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A Dynamic Analysis of Urban Growth

by

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I. Introduction

It is well recognized that the physical and demographic expansion of urban sector (or city) is oriented by and also accompanied with industrialization and with per-capita income increase, but on the other hand this urban growth often created urban miseries in many cities. Main concern of this paper is to develop a growth model of urban sector, and to classify the possible patterns of urban growth, and also clarify the condition for immiserizing urban growth. Finally the model is applied to the recent trends of statistical data in Japan to point out that the urbanization in the rapid growth period (1965-70) resulted in an immiserizing urban growth, and recent trend of urbanization was excessive from the point of view of a rational urban model.

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II. Selection Among Urban and Local Sectors

We write the utility function of each worker (and consumer) as

$$U = \beta_0 L^{\beta_1} \cdot S^{\beta_2} \cdot H^{\beta_3} \qquad (0<\beta_1, \beta_2, \beta_3<1) \qquad (1)$$

L, S, H stand for residential land service, leisure time and other consumer goods. Worker will choose the job opportunity maximizing this utility with two constraints, i.e., income and time constraints.

We consider an economy which consists of circular urban production sector (CBD) with a radius (e), surrounding residential area and local sector. When each worker works at urban sector he resides in residential area with time distance D from the central point and commutes (D-e) hours everyday. We write land (service), leisure time, other consumer goods, total available time, land price, price of consumer's goods, transport cost as a function of (D-e), wage level as L*, S*, H*, T, PL*(D), P_H, m(D-e), w* where * symbol denotes urban sector. Then his constraints are

$$PL^*(D)L^* + 2m(D-e) + P_H H^* = w^*E^* \qquad (2)$$

$$E^* + 2(D-e) + S^* = T \qquad (3)$$

When he works at local sector (with ~ symbol) we assume that he can work at same residential space with a lower wage, and that land price is fixed at Ω. Then the two restraints will be

$$\Omega\tilde{L} + P_H\tilde{H} = \tilde{w}\tilde{E} \tag{4}$$

$$\tilde{E} + \tilde{S} = T \tag{5}$$

and

$$\tilde{w} < w^* \tag{6}$$

We specify the transport cost as

$$m(D-e) = w^*q(D-e)^h \tag{7}$$

where q stands for real transport cost of unit time distance in terms of urban wage.

III. Employment at Urban Sector

Maximizing (1) with constraints (2) and (3)

$$\max_{\{L^*,S^*,H^*,D,E^*\}} U - \lambda\{P_L{}^*L^* + 2m + H^* - w^*E^*\} - \mu\{E^* + 2(D-e) + S^* - T\} \tag{8}$$

differentiation gives us

$$\beta_1 U = \lambda P_L{}^*(D)L^* \tag{9}$$

$$\beta_2 U = \mu S^* \tag{10}$$

$$\beta_3 U = \lambda P_H H^* \tag{11}$$

$$-\lambda\{P_L{}^*(\acute{D})' L^* + 2m'\} - 2\mu = 0 \tag{12}$$

$$\lambda w^* = \mu \tag{13}$$

where dash (′) implies differentiation by distance.

From (9)(10)(12)(13) and (3)

$$P_L{}^*(D)L^* = \frac{\beta_1 U}{\lambda} = \frac{\beta_1 \mu S^*}{\lambda\beta_2} = \frac{\beta_1 w^* S^*}{\beta_2}$$

$$= \frac{\beta_1 w^*}{\beta_2}\{T - E^* - 2(D-e)\} \tag{14}$$

From (10) and (11)

$$H^* = \frac{\beta_3 w^* S^*}{\beta_2 P_H} \tag{15}$$

We define the disposable time at distance D, DT(D), as

$$DT(D) \equiv T - 2(D-e) - \frac{2m}{w^*} \tag{16}$$

From (2)(14)(15)(16)

$$E^* = \frac{\beta_{13}}{\beta_{123}} DT(D) + \frac{2m(D-e)}{w^*} \tag{17}$$

where βij and βijk denote $(\beta i + \beta j)$ and $(\beta i + \beta j + \beta k)$ respectively. Similarly from (3)(17)(15)

$$S^* = \left(\frac{\beta_2}{\beta_{123}}\right) DT(D) \tag{18}$$

$$H^* = \left(\frac{\beta_3 w^*}{\beta_{123} P_H}\right) DT(D) \tag{19}$$

From (12)(2)

$$\frac{P_L{}^*(\acute{D})}{P_L{}^*(D)} = \frac{-2(w^* + m')}{w^*E^* - P_H H - 2m} = \frac{\beta_{123}DT(\acute{D})}{\beta_1 DT(D)} \tag{20}$$

Therefore

$$P_L{}^*(D) = C\{DT(D)\}^{(\beta_{123}/\beta_1)} \tag{21}$$

$$L^* = \frac{w^*E^* - 2m - H^*}{P_L{}^*(D)} = \left(\frac{\beta_1 w^*}{\beta_{123}C}\right)\{DT(D)\}^{(-)\beta_{23}/\beta_1} \tag{22}$$

The density of population in residencial area (N*) can be defined as

$$N^*(D) = \frac{2\pi D}{L^*(D)} = \frac{2\pi\beta_{123}CD}{\beta_1 w^*}\{DT(D)\}^{(\beta_{23}/\beta_1)} \tag{23}$$

From (1)(18)(19)(22)

$$U^* = M\cdot C^{(-\beta_1)} \cdot w^{*\beta_{13}} \cdot P_H{}^{(-\beta_3)} \tag{24}$$

$$M = \beta_0\left(\frac{\beta_1{}^{\beta_1} \cdot \beta_2{}^{\beta_2} \cdot \beta_3{}^{\beta_3}}{\beta_{123}{}^{\beta_{123}}}\right) \tag{25}$$

IV. Employment at Local Sector

Maximizing (1) with constraints (4) and (5)

$$\underset{\{\tilde{L},\tilde{S},\tilde{H},\tilde{E},\}}{\mathrm{MAX}} \cdot U - \xi\{\Omega\tilde{L} + P_H\tilde{H} - \tilde{w}\tilde{E}\} - \rho\{\tilde{E} + \tilde{S} - T\} \tag{26}$$

differentiation gives us

$$\beta_1 U = \xi\Omega\tilde{L} \tag{27}$$

$$\beta_2 U = \rho \tilde{S} \tag{28}$$

$$\beta_3 U = \xi P_H \tilde{H} \tag{29}$$

$$\xi \tilde{w} = \rho \tag{30}$$

Therefore

$$U = \frac{\xi \Omega \tilde{L}}{\beta_1} = \frac{\rho \tilde{S}}{\beta_2} = \frac{\xi P_H \tilde{H}}{\beta_3} \tag{31}$$

From (31)(4)(5)

$$\tilde{S} = \left(\frac{\beta_2}{\beta_{123}} \right) T \tag{32}$$

$$\tilde{E} = \left(\frac{\beta_{13}}{\beta_{123}} \right) T \tag{33}$$

$$\tilde{H} = \left(\frac{\beta_3 \tilde{w}}{\beta_{123} P_H} \right) T \tag{34}$$

$$\tilde{L} = \left(\frac{\beta_1 \tilde{w}}{\beta_{123} \Omega} \right) T \tag{35}$$

From (1) and (32)(34)(35)

$$\tilde{U} = M \cdot T^{\beta_{123}} \left(\frac{\tilde{w}}{\Omega} \right)^{\beta_1} \cdot \left(\frac{\tilde{w}}{P_H} \right)^{\beta_3} \tag{36}$$

V. Arbitration of Labor and Land Markets

Demand price of land for residencial use $P_L{}^*$ (21) decreases with distance, and is equal with Ω at a distance $(\hat{D})$. Beyond $\hat{D}$ land will be utilized by local sector. On the other hand U^* at (24) is constant while $P_L{}^*$ decreases, and will decrease if $P_L{}^*$ becomes constant beyond $\hat{D}$. So we assume that U^* is equal to $\tilde{U}$ at $\hat{D}$, and two markets of labor and land are arbitrated simultaneously at $\hat{D}$.

$$P_L^*(\hat{D}) = \Omega \tag{37}$$

$$U^* = \tilde{U} \Big|_{D = \hat{D}} \tag{38}$$

From (21) (37)

$$C = \Omega\{DT^*(\hat{D})\}^{(-\beta_{123}/\beta_1)} \tag{39}$$

$$\therefore P_L^*(D) = \Omega \cdot \left\{\frac{DT^*(D)}{DT^*(\hat{D})}\right\}^{(\beta_{123}/\beta_1)}$$

$$= \Omega\left(\frac{w^*}{\tilde{w}}\right)^{(\beta_{13}/\beta_1)} \cdot \left(\frac{DT^*[D]}{T}\right)^{(\beta_{123}/\beta_1)} \tag{40}$$

$$U^* = M \cdot \left(\frac{w^*}{\Omega}\right)^{\beta_1} \cdot \left(\frac{w^*}{P_H}\right)^{\beta_3} \cdot \{DT^*(\hat{D})\}^{\beta_{123}} \tag{41}$$

From (36)(38)(41)

$$DT^*(\hat{D}) = T - 2(\hat{D}-e) - \frac{2m(\hat{D}-e)}{w^*} = T\left(\frac{\tilde{w}}{w^*}\right)^{\frac{\beta_{13}}{\beta_{123}}} \tag{42}$$

Now

$$L^*(D) = \left\{\frac{\beta_1 \cdot \tilde{w}^{(\beta_{13}/\beta_1)} \cdot T^{(\beta_{123}/\beta_1)}}{\beta_{123} \cdot \Omega \cdot w^{*(\beta_3/\beta_1)}}\right\}\{DT^*(D)\}^{(-)\beta_{23}/\beta_1} \tag{43}$$

$$N^*(D) = Q \cdot D \cdot \{DT^*(D)\}^{(\beta_{23}/\beta_1)} \tag{44}$$

$$Q \equiv \frac{2\Pi\beta_{123}\Omega w^{*(\beta_3/\beta_1)}}{\beta_1 T^{(\beta_{123}/\beta_1)}\tilde{w}^{(\beta_{13}/\beta_1)}} \tag{45}$$

and total population ($\bar{N}^*$) is calculated as

$$\bar{N}^* = \int_e^{\hat{D}} N^*(D)\ dD \tag{46}$$

This population can be understood as the possible supply of labor to urban sector.

Let us assume the production function of urban sector to produce urban goods (Y) as

$$Y = \gamma_0 \bar{N}^{*\gamma_1} \cdot K^{*\gamma_2} \cdot (\pi e^2)^{\gamma_3} \tag{47}$$

where K denotes capital stock in urban sector.[1)]

If we postulate

$$\frac{\partial Y}{\partial(\pi e^2)} \bigg/ \frac{\partial Y}{\partial N^*} = P_L(e) \ / \ w^* \tag{48}$$

and land price in CBD is fixed with $P_L(e)$, then from (47)(48)

$$\bar{N}^* = \frac{\gamma_1 \pi e^2 P_L^*(e)}{\gamma_3 w^*} = \frac{\pi \gamma_1 \Omega w^{*(\beta_3/\beta_1)} e^2}{\gamma_3 \tilde{w}^{(\beta_{13}/\beta_1)}} \tag{49}$$

From (42)(46)(49) we can solve $\bar{N}^*$, $\hat{D}$ and e simultaneously.

We verify the uniqueness of $\bar{N}^*$ as follows.

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1) Based upon "small-city-assumption" we do not discuss the balance between Y and demand of consumption goods (H).

From (7)(42), $(\hat{D}-e)$ is solved uniquely. Then from (44)(46)(16)

$$\bar{N}^* = Q\int_e^{\hat{D}} D\{DT^*(D)\}^{(\beta_{23}/\beta_1)}\,dD$$

$$= Q\int_0^{(\hat{D}-e)} (D+e)\{DT^*(D+e)\}^{(\beta_{23}/\beta_1)}\,dD$$

$$= A_1 + A_2 e \qquad (50)$$

$$A_1 \equiv \int_0^{(\hat{D}-e)} D\cdot\{T - 2D - \frac{2m(D)}{w^*}\}^{(\beta_{23}/\beta_1)}\,dD \quad (> 0) \qquad (51)$$

$$A_2 \equiv \int_0^{(\hat{D}-e)} \{T - 2D - \frac{2m(D)}{w^*}\}^{(\beta_{23}/\beta_1)}\,dD \quad (> 0) \qquad (52)$$

So the equality between (49) and (50) uniquely determines $\bar{N}^*$ and e, then $\hat{D}$ by (42).

Postulating

$$P_Y(\partial Y/\partial\bar{N}^*) = w^* \qquad (53)$$

Writing the price of Y as P_Y, we get

$$K^* = \left\{\frac{w^*\bar{N}^{*(1-\gamma_1)}}{\gamma_0\gamma_1(\pi e^2)^{\gamma_3}}\right\}^{(1/\gamma_2)} \qquad (54)$$

Let us define our general economy as follows:

Definition: G-economy is an economy consisting of CBD, residencial ring and surrounding local sector, and is described by eighteen variables, eleven scalar variables $(\tilde{U}, \tilde{L}, \tilde{S}, \tilde{H}, \tilde{E}, U^*, Y, K^*, \bar{N}^*, \hat{D}, e)$ and seven variables as functions of distance (DT^*, L^*, S^*, H^*, E^*, P_L^*, N^*). These variables are solved by eighteen equations ((36)(35)(32)(34)(33)(38)(47)(54)(49)(42)(46)(16)(43)(18)(19)(17)(40)(44)) in terms of three exogenous variables (w^*, $\tilde{w}$, Ω) and distance.[2)]

VI. Analysis of Prototype Economy (1)

In this section we simplify our general model in three points and develop an explicit dynamic analysis of urban growth. These three points are as follows:

(a) We neglect H-goods, and eliminate H, P_H and β_3 in (1)(2)(4) and others equations.

(b) We assume constant marginal transport cost and put h in (7) and other relevant equations as unity.

(c) We assume that CBD concentrates at centre point, and put e in (2)(3)(42)(46) and relevant equations as zero, and neglect the equations (47)-(54).

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2) We can interpret our economy in another way: (i) To reinterpret (54) to decide w^*, and (ii) to assume that $\tilde{w}$ and Ω are functions of capital stock per-worker in local sector ($\tilde{K}$). Then we have twenty endogenous variables (replacing K^* with w^*, and adding $\tilde{w}$ and Ω) and two exogenous variables (K^* and $\tilde{K}$).

We define a prototype economy as follows:

Definition: P-economy is a simplified version of G-economy modified by additional assumptions (a)(b) and (c), and is described by thirteen variables ($\tilde{U}$, $\tilde{L}$, $\tilde{S}$, $\tilde{E}$, U*, $\bar{N}$*, $\hat{D}$, DT*, L*, S*, E*, P_L*, N*). These variables are solved by thirteen equations (36)(35)(32)(33)(38)(46)(42)(16)(43)(18)(17)(40)(44) in terms of three exogenous variables (w*, $\tilde{w}$, Ω) with neccessary amendment of putting β_3 and e as zero.

Especially we note that $\hat{D}$ and $\bar{N}$* are solved explicitly as follows:

$$\hat{D} = \frac{T}{2(1+q)}\left\{1 - \left(\frac{\tilde{w}}{w^*}\right)^a\right\} \tag{55}$$

$$\bar{N}^* = S\hat{N}^* \tag{56}$$

$$\hat{N}^* \equiv \frac{a\pi\Omega T}{2(1+a)(1+q)^2\tilde{w}} \tag{57}$$

$$S \equiv 1 - \left(\frac{\tilde{w}}{w^*}\right)\left\{1 + \frac{1}{a}\left[1 - \left(\frac{\tilde{w}}{w^*}\right)^a\right]\right\} \tag{58}$$

$$a \equiv \frac{\beta_1}{\beta_{12}} \tag{59}$$

Let us write the overtime growth rate of X as R_X, then growth rates of thirteen endogenous variables and DT*($\hat{D}$) are as follows:

$$R_{\tilde{S}} = R_{\tilde{E}} = R_{DT^*(D)} = R_{S^*(D)} = R_{E^*(D)} = 0 \tag{60}$$

$$\begin{Bmatrix} R_{U^*} \text{ or } R_{\tilde{U}} \\ R_{\tilde{L}} \\ R_{\bar{N}^*} \\ R_{\hat{D}} \\ R_{L^*(D)} \\ R_{P_L^*(D)} \\ R_{N^*(D)} \\ R_{DT^*(\hat{D})} \end{Bmatrix} = \begin{pmatrix} 0 & \beta_1 & -\beta_1 \\ 0 & 1 & -1 \\ F & (-)(1+F) & 1 \\ d & -d & 0 \\ 0 & 1 & -1 \\ 1 & -1 & 1 \\ 0 & -1 & 1 \\ (-)a & a & 0 \end{pmatrix} \begin{Bmatrix} R_{w^*} \\ R_{\tilde{w}} \\ R_{\Omega} \end{Bmatrix} \tag{61}$$

where $F \equiv (-)\dfrac{d\ln S}{d\ln\left(\frac{\tilde{w}}{w^*}\right)} = (-)\dfrac{d\ln S \;/\; d\left(\frac{\tilde{w}}{w^*}\right)}{d\ln\left(\frac{\tilde{w}}{w^*}\right)/d\left(\frac{\tilde{w}}{w^*}\right)} = (-)\left(\dfrac{\tilde{w}}{w^*S}\right)\left\{\dfrac{dS}{d\left(\frac{\tilde{w}}{w^*}\right)}\right\}$

$$= \frac{(1+a)\tilde{w}}{aw^*S}\left(1 - \left[\frac{\tilde{w}}{w^*}\right]^a\right)\ (\ >0) \tag{62}$$

$$d \equiv \frac{a\ DT^*(\hat{D})}{2(1+q)\hat{D}} \tag{63}$$

We summarize the results about three important variables in the next theorem:

<u>Theorem I</u>: In P-economy, the signs of growth rates of U^*, $\hat{D}$, $\bar{N}^*$ are expressed as

Table. I. Three Growth Rates

sign of growth rate / patterns		R_{U^*}	$R_{\hat{D}}$	$R_{\bar{N}^*}$
I. $R_{W^*}>R_{\tilde{W}}>R_{\Omega}$	I_1	+	+	+
	I_2	+	+	-
II. $R_{W^*}>R_{\Omega}>R_{\tilde{W}}$		-	+	+
III. $R_{\Omega}>R_{W^*}>R_{\tilde{W}}$		-	+	+
IV. $R_{\Omega}>R_{\tilde{W}}>R_{W^*}$	IV_1	-	-	+
	IV_2	-	-	-
V. $R_{\tilde{W}}>R_{\Omega}>R_{W^*}$		+	-	-
VI. $R_{\tilde{W}}>R_{W^*}>R_{\Omega}$		+	-	-

$$\text{sign}(R_{U*}) = \text{sign}(R_{\tilde{W}} - R_{\Omega}) \tag{64}$$

$$\text{sign}(R_{\hat{D}}) = \text{sign}(R_{W*} - R_{\tilde{W}}) \tag{65}$$

$$\text{sign}(R_{\bar{N}*}) = \text{sign}(FR_{W*} - [1+F]R_{\tilde{W}} + R_{\Omega}) \tag{66}$$

Therefore the overtime changes of utility level of urban worker (U*), physical size of city ($\hat{D}$) and total population of city ($\bar{N}$*) are classified in eight patterns as shown in Table I.

These cases are shown in Fig. I on R_{W*} - $R_{\tilde{W}}$ plane. The line AA′ denotes the equation:

$$R_{W*} = \left(\frac{1+F}{F}\right)R_{\tilde{W}} - \left(\frac{1}{F}\right) R_{\Omega} \tag{67}$$

Let us define two concepts as follows:

Definition: Balanced-Urban-Growth (BUG) as

$$R_{U*},\ R_{\hat{D}},\ R_{\bar{N}*} > 0 \tag{68}$$

Definition: Immiserizing-Urban-Growth(IUG) as

$$R_{U*} < 0,\ R_{\hat{D}},\ R_{\bar{N}*} > 0 \tag{69}$$

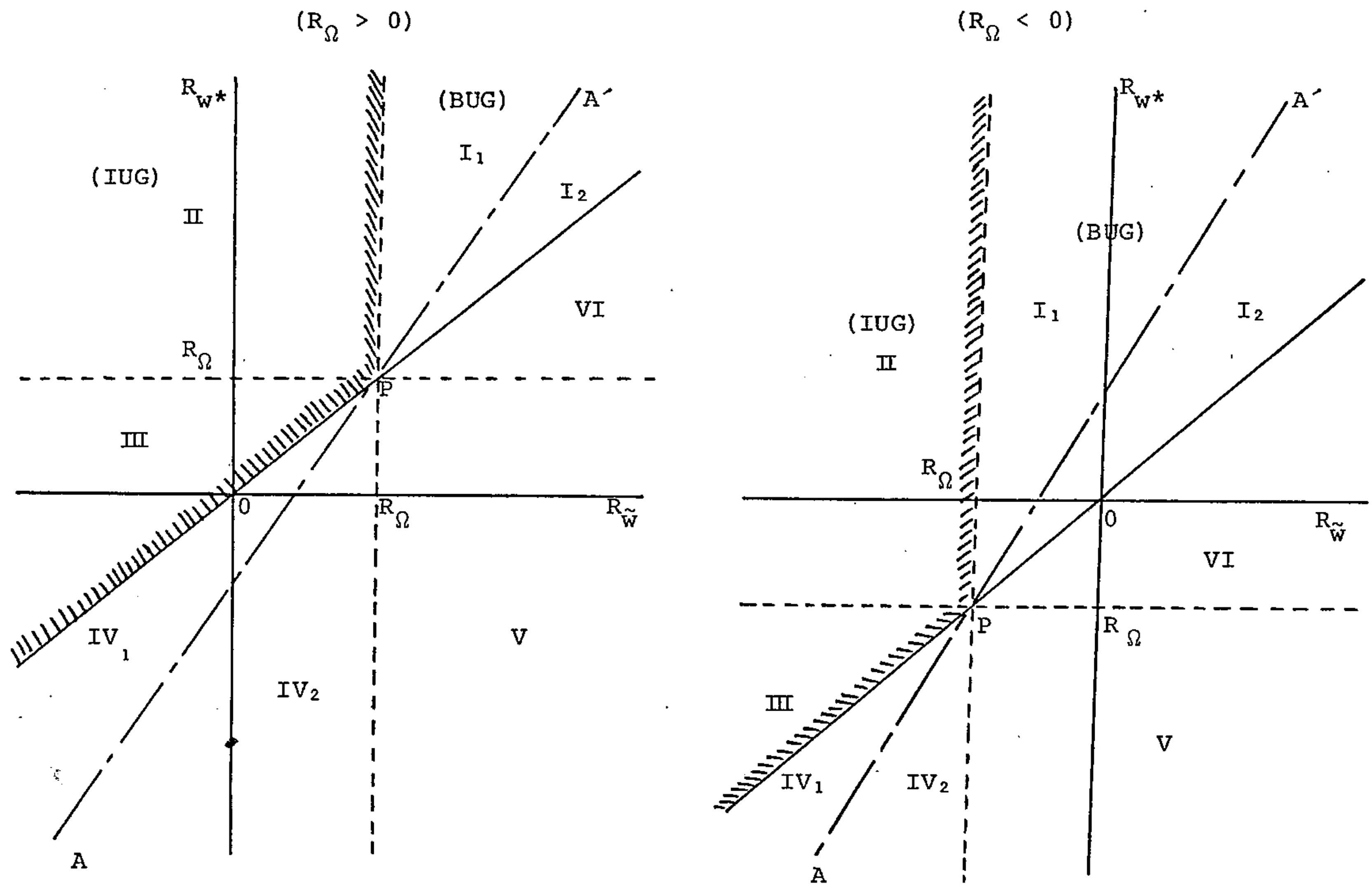
Then we have following theorem:

Theorem II : In P-economy, (a) the N & S condition for BUG is

(i) $R_{W*} > R_{\tilde{W}} > R_{\Omega}$ and

and (ii) $(R_{W*} - R_{\tilde{W}}) > \left(\frac{1}{F}\right)(R_{\tilde{W}} - R_{\Omega})$ (70)

Fig.I. Classification of Cases



and (b) the N & S condition for IUG is

$$\mathrm{MIN}\ (R_{W^*},\ R_{\Omega}) > R_{\tilde{W}} \qquad (71)$$

Usually higher growth of wage in urban sector ($R_{W^*} > R_{\tilde{W}}$) is interpreted as neccessary condition for urban growth, but discussion above suggests that this is neccessary for physical expansion ($R_{\hat{D}}>0$) but is not neccessary nor sufficient for urban population growth ($R_{\bar{N}^*}>0$) and for improvement of urban standard of living ($R_{U^*}>0$).[3)]

Fig. II shows the short-term changes of DT*, P_L*, L* and N* in case.I ($R_{W^*}>R_{\tilde{W}}>R_{\Omega}$), in which $\hat{D}$, $P_L(\hat{D})$, $L^*(\hat{D})$ increase, and $DT^*(\hat{D})$ and $N^*(\hat{D})$ decrease overtime.[4)]

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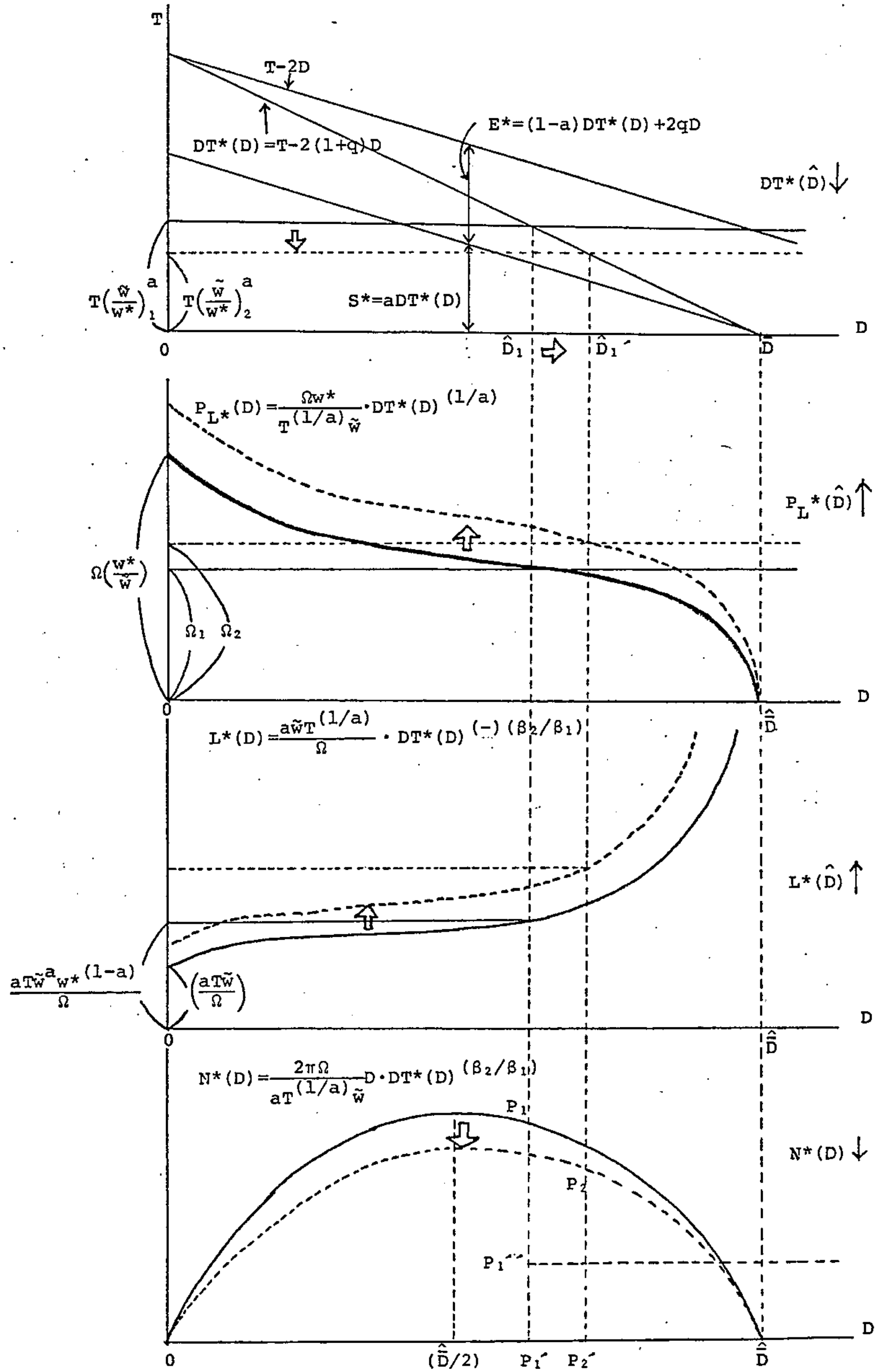
3) In this sense IUG is a similar concept with immiserizing growth (Bhagwati (1)) and immiserizing export growth (Fukuchi (2), p.46). These concepts suggests the possibilities of decreasing income growth by favorable terms of trade and export promotion.

4) Trend of $\bar{N}^*$ depends on comparison between negative effect of decreasing population density from OP_1 to OP_2 and positive effect of increasing distance from P_1' to P_2'.

Fig.II . Case.I ($R_{w*} > R_{\tilde{w}} > R_{\Omega}$)



(Remark) P_L*(and L*) curves has a point of torsion at DT*=1.
$\hat{D}$ is defined in (78). N* is maximized at ($\hat{D}$/2).
P_1' P_1'' shows a constant density in rural sector ($\tilde{L}^{-1}$).

VII. Analysis of Prototype Economy (2)

In this section we deal with long-term trend starting an initial condition $(w^*(0) > \tilde{w}(0))$. We define

$$y \equiv (\tilde{w} / \Omega), \quad \chi \equiv (w^* / \Omega), \quad (y/\chi \leqq 1) \tag{72}$$

Then from (61)

$$R_{U^*} = \beta_1 R_y \tag{73}$$

From (55)

$$\hat{D} = \frac{T}{2(1+q)} \{1 - (y/\chi)^a\} \tag{74}$$

From (56)(57)(58)

$$\bar{N}^* = \frac{a\pi T}{2(1+a)(1+q)^2 y} \left\{1 - \left(\frac{y}{\chi}\right)\left[1 + \frac{1}{a} - \frac{1}{a}\left(\frac{y}{\chi}\right)^a\right]\right\} \tag{75}$$

Then from (75)

$$\left.\frac{\partial \bar{N}^*}{\partial (y/\chi)}\right|_{\bar{y}} = \frac{\text{const}}{y}\left(1 + \frac{1}{a}\right)\left\{\left(\frac{y}{\chi}\right)^a - 1\right\} \qquad (\leqq 0) \tag{76}$$

$$\left.\frac{\partial \bar{N}^*}{\partial y}\right|_{\left(\overline{\frac{y}{\chi}}\right)} = (-)\frac{\text{const}}{y^2} \qquad (< 0) \tag{77}$$

So the iso-population curves are concave to east and the iso-distance lines are half-lines as shown in Fig.III. If current position is at point P, then eight patterns in Table. I. are described as shown. $\hat{\bar{D}}$ stands for the saturated size of city defined by

$$\hat{\hat{D}} = \frac{T}{2(1 + q)} \tag{78}$$

Now from (42)(43).

$$L^*(\hat{D}) = \frac{aT^{(1/a)}\tilde{w}}{\Omega} \quad DT^*(\hat{D}) = a \cdot T \cdot y^a \cdot \chi^{(1-a)} \tag{79}$$

So the iso-density-of-population curve is defined as

$$y = \left\{ \frac{L^*(\hat{D}) \text{ at } P}{aT} \right\}^{(1/a)} \cdot \chi^{(-)(\beta_2/\beta_1)} \tag{80}$$

and shown by BPB′ in Fig. III.[5] When a city is expanding physically and demographically ($R_{\hat{D}}$, $R_{\tilde{N}*} > 0$), this curve divides the case into two:(i)widening urbanisation with decreasing density (shadowded area) and (ii) deepening urbanisation with increasing density (dotted area).

When we assume the constant growth rates of w*, $\tilde{w}$ and Ω as

$$\delta_1 \equiv R_{w*}, \quad \delta_2 \equiv R_{\tilde{w}}, \quad \delta_3 \equiv R_{\Omega} \tag{81}$$

The conditions for existence of city in the long-run are

$$y < \chi \quad \text{and} \quad y < \infty \tag{82}$$

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5) $L^*(\hat{D})$ at P shows the inverse of current density of population at $\hat{D}$. From (43) population density at any distance $\{L^*(D)\}^{-1}$ moves in parallel with the one at $\hat{D}$.

Fig. III. Process of Urban growth

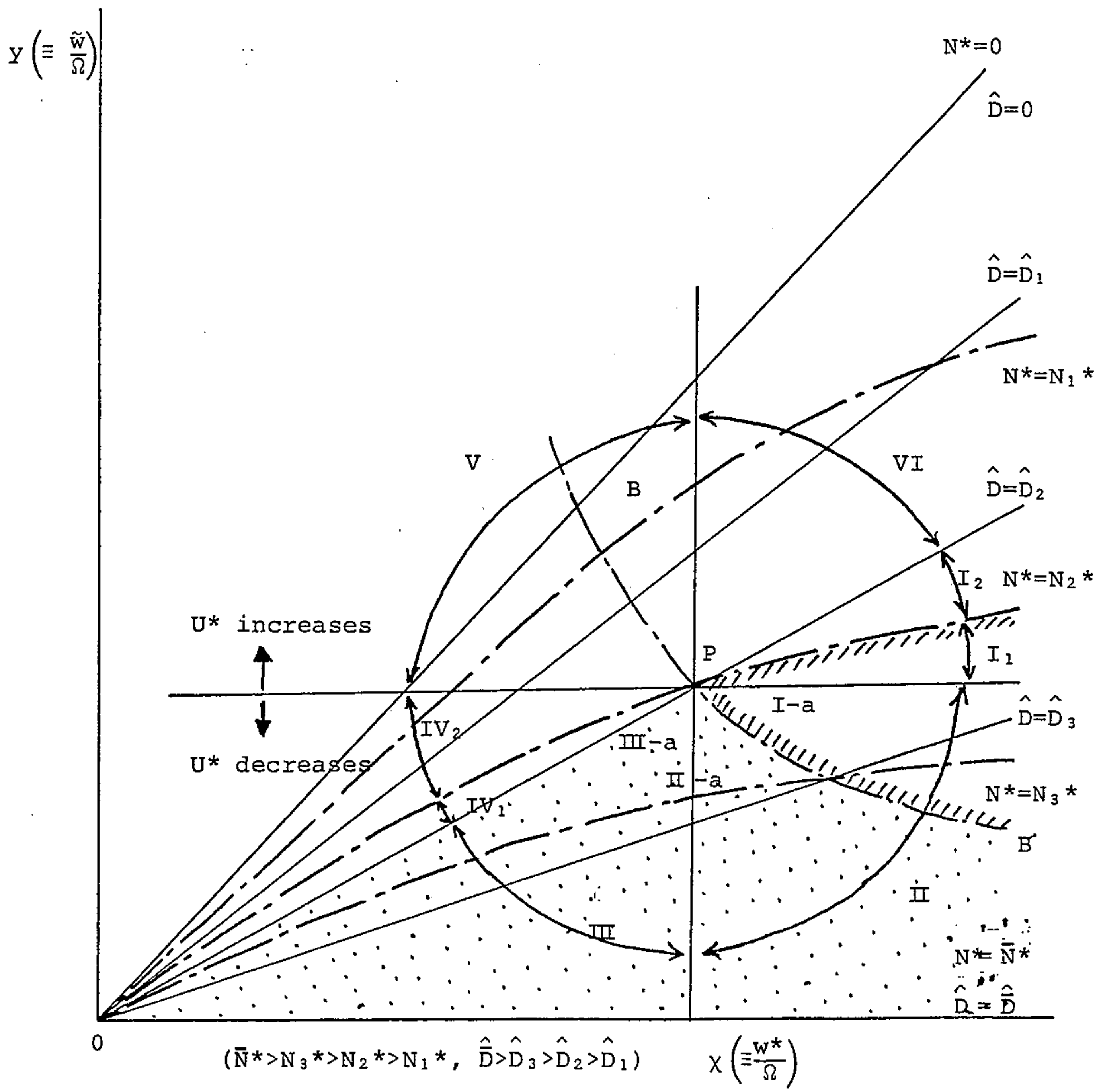


Table. II . Constant-Growth-Rates Cases

case \ variable		$DT^*(\hat{D})$	$L^*(\hat{D})$	$P_L^*(D)$	U^*	$\hat{D}$	$\bar{N}^*$
I - a.	$\delta_1 > \delta_2 = \delta_3$	0	const	∞	const	T/2(1+q)	const
II .	$\delta_1 > \delta_3 > \delta_2$	0	0	∞	0	T/2(1+q)	∞
II - a.	$\delta_1 = \delta_3 > \delta_2$	0	0	∞	0	T/2(1+q)	∞
III.	$\delta_3 > \delta_1 > \delta_2$	0	0	∞	0	T/2(1+q)	∞
III - a.	$\delta_3 > \delta_1 = \delta_2$	const	0	∞	0	const	∞
P.	$\delta_1 = \delta_2 = \delta_3 (>0)$	const	const	∞	const	const	const

The cases satisfying (82) are compiled in Table. II . Especially the case P suggests a "neutral urban growth" in which $DT^*(\hat{D})$ $L^*(D)$, U^* and $\hat{D}$ are held constant and $P_L^*(D)$ grows by constant rate.

IIX. Effects of Transport Cost

In this section we analyze the effects of changing transport cost system (expressed by q and h) in P-economy.

(A) First we calculate the effects of change in q upon urban variables:

$$\text{sign.}\,\partial\{U^*, \bar{N}^*, \hat{D}, DT^*(D), L^*(D), S^*(D), E^*(D), P_L^*(D), N^*(D)\}/\partial q$$
$$= (0, -, -, -, +, -, +, -, -) \qquad (83)$$

Therefore the decrease in average transport cost results in increase of urban population ($\bar{N}^*$), physical size of city ($\hat{D}$) and urban land price (P_L^*) while utility level of urban worker (U^*) is held constant.

(B) Let us consider the G-economy with additional assumptions (a)(c), and assume $h \neq 1$ in general. First from (41)(42)(16)

$$\frac{\partial U^*}{\partial h} = \frac{\partial DT^*(\hat{D})}{\partial h} = 0 \qquad (84)$$

$$\frac{\partial DT^*(D)}{\partial h} = (-)\,2qD^h \cdot \ln D \qquad (85)$$

similarly we obtain

$$\text{sign}(D-1)\,\partial\{DT^*(D),\ L^*(D),\ S^*(D),\ E^*(D),\ P_L^*(D),\ N^*(D)\}\ /\ \partial h = (-,\ +,\ -,\ +,\ -,\ -) \tag{86}$$

From (42)

$$\ln\left\{T\left(1-\left[\frac{\tilde{w}}{w^*}\right]^a\right)-2\hat{D}\right\} = \ln 2q + h\ln\hat{D}$$

$$\therefore (-)\frac{1}{qDh}\left(\frac{\partial\hat{D}}{\partial h}\right) = \ln\hat{D} + \frac{h}{\hat{D}}\left(\frac{\partial\hat{D}}{\partial h}\right)$$

$$\therefore \frac{\partial\hat{D}}{\partial h} = (-)\left(\frac{h}{\hat{D}} + \frac{1}{q\hat{D}^h}\right)\ln\hat{D} \tag{87}$$

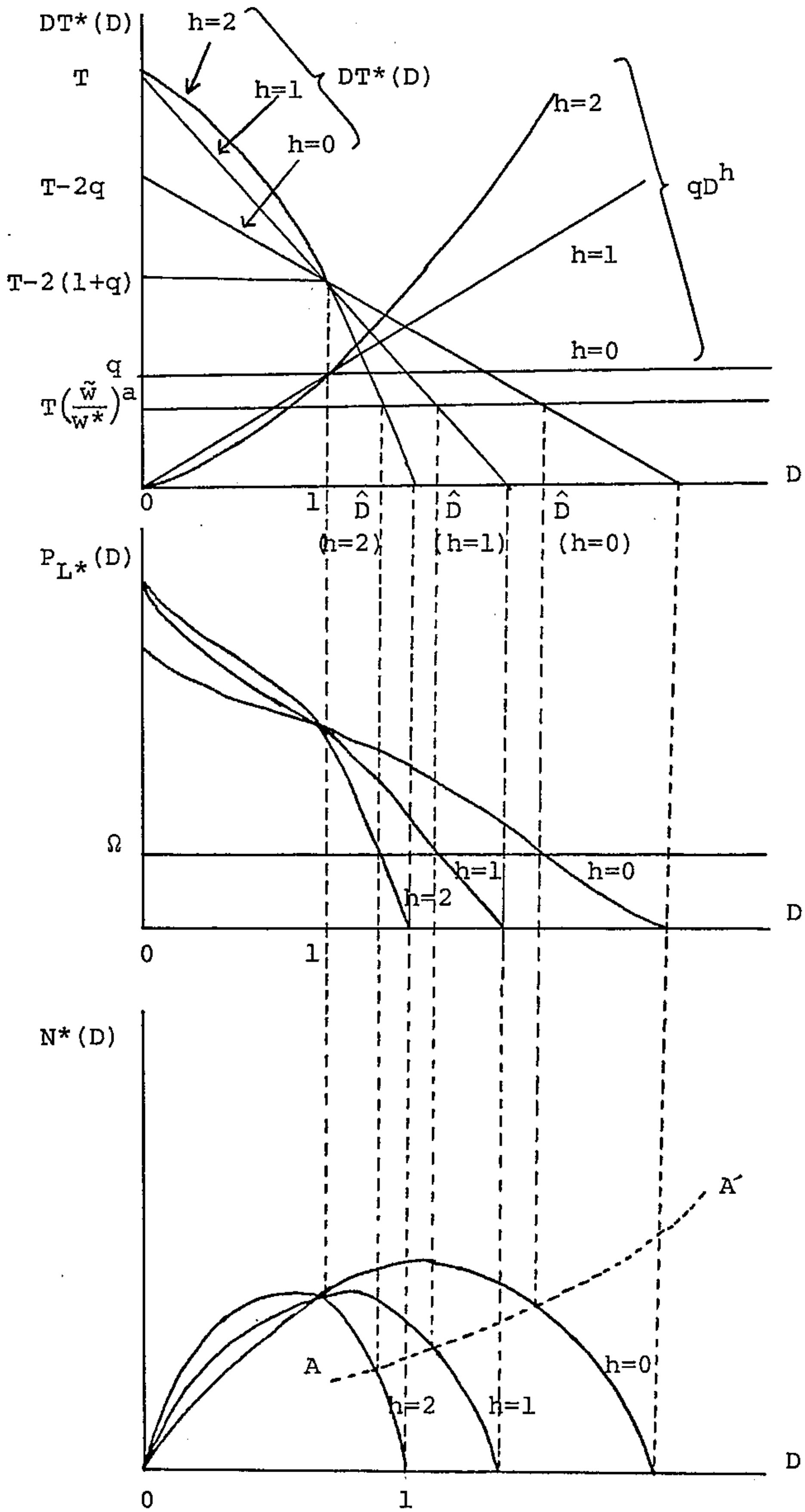
$$(\hat{D}-1)\left(\frac{\partial\hat{D}}{\partial h}\right) < 0 \tag{88}$$

Therefore the increase of elasticity of transport cost to distance (h) results in decrease (increase) of physical size of city ($\hat{D}$) if $\hat{D}>1$ ($\hat{D}<1$), and decrease of urban land price (P_L^*) as well as population density (N^*) if $D>1(D<1)$, keeping the utility level of urban worker (U^*) as constant. The effect of increase in h is shown in Fig.IV. We note, from (42) and (44), that

$$\partial N^*(\hat{D})\ /\ \partial\hat{D} > 0 \tag{89}$$

which is shown by upward curve AA′ in Fig.IV. The increase of h accelerate the concentration of distancial population

Fig.IV. Effects of increase of h



(Remark) We assume $\hat{D}>1$ in this figure.

distribution to centre by increasing and decreasing $N^*(D)$ according to $D<1$ and $D>1$. Therefore the effect upon total urban population is indecisive.

IX. Existence of CBD with Finite Size

In this section we analyze the effect of existence of urban sector ($e>0$). Let us consider the G-economy with additional assumption (a)(b), and assume $e\neq 0$. In this case $(\hat{D}-e)$ is determined by (42) as

$$(\hat{D}-e) = \frac{T}{2(1+q)}\left\{1-\left(\frac{\tilde{w}}{w^*}\right)^a\right\} \tag{90}$$

From (49)

$$\bar{N}^* \equiv Re^2 \tag{91}$$

$$R \equiv \frac{\pi\gamma_1\Omega}{\gamma_3\tilde{w}} \tag{92}$$

From (50)(90)

$$\bar{N}^* = Q'\int_e^{\hat{D}} D\cdot\{DT^*(D)\}^{(\beta_2/\beta_1)}\cdot dD$$

$$= \frac{aQ'}{2(1+q)}\{e(DT^*[e])^{(1/a)} - \hat{D}(DT^*[\hat{D}])^{(1/a)}$$

$$+ \frac{a}{2(1+a)(1+q)}((DT^*[e])^{(1+a)/a} - (DT^*[\hat{D}])^{(1+a)/a})\} \tag{93}$$

where

$$Q' \equiv \frac{2\pi\Omega}{aT^{(1/a)}\tilde{w}} \tag{94}$$

From (42)

$$DT^*(e) = T \tag{95}$$

So, from (42)(57)(58)(95)

$$\bar{N}^* = Be + J \tag{96}$$

$$B \equiv \frac{Q' aT^{(1/a)}}{2(1+q)}\left(1 - \frac{\tilde{w}}{w^*}\right) \qquad (>0) \tag{97}$$

$$J \equiv S\hat{N}^* \tag{98}$$

From (91)(96)

$$Re^2 - Be - J = 0 \tag{99}$$

So[6]

$$e = \left(\frac{B}{2R}\right) + \sqrt{\left(\frac{B}{2R}\right)^2 + \left(\frac{J}{R}\right)} \tag{100}$$

$$\partial\left(\frac{B}{R}\right) \Big/ \partial\left(\frac{\tilde{w}}{w^*}\right) < 0, \quad \partial J \Big/ \partial\left(\frac{\tilde{w}}{w^*}\right) < 0 \tag{101}$$

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6) One of the roots of (99) is negative as J>0. So we take the positive root.

Thus from (90)(100)(101)(91)

$$\text{sign}\, R_{(\hat{D}-e)} = \text{sign}\, R_{\hat{D}} = \text{sign}\, R_{e} = \text{sign}\, R_{\bar{N}*}$$
$$= \text{sign}(R_{w*} - R_{\tilde{w}}) \Big|_{\bar{\Omega}} \qquad (102)$$

$$\text{sign}\, R_{\bar{N}*} = \text{sign}(R_{\Omega} - R_{\tilde{w}}) \Big|_{\bar{w}*} \qquad (103)$$

Fig. V shows the case. I ($R_{w*}>R_{\tilde{w}}>R_{\Omega}$) in which

$$R_{\hat{D}},\ R_{\bar{e}},\ R_{(\hat{D}-e)},\ R_{P_L*(\hat{D})},\ R_{P_L*(e)},\ R_{L*(\hat{D})},\ R_{\bar{N}*} > 0 \qquad (104)$$

$$R_{DT*(\hat{D})},\ R_{Nd*(e)},\ R_{Nn*(\hat{D})} < 0 \qquad (105)$$

where Nd* and Nn* stand for density of population at daytime and nighttime.

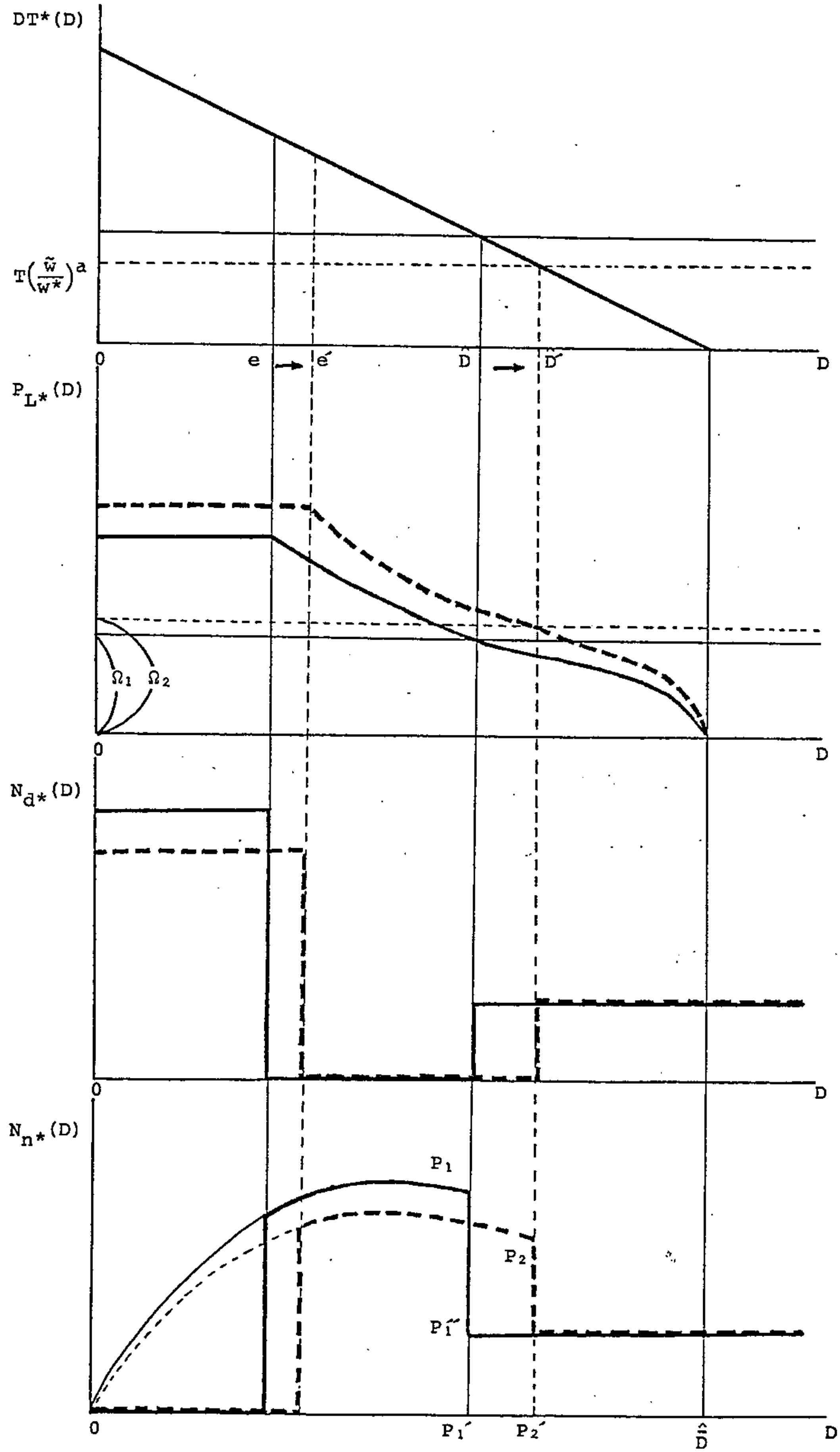
In G-economy modified only by (b) with H-goods and CBD, still (102) holds true. In this case from (46)(49) e is solved as:

$$e = V_1\{1 + \sqrt{1 + V_2}\ \} \qquad (106)$$

$$V_1 \equiv \frac{\gamma_3}{2\gamma_1(1+q)} \left\{1 - \left(\frac{\tilde{w}}{w*}\right)^{(\beta_{13}/\beta_1)}\right\} \qquad (107)$$

$$V_2 \equiv \frac{2\gamma_1\alpha' \left\{1 - \left(\frac{\tilde{w}}{w*}\right)^{(\beta_{13}/\beta_1)} \left[1 + \frac{1}{\alpha}\left(1 - \left[\frac{\tilde{w}}{w*}\right]^{(\beta_{13}/\beta_{123})}\right)\right]\right\}}{\gamma_3(1 + \alpha')\left\{1 - \frac{\tilde{w}}{w*}^{(\beta_{13}/\beta_1)}\right\}^2} \qquad (108)$$

Fig.V. Case.I ($R_{w*} > R_{\tilde{w}} > R_{\Omega}$)



where $$\alpha' \equiv \frac{\beta_1}{\beta_{123}} \tag{109}$$

One of the sufficient condition for urban population growth ($R_{\bar{N}*}>0$) in this case is derived from (49) and (109):

$$R_{W*}>R_{\tilde{W}} \text{ and } R_{\Omega} + \left(\frac{\beta_3}{\beta_1}\right)R_{W*} - \left(\frac{\beta_{13}}{\beta_1}\right)R_{\tilde{W}} > 0 \tag{110}$$

or $$R_{W*} > \text{MAX}\left\{R_{\tilde{W}},\ \left(\frac{\beta_{13}}{\beta_3}\right)R_{\tilde{W}} - \left(\frac{\beta_1}{\beta_3}\right)R_{\Omega}\right\} \tag{111}$$

X. Application to Recent Urban Growth in Japan

In this section we want to empirically test the relevancy or fitness of our simple growth model of P-economy to the actual tendency of urbanization in Japan through comparison of calculated and actual growth rates of urban land price. Basic statistics are compiled in statistical table in Appendex. Based upon these data we showed the classification of urbanization in each period according to Table. I. Classification was made to all Japanese cities and also for Tokyo Metropolitan Area.

The results are shown in Table. III (all cities) and in Table. IV (Tokyo Metropolitan Area , TMA). We calculated the theoretical growth rate of urban land price ($\hat{R}_{P_L*}$) based upon equation (61):

$$\hat{R}_{P_L*(D)} = R_{W*} - R_{\tilde{W}} + R_{\Omega} \tag{112}$$

Table. III. Analysis of All Cities

variables / period	comparison of (1)(2)(3)		signs of expected growth rates			growth rate of urban land price	
	comparison of growth rates	pattern	$\hat{R}_{U^*}$	$\hat{R}_{\hat{D}}$	$\hat{R}_{N^*}$	estimated ($R_{P_L^*}$)	actual ($R_{P_L^*}$)
1960 - 65	$R_{\tilde{W}} \geqq R_{\Omega} \geqq R_{W^*}$	V	+	-	-	10.78	22.36
1965 - 70	$R_{W^*} > R_{\Omega} > R_{\tilde{W}}$	II	-	+	+	13.39	12.68
1970 - 75	$R_{\tilde{W}} > R_{W^*} > R_{\Omega}$	VI	+	-	-	11.12	14.04
1965 - 75	$R_{W^*} > R_{\tilde{W}} > R_{\Omega}$	I	+	+	?	12.35	13.35
1960 - 75	$R_{W^*} > R_{\tilde{W}} > R_{\Omega}$	I	+	+	?	12.37	16.28

Table. IV. Analysis of Tokyo Metropolitan Area

variables / period	comparison of (1)(2)(4)		signs of expected growth rates			growth rate of urban land price	
	comparison of growth rates	pattern	$\hat{R}_{U*}$	$\hat{R}_{\hat{D}}$	$\hat{R}_{N*}$	estimated $(\hat{R}_{P_L*})$	actual (R_{P_L*})
1960 - 65	$R_{\tilde{W}} \geqq R_{\Omega} > R_{W*}$	V	+	-	-	9.87	29.77
1965 - 70	$R_{W*} > R_{\Omega} > R_{\tilde{W}}$	II	-	+	+	11.39	9.35
1970 - 75	$R_{\tilde{W}} > R_{W*} > R_{\Omega}$	VI	+	-	-	11.05	13.32
1965 - 75	$R_{\tilde{W}} > R_{W*} > R_{\Omega}$	VI	+	-	-	11.31	11.32
1960 - 75	$R_{\tilde{W}} > R_{W*} > R_{\Omega}$	VI	+	-	-	10.82	17.16

(Remark) (1)(2)(3)(4) refer to number of columns in Appendex statistical table.

As for the periods 1965 - 70 and 1965 - 75 the calculated growth rates coincide with actual average growth rates of all cities with minor error not exceeding one percent. In case of TMA the calculated growth fitted to actual tendency very well. If we include the period 1960 - 65, when there existed a strong speculative demand and a rapid growth of urban land price, the fittness of course becomes worse. But on the whole we can interpret that these results, in general, empirically support our model. It is also interesting to note that in a period of rapid national growth and strong urbanization, 1965 - 70, the pattern of growth of all cities and TMA was pattern II , i.e., Immiserizing-Urban-Growth. Therefore this suggests that in this period the rapid physical and demographic expansion of Japanese urban area was accompanied with deterioration of standard of living of urban workers. The three growth rates (R_{W*}, $R_{\tilde{W}}$, R_{Ω}) were approximately same for all cities in 1960 - 65. This suggests that the urbanization in 1960 - 65 was Balanced-Urban-Growth. So these results suggest that the type of urbanization changed from BUG in early rapid growth period to IUG in late rapid growth period and this change created urban miseries and accelerated the social frustration in cities.

The comparison between theoretical and actual growth rates suggests that the divergence may due to disturvances due to expectations and other irrational factors. So we define <u>Definition</u>: Excessively-Growing-City (EGC) as

$$R_{\bar{N}*} > \hat{R}_{\bar{N}*} \tag{113}$$

Then we can conclude that Japanese cities were of EGC type at 1960 - 65 and 1970 - 75, as actual growth rate was higher than natural growth rate of population (1.1 percent) and theoretical rate was negative. For TMA the expected growth rate ($\hat{R}_{\bar{N}*}$) was always negative for 1965 - 75 or 1960 - 75 but actual growth rate was greater them 2.5 percent. So our analysis strongly suggests the need of decentralization of population in TMA after being EGC for two decades.

XI. Summary and Conclusions

In this paper we presented a simple growth model and discussed the changes of basic variables, especially physical size of city ($\hat{D}$), urban population ($\bar{N}*$) and utility level of urban workers ($U*$), then discussed the neccessary improvement after relaxing rigid assumptions. Our simple growth model showed good fittness to recent urbanization in Japan, and is expected to work as a basic theoretical framework for further analysis, though of course it must be improved in many aspects into the future.

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Appendex: Basic Statistical Data

(percent)

variables / period	(1) Income of Agriculture Household ($R_{\tilde{W}}$)	(2) Price of Agricultural Land (R_{Ω})	(3) Income of Urban Worker (All Cities) (R_{W*}) (1)	(4) same (Tokyo) ($R_{W*'}$)	(5) Urban Population (Three Big Areas) (R_{N*})	(6) same (Tokyo Area) ($R_{N*'}$)	(7) Price of Urban Land (All Cities) (R_{P_L*})	(8) same (Six Big Cities) ($R_{P_L*'}$)
1960 - 1965	11.66	11.42	11.02	10.11	2.80	3.30	22.36	29.77
1965 - 1970	7.86	8.81	12.44	10.44	2.37	2.79	12.68	9.35
1970 - 1975	19.74	14.73	16.13	16.06	1.97	2.32	14.04	13.32
1965 - 1975	13.64	11.73	14.26	13.22	2.17	2.55	13.35	11.32
1960 - 1975	12.98	11.63	13.72	12.17	2.38	2.80	16.28	17.16

Remark: (1) Showing the figures for 1963 - 65 and 1963 - 1975 in stead of 1960 - 65 and 1960 - 75.