma L = \frac{\gamma}{6}(L_1 - 3L), < \frac{\gamma}{6}(7L_1 + 3L), = \frac{p_2^* + \gamma L_1}{2}$$

and

$$\begin{aligned} \pi_3^*(p_1^*, p_2^*, p_2^* - \gamma L) = & \ \frac{\gamma}{144} \ (10L^2 + 12LL_2 - 126L_2^2) \\ \geq & \ \frac{\gamma}{144} \ (L^2 + 18LL_2 + 81L_2^2) = \pi_3^*(p_1^*, p_2^*, \frac{p_2^* + \gamma L_1}{2}) \end{aligned}$$

is equivalent to $\frac{4\sqrt{13}+1}{3}L \leq L_1$. For $\frac{4\sqrt{13}+1}{3}L \leq L_1$, its best reply is $p_2^* - \gamma L$. That is, its optimal strategy is to everywhere undercuts the second one. In this case, the second government would have an incentive to lower its tax, thus yielding no equilibrium. Otherwise, its best reply is $\frac{p_2^*+\gamma L}{2}$. For fixed $p^*{}_1$ and $p^*{}_2$ such that $p^*{}_1 < p^*{}_2$, both $p^*{}_1$ and $p^*{}_2$ are so small that the third government cannot undercut any other competitors. Accordingly, its best reply is $\frac{p_2^*+\gamma L_1}{2}$. After all, if $\frac{4\sqrt{13}+1}{3}L \leq L_1$ no equilibrium exists, otherwise the best reply of the third government against p_1^* and p_2^* is $\frac{p_2^*+\gamma L_1}{2}$, i.e., p_3^*. □

Substituting (16) into (6) yields

$$\pi_b{}^*{}_1 = \frac{\gamma}{144}(7L_1 + 3L)^2, \qquad \pi_b{}^*{}_2 = \frac{\gamma}{18}(L_1 + 3L)^2, \qquad \pi_b{}^*{}_3 = \frac{\gamma}{144}(L_1 + 9L)^2, \tag{21}$$

provided that $L < \frac{1}{3}$.

Three major properties can be drawn from the previous two propositions:

(n1) When the sizes of the preferal governments are the same, the existance of an equilibrium is guaranteed. Contrary this, when the sizes of the adjoining governments are the same, an equilibrium exists if and only if $\frac{4\sqrt{13}+1}{3}L \leq L_1$.

(n2) By denoting $\delta_a \equiv \frac{L_2}{L_1+L_2+L_3}$, the revenues in equilibrium displayed in (15) are shown in Figure 16 as the dashed lines. By denoting $\delta_2 \equiv \frac{L_1}{L_1+L_2+L_3}$, the revenues in equilibrium displayed in (21) are shown in Figure 16 as the solid lines. This figure will help to clarify how the relative position has an influence on the revenue ratios when two governments have the same size. As seen in the figure,

$$\begin{aligned} \pi_{a2}^*(\delta) &> \pi_{b1}^*(\delta), &&\text{for small } \delta \\ \pi_{a2}^*(\delta) &< \pi_{b1}^*(\delta), &&\text{for large } \delta. \end{aligned}$$

This results states that if the size of the remainder is smaller, then its best location is interior, otherwise it is peripheral.

(n3) When the sizes of peripheral governments are the same, we have

$$\lambda_a(\delta_a) \stackrel{\text{def}}{=} \frac{\pi_a{}^*{}_2}{\pi_a{}^*{}_1 + \pi_a{}^*{}_2 + \pi_a{}^*{}_3} = \frac{(1+\delta_a)^2}{(2-\delta_a)^2 + (1+\delta_a)^2}.$$

On the other hand, when the sizes of the adjoining governments are the same, we have

$$\lambda_2(\delta_2) \stackrel{\text{def}}{=} \frac{\pi_b{}^*{}_1}{\pi_b{}^*{}_1 + \pi_b{}^*{}_2 + \pi_b{}^*{}_3} = \frac{(3+11\delta_b)^2}{(3+11\delta_b)^2 + 2(6-2\delta_b)^2 + (9-7\delta_b)^2}.$$

For comparative purposes, the revenue ratios $\lambda_a(\delta)$ and $\lambda_b(\delta)$ are presented in Figure 17. We also place the dotted line at 45° to each axis in Figure 17. The revenue ratio $\lambda_a(\delta)$ lies above the dotted line for $\delta < \frac{1}{2}$. This states that the interior government can get more revenue than the size ratio multiplied by the aggregate revenue of all governments if its size is less than half the size of the whole market. On the other hand, the revenue ratio $\lambda_b(\delta)$ lies above the dotted line only for very small δ. This means that the peripheral government can get more revenue than the size ratio multiplied by the aggregate revenue only if its size is relatively very small.

Equilibrium for N Governments

We now look at the model when there are more than three governments, with the restriction that the sizes of all governments are the same, i.e., $L_i \equiv L(i = 1, \cdots, N)$.

Proposition 3. *When the sizes of all governments are the same, there exists a unique tax vector in equilibrium. It is given by*

$$p^*{}_i = \frac{\gamma L}{A(N)} \sum_{k=1}^{N} \min\{\alpha_k, \alpha_i\} \min\{\alpha_{N+1-k}, \alpha_{N+1-i}\}. \tag{22}$$

where, $\alpha_k \overset{\text{def}}{=} \frac{1}{2}\{(2+\sqrt{3})^{k-1} + (2-\sqrt{3})^{k-1}\}, A(N) \overset{\text{def}}{=} \alpha_N + 2\sum_{k=1}^{N-2} \alpha_{N-k} + \alpha_1$.

Proof. Following (4), if an equilibrium exists, $p^*{}_i$'s have to satisfy the following simultaneous system of linear equations:

$$\begin{aligned} 2p^*{}_1 - p^*{}_2 &= \gamma L, \\ -p^*{}_{i-1} + 4p^*{}_i - p^*{}_{i+1} &= \gamma L, \qquad i \in \{2, \cdots, N-1\}; \\ -p^*{}_{N-1} + 2p^*{}_N &= \gamma L. \end{aligned} \tag{23}$$

Some simple calculation shows that

$$p^*{}_i < \gamma L, \qquad 1 \le i \le N. \tag{24}$$

It follows from the restriction $p_i > 0$ that

$$\begin{aligned} p_i &> p^*{}_{i+1} - \gamma L, \qquad i \in \{1, \cdots, N-1\}; \\ p_i &> p^*{}_{i-1} - \gamma L, \qquad i \in \{2, \cdots, N\}. \end{aligned}$$

These inequalities state that the i-th government can not undercut any other competitors by oneself. It remains to check that each government would not undercut its adjoining competitors with its next-adjoining governments. We know from the argument developed in the first part of the proof of Proposition 2b that the undercutting is not optimal for them. It can be concluded that each government would not undercut any competitors with any other competitors. Therefore, the best reply is p^*_i. The derivation of the analytical solution (22) is given in Appendix. □

Based on examining the proposition, the following five characteristics have been defined:

(n1) If the sizes of all governments are the same, a Nash equilibrium necessarily and uniquely exists, whatever the number of governments.

(n2) We see that $p^*_i > 0$, thus, $\pi^*_i > 0$. This means that competition among governments does not drive price down to the competitive level, thus yielding positive revenues.

(n3) Transport cost γ decreases, leading to more competition among governments. This, then, causes a drop in revenues.

(n4) In the case where $\gamma L_i = 1 (i = 1, \cdots, N)$, the taxes in equilibrium p^*_i's for $N = 3, \cdots, 8$ are illustrated in Table 1, containing two entries in each cell. The upper entry in each cell shows p^*_i's by fractional numbers. The lower entry in each cell represents p^*_i's by three places of decimals for comparative purposes. The revenues in equilibrium π^*_i's are given in Table 2. The data in Table 1 tell us that the farther government exists from the boundary of the linear market, the lower its tax in equilibrium. However, the data in Table 2 state that the second and $(N-1)$-th governments can obtain the largest revenue. Thus, the best location is the second and $(N-1)$-th positions.

(n5) Figure 18 shows the aggregate revenues of all governments when $L_i = \frac{1}{\gamma N}$ for $i = 1, \cdots, N$. We see from this figure that as the number of governments increase, the comparative process become more stringent.

3. The model for transportation firms

Statement of the model

The new conceptual feature introduced here is a trip. This model differs from the model in the previous sections in that the service provided by the firms is consumed by trips, with associated origins and destinations.

In here we confine our discussion to the model for two governments because the extension of the model to more than two governments is quite complex. We assume that the length of all trips are the same, defined as T. In addition, we assume that they are spread evenly in the whole market segment at uniform density. In a linear market, this implies that the middle point between the origin point and the corresponding destination point is evenly distributed between $\frac{T}{2}$ and $L_1 + L_2 - T$. Because the density of trips is taken as unity without loss of generality, the number of trips is $L_1 + L_2 - T$. It should be noted that when T is zero, this model agrees with the model examined in the previous two sections. Here we consider only the case where

$$0 \leq T < \min\{L_1, L_2\}. \tag{25}$$

This is because the case of small T is important for applications, and that the same story holds for the case of $T \geq \min\{L_1, L_2\}$.

Equilibrium

As is the case with the ordinary firms, the first government's demand function for a given p_2 is expressed as follows:

$$D_1(p_1, p_2) = \begin{cases} L_1 + L_2 - T, & 0 \leq p_1 < p_2 - \gamma(L_2 - T); \\ L_1 + \frac{1}{\gamma}(p_2 - p_1), & p_2 - \gamma(L_2 - T) \leq p_1 < p_2; \\ L_1 + \frac{1}{\gamma}(p_2 - p_1) - T, & p_2 < p_1 < p_2 + \gamma(L_1 - T); \\ 0, & p_2 + \gamma(L_1 - T) \leq p_1. \end{cases} \tag{26}$$

The derivation of this function is illustrated in Figure 19. For $0 \leq p_1 < p_2 - \gamma(L_2 - T)$, the first government everywhere undercuts the second one in terms of full price. The case of $p_2 + \gamma(L_1 - T) \leq p_1$ is the converse to this. Otherwise, the marginal customer lies between O and $L_1 + L_2$. Hence, the first government's revenue on completing the square on p_1 is written as:

$$\pi_1(p_1, p_2)$$
$$= \begin{cases} (L_1 + L_2 - T)p_1, & 0 \leq p_1 < p_2 - \gamma(L_2 - T); \\ -\frac{1}{\gamma}\{p_1 - \frac{1}{2}(p_2 + \gamma L_1)\}^2 + \frac{1}{4\gamma}\{p_2 + \gamma L_1\}^2, & p_2 - \gamma(L_2 - T) \leq p_1 < p_2; \\ -\frac{1}{\gamma}\{p_1 - \frac{1}{2}(p_2 + \gamma(L_1 - T))\}^2 + \frac{1}{4\gamma}\{p_2 + \gamma(L_1 - T)\}^2, & p_2 < p_1 < p_2 + \gamma(L_1 - T). \\ 0, & p_2 + \gamma(L_1 - T) \leq p_1. \end{cases} \tag{27}$$

The revenue function $\pi_1(p_1, p_2)$ is plotted for a fixed $p_2 > 0$ in Figures 20, 21 and 22. The most striking feature of this function is the presence of discontinuity at $p_1 = p_2$. The second government's revenue is similarly defined.

Proposition 4. *For the transportation firms and $L_1 \geq L_2$, there exists a unique tax vector in equilibrium if and only if*

$$L_1 \geq L_2 + 3\sqrt{TL_1} - 2T. \tag{28}$$

Whenever it exists, it is given by (29):

$$p_1{}^* = \frac{\gamma}{3}(2L_1 + L_2 - 2T), \qquad p_2{}^* = \frac{\gamma}{3}(L_1 + 2L_2 - T). \tag{29}$$

Proof. Given the revenue function (27), we consider four cases with respect to p_2. For sufficient high values of p_2, i.e., $0 \leq p_1 < p_2 - \gamma(L_2 - T)$, the first government would just undercut its competitor. In this case, no equilibrium exists. This is because no trips can never be the best reply of the second government against this reply. For high values of p_2, i.e., $p_2 - \gamma(L_2 - T) \leq p_1 < p_2$, the best reply of the first government would be $\frac{p_2+\gamma L_1}{2}$, as shown in Figure 20. In this case the first government capture all trips crossing the boundary. For lower values of p_2, i.e., $p_2 < p_1 < p_2 + \gamma(L_1 - T)$ the first government would just capture all trips crossing the boundary. That is, its best reply is $p_2 - \epsilon$ with $\epsilon > 0$ arbitrarily small, as can be seen from Figure 21. However, in this case, an equilibrium can never arise. This is because the best reply of the second government against this reply is $p_2 - 2\epsilon$. For sufficiently low values of p_2, i.e., $p_2 + \gamma(L_1 - T) \leq p_1$, the best reply of the first government would be $\frac{p_2+\gamma(L_2-T)}{2}$, as illusterated in Figure 22. In this case the first government capture no trips crossing the boundary. By interchanging the indices 1 and 2, similar conditions are obtained for the second government.

Hence if an equilibrium exists, there are two cases whether the first government can capture all trips crossing the boundary or not. If the first government can capture these trips, under the restriction that $\gamma L_1 \leq p_2^* < \gamma(L_1 + 2L_2 - 2T)$ and $0 < p_1^* \leq \gamma(\sqrt{L_2} - \sqrt{T})^2$

$$p_1{}^* = \frac{p_2{}^* + \gamma L_1}{2}, \qquad p_2{}^* = \frac{p_1{}^* + \gamma(L_2 - T)}{2}. \tag{30}$$

The above-mentioned inequalities and equalities are is equivalent to (28) and (29). Otherwise, an argument similar to the above one, but interchanging the indices 1 and 2 shows that $L_2 \geq L_1 + 3\sqrt{TL_2} - 2T$. It follows from the assumption (25) that $L_2 > L_1 + 3T - 2T \geq L_1$, this contradicts that $L_1 \geq L_2$. □

The substitution (29) into (27) yields

$$\pi_1{}^* = \frac{\gamma}{9}(2L_1 + L_2 - 2T)^2, \qquad \pi_2{}^* = \frac{\gamma}{9}(L_1 + 2L_2 - T)^2, \tag{31}$$

provided that an equilibrium exists. It should be noted that if $T = 0$, then the above-mentioned proposition agrees with proposition 1. We see from (29) and (31) that $p_1{}^* \geq p_2{}^*$ and $\pi_1{}^* \geq \pi_2{}^*$, irrespective of T.

The proceeding proposition leads to the following properties:

(n1) The existence of a Nash equilibrium is guaranteed when either the difference of the sizes of two governments $|L_1 - L_2|$ is large or the length of trips T is short. Otherwise, for both governments it is important to capture the trips crossing the boundary. Thus, the incentive to undercut will be greater.

(n2) As the length of trips T varies, the taxes in equilibrium also change. Let $\delta \equiv \frac{L_1}{L_1+L_2}$ and $\tau \equiv \frac{T}{L_1+L_2}$. We have

$$\frac{\pi^*_1}{\pi^*_1 + \pi^*_2} = \begin{cases} \frac{(1+\delta-\tau)^2}{(1+\delta-\tau)^2+(2-\delta-2\tau)^2}, & \text{if } \delta < \frac{1}{2}; \\ \frac{(1+\delta-2\tau)^2}{(1+\delta-2\tau)^2+(2-\delta-\tau)^2}. & \text{if } \delta > \frac{1}{2}, \end{cases} \quad (32)$$

provided that a Nash equilibrium exists. Three different functions $\frac{\pi_1^*}{\pi_1^*+\pi_2^*}$ are plotted, corresponding to the three values of τ ($\tau = 0.0$, $\tau = 0.1$, $\tau = 0.2$), in Figure 23. It should be noted that as τ increases, the smaller the range of δ such that a Nash equilibrium exists. In particular, if τ is more than $\frac{1}{4}$, no Nash equilibrium exists, irrespective of δ. Using (32) we have

$$\frac{\partial}{\partial \tau}\left\{\frac{\pi^*_1}{\pi^*_1 + \pi^*_2}\right\} = \begin{cases} > 0 & \text{if } \delta < \frac{1}{2}; \\ < 0 & \text{if } \delta > \frac{1}{2}. \end{cases}$$

Accordingly, for $L_1 < L_2$, as T increases, so does $\frac{\pi^*_1}{\pi^*_1+\pi^*_2}$. Contrary to this, the greater T, the smaller will be $\frac{\pi^*_2}{\pi^*_1+\pi^*_2}$. This results mean that for small governments, the transportation firms are more profitable than ordinary firms, provided that a Nash equilibrium exists. Moreover, as T increases, this effect becomes marked.

4. Conclusions

In summary, the following conclusions can be drawn from the preceding discussion:

(c1) When two government compete for revenues, for ordinary firms a Nash equilibrium necessarily and uniquely exists. For transportation firms the existence of a Nash equilibrium depends on the spatial configurations of governments.

(c2) The taxes and the revenues in equilibrium that we have obtained suggests that the spatial configuration and the sizes of contries are essential for governments. This is because, for interior governments, encroachments on the market areas of neighbours become profitable to the government, thus making the competitive process more stringent. While, the peripheral government enjoys a local monopoly.

(c3) Considering the size ratios of countries, smaller countries, like Luxembourg, will generate more revenues than bigger countries like France and Germany when the European market is opened. This conclusion is more apparent when the length of trips is increased.

A. Mathematical appendix

Property. *The solution to the system (4) is given by*

$$p^*_i = \frac{\gamma}{A(N)} \sum_{k=1}^{N} \min\{\alpha_k, \alpha_i\} \min\{\alpha_{N+1-k}, \alpha_{N+1-i}\} L_i. \tag{33}$$

Proof. First, α_i's satisfy the following recurring formula:

$$4\alpha_{k+1} = \alpha_{k+2} + \alpha_k, \qquad k \geq 1. \tag{34}$$

In addition, the following identities hold:

$$4\alpha_k\alpha_{N+1-k} - \alpha_k\alpha_{N-k} - \alpha_{k-1}\alpha_{N+1-k} = A(N), \qquad k = 1, \cdots, N-1. \tag{35}$$

We now prove (35) by induction. For $k = 2$, by noting that $\alpha_1 = 1$ and $\alpha_2 = 2$

$$\begin{aligned}
4\alpha_2\alpha_{N-1} - \alpha_2\alpha_{N-2} - \alpha_1\alpha_{N-1} &= 7\alpha_{N-1} - 2\alpha_{N-2} \\
&= \alpha_N + 3\alpha_{N-1} - \alpha_{N-2} \\
&= \alpha_N + 2\alpha_{N-1} + 3\alpha_{N-2} - \alpha_{N-3} \\
&= \alpha_N + 2(\alpha_{N-1} + \cdots + \alpha_3) + 3\alpha_2 - \alpha_1 \\
&= \alpha_N + 2(\alpha_{N-1} + \cdots + \alpha_3 + \alpha_2) + \alpha_1 \\
&= A(N).
\end{aligned}$$

Assuming it is true for k, we prove it is true for $k+1$. For $k+1$, making use of (34) and (35), we have

$$\begin{aligned}
4\alpha_{k+1}\alpha_{N-k} - \alpha_{k+1}\alpha_{N-k-1} - \alpha_k\alpha_{N-k} &= \alpha_{k+1}(4\alpha_{N-k} - \alpha_{N-k-1}) - \alpha_k\alpha_{N-k} \\
&= (4\alpha_k - \alpha_{k-1})\alpha_{N-k+1} - \alpha_k\alpha_{N-k} \\
&= 4\alpha_k\alpha_{N+1-k} - \alpha_k\alpha_{N-k} - \alpha_{k-1}\alpha_{N+1-k}. \\
&= A(N).
\end{aligned}$$

From (34) and (35), we have

$$2\alpha_N - \alpha_{N-1} = 7\alpha_{N-1} - 2\alpha_{N-2} = 4\alpha_2\alpha_{N-1} - \alpha_2\alpha_{N-2} - \alpha_1\alpha_{N-1} = A(N). \tag{36}$$

On the other hand, for $i = 2, \cdots, N-1$ rearranging (22) gives

$$\begin{aligned}
\frac{4A(N)}{\gamma} p^*_i &= 4\sum_{k=1}^{N} \min\{\alpha_k, \alpha_i\} \min\{\alpha_{N+1-k}, \alpha_{N+1-i}\} L_k; \\
-\frac{A(N)}{\gamma} p^*_{i+1} &= -\sum_{k=1}^{N} \min\{\alpha_k, \alpha_{i+1}\} \min\{\alpha_{N+1-k}, \alpha_{N-i}\} L_k; \\
-\frac{A(N)}{\gamma} p^*_{i-1} &= -\sum_{k=1}^{N} \min\{\alpha_k, \alpha_{i-1}\} \min\{\alpha_{N+1-k}, \alpha_{N+2-i}\} L_k.
\end{aligned}$$

Taking account of (34) and (35), we obtain

$$\begin{aligned}
\frac{A(N)}{\gamma} & (4p^*_i - p^*_{i+1} - p^*_{i-1}) \\
= & \sum_{k<i}^{N} \alpha_k(4\alpha_{N+1-i} - \alpha_{N-i} - \alpha_{N+2-i})L_k + (4\alpha_i\alpha_{N+1-i} - \alpha_i\alpha_{N-i} - \alpha_{i-1}\alpha_{N+1-i})L_i \\
& + \sum_{k>i}^{N} \alpha_{N+1-k}(4\alpha_i - \alpha_{i+1} - \alpha_{i-1})L_k \\
= & (4\alpha_i\alpha_{N+1-i} - \alpha_i\alpha_{N-i} - \alpha_{i-1}\alpha_{N+1-i})L_i \\
= & A(N)L_i.
\end{aligned}$$

This together with $A(N) \neq 0$ gives the rest equation of (22).

Moreover, rewriting (22) gives

$$\begin{aligned}
\frac{2A(N)}{\gamma}p^*_1 &= 2\alpha_1 \sum_{k=1}^{N} \alpha_{N+1-k}L_k; \\
-\frac{A(N)}{\gamma}p^*_2 &= -\alpha_1\alpha_{N-1}L_1 - \alpha_2 \sum_{k=2}^{N} \alpha_{N+1-k}L_k; \\
-\frac{A(N)}{\gamma}p^*_{N-1} &= -\alpha_2 \sum_{k=1}^{N-1} \alpha_k L_k - \alpha_1\alpha_{N-1}L_N; \\
\frac{2A(N)}{\gamma}p^*_N &= 2\alpha_1 \sum_{k=1}^{N} \alpha_k L_k.
\end{aligned}$$

As a result, noting that $2\alpha_1 = \alpha_2$, $\alpha_1 = 1$ and using (36), we have

$$\begin{aligned}
\frac{2A(N)}{\gamma}p^*_1 - \frac{A(N)}{\gamma}p^*_2 &= \alpha_1(2\alpha_N - \alpha_{N-1})L_1 + (2\alpha_1 - \alpha_2)\sum_{k=2}^{N} \alpha_{N+1-k}L_k = A(N)L_1, \\
\frac{2A(N)}{\gamma}p^*_N - \frac{A(N)}{\gamma}p^*_{N-1} &= (2\alpha_1 - \alpha_2)\sum_{k=1}^{N-1} \alpha_k L_k + \alpha_1(2\alpha_N - \alpha_{N-1})L_N = A(N)L_N.
\end{aligned}$$

As $A(N) \neq 0$, we obtain the first and the last equations of (22). □

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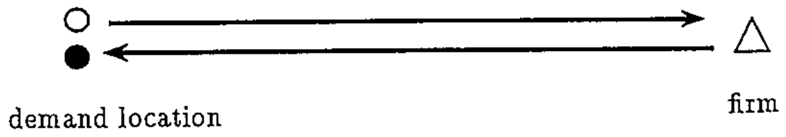

Figure 1: A typical flow pattern for ordinary firms

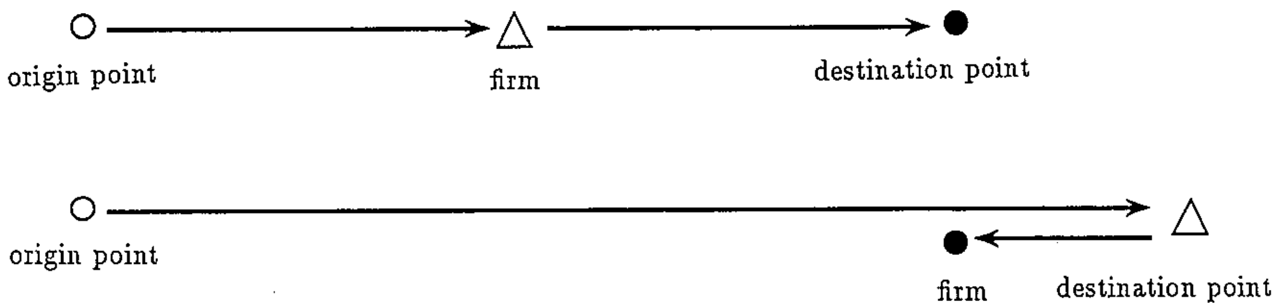

Figure 2: A typical flow pattern for transportation firms

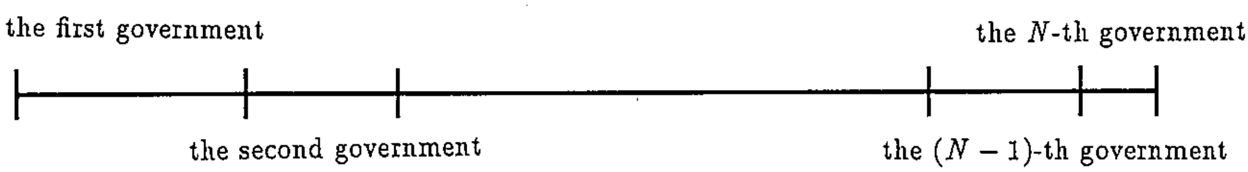

Figure 3: Linear market

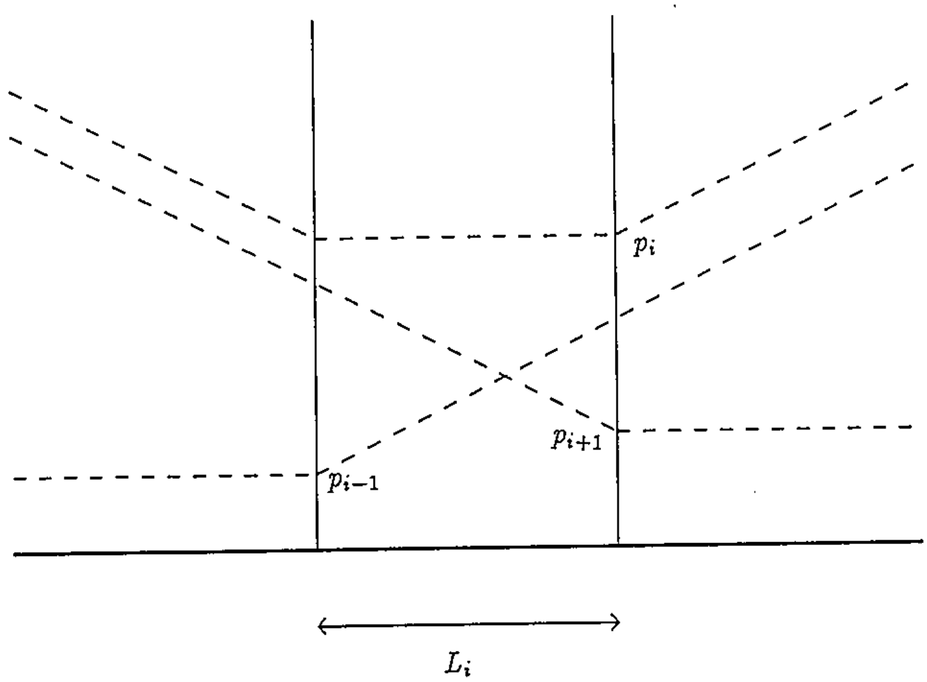

Figure 4: Price undercutting by two governments

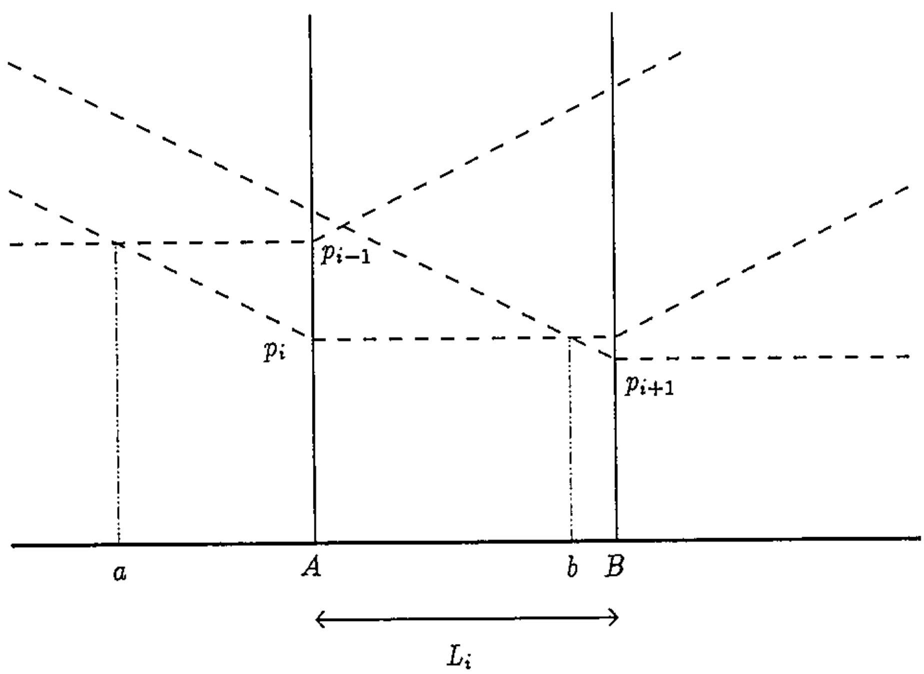

Figure 5: Market area of the i-th government

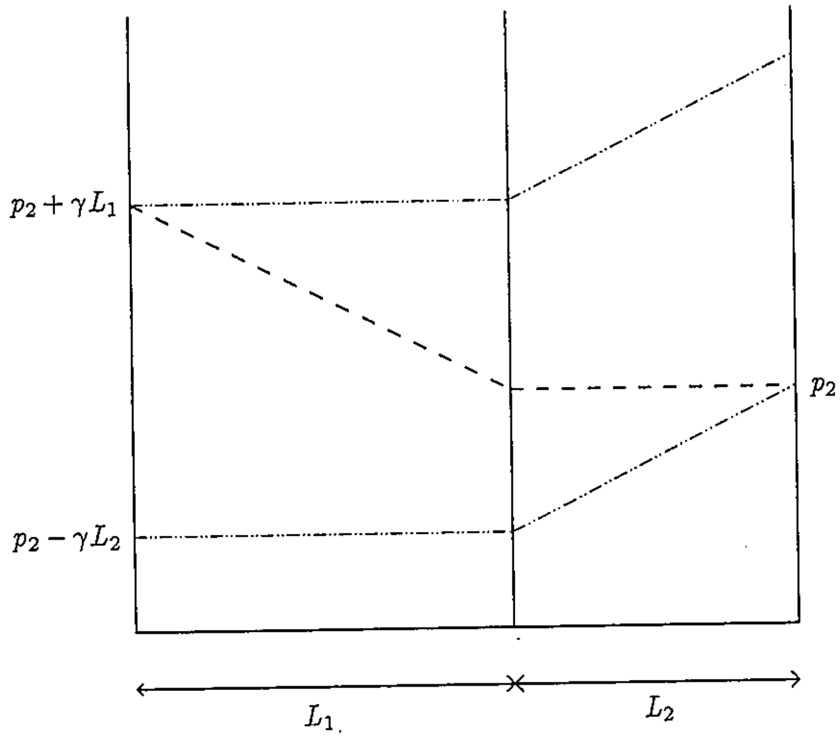

Figure 6: Derivation of the demand function of the first government

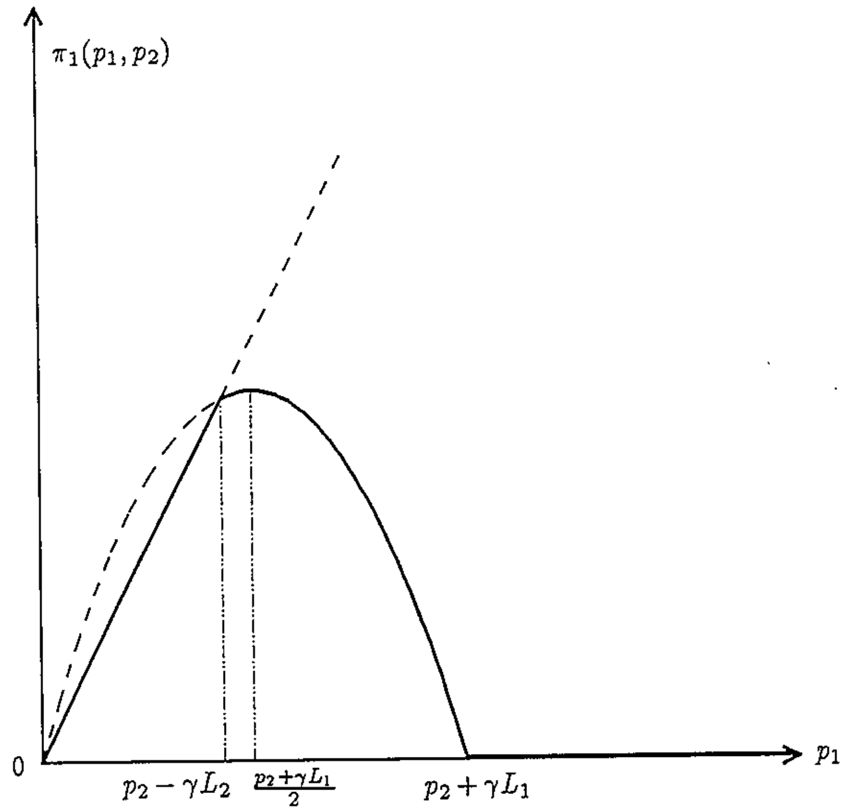

Figure 7: Revenue function of the first government

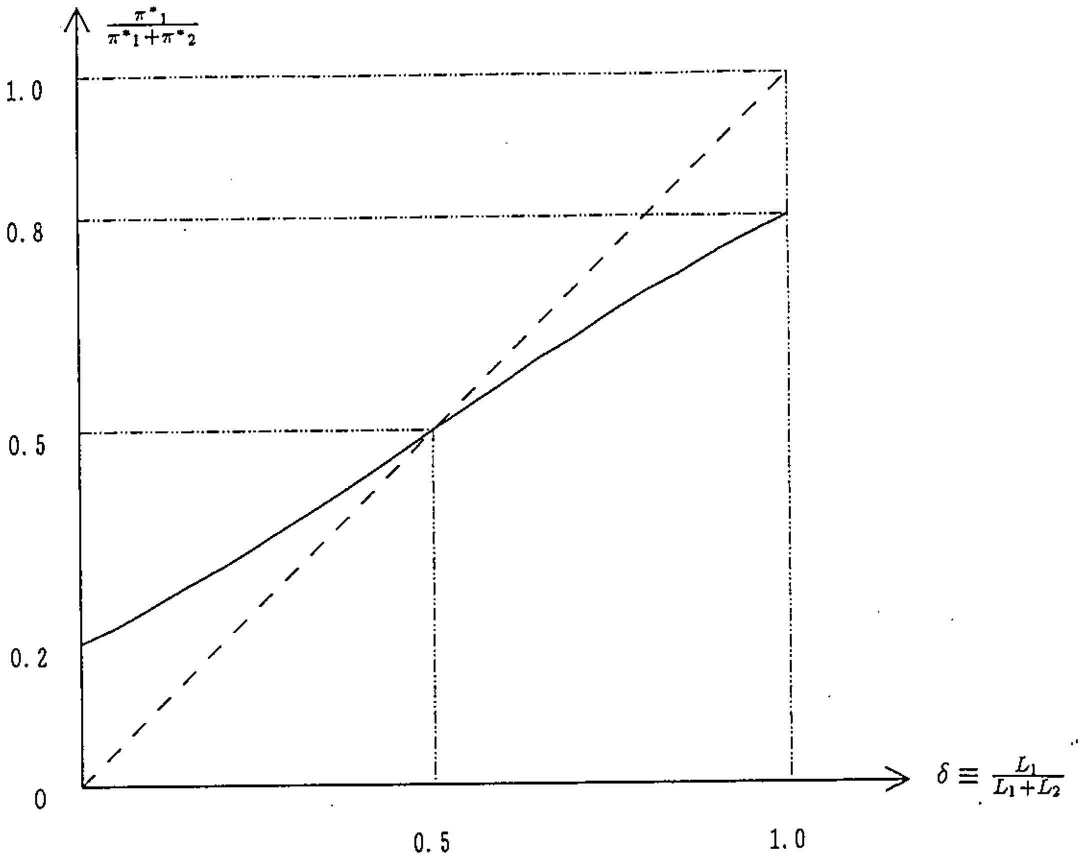

Figure 8: Relationship between size ratios and revenue ratios

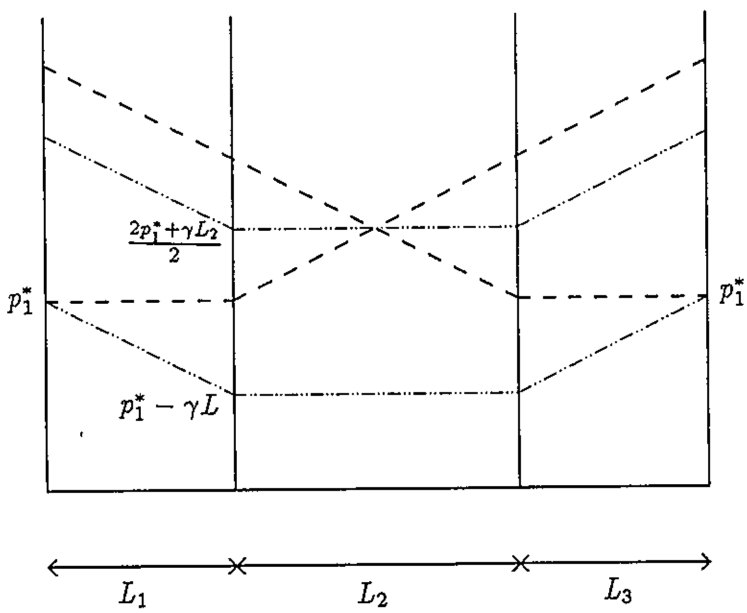

Figure 9: Derivation of the revenue function of the second government

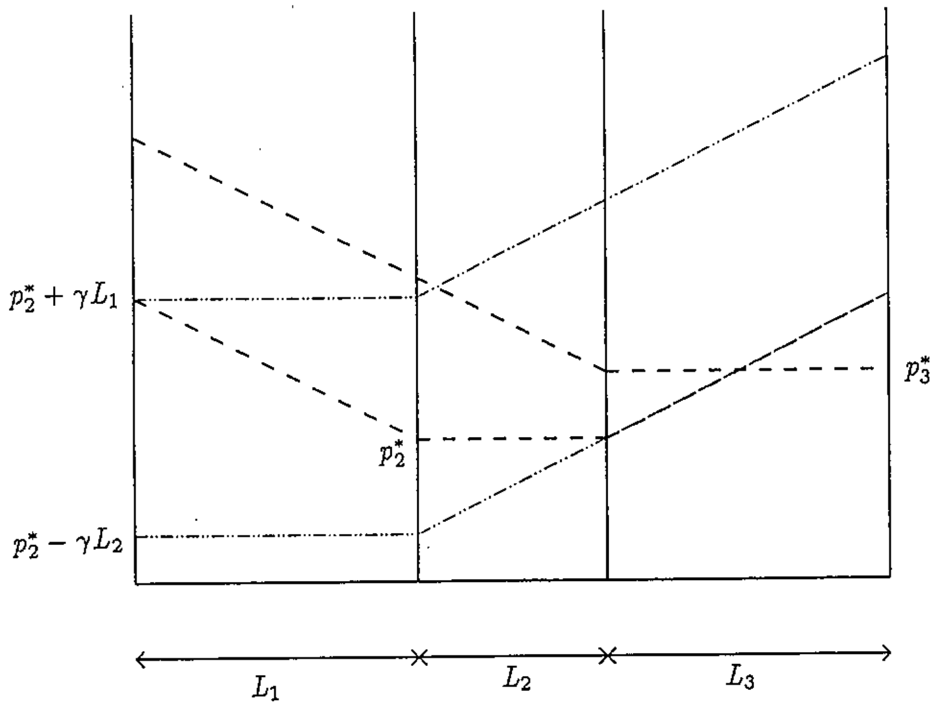

Figure 10: Derivation of the revenue function of the first government

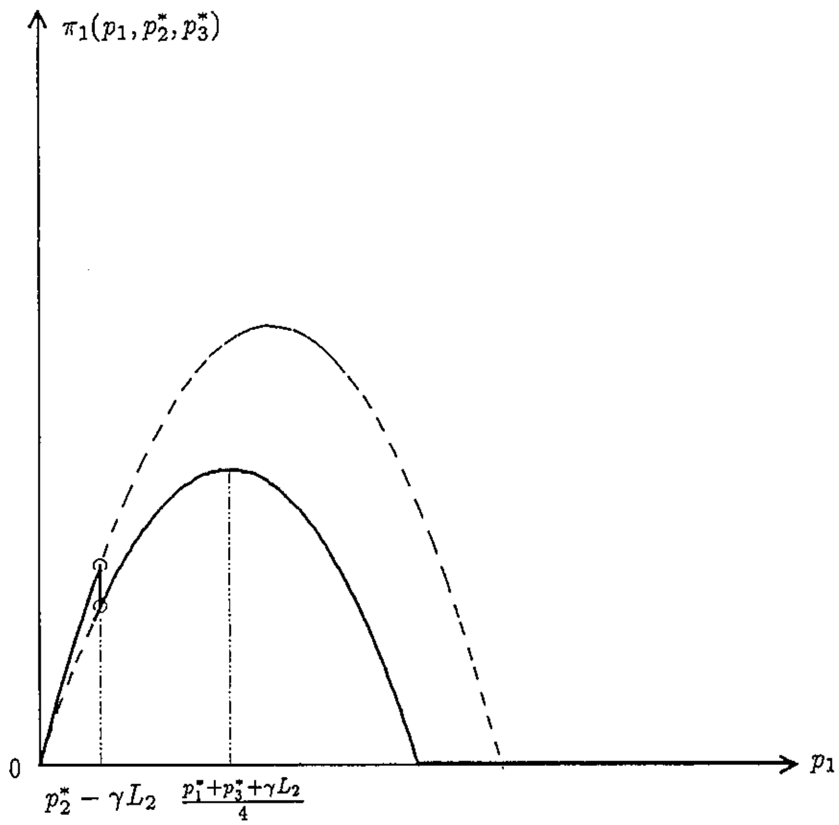

Figure 11: Revenue function of the first government

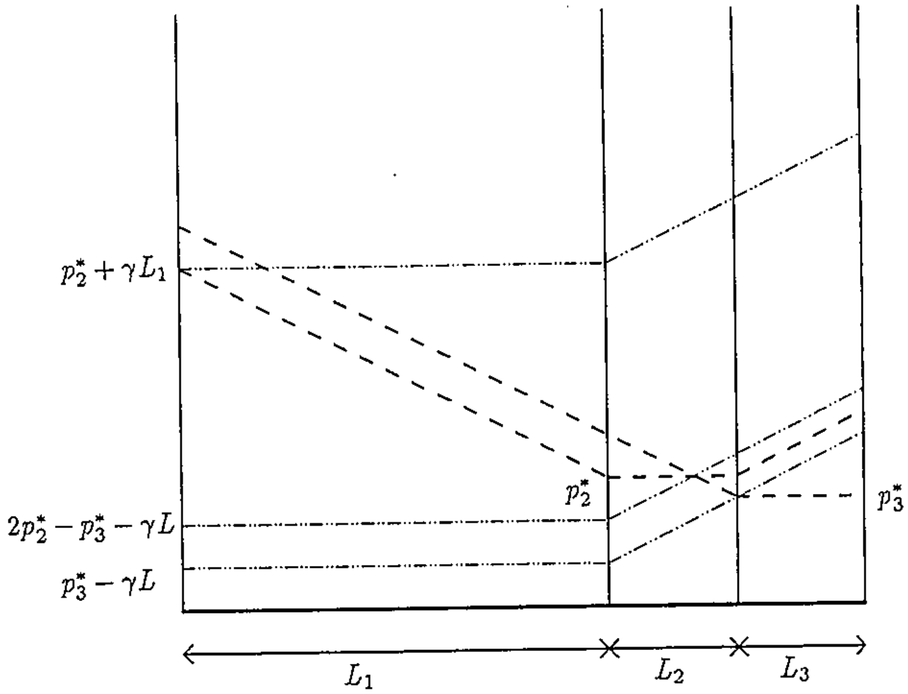

Figure 12: Derivation of the revenue function of the first government

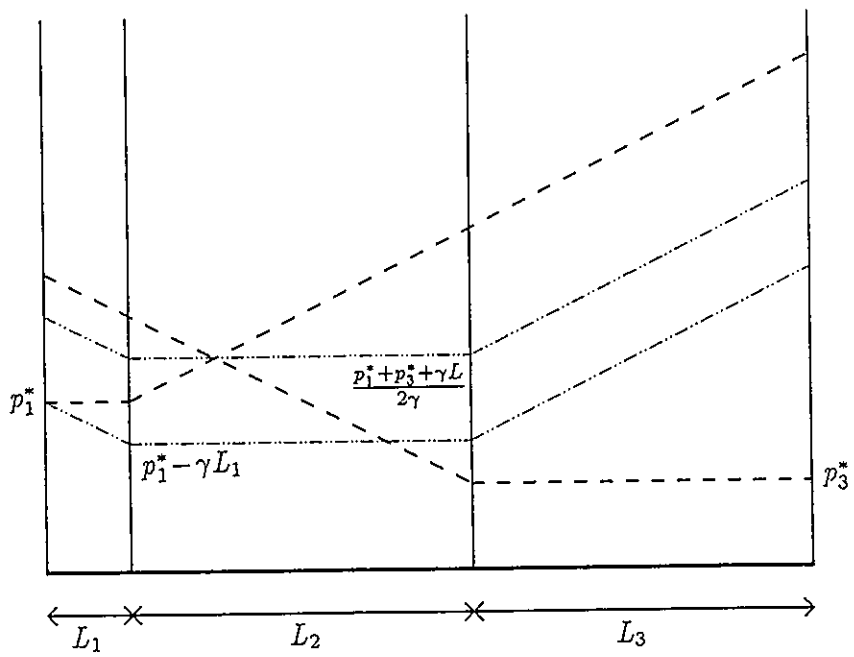

Figure 13: Derivation of the revenue function of the second government

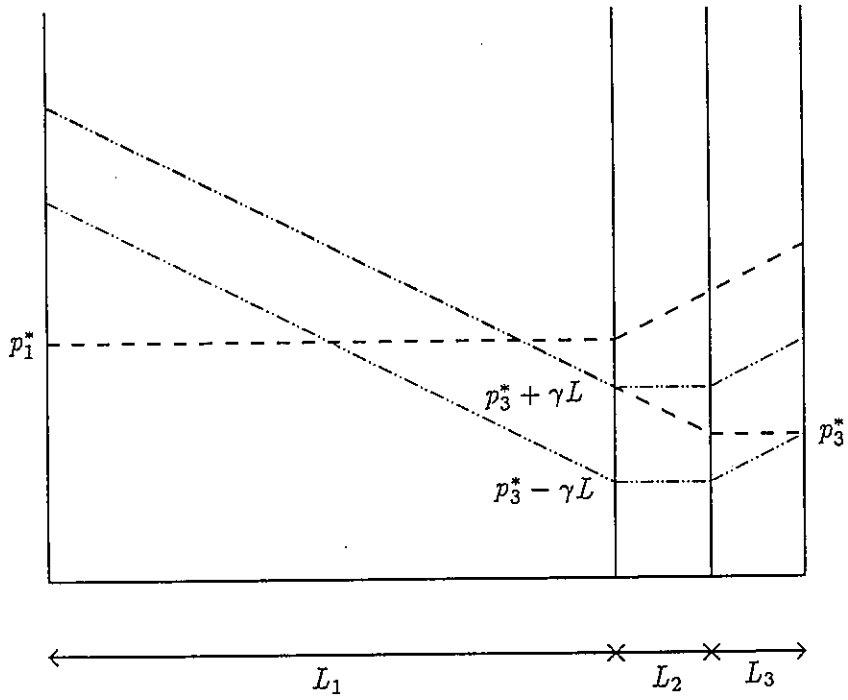

Figure 14: Derivation of the revenue function of the second government

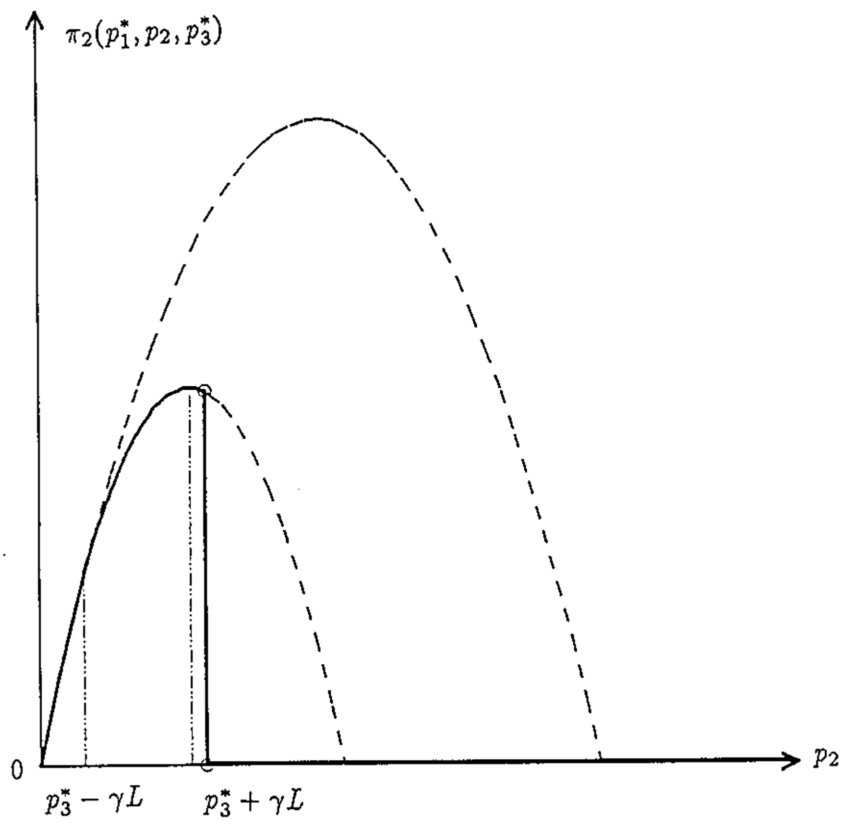

Figure 15: Revenue function of the second government

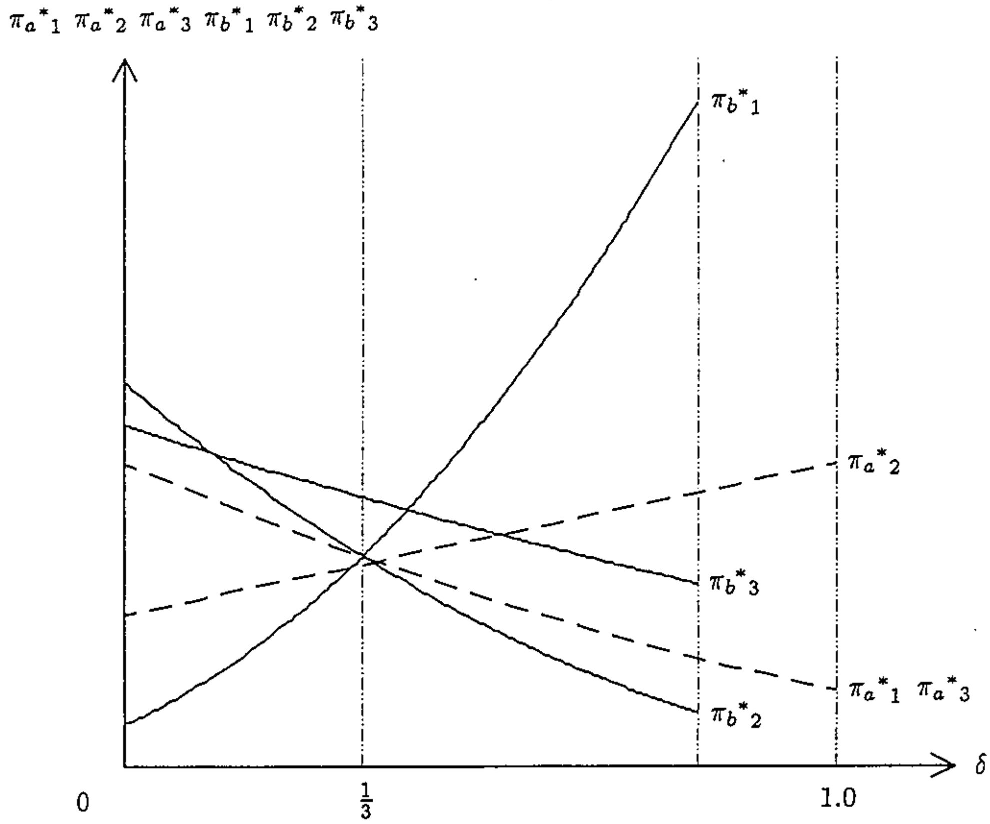

Figure 16: Revenue functions in equilibrium of three governments

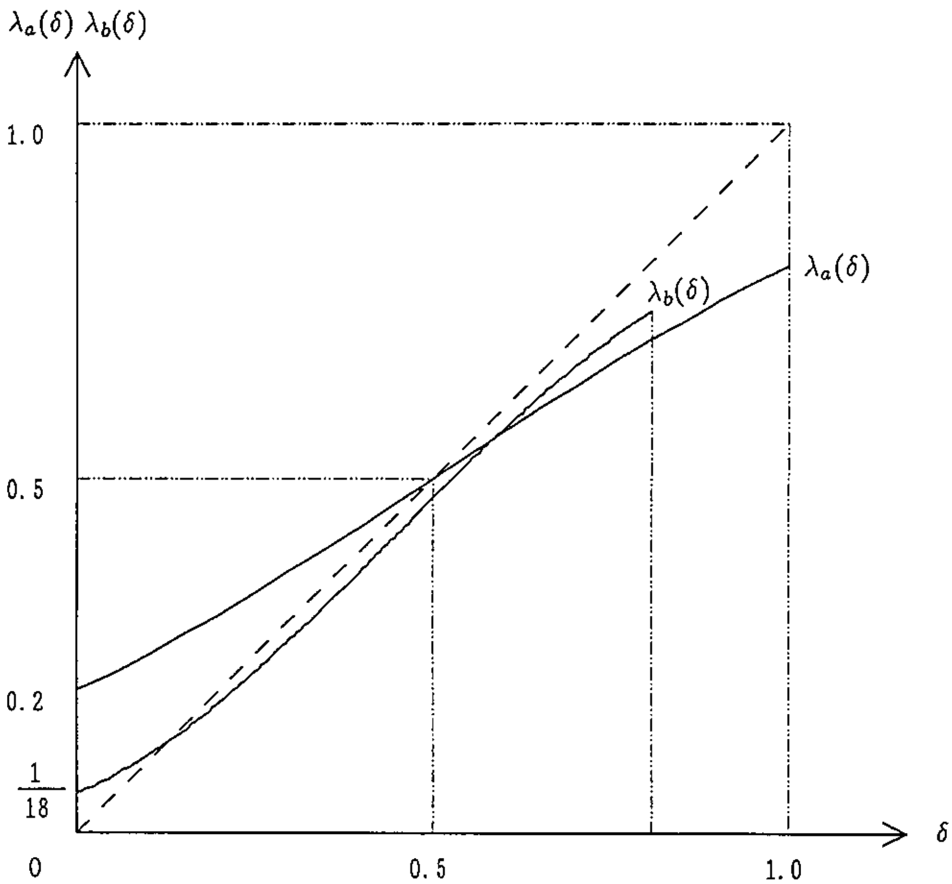

Figure 17: Relationship between size ratios and revenue ratios

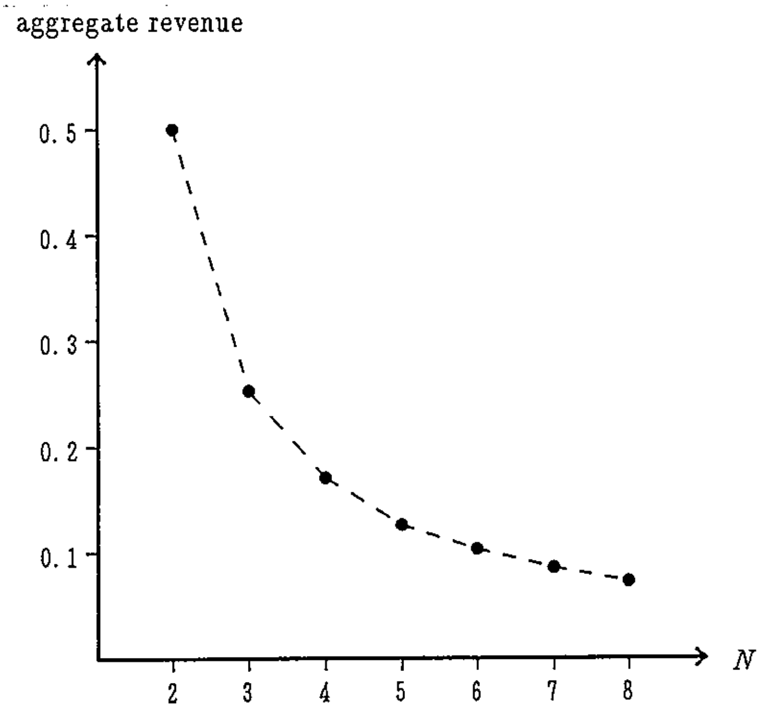

Figure 18: Aggregate revenues when the sizes of all firms are the same

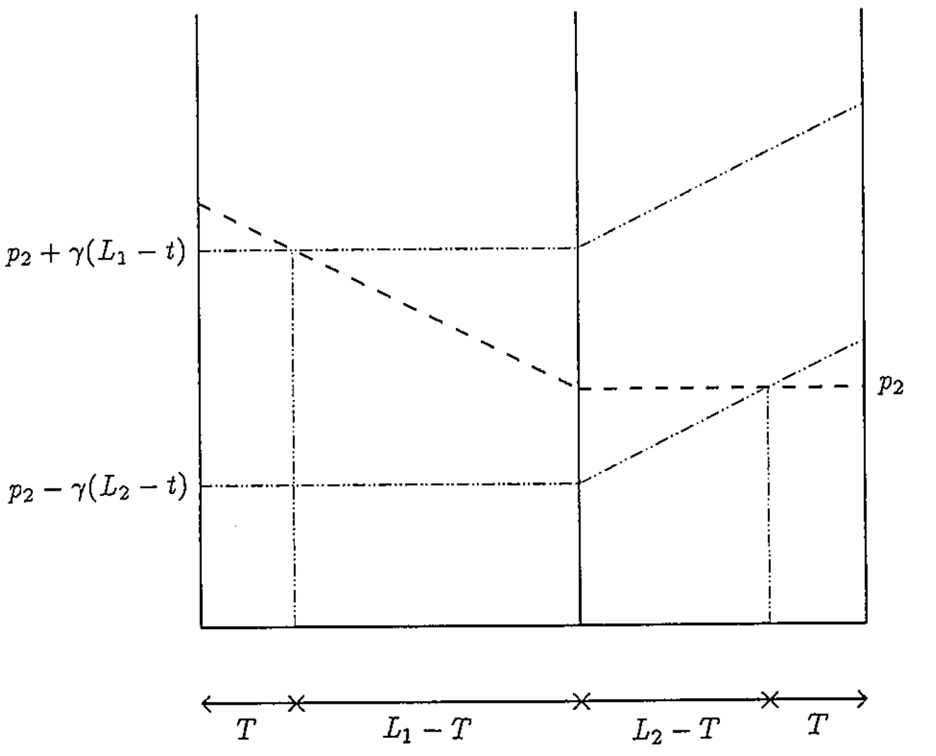

Figure 19: Derivation of the demand function for transportation firms

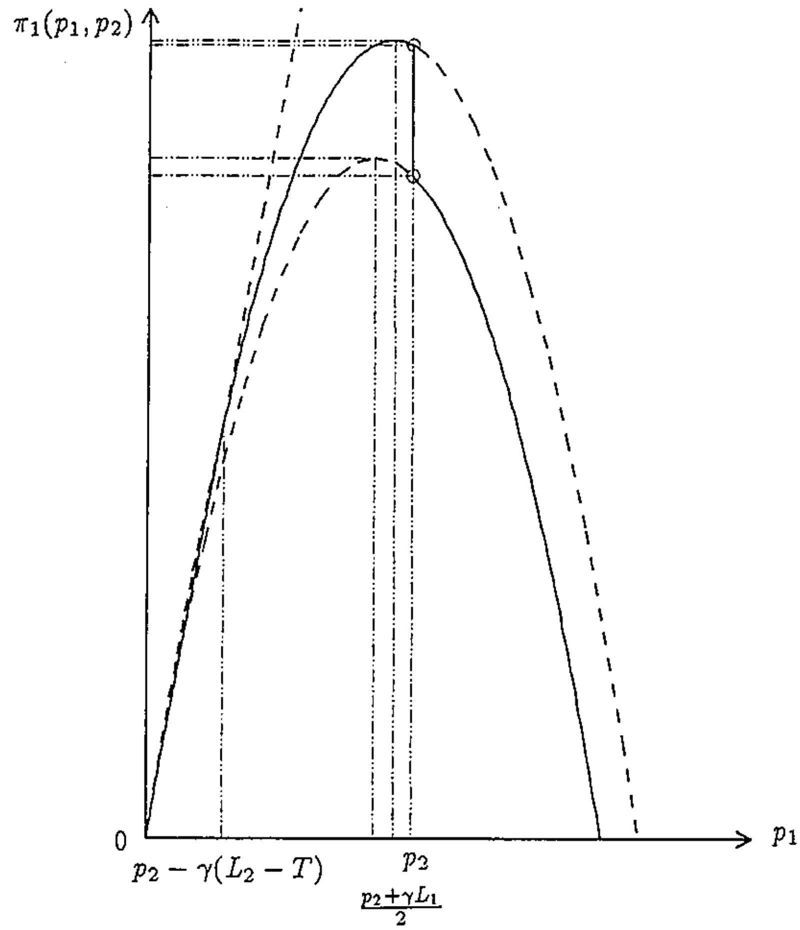

Figure 20: Revenue function for high value of p_2

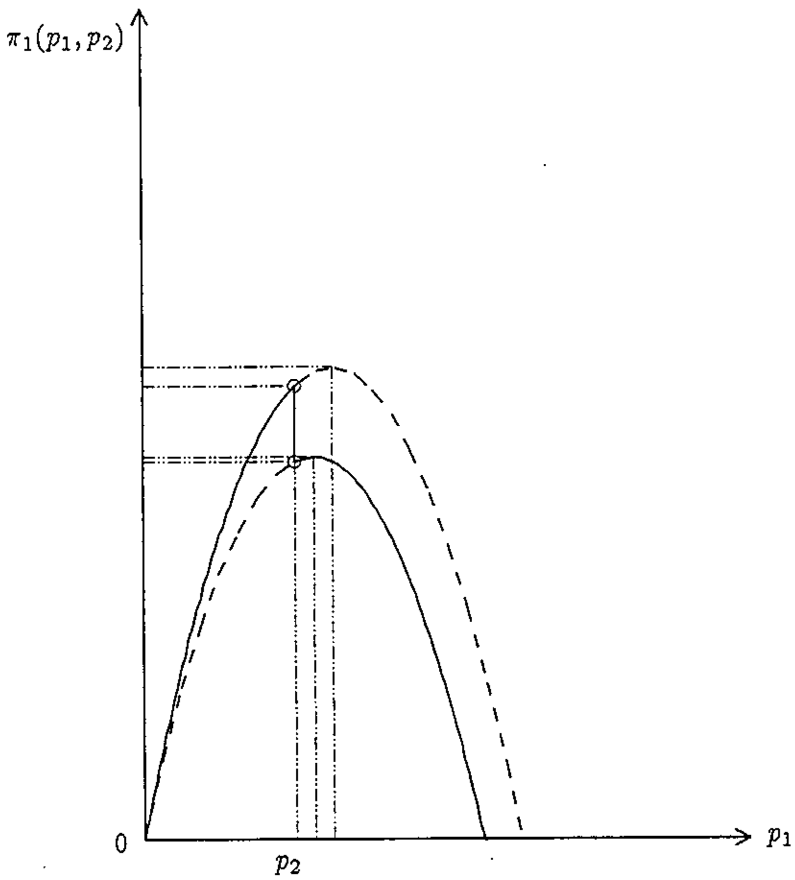

Figure 21: Revenue function for lower value of p_2

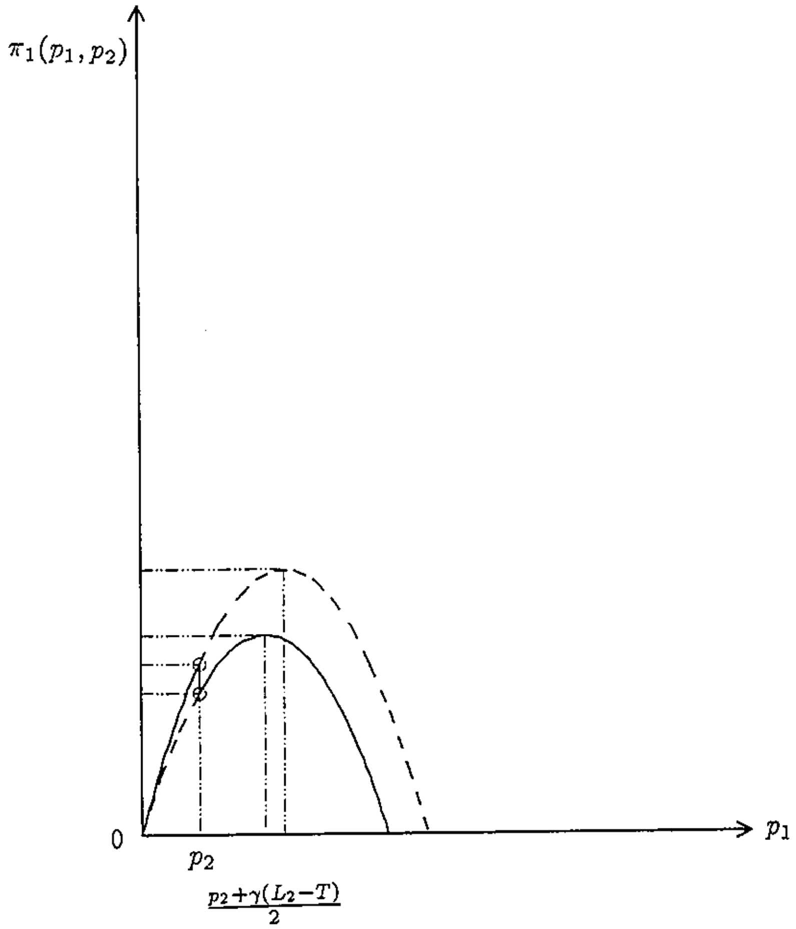

Figure 22: Revenue function for sufficiently low value of p_2

Table 1. Mill prices in equilibrium on a linear market

N	1	2	3	4	5	6	7	8
3	$\frac{10}{12}$ 0.833	$\frac{8}{12}$ 0.667	$\frac{10}{12}$ 0.833	- -	- -	- -	- -	- -
4	$\frac{36}{45}$ 0.800	$\frac{27}{45}$ 0.600	$\frac{27}{45}$ 0.600	$\frac{36}{45}$ 0.800	- -	- -	- -	- -
5	$\frac{133}{168}$ 0.792	$\frac{98}{168}$ 0.583	$\frac{84}{168}$ 0.500	$\frac{98}{168}$ 0.583	$\frac{133}{168}$ 0.792	- -	- -	- -
6	$\frac{495}{627}$ 0.789	$\frac{363}{627}$ 0.579	$\frac{330}{627}$ 0.526	$\frac{330}{627}$ 0.526	$\frac{363}{627}$ 0.579	$\frac{495}{627}$ 0.789	- -	- -
7	$\frac{1846}{2340}$ 0.789	$\frac{1352}{2340}$ 0.578	$\frac{1222}{2340}$ 0.522	$\frac{1196}{2340}$ 0.511	$\frac{1222}{2340}$ 0.522	$\frac{1352}{2340}$ 0.578	$\frac{1846}{2340}$ 0.789	- -
8	$\frac{6888}{8733}$ 0.789	$\frac{5043}{8733}$ 0.577	$\frac{4551}{8733}$ 0.521	$\frac{4428}{8733}$ 0.507	$\frac{4428}{8733}$ 0.507	$\frac{4551}{8733}$ 0.521	$\frac{5043}{8733}$ 0.577	$\frac{6888}{8733}$ 0.789

Table 2. Profits in equilibrium on a linear market

N	1	2	3	4	5	6	7	8
3	0.694	0.889	0.694	-	-	-	-	-
4	0.640	0.720	0.720	0.640	-	-	-	-
5	0.627	0.681	0.500	0.681	0.627	-	-	-
6	0.623	0.670	0.554	0.554	0.670	0.623	-	-
7	0.622	0.668	0.545	0.522	0.545	0.668	0.622	-
8	0.622	0.667	0.543	0.514	0.514	0.543	0.667	0.622