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Skew-symmetric Additive Representations of Preferences

by

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Abstract

This paper extends Fishburn's Savage–type axiomatization for skew-symmetric additive representations of preferences in decision making under uncertainty to non-simple acts. It is shown that the representation that covers all acts follows if we modify Savage's P2, P6 and P7 and drop P1.

1 Introduction

This paper is concerned with an extension of Fishburn's Savage–type axiomatization for skew-symmetric additive (SSA) representations of preferences in decision making under uncertainty (see Savage (1954) and Fishburn (1970)). An SSA representation is described as follows. Let S and X be non-empty sets of states and consequences, respectively. Let $\mathcal{F}$ be a set of acts which are functions from S into X. By $\succ$, we denote a binary *is preferred to* relation on $\mathcal{F}$. Then we say that $(\mathcal{F}, \succ)$ has an SSA representation if there exist a real valued function ϕ on $X \times X$ and a probability measure π on 2^S such that for all $f, g \in \mathcal{F}$,

$$f \succ g \iff \int_S \phi(f(s), g(s)) d\pi(s) > 0,$$

wherer ϕ is skew-symmetric (i.e., $\phi(x,y) + \phi(y,x) = 0$ for all $x, y \in X$). The SSA representation reduces to subjective expected (SEU) repersentations when ϕ is separable as $\phi(x,y) = u(x) - u(y)$ for some real valued function u on X.

In the last decade, several axiomatizations for the SSA representation have appeared. Fishburn (1984) and Fishburn and LaValle (1987) used lottery acts for that axiomatization to generalize Anscombe and Aumann's SEU axiomatization (see Anscombe and Aumann (1963)). A general axiomatization with finite states was obtained by Fishburn (1990). Savage–type axiomatizations were developed by Fishburn (1988a) and Sugden (1993). However, their axiomatizations are not fully more general than Savage's SEU, since they

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considered SSA representations for simple acts, i.e., the resulting consequences are finite. The aim of this paper is to extend their axiomatizations to cover all non-simple acts. Our key axioms are modifications of Savage's P2, P6 and P7.

The paper organized as follows. Section 2 reviews Savage's SEU representation and Fishburn's SSA representation. Then in Section 3 we state axioms for our extended SSA representation. Section 4 provides the proof of the extended SSA theorem.

2 Savage and SSA Axioms

Throughout the paper, let $\mathcal{F}$ be the set of all acts, defined as functions from S into the consequence space X. Subsets of S are called events. By A^c we denote the complement $S \setminus A$ of an event A. A constant act is an act f such that $f(s) = x$ for all $s \in S$ and some $x \in X$. Every $x \in X$ will be identified with a constant act. A simple act is an act f such that $\{f(s) : s \in S\}$ is finite. Let $\mathcal{F}^s$ be the set of all simple acts, so $\mathcal{F}^s \subseteq \mathcal{F}$.

For $f, g \in \mathcal{F}$ and an event $A \in 2^S$, let $f \bigcirc_A g$ denote the act h such that $h(s) = f(s)$ for all $s \in A$, and $f(s) = g(s)$ for all $s \in A^c$. Then any simple acts can be represented as follows: for $A_1, \ldots, A_{n-1} \in 2^S$ and $x_1, \ldots, x_n \in X$,

$$(\ldots((x_1 \bigcirc_{A_1} x_2) \bigcirc_{A_2} x_3) \ldots \bigcirc_{A_{n-2}} x_{n-1}) \bigcirc_{A_{n-1}} x_n.$$

When $A_1 \subseteq A_2 \subseteq \ldots \subseteq A_{n-1}$, we shall simply write $x_1 \bigcirc_{A_1} x_2 \ldots x_{n-1} \bigcirc_{A_{n-1}} x_n$.

Let $\sim$ and $\succeq$ be defined in the usual way: for $f, g \in \mathcal{F}$, $f \sim g$ if $f \succeq g$ and $g \succeq f$; $f \succ g$ if not$(g \succeq f)$. An event A is said to be null if $f \bigcirc_A h \sim g \bigcirc_A h$ for all $f, g, h \in \mathcal{F}^s$. Let $\mathcal{N}$ be the set of all null events. A partition of a nonnull event A, denoted by $\omega(A)$, is a set of a finite number of mutually disjoint nonempty events whose union equals A. An n-partition $\omega(A)$ will be denoted by $\{A_1, \ldots, A_n\}$ for $A_i \in 2^A, i = 1, \ldots, n$. A comparative probability relation $\succ^*$ on 2^S, read as "is more probable than", is induced by $\succ$ as follows. For all $A, B \in 2^S$,

$$A \succ^* B \text{ if for all } x, y \in X,\ x \bigcirc_A y \succ x \bigcirc_B y \text{ whenever } x \succ y.$$

Let $\sim^*$ and $\succeq^*$ be defined in the usual way.

Definition 1 $(\mathcal{F}, \succ)$ *is said to fulfill Savage's axiom system if* $\succ$ *on* $\mathcal{F}$ *satisfies the following seven axioms, which apply to all* $f, g, h, h' \in \mathcal{F}$, *all* $x, y, z, w \in X$, *and all* $A, B \in 2^S$.

P1. $\succ$ *on* $\mathcal{F}$ *is a weak order.*

P2. *If* $f \bigcirc_A h \succeq g \bigcirc_A h$, *then* $f \bigcirc_A h' \succeq g \bigcirc_A h'$.

P3. *If* $A \notin \mathcal{N}$, *then* $x \succ y \iff x \bigcirc_A f \succ y \bigcirc_A f$.

P4. *If* $x \succ y$ *and* $z \succ w$, *then* $x \bigcirc_A y \succ x \bigcirc_B y \iff z \bigcirc_A w \succ z \bigcirc_B w$.

P5. $a \succ b$ *for some* $a, b \in X$.

P6. *If $f \succ g$, then for each $x \in X$, there is a partition $\omega(S)$ such that for all $C \in \omega(S)$, $x \bigcirc_C f \succ g$ and $f \succ x \bigcirc_C g$.*

P7. *If $f \bigcirc_A h \succ g(s) \bigcirc_A h$ for all $s \in A$, then $f \bigcirc_A h \succeq g \bigcirc_A h$; if $f(s) \bigcirc_A h \succ g \bigcirc_A h$ for all $s \in A$, then $f \bigcirc_A h \succeq g \bigcirc_A h$.*

The meanings of P1–P7 are discussed in many places in the literature (e.g., Fishburn (1970, 1988b)), so that we do not repeat them here. However, it may be deserved to be mentioned here for the later development of our axioms that, in P6, at least one of the acts f and g can be simple.

Savage's representation theorem is stated as follows.

Savage's SEU Theorem. *If $(\mathcal{F}, \succ)$ fulfills Savage's axiom system, then there exist a real valued bounded function u on X and a finitely additive probability measure π on 2^S such that for all $f, g \in \mathcal{F}$, all $A, B \subseteq S$, and all $0 < \lambda < 1$,*

(1) $f \succ g \iff \int_S u(f(s))d\pi(s) > \int_S u(g(s))d\pi(s)$.
(2) $A \succ^* B \iff \pi(A) > \pi(B)$.
(3) $A \in \mathcal{N} \iff \pi(A) = 0$.
(4) $\pi(C) = \lambda\pi(A)$ *for some* $C \subset A$.

Moreover, u is unique up to a positive linear transformation and π is unique.

Although u must be bounded in Savage's SEU theorem, it is not the case when only feasible acts are simple.

Let $\mathcal{F}_{xy}$ denote the set of acts whose only components are x and y, i.e.,

$$\mathcal{F}_{xy} = \{f \in \mathcal{F} : f(s) \in \{x, y\} \text{ for all } s \in S\}.$$

Definition 2 *$(\mathcal{F}, \succ)$ is said to fulfill Fishburn's SSA axiom system if $\succ$ on $\mathcal{F}$ satisfies P2–P5 in Definition 1 and the following three axioms, which apply to all $x, y \in X$, all $f, g, h \in \mathcal{F}$, and all $A, B \in 2^S$.*

P1*. *$\succ$ on $\mathcal{F}$ is asymmetric, and $\succ$ on $\mathcal{F}_{xy}$ is a weak order.*

P2*. *If $A \cap B = \emptyset$, $f \bigcirc_A h \succeq g \bigcirc_A h$, and $f \bigcirc_B h \succeq g \bigcirc_B h$, then $f \bigcirc_{A\cup B} h \succeq g \bigcirc_{A\cup B} h$, and if, in addition, $f \bigcirc_A h \succ g \bigcirc_A h$, then $f \bigcirc_{A\cup B} h \succ g \bigcirc_{A\cup B} h$.*

P6*. *If $f \succ g$, then given $x, y \in X$, there is a partition $\omega(S)$ such that, for all $C \in \omega(S)$, $x \bigcirc_C f \succ y \bigcirc_C g$, $x \bigcirc_C f \succ g$, and $f \succ y \bigcirc_C g$.*

Those axioms $P1^*, P2^*$, and $P6^*$ were proposed by Fishburn (1988a). A recent axiomatization by Sugden (1993) modifies P4 and replaces $P1^*$ by the completeness of $\succeq$, i.e., for all $f, g \in \mathcal{F}$, $f \succeq g$ or $g \succeq f$. However, their axiom systems cannot apply to $(\mathcal{F}, \succ)$ when $\mathcal{F} \setminus \mathcal{F}^s$ is not empty.

Fishburn's SSA representation theorem is stated as follows.

Fishburn's SSA Theorem. *If $(\mathcal{F}^s, \succ)$ fulfills Fishburn's SSA axiom system, then there exist a skew-symmetric, real valued function ϕ on $X \times X$ and a finitely additive probability measure π on 2^S such that for all $f, g \in \mathcal{F}^s$, all $A, B \subseteq S$, and all $0 < \lambda < 1$, (2), (3), and (4) of Savage's SEU theorem hold along with*

$$f \succ g \iff \sum_{(x,y) \in Y_{fg}} \phi(x,y)\pi(\{s \in S : f(s) = x \text{ and } g(s) = y\}) > 0,$$

where $Y_{fg} = \{(x,y) \in X \times X : f(s) = x \text{ and } g(s) = y \text{ for some } s \in S\}$ is finite. Moreover, ϕ is unique up to multiplication by a positive constant, and π is unique.

3 Axioms and a Theorem

This section presents the main result. We introduce three new axioms and state our extended SSA representation theorem which covers all non-simple acts. One of our axioms is a cancellation axiom whose original idea appeared in nontransitive additive conjoint measurement by Fishburn (1990, 1991). The other two axioms are respectively modifications of P6 and P7.

To state our cancellation axiom, we need the following notation. Given a finite partition ω of S and an integer $n > 1$, we shall define a binary relation E^n_ω on $\mathcal{F}^n = \mathcal{F} \times \ldots \times \mathcal{F}$ (n times) as follows: for all $(f^1, \ldots, f^n), (g^1, \ldots, g^n) \in \mathcal{F}^n$,

$$\begin{aligned} &(f^1, \ldots, f^n) E^n_\omega (g^1, \ldots, g^n) \\ &\iff \text{it is true for each } A \in \omega \text{ that for all } f, g \in \mathcal{F} \\ &\quad |\{k : (f^k(s), g^k(s)) = (f(s), g(s)) \text{ for all } s \in A\}| \\ &\quad = |\{k : (f^k(s), g^k(s)) = (g(s), f(s)) \text{ for all } s \in A\}|. \end{aligned}$$

The following axioms apply to all $f, g, f^1, \ldots, f^4, g^1, \ldots, g^4 \in \mathcal{F}$, all $x, y \in X$, all $A \in 2^S$, and all 3-partitions ω of S.

P2.** *If $(f^1, f^2, f^3, f^4) E^4_\omega (g^1, g^2, g^3, g^4)$ and $f^k \succeq g^k$ for $k = 1, 2, 3$, then $g^4 \succeq f^4$.*

P6.** *If $f \succ g$ and $x \succeq y$, then for every event A, there is a partition $\omega(A)$ such that, for all $B \in \omega(A)$, $y \bigcirc_B f \succ x \bigcirc_B g$.*

P7*. *If $f(s) \bigcirc_A f \succeq g(s) \bigcirc_A g$ for all $s \in A$, then $f \succeq g$.*

*P2*** is a cancellation axiom. When S is finite, this is tantamount to the cancellation axiom proposed by Fishburn (1990, 1991). Suppose that the SSA represntation is to hold and $(f^1, f^2, f^3, f^4) E^4_\omega (g^1, g^2, g^3, g^4)$. Then

$$\sum_{k=1}^{4} \sum_{A \in \omega(S)} \int_A \phi(f^k(s), g^k(s)) d\pi(s) = 0.$$

If $f^k \succeq g^k$ for $k = 1, 2, 3$, then

$$\sum_{A \in \omega(S)} \int_A \phi(f^k(s), g^k(s)) d\pi(s) \geq 0 \text{ for } k = 1, 2, 3,$$

so that

$$\sum_{A \in \omega(S)} \int_A \phi(f^4(s), g^4(s)) d\pi(s) \leq 0.$$

This implies that $g^4 \succeq f^4$, so $P2^{**}$ is necesarry for the SSA represntation. We also note that at most 3-partitions are necessary for the representation.

$P6^{**}$ asserts that any event A can be partitioned into small-probability events such that the conclusion of the axiom obtains. Together with the other axioms, it will be shown later in Lemma 2 that the claim of $P6^{**}$ can be valid without the restriction of $x \succeq y$. $P7^*$ is a conditional monotone dominance axiom. $P6^{**}$ and $P7^*$ are also necessary for the SSA represntation.

Our main result is stated as follows.

Extended SSA Theorem. *If $P2^{**}, P3, P4, P5, P6^{**}$, and $P7^*$ hold, then there exist a bounded skew-symmetric real-valued function ϕ on $X \times X$ and a finitely additive probability measure π on 2^S such that for all $f, g \in \mathcal{F}$, all $A, B \subseteq S$, and all $0 < \lambda < 1$, (2), (3), and (4) of Savage's SEU theorem hold along with*

$$f \succ g \iff \int_S \phi(f(s), g(s)) d\pi(s) > 0.$$

Moreover, ϕ is unique up to multiplication by a positive constant, and π is unique.

The proof will appear in the next section.

4 Proof of Extended SSA Theorem

Throughout the section we shall assume that Axioms $P2^{**}, P3, P4, P5, P6^{**}$, and $P7^*$ hold. We prove the extended SSA theorem in four steps. The first step shows that $(\mathcal{F}^s, \succ)$ has Fishburn's SSA representation, i.e., there exist a skew-symmetric real valued function ϕ on $X \times X$ and a probability measure π on 2^S that satisfy the numerical representation of $\succ$ in Fishburn's SSA theorem. The second step shows that ϕ must be bounded. In the third step, we shall introduce binary conditional relations $\succ_A$ on $\mathcal{F} \times \mathcal{F}$ for events A with $0 < \pi(A) \leq \frac{1}{2}$, and prove that $(\mathcal{F} \times \mathcal{F}, \succ_A)$ has Savage's SEU representation. In the final step, we complete the proof of the theorem.

Step 1. We prove the following claim.

Claim 1 *$(\mathcal{F}^s, \succ)$ has Fishburn's SSA representation.*

The proof will appear at the end of this step.

Without restricting our attention to $\mathcal{F}^s$, we obtain the following lemma.

Lemma 1 *$\succ$ on $\mathcal{F}$ is asymmetric, and P2 and P2* hold.*

Proof. Since $(h,h,h,h)E^4_\omega(h,h,h,h)$ for any act h and any partition ω, $P2^{**}$ requires $h \sim h$. To show that $\succ$ on $\mathcal{F}$ is asymmetric, we suppose $f \succ g$ for $f, g \in \mathcal{F}$. Then for any $h \in \mathcal{F}$, $(h,h,f,g)E^4_\omega(h,h,g,f)$. Since $h \sim h$ and $f \succ g$, it follows from $P2^{**}$ that not$(g \succ f)$. Hence $\succ$ is asymmetric.

To show $P2$, let $f^i = f \bigcirc_A h_i$, $g^i = g \bigcirc_A h_i$, and $h^i = h \bigcirc_A h_i$ for $i = 1,2$. Then for $\omega(S) = \{A, A^c\}$, $(f^1, h^1, h^2, g^2)E^4_\omega(g^1, h^1, h^2, f^2)$. If $f^1 \succeq g^1$, then by $P2^{**}$, $f^2 \succeq g^2$, so $P2$ holds.

To show $P2^*$, suppose that the hypotheses of $P2^*$ hold. Let $\omega(S) = \{A, A^c\}$ and

$$\begin{aligned} f^1 &= h, & g^1 &= h, \\ f^2 &= f \bigcirc_A h, & g^2 &= g \bigcirc_A h, \\ f^3 &= f \bigcirc_B h, & g^3 &= g \bigcirc_B h, \\ f^4 &= g \bigcirc_{A\cup B} h, & g^4 &= f \bigcirc_{A\cup B} h, \end{aligned}$$

so $(f^1, f^2, f^3, f^4)E^4_\omega(g^1, g^2, g^3, g^4)$. If $f^2 \succeq g^2$ and $f^3 \succeq g^3$, then by $P2^{**}$, $g^4 \succeq f^4$. Hence the first part of $P2^*$ obtains. Assume next that $f^2 \succ g^2$, and $f^3 \succeq g^3$. If $f^4 \sim g^4$, then by $P2^{**}$, $g^2 \succeq f^2$. This is a contradiction. Hence we must have $g^4 \succ f^4$, so the second part of $P2^*$ follows. □

The claim of Axiom $P6^{**}$ can be strengthened as in the following lemma, which will be used in place of $P6^{**}$.

Lemma 2 *If $f \succ g$ and $x \succeq y$, then for every event A, there is a partition $\omega(A)$ such that for all $B \in \omega(A)$, $y \bigcirc_B f \succ x \bigcirc_B g$ and $x \bigcirc_B f \succ y \bigcirc_B g$.*

Proof. Suppose that $f \succ g$ and $x \succeq y$. Then by $P6^{**}$, there is a partition $\omega(A)$ such that for all $B \in \omega(A)$, $y \bigcirc_B f \succ x \bigcirc_B g$. For some $a \in X$ and $B \in \omega(A)$, let $\omega(S) = \{B, B^c\}$ and

$$\begin{aligned} f^1 &= y \bigcirc_B f, & g^1 &= x \bigcirc_B g, \\ f^2 &= x \bigcirc_B a, & g^2 &= y \bigcirc_B a, \\ f^3 &= x \bigcirc_B a, & g^3 &= y \bigcirc_B a, \\ f^4 &= y \bigcirc_B g, & g^4 &= x \bigcirc_B f, \end{aligned}$$

so $(f^1, f^2, f^3, f^4)E^4_\omega(g^1, g^2, g^3, g^4)$. Since $f^1 \succ g^1, f^2 \succeq g^2$, and $f^3 \succeq g^3$, $P2^{**}$ impies that $x \bigcirc_B f \succ y \bigcirc_B g$. □

Proof of Claim 1. If $P1^*$ and $P6^*$ are to hold, then $(\mathcal{F}^s, \succ)$ fulfills Fishburn's SSA axiom system, so that the desired result of the claim obtains. However, this is not the case, i.e., $P6^*$ may not be drived from our axiom system. We note that requirements of $P6^*$ can be divided into two parts (i) and (ii) as follows: if $f \succ g$, then given $x, y \in X$, there is a partition $\omega(S)$ such that for all $C \in \omega(S)$,

(i) $x \bigcirc_C f \succ g$ and $f \succ y \bigcirc_C g$,
(ii) $x \bigcirc_C f \succ y \bigcirc_C g$.

In Fishburn's proof of his SSA theorem, the requirement (i) is needed to show that $\succ$ on $\mathcal{F}_{xy}$ satisfies P6. In other part of the proof, only the requirement (ii), which is a direct consequence of Lemma 2, is applied. Therefore, it suffices to show that our axiom system implies that $P1^*$ holds and $\succ$ on $\mathcal{F}_{xy}$ satisfies P6.

First, we show $P1^*$. By Lemma 1, $\succ$ on $\mathcal{F}^s$ is asymmetric. Suppose that $x \succ y$ for $x, y \in X$ as assured by P5. It suffices to show that $\succ$ on $\mathcal{F}_{xy}$ is negatively transitive. Let $\omega(S) = \{(A \setminus (B \cup C)) \cup (A \cap B \cap C) \cup (A^c \cap B^c \cap C^c), (B \setminus (A \cup C)) \cup ((C \cap A) \setminus B), (C \setminus (A \cup B)) \cup ((A \cap B) \setminus C)\}$ and

$$\begin{aligned} f^1 &= x, & g^1 &= x, \\ f^2 &= x \bigcirc_A y, & g^2 &= x \bigcirc_B y, \\ f^3 &= x \bigcirc_B y, & g^3 &= x \bigcirc_C y, \\ f^4 &= x \bigcirc_C y, & g^4 &= x \bigcirc_A y, \end{aligned}$$

so $(f^1, f^2, f^3, f^4) E^4_\omega (g^1, g^2, g^3, g^4)$. Assume that $g^2 \succeq f^2$ and $g^3 \succeq f^3$. Then by $P2^{**}$, $f^4 \succeq g^4$. Hence $\succ$ is negatively transitive.

Next we show that $\succ$ on $\mathcal{F}_{xy}$ satisfies P6. Suppose that $x \succ y$ and $f \succ g$ for $f, g \in \mathcal{F}_{xy}$. Then $f = x \bigcirc_A y$ and $g = x \bigcirc_B y$ for some event A and B. Given x, it follows from Lemma 2 that there are partitions $\omega(A \setminus B)$ and $\omega(B \setminus A)$ such that, for all $C \in \omega(A \setminus B) \cup \omega(B \setminus A)$, $x \bigcirc_C f \succ x \bigcirc_C g$. Since $f = x \bigcirc_C f$ for $C \in \omega(A \setminus B)$ and $g = x \bigcirc_C f$ for $C \in \omega(B \setminus A)$, we obtain that for all $C \in \omega(A \setminus B) \cup \omega(B \setminus A)$, $f \succ x \bigcirc_C g$ and $x \bigcirc_C f \succ g$.

For $A^c \cap B^c$, Lemma 2 implies that there is a partition $\omega(A^c \cap B^c)$ such that, for all $C \in \omega(A^c \cap B^c)$, $y \bigcirc_C f \succ x \bigcirc_C g$ and $x \bigcirc_C f \succ y \bigcirc_C g$, so $f \succ x \bigcirc_C g$ and $x \bigcirc_C f \succ g$. Since $f \succ g$, $f = x \bigcirc_{A \cap B} f \succ g = x \bigcirc_{A \cap B} g$. Thus we let $\omega(S) = \omega(A \setminus B) \cup \omega(B \setminus A) \cup \omega(A^c \cap B^c) \cup \{A \cap B\}$. Then for all $C \in \omega(S)$, $x \bigcirc_C f \succ g$ and $f \succ x \bigcirc_C g$.

When x is replaced by y in the preceding paragraphs, it readily follows that there is a partition $\omega(S)$ such that, for all $C \in \omega(S)$, $y \bigcirc_C f \succ g$ and $f \succ y \bigcirc_C g$. Hence $\succ$ on $\mathcal{F}_{xy}$ satisfies P6. □

Step 2. Given an event A, we define a preference subset $P_A \subseteq \mathcal{F} \times \mathcal{F}$ and an indifference subset $I_A \subseteq \mathcal{F} \times \mathcal{F}$ as follows: for all $f, g \in \mathcal{F}$,

$$\begin{aligned} (f, g) \in P_A &\iff f \bigcirc_A h \succ g \bigcirc_A h \text{ for some } h \in \mathcal{F}, \\ (f, g) \in I_A &\iff f \bigcirc_A h \sim g \bigcirc_A h \text{ for some } h \in \mathcal{F}. \end{aligned}$$

The inverse R^{-1} of a subset $R \subseteq \mathcal{F} \times \mathcal{F}$ is defined by $R^{-1} = \{(f, g) \in \mathcal{F} : (g, f) \in R\}$. We note by P2 that $I_A = I_A^{-1}$, $P_A \cap P_A^{-1} = P_A \cap I_A = P_A^{-1} \cap I_A = \emptyset$. When A is not null, P_A is not empty. We shall write P and I in place of P_S and I_S, respectively.

In what follows, we shall denote elements of $\mathcal{F}^2 = \mathcal{F} \times \mathcal{F}$ by bold faced letters, and the first and second components of $\boldsymbol{f} \in \mathcal{F}^2$ will be denoted by f_1 and f_2, respectively. Each

f is regarded as a mapping from S into $X^2 = X \times X$. Thus we let $f(s) = (f_1(s), f_2(s))$ for all $s \in S$. The inverse f^{-1} of $f = (f_1, f_2)$ is defined by $f^{-1} = (f_2, f_1)$. We say that f is simple if f_1 and f_2 are simple. Let $f \bigcirc_A g$ denote a pair of act $(f_1 \bigcirc_A g_1, f_2 \bigcirc_A g_2)$. A pair of constant acts will be denoted by $x \in X^2$. A pair of identical constant acts will be denoted by x^*, i.e., $x_1 = x_2$. When $(f_1^1, \ldots, f_1^m) E_\omega^m (f_2^1, \ldots, f_2^m)$, we shall write $(f^1, \ldots, f^m) \in E_\omega^m$.

Since $P2^*$ will be frequently used in our proof, we rewrite it using our new notations as follows.

P2*. *If $A \cap B = \emptyset$, $f \in P_A \cup I_A$, and $f \in P_B \cup I_B$, then $f \in P_{A\cup B} \cup I_{A\cup B}$, and if, in addition, $f \in P_A$, then $f \in P_{A\cup B}$.*

By Fishburn's SSA theorem, there exist a skew-symmetric real valued function ϕ on X^2 and a finitely additive probability measure π on 2^S such that (2)–(4) of Savage's SEU theorem hold along with, for all simple $f \in \mathcal{F}^2$,

$$f \in P \Longleftrightarrow \sum_{x \in Y_f} \phi(x)\pi(\{s : f(s) = x\}),$$

where $Y_f = \{x \in X^2 : f(s) = x \text{ for some } s \in S\}$ is finite. Moreover, ϕ is unique up to multiplication by a positive constant, and π is unique.

Given an event A with $0 \leq \pi(A) \leq \frac{1}{2}$, let B be any event such that $A \cap B = \emptyset$ and $\pi(A) = \pi(B)$. Then each pair of acts in $\mathcal{F}^2$ belongs to exactly one of the following classes:

(1) f is *big* on $A \Longleftrightarrow f \bigcirc_A x \in P_{A\cup B}$ for all $x \in X^2$.
(2) f is *little* on $A \Longleftrightarrow f \bigcirc_A x \in P^{-1}_{A\cup B}$ for all $x \in X^2$.
(3) f is *normal* on $A \Longleftrightarrow f \bigcirc_A a \notin P_{A\cup B}$ and $f \bigcirc_A b \notin P^{-1}_{A\cup B}$ for some $a, b \in X^2$.

Note that f is big on A iff f^{-1} is little on A. If f is simple on A, then f is normal on A whenever $\pi(A) \leq \frac{1}{2}$. It follows from the following lemma that the above definitions are independent of choice of B, so they are well defined.

Lemma 3 *If $A \cap B = A \cap C = \emptyset, \pi(A) = \pi(B) = \pi(C) \neq 0$, and $f \bigcirc_A x \in P_{A\cup B}$ for all $x \in X^2$, then $f \bigcirc_A x \in P_{A\cup C}$ for all $x \in X^2$.*

Proof. Suppose that the hypotheses of the lemma hold. Assume that $f \bigcirc_A a \notin P_{A\cup C}$ for some $a \in X^2$. For some b^*, let $\omega(S) = \{A, A^c\}$ and

$$\begin{aligned} f^1 &= (f \bigcirc_A x) \bigcirc_{A\cup B} b^*, \\ f^2 &= (f^{-1} \bigcirc_A a^{-1}) \bigcirc_{A\cup C} b^*, \\ f^3 &= (a^{-1} \bigcirc_A a) \bigcirc_{A\cup C} b^*, \\ f^4 &= (a \bigcirc_A x^{-1}) \bigcirc_{A\cup B} b^*, \end{aligned}$$

so $(f^1, f^2, f^3, f^4) \in E_\omega^4$. Since $f^1 \in P, f^2 \in P \cup I$, and $f^3 \in I$, $P2^{**}$ implies $f^4 \in P^{-1}$. Therefore we obtain

$$f^4 \in P^{-1} \Longleftrightarrow \phi(a) < \phi(x).$$

Since x is arbitrary, this is a contradiction. Hence $f \bigcirc_A x \in P_{A\cup C}$ for all $x \in X^2$. □

The aim of this step is to prove the following claim.

Claim 2 *ϕ is bounded. Furthermore, if there is an $f \in \mathcal{F}^2$ which is either big or little on some event A with $0 < \pi(A) \leq \frac{1}{2}$, then for all $x \in X^2$, $\phi(x) < \sup\{\phi(y) : y \in X^2\}$.*

To prove the claim, we need the following lemma.

Lemma 4 *Suppose that $A \cap B = \emptyset, \pi(A) = \pi(B) \neq 0$, and f is big (respectively, little) on A. Then g is little (respectively, big) on B if and only if $f \bigcirc_A g \in I_{A\cup B}$.*

Proof. Suppose that $A \cap B = \emptyset, \pi(A) = \pi(B) \neq 0$, and f is big on A. When f is little on A, the proof is similar. Then $f \bigcirc_A g(s) \in P_{A\cup B}$ for all $s \in B$. Thus by $P7^*$, $f \bigcirc_A g \in P_{A\cup B} \cup I_{A\cup B}$.

First we assume that g is little on B. Then $f(s) \bigcirc_A g \in P^{-1}_{A\cup B}$ for all $s \in A$, so by $P7^*$, $f \bigcirc_A g \in P^{-1}_{A\cup B} \cup I_{A\cup B}$. Hence $f \bigcirc_A g \in I_{A\cup B}$.

Assume next that $f \bigcirc_A g \in I_{A\cup B}$. Suppose on the contrary that g is not little on B. Then by definition, $a \bigcirc_A g \in P_{A\cup B} \cup I_{A\cup B}$ for some $a \in X^2$. Let $\omega(S) = \{A, A^c\}$ and

$$\begin{aligned} f^1 &= (f^{-1} \bigcirc_A g^{-1}) \bigcirc_{A\cup B} b^*, \\ f^2 &= (a \bigcirc_A g) \bigcirc_{A\cup B} b^*, \\ f^3 &= (a^{-1} \bigcirc_A a) \bigcirc_{A\cup B} b^*, \\ f^4 &= (f \bigcirc_A a^{-1}) \bigcirc_{A\cup B} b^*, \end{aligned}$$

so $(f^1, f^2, f^3, f^4) \in E^4_\omega$. Since $f^1, f^3 \in I$ and $f^2 \in P \cup I$, $P2^{**}$ implies that $f^4 \in P^{-1}_{A\cup B} \cup I_{A\cup B}$. This contradicts bigness of f on A. Hence g must be little on B. □

Proof of Claim 2. Suppose on the contrary that ϕ is unbounded. Let $\{A_0, B_0\}$ be a partition of S with $\pi(A_0) = \pi(B_0) = \frac{1}{2}$. We then construct denumerable partitions $\omega(A_0) = \{B_1, B_2, \ldots\}$ and $\omega(B_0) = \{A_1, A_2, \ldots\}$ with $\pi(A_i) = \pi(B_i) = 2^{-(i+1)}$ for $i = 1, 2, \ldots$. Since ϕ is unbounded, for $i = 1, 2, \ldots$, we can take $\phi(x_{i-1}) \geq 2^i$ for some $x_{i-1} \in X^2$. Then let

$$\begin{aligned} f(s) &= x_i \text{ for all } s \in A_i \text{ and } i = 0, 1, 2, \ldots, \\ g(s) &= x_i \text{ for all } s \in B_i \text{ and } i = 0, 1, 2, \ldots. \end{aligned}$$

Clearly we have

$$\int_S \phi(f(s))d\pi(s) = \int_S \phi(g(s))d\pi(s) = +\infty.$$

By the construction of f, $f \in P_{A_0}$ and $f(s) \in P_{B_0}$ for all $s \in B_0$. By $P7^*$, $f \in P_{B_0} \cup I_{B_0}$. Thus by $P2^*$, $f \in P_{A_0\cup B_0} = P$. Similarly, $g \in P$.

We show that f is big on B_0. Note that

$$\lim_{n\to\infty} \sum_{i=1}^{n} \phi(x_i)\pi(A_i) = +\infty.$$

Then for each $x \in X^2$, there is a natural number m such that

$$\sum_{i=1}^{m} \phi(x_i)\pi(A_i) + \pi(A_0)\phi(x) > 0.$$

Thus let a simple f^1 be defined by

$$\begin{aligned} f^1(s) &= x \quad \text{if } s \in A_0, \\ &= x_i \quad \text{if } s \in A_i \quad (i = 1, \ldots, m), \\ &= a^* \quad \text{otherwise,} \end{aligned}$$

so $f^1 \in P_{\cup_{i=0}^m A_i}$. Let

$$\begin{aligned} f^2(s) &= x_i \quad \text{if } s \in A_i \quad (i = m+1, m+2, \ldots), \\ &= a^* \quad \text{otherwise,} \end{aligned}$$

so by $P7^*$, $f^2 \in P_{\cup_{i>m} A_i} \cup I_{\cup_{i>m} A_i}$. Since $f \bigcirc_{B_0} x = f^1 \bigcirc_{\cup_{i=0}^m A_i} f^2$, it follows from $P2^*$ that $f \bigcirc_{B_0} x \in P$. Since x is arbitrary, f is big on B_0. It is similarly shown that g is big on A_0 and $x \bigcirc_{A_1} f$ is big on B_0 for any x.

Take $x \in X^2$ to satisfy $(x \bigcirc_{A_1} x_1^{-1}) \bigcirc_{B_0} a^* \in P$. Then let $\omega(S) = \{A_1, A_1^c\}$ and

$$\begin{aligned} f_1 &= (x \bigcirc_{A_1} x_1^{-1}) \bigcirc_{B_0} a^*, \\ f_2 &= (x_1^{-1} \bigcirc_{A_1} x_1) \bigcirc_{B_0} a^*, \\ f_3 &= (x_1 \bigcirc_{A_1} f) \bigcirc_{B_0} g^{-1}, \\ f_4 &= (x^{-1} \bigcirc_{A_1} f^{-1}) \bigcirc_{B_0} g. \end{aligned}$$

Then $(f_1, f_2, f_3, f_4) \in E_\omega^4$. By Lemma 4, $f_3 \in I$, since $x_1 \bigcirc_{A_1} f$ is big on B_0 and g^{-1} is little on A_0. Since $f_1 \in P$ and $f_2 \in I$, $P2^{**}$ implies $f_4 \in P^{-1}$. However, by Lemma 4, we must have $f_4 \in I$, since $x \bigcirc_{A_1} f$ is big on B_0. This is a contradiction. Hence ϕ must be bounded.

Suppose that there is an f that is big on an event A with $0 < \pi(A) \leq \frac{1}{2}$. When f is little on A, the proof is similar. Let B be an event such that $A \cap B = \emptyset$ and $\pi(A) = \pi(B)$. Suppose on the contrary that $\phi(a) = \sup\{\phi(x) : x \in X^2\}$ for some $a \in X^2$. Then by Claim 1, $f^{-1}(s) \bigcirc_A a \in P_{A \cup B} \cup I_{A \cup B}$ for all $s \in A$. By $P7^*$, $f^{-1} \bigcirc_A a \in P_{A \cup B} \cup I_{A \cup B}$, so $f \bigcirc_A a^{-1} \notin P_{A \cup B}$. This contradicts bigness of f. Hence $\phi(x) < \sup\{\phi(y) : y \in X^2\}$ for all $x \in X^2$. □

Step 3. Given an event A with $\pi(A) = \frac{1}{2}$, we define a binary relation $\succeq_A$ on $\mathcal{F}^2$ by

$$\begin{aligned} f \succeq_A g \quad &\text{iff} \quad f \bigcirc_A h \in P \cup I \text{ and } g \bigcirc_A h \in P^{-1} \cup I \text{ for some simple } h, \\ &\text{or} \quad f \text{ and } g \text{ are either big or little on } A. \end{aligned}$$

It is easy to see from $P2^{**}$ that this definition does not depend on choice of specific h. Let $\sim_A$ and $\succ_A$ be defined as usual: for $f, g \in \mathcal{F}^2$, $f \sim_A g$ iff $f \succeq_A g$ and $g \succeq_A f$; $f \succ_A g$ iff not$(g \succeq_A f)$.

Now we have the following claim whose proof is given at the end of this step.

Claim 3 *Suppose that $\pi(A) = \frac{1}{2}$. Then $(\mathcal{F}^2, \succ_A)$ has Savage's SEU representation.*

Before establishing the proof of the claim, we need the following three lemmas.

Lemma 5 *Suppose that $A \cap B = \emptyset$ and $\pi(A) = \pi(B) \neq 0$. Then $\boldsymbol{f} \in \mathcal{F}^2$ is normal on A if and only if $\boldsymbol{f} \bigcirc_A \boldsymbol{h} \in I_{A\cup B}$ for some simple $\boldsymbol{h}$.*

Proof. Suppose that $A \cap B = \emptyset$ and $\pi(A) = \pi(B) \neq 0$. First we assume that $\boldsymbol{f}$ is normal on A. Then $\boldsymbol{f} \bigcirc_A \boldsymbol{a} \in P_{A\cup B} \cup I_{A\cup B}$ and $\boldsymbol{f} \bigcirc_A \boldsymbol{b} \in P^{-1}_{A\cup B} \cup I_{A\cup B}$ for some $\boldsymbol{a}, \boldsymbol{b} \in X^2$. If $\boldsymbol{f} \bigcirc_A \boldsymbol{a} \in I_{A\cup B}$ or $\boldsymbol{f} \bigcirc_A \boldsymbol{b} \in I_{A\cup B}$, then the desired result obtains. Thus we assume that $\boldsymbol{f} \bigcirc_A \boldsymbol{a} \in P_{A\cup B}$ and $\boldsymbol{f} \bigcirc_A \boldsymbol{b} \in P^{-1}_{A\cup B}$. Since $\pi(A) = \pi(B) \neq 0$, we have $\boldsymbol{a} \bigcirc_A \boldsymbol{a}^{-1} \in I_{A\cup B}$. For $\boldsymbol{c}^*$, let $\omega(S) = \{A, A^c\}$ and

$$\begin{aligned}
\boldsymbol{f}^1 &= (\boldsymbol{f} \bigcirc_A \boldsymbol{a}) \bigcirc_{A\cup B} \boldsymbol{c}^*,\\
\boldsymbol{f}^2 &= (\boldsymbol{f}^{-1} \bigcirc_A \boldsymbol{b}^{-1}) \bigcirc_{A\cup B} \boldsymbol{c}^*,\\
\boldsymbol{f}^3 &= (\boldsymbol{a} \bigcirc_A \boldsymbol{a}^{-1}) \bigcirc_{A\cup B} \boldsymbol{c}^*,\\
\boldsymbol{f}^4 &= (\boldsymbol{a}^{-1} \bigcirc_A \boldsymbol{b}) \bigcirc_{A\cup B} \boldsymbol{c}^*.
\end{aligned}$$

Then $\boldsymbol{f}^1 \in P, \boldsymbol{f}^2 \in P, \boldsymbol{f}^3 \in I$, and $(\boldsymbol{f}^1, \boldsymbol{f}^2, \boldsymbol{f}^3, \boldsymbol{f}^4) \in E^4_\omega$. By $P2^{**}$, $\boldsymbol{f}^4 \in P^{-1}$, so we have

$$\begin{aligned}
\boldsymbol{f}^4 \in P^{-1} &\implies \pi(A)\phi(\boldsymbol{a}) + \pi(B)\phi(\boldsymbol{b}^{-1}) > 0\\
&\implies \phi(\boldsymbol{a}) > \phi(\boldsymbol{b}).
\end{aligned}$$

Suppose that $C, D \subseteq B$, $0 < \pi(C) < \pi(B)$, and $0 < \pi(D) < \pi(B)$. In what follows, we are to show that if $\boldsymbol{f} \bigcirc_A (\boldsymbol{a} \bigcirc_C \boldsymbol{b}) \in P_{A\cup B}$ and $\boldsymbol{f} \bigcirc_A (\boldsymbol{a} \bigcirc_D \boldsymbol{b}) \in P^{-1}_{A\cup B}$, then $\pi(D) < \pi(C)$. Hence by Lemma 2 and the property of π, there is a unique number $0 < \alpha < \pi(B)$ such that $\boldsymbol{f} \bigcirc_A (\boldsymbol{a} \bigcirc_E \boldsymbol{b}) \in I_{A\cup B}$ for all events $E \subset B$ with $\pi(E) = \alpha$. Note that $\boldsymbol{a} \bigcirc_E \boldsymbol{b}$ is simple. Hence the desired result obtains.

Assume that $\boldsymbol{f} \bigcirc_A (\boldsymbol{a} \bigcirc_C \boldsymbol{b}) \in P_{A\cup B}$ and $\boldsymbol{f} \bigcirc_A (\boldsymbol{a} \bigcirc_D \boldsymbol{b}) \in P^{-1}_{A\cup B}$. Take any event C' such that $\pi(C') = \pi(C)$ and $C' \subseteq A$. Then it is easy to see that $(\boldsymbol{a} \bigcirc_{C'} \boldsymbol{b}) \bigcirc_A (\boldsymbol{a}^{-1} \bigcirc_C \boldsymbol{b}^{-1}) \in I_{A\cup B}$. For $\boldsymbol{c}^*$, let $\omega(S) = \{A, A^c\}$ and

$$\begin{aligned}
\boldsymbol{f}^1 &= (\boldsymbol{f} \bigcirc_A (\boldsymbol{a} \bigcirc_C \boldsymbol{b})) \bigcirc_{A\cup B} \boldsymbol{c}^*,\\
\boldsymbol{f}^2 &= (\boldsymbol{f}^{-1} \bigcirc_A (\boldsymbol{a}^{-1} \bigcirc_D \boldsymbol{b}^{-1})) \bigcirc_{A\cup B} \boldsymbol{c}^*,\\
\boldsymbol{f}^3 &= ((\boldsymbol{a} \bigcirc_{C'} \boldsymbol{b}) \bigcirc_A (\boldsymbol{a}^{-1} \bigcirc_C \boldsymbol{b}^{-1})) \bigcirc_{A\cup B} \boldsymbol{c}^*,\\
\boldsymbol{f}^4 &= ((\boldsymbol{a}^{-1} \bigcirc_{C'} \boldsymbol{b}^{-1}) \bigcirc_A (\boldsymbol{a} \bigcirc_D \boldsymbol{b})) \bigcirc_{A\cup B} \boldsymbol{c}^*.
\end{aligned}$$

Then $\boldsymbol{f}^1 \in P, \boldsymbol{f}^2 \in P, \boldsymbol{f}^3 \in I$, and $(\boldsymbol{f}^1, \boldsymbol{f}^2, \boldsymbol{f}^3, \boldsymbol{f}^4) \in E^4_\omega$. By $P2^{**}$, $\boldsymbol{f}^4 \in P^{-1}$. Noting that $\phi(\boldsymbol{a}) > \phi(\boldsymbol{b})$, we have

$$\begin{aligned}
&(\boldsymbol{a} \bigcirc_{C'} \boldsymbol{b}) \bigcirc_A (\boldsymbol{a}^{-1} \bigcirc_D \boldsymbol{b}^{-1}) \in P_{A\cup B}\\
&\implies \pi(C')\phi(\boldsymbol{a}) + (\pi(A) - \pi(C'))\phi(\boldsymbol{b}) + \pi(D)\phi(\boldsymbol{a}^{-1}) + (\pi(B) - \pi(D))\phi(\boldsymbol{b}^{-1}) > 0\\
&\implies (\pi(C') - \pi(D))(\phi(\boldsymbol{a}) - \phi(\boldsymbol{b})) > 0\\
&\implies \pi(C') > \pi(D)\\
&\implies \pi(C) > \pi(D).
\end{aligned}$$

Next we assume that $f \bigcirc_A h \in I_{A \cup B}$ for some simple h. Then we are to show that f is normal on A. Suppose on the contrary that f is big on A. We then derive a contradiction. When f is little on A, a similar contradiction obtains. Hence f must be normal on A.

Since h is simple, for every $s \in B$, $\phi(h^{-1}(s)) \leq \phi(a)$ for some $a \in X^2$, so by Claim 1, $a \bigcirc_A h \in P_{A\cup B} \cup I_{A \cup B}$. For c^*, let $\omega(S) = \{A, A^c\}$ and

$$\begin{aligned} f^1 &= (f^{-1} \bigcirc_A h^{-1}) \bigcirc_{A \cup B} c^*, \\ f^2 &= (a \bigcirc_A h) \bigcirc_{A \cup B} c^*, \\ f^3 &= (a^{-1} \bigcirc_A a) \bigcirc_{A \cup B} c^*, \\ f^4 &= (f \bigcirc_A a) \bigcirc_{A \cup B} c^*, \end{aligned}$$

so $(f^1, f^2, f^3, f^4) \in E^4_\omega$. Since $f^1 \in I, f^2 \in P \cup I$, and $f^3 \in I$, $P2^{**}$ implies $f^4 \in P^{-1} \cup I$. This contradicts the bigness of f. □

Lemma 6 *Suppose that $A, B \notin \mathcal{N}, A \cap B = \emptyset$, and $0 < \pi(A \cup B) \leq \frac{1}{2}$. Then we have that for all $f \in \mathcal{F}^2$,*
(1) if f is big on A and g is big on B, then $f \bigcirc_A g$ is big on $A \cup B$.
(2) if f is little on A and g is little on B, then $f \bigcirc_A g$ is little on $A \cup B$.
(3) if f is big on A and g is not big on B, then $f \bigcirc_A g$ is normal on $A \cup B$.
(4) if f is little on A and g is not little on B, then $f \bigcirc_A g$ is normal on $A \cup B$.

Proof. Suppose that $A, B \notin \mathcal{N}$, $A \cap B = \emptyset$, and $0 < \pi(A \cup B) \leq \frac{1}{2}$. Let A' and B' be events such that $\pi(A) = \pi(A'), \pi(B) = \pi(B'), A' \cap B' = \emptyset$, and $(A \cup B) \cap (A' \cup B') = \emptyset$.

(1) Suppose that f is big on A and g is big on B. Then for all $x \in X^2$, $f \bigcirc_A x \in P_{A \cup A'}$ and $g \bigcirc_B x \in P_{B \cup B'}$. By $P2^*$, $(f \bigcirc_A g) \bigcirc_{A \cup B} x \in P_{A \cup B \cup A' \cup B'}$, so $f \bigcirc_A g$ is big on $A \cup B$.

(2) The proof is similar to (1).

(3) Suppose that f is big on A and g is not big on B. Then by Claim 2, ϕ is bounded and $|\phi(x)| < 1$ for all $x \in X^2$. With no loss of generality we assume $\sup \phi = 1$. We are to show that there are $a, b \in X^2$, $C \subset B'$, and $C' \subset A'$ such that $0 < \pi(C) < \pi(B')$, $0 < \pi(C') < \pi(A')$, and

(a) $x \bigcirc_A a \in P^{-1}_{A \cup A' \cup C}$ and $g \bigcirc_B a \in P^{-1}_{B \cup (B' \setminus C)}$ for all $x \in X^2$,
(b) $f \bigcirc_A b \in P_{A \cup (A' \setminus C')}$ and $x \bigcirc_B b \in P_{B \cup B' \cup C'}$ for all $x \in X^2$.

Thus by $P2^*$ and $P7^*$, $(f \bigcirc_A g) \bigcirc_{A \cup B} a \in P^{-1}_{A \cup B \cup A' \cup B'}$ and $(f \bigcirc_A g) \bigcirc_{A \cup B} b \in P_{A \cup B \cup A' \cup B'}$, so that $f \bigcirc_A g$ is normal on $A \cup B$.

First we show (a). Since g is not big on B, $g \bigcirc_B b \in P^{-1}_{B \cup B'} \cup I_{B \cup B'}$ for some $b \in X^2$. Take any C such that $C \subset B'$ and $0 < \pi(C) < \pi(B')$. Then for any $a, c^* \in X^2$, let $\omega(S) = \{B, B^c\}$ and

$$\begin{aligned} f^1 &= (g^{-1} \bigcirc_B b^{-1}) \bigcirc_{B \cup B'} c^*, \\ f^2 &= (b^{-1} \bigcirc_B b) \bigcirc_{B \cup B'} c^*, \end{aligned}$$

$$f^3 = (b \bigcirc_B a^{-1}) \bigcirc_{B\cup(B'\setminus C)} c^*,$$
$$f^4 = (g \bigcirc_B a) \bigcirc_{B\cup(B'\setminus C)} c^*,$$

so $(f^1, f^2, f^3, f^4) \in E^4_\omega$. Since $C \subset B'$ is arbitrary, we can take such a C such that $0 < \pi(C) < \pi(B)(1 - |\phi(b)|)$. Thus we have

$$\frac{\pi(B)}{\pi(B) - \pi(C)}|\phi(b)| < 1.$$

Then since a is arbitrary, we let $a \in X^2$ satisfy

$$-1 < \phi(a) < \min\{\frac{-\pi(A)}{\pi(A) + \pi(C)}, \quad \frac{-\pi(B)}{\pi(B) - \pi(C)}|\phi(b)|\}.$$

Therefore, we have that for all $x \in X^2$,

$$\begin{aligned}
\phi(a) < -\frac{\pi(A)}{\pi(A) + \pi(C)} &\implies \pi(A)\phi(x) + (\pi(A') + \pi(C))\phi(a) < 0 \\
&\implies x \bigcirc_A a \in P^{-1}_{A\cup A'\cup C}. \\
\phi(a) < -\frac{\pi(B)}{\pi(B) - \pi(C)}|\phi(b)| &\implies \phi(a) < \frac{\pi(B)}{\pi(B) - \pi(C)}\phi(b) \\
&\implies \pi(B)\phi(b) + (\pi(B') - \pi(C))\phi(a^{-1}) > 0 \\
&\implies f^3 = (b \bigcirc_B a^{-1}) \bigcirc_{B\cup(B'\setminus C)} c^* \in P.
\end{aligned}$$

Since $f^1 \in P \cup I$ and $f^2 \in I$, $P2^{**}$ implies $f^4 \in P^{-1}$, so $g \bigcirc_B a \in P^{-1}_{B\cup(B'\setminus C)}$.

Next we show (b). Take any $C'' \subset A'$ such that $0 < \pi(C'') < \pi(A')$. Then consider any b which satisfies

$$\frac{\pi(B)}{\pi(B') + \pi(C'')} < \phi(b) < 1.$$

Since $f \bigcirc_A b \in P_{A\cup A'}$, Lemma 2 implies that there is a partition $\omega(C'')$ such that for all $C' \in \omega(C'')$, $f \bigcirc_A b \in P_{A\cup(A'\setminus C')}$. We restrict b to satisfy

$$\frac{\pi(B)}{\pi(B') + \pi(C')} < \phi(b) < 1.$$

Hence $x \bigcirc_B b \in P_{B\cup B'\cup C'}$ and $f \bigcirc_B b \in P_{A\cup(A'\setminus C')}$.

(4) The proof is similar to (3). □

Lemma 7 *Let $\pi(A) = \frac{1}{2}$. Then we have*
(1) $f \sim_A g$ if and only if $f \bigcirc_A h \in I$ and $g \bigcirc_A h \in I$ for some simple h, or f and g are either big or little on A.
(2) $f \succ_A g$ if and only if $f \bigcirc_A h \in P$ and $g \bigcirc_A h \in P^{-1}$ for some simple h.

Proof. (1) Suppose that $f \bigcirc_A h \in I$ and $g \bigcirc_A h \in I$ for some simple h, or f and g are either big or little on A. Then it readily follows from the definition that $f \sim_A g$.

Suppose next that $f \sim_A g$. Then by definition, $f \succeq_A g$ and $g \succeq_A f$. It follows from the definition of $\succeq_A$ that there are simple h_1 and h_2 such that $f \bigcirc_A h_1 \in P \cup I, g \bigcirc_A h_1 \in P^{-1} \cup I, g \bigcirc_A h_2 \in P \cup I$ and $f \bigcirc_A h_2 \in P^{-1} \cup I$, or f and g are either big or little on A. Assume that f and g are neither big nor little on A. Then let $\omega(S) = \{A, A^c\}$ and

$$\begin{aligned} f^1 &= f \bigcirc_A h_1 \\ f^2 &= g^{-1} \bigcirc_A h_1^{-1} \\ f^3 &= g \bigcirc_A h_2 \\ f^4 &= f^{-1} \bigcirc_A h_2^{-1}, \end{aligned}$$

so $(f^1, f^2, f^3, f^4) \in E^4_\omega$. Since f^1, f^2, and f^3 are in $P \cup I$, $P2^{**}$ implies that $f^4 \in P^{-1} \cup I$, so $f \bigcirc_A h_2 \in P \cup I$. Thus $f \bigcirc_A h_2 \in I$. Similarly we have $g \bigcirc_A h_2 \in I$. Hence the desired result obtains.

(2) Suppose that $f \bigcirc_A h \in P$ and $g \bigcirc_A h \in P^{-1}$ for some simple h. Then by definition, $f \succeq_A g$. We are to show that $f \sim_A g$ is false.

If f is little on A, then $f \bigcirc_A h(s) \in P^{-1}$ for all $s \in A^c$. By $P7^*$, $f \bigcirc_A h \in P^{-1} \cup I$. This is a contradiction, so that f is not little on A. Similarly, g is not big on A. If f is big on A, then by Lemma 5, there are no simple h_1 such that $f \bigcirc_A h_1 \in I$. Similarly, littleness of g on A implies that $g \bigcirc_A h_2 \in I$ for no simple h_2. Hence, if f is big on A or g is little on A, then by (1), $f \sim_A g$ is false.

Assume that f is normal on A. When g is normal on A, a similar analysis leads to the desired conclusion. By Lemma 5, $f \bigcirc_A h_1 \in I$ for a simple h_1. It then follows from $P2^{**}$ that $g \bigcirc_A h_1 \in P^{-1}$. Therefore, there is no simple h such that $f \bigcirc_A h \in I$ and $g \bigcirc_A h \in I$. Hence $f \sim_A g$ is false.

Suppose next that $f \succ_A g$. Then there is a simple h such that either $f \bigcirc_A h \in P$ and $g \bigcirc_A h \in P^{-1} \cup I$, or $f \bigcirc_A h \in P \cup I$ and $g \bigcirc_A h \in P^{-1}$. Assume that $f \bigcirc_A h \in P$ and $g \bigcirc_A h \in P^{-1} \cup I$. A similar proof applies to the other case. If $g \bigcirc_A h \in P^{-1}$, then the desired result obtains. Thus we assume that $g \bigcirc_A h \in I$. Since h is simple, there are two partitions, $\omega(A) = \{A_1, \ldots, A_n\}$ and $\omega(A^c) = \{B_1, \ldots, B_n\}$, and a simple h_0 such that $\pi(A_i) = \pi(B_i)$ for $i = 1, \ldots, n$, and for some $a_1, \ldots, a_n \in X^2$,

$$\begin{aligned} h(s) &= a_i \quad \text{if } s \in B_i, \\ h_0(s) &= a_i \quad \text{if } s \in A_i. \end{aligned}$$

Thus $h_0^{-1} \bigcirc_A h \in I$.

We assume that $h_0 \bigcirc_A h_1^{-1} \in P$ for some simple h_1. Then let $\omega(S) = \{A, A^c\}$ and

$$\begin{aligned} f^1 &= g^{-1} \bigcirc_A h^{-1}, \\ f^2 &= h_0^{-1} \bigcirc_A h, \\ f^3 &= h_0 \bigcirc_A h_1^{-1}, \\ f^4 &= g \bigcirc_A h_1, \end{aligned}$$

so $(f^1, f^2, f^3, f^4) \in E^4_\omega$. Since $f^1 \in I, f^2 \in I$, and $f^3 \in P$, $P2^{**}$ implies that $g \bigcirc_A h_1 \in P^{-1}$. Therefore, it suffices to show that such an h_1 exists and satisfies $f \bigcirc_A h_1 \in P$.

With no loss of generality we assume that $\sup \phi = 1, \phi(a_1) \geq \ldots \geq \phi(a_n)$, and $\pi(B_i) > 0$ for $i = 1, \ldots, n$. If $\phi(a_1) = -1$, then for all $x \in X^2$, $x \bigcirc_A h \in P^{-1} \cup I$. Thus applying $P2^{**}$, it is easy to see that $f \bigcirc_A x^{-1} \in P$, so f is big on A. By Claim 2, $|\phi(a_1)| < 1$, a contradiction. Hence we can take some $b \in X^2$ such that $\phi(a_1) > \phi(b)$. By Lemma 2, there is a partition $\omega(A_1)$ such that for all $C \in \omega(A_1)$,

$$f \bigcirc_A ((b \bigcirc_C a_1) \bigcirc_{A_1} h) \in P.$$

Let $h_1 = (b \bigcirc_C a_1) \bigcirc_{A_1} h$. Then it readily follows that $h_0 \bigcirc_A h_1^{-1} \in P$. □

Proof of Claim 3. We say that $B \subseteq A$ is null with respect to (w.r.t.) $\succsim_A$ if for all $f, g, h \in \mathcal{F}^2$, $f \bigcirc_B h \sim_A g \bigcirc_B h$. We are to show that $\succ_A$ satisfies the following seven axioms, which are understood as applying to all $f, g, h, h' \in \mathcal{F}^2$, all $x, y, z, w \in X^2$, and all $B, C \in 2^A$.

B1. $\succ_A$ is a weak order.
B2. If $f \bigcirc_B h \succsim_A g \bigcirc_B h$, then $f \bigcirc_B h' \succsim_A g \bigcirc_B h'$.
B3. If B is not null w.r.t. $\succsim_A$, then $x \succ_A y \iff x \bigcirc_B f \succ_A y \bigcirc_B f$.
B4. If $x \succ_A y$ and $z \succ_A w$, then $x \bigcirc_B y \succ_A x \bigcirc_C y \iff z \bigcirc_B w \succ_A z \bigcirc_C w$.
B5. $a \succ_A b$ for some $a, b \in X^2$.
B6. If f or g are simple and $f \succ_A g$, then for each $x \in X^2$, there is a partition $\omega(A)$ such that for all $C \in \omega(A)$, $x \bigcirc_C f \succ_A g$ and $f \succ_A x \bigcirc_C g$.
B7. If $f \bigcirc_B h \succ_A g(s) \bigcirc_B h$ for all $s \in B$, then $f \bigcirc_B h \succsim_A g \bigcirc_B h$; if $f(s) \bigcirc_B h \succ_A g \bigcirc_B h$ for all $s \in B$, then $f \bigcirc_B h \succsim_A g \bigcirc_B h$.

If f and g in Axiom B6 can be arbitrary, then $(\mathcal{F}^2, \succ_A)$ fulfills Savage's axiom system, so that it has Savage's SEU representation. However, it follows from the proof of that representation (see Fishburn, 1970, Chapter 10) that it suffices to assume that f or g is simple. Thus B1–B7 implies that $(\mathcal{F}^2, \succ_A)$ has Savage's SEU representation.

(B1) First we show that $\succ_A$ is asymmetric. Suppose on the contrary that $g \succ_A f$ and $f \succ_A g$. Then by Lemma 7, there are simple h_1, h_2 such that $g \bigcirc_A h_1 \in P, f \bigcirc_A h_1 \in P^{-1}, f \bigcirc_A h_2 \in P$, and $g \bigcirc_A h_2 \in P^{-1}$. Those four relations obviously violates $P2^{**}$. Hence $\succ_A$ must be asymmetric.

To show negative transitivity of $\succ_A$, we suppose that not$(f \succ_A g)$ and not$(g \succ_A h)$. Then $g \succsim_A f$ and $h \succsim_A g$. We are to show that $h \succsim_A f$, i.e., not$(f \succ_A h)$. When at least one of f, g, and h is not normal on A, the desired result easily obtains. Thus we assume that they are normal on A. Then by definition, there are simple h_1, h_2 such that $h \bigcirc_A h_1 \in P \cup I, g \bigcirc_A h_1 \in P^{-1} \cup I, g \bigcirc_A h_2 \in P \cup I$, and $f \bigcirc_A h_2 \in P^{-1} \cup I$. Let $\omega(S) = \{A, A^c\}$ and

$$\begin{aligned} f^1 &= h \bigcirc_A h_1, \\ f^2 &= g^{-1} \bigcirc_A h_1^{-1}, \end{aligned}$$

$$f^3 = g \bigcirc_A h_2,$$
$$f^4 = h^{-1} \bigcirc_A h_2^{-1},$$

so $(f^1, f^2, f^3, f^4) \in E^4_\omega$. Since f^1, f^2 and f^3 are in $P \cup I$, $P2^{**}$ implies $h \bigcirc_A h_2 \in P \cup I$. Hence $h \succeq_A f$.

(B2) We have three cases to examine.

Case 1. (f and g are normal on B) It follows from Lemma 6 that for all $h \in \mathcal{F}^2$, $f \bigcirc_B h$ and $g \bigcirc_B h$ are normal on A. Suppose that $f \bigcirc_B h_1 \succeq_A g \bigcirc_B h_1$. Then $(f \bigcirc_B h_1) \bigcirc_A h_3 \in P \cup I$ and $(g \bigcirc_B h_1) \bigcirc_A h_3 \in P^{-1} \cup I$ for some simple h_3. By Lemma 5, $(g \bigcirc_B h_2) \bigcirc_A h_4 \in I$ for some simple h_4. Let $\omega(S) = \{B, B^c\}$ and

$$f^1 = (f \bigcirc_B h_1) \bigcirc_A h_3,$$
$$f^2 = (g^{-1} \bigcirc_B h_1^{-1}) \bigcirc_A h_3^{-1},$$
$$f^3 = (g \bigcirc_B h_2) \bigcirc_A h_4,$$
$$f^4 = (f^{-1} \bigcirc_B h_2^{-1}) \bigcirc_A h_4^{-1}.$$

so $(f^1, f^2, f^3, f^4) \in E^4_\omega$. Since $f^1, f^2 \in P \cup I$ and $f^3 \in I$, it follows from $P2^{**}$ that $f^4 \in P^{-1} \cup I$. Hence $(f \bigcirc_B h_2) \succeq_A g \bigcirc_B h_2$.

Case 2. (either f is big on B and g is not big on B, or f is not little on B and g is little on B) Suppose that f is big on B and g is not big on B. The proof for the other case is similar. It suffices to show that $f \bigcirc_B h \succ_A g \bigcirc_B h$ for all $h \in \mathcal{F}^2$.

Take any B' such that $\pi(B') = \pi(B)$ and $B' \subseteq A^c$. Let $C = A \setminus B$ and $C' = A^c \setminus B'$. If h is big on C, then by Lemma 6, $f \bigcirc_B h$ is big on A and $g \bigcirc_B h$ is normal on A. Hence, $f \bigcirc_B h \succ_A g \bigcirc_B h$.

We assume next that h is little on C. Then $f \bigcirc_B h$ is normal on A. If g is little on B, then by Lemma 6, $g \bigcirc_B h$ is little on A. Thus $f \bigcirc_B h \succ_A g \bigcirc_B h$. If g is normal on B, then by Lemma 6, $g \bigcirc_B h$ is normal on A. Thus it follows from Lemma 5 that there are simple h_1, h_2, and h_3 such that $(g \bigcirc_B h) \bigcirc_A (h_1 \bigcirc_{B'} h_2) \in I$ and $(g \bigcirc_B h_1) \bigcirc_{B \cup B'} h_3 \in I$. Let $\omega(S) = \{B, B^c\}$ and

$$f^1 = (g \bigcirc_B h) \bigcirc_A (h_1 \bigcirc_{B'} h_2),$$
$$f^2 = (g^{-1} \bigcirc_B h_1^{-1}) \bigcirc_{B \cup B'} h_3^{-1},$$
$$f^3 = (f \bigcirc_B h_1) \bigcirc_{B \cup B'} h_3,$$
$$f^4 = (f^{-1} \bigcirc_B h^{-1}) \bigcirc_A (h_1^{-1} \bigcirc_{B'} h_2^{-1}),$$

so $(f^1, f^2, f^3, f^4) \in E^4_\omega$. Since $f^1, f^2 \in I$, by $P2^{**}$, $f^3 \in P$ implies $f^4 \in P^{-1}$. Thus $f \bigcirc_B h \succ_A g \bigcirc_B h$. It remains to show that $f^3 \in P$. Since f is big on B and g is normal on B, $f \bigcirc_B a \in P_{B \cup B'}$ and $g \bigcirc_B a \notin P_{B \cup B'}$ for some $a \in X^2$. For b^*, let $\omega(S) = \{B, B^c\}$ and

$$g^1 = (g \bigcirc_B h_1) \bigcirc_{B \cup B'} h_3,$$

$$
\begin{aligned}
g^2 &= (f \bigcirc_B a) \bigcirc_{B\cup B'} b^*, \\
g^3 &= (g^{-1} \bigcirc_B a^{-1}) \bigcirc_{B\cup B'} b^*, \\
g^4 &= (f^{-1} \bigcirc_B h_1^{-1}) \bigcirc_{B\cup B'} h_3^{-1},
\end{aligned}
$$

so $(g^1, g^2, g^3, g^4) \in E^4_\omega$. Since $g^1 \in I$ and $g^2, g^3 \in P$, $P2^{**}$ implies that $g^4 \in P^{-1}$.

Last we assume that h is normal on C. Then by Lemma 5, $h \bigcirc_C h_1 \in I_{C\cup C'}$ for some simple h_1. Since $f \bigcirc_B x^{-1} \in P_{B\cup B'}$ and $x \bigcirc_B x^{-1} \in I_{B\cup B'}$, $P2^*$ implies that

$$
(x \bigcirc_B h) \bigcirc_A (x^{-1} \bigcirc_{B'} h_1) \in I \text{ and } (f \bigcirc_B h) \bigcirc_A (x^{-1} \bigcirc_{B'} h_1) \in P.
$$

Therfore, $f \bigcirc_B h \succ_A x \bigcirc_B h$ for all $x \in X^2$. Since $g \bigcirc_B a^{-1} \notin P_{B\cup B'}$ for some $a \in X^2$, it follows similarly that $a \bigcirc_B h \succeq_A g \bigcirc_B h$. By transitivity of $\succeq_A$, $f \bigcirc_B h \succ_A g \bigcirc_B h$.

Case 3. (f and g are either big or little on B) Suppose that f and g are big on B. When f and g are little on B, the proof is similar. Then it suffices to show that $f \bigcirc_B h \sim_A g \bigcirc_B h$ for all h.

Suppose on the contrary that $f \bigcirc_B h \succ_A g \bigcirc_B h$ for some h. Then by Lemma 7, $(f \bigcirc_B h) \bigcirc_A h_1 \in P$ and $(g \bigcirc_B h) \bigcirc_A h_1 \in P^{-1}$ for some simple h_1. Take any event B' such that $B \cap B' = \emptyset$ and $\pi(B) = \pi(B')$. For $x, a^* \in X^2$, let $\omega(S) = \{B, B^c\}$ and

$$
\begin{aligned}
f^1 &= (g^{-1} \bigcirc_B h^{-1}) \bigcirc_A h_1^{-1}, \\
f^2 &= (g \bigcirc_B x) \bigcirc_{B\cup B'} a^*, \\
f^3 &= (x \bigcirc_B x^{-1}) \bigcirc_{B\cup B'} a^*, \\
f^4 &= (x^{-1} \bigcirc_B h) \bigcirc_A h_1,
\end{aligned}
$$

so $(f^1, f^2, f^3, f^4) \in E^4_\omega$. Since g is big on B, we have $f^2 \in P$. Since $f^1 \in P$ and $f^3 \in I$, $P2^{**}$ implies $f^4 \in P^{-1}$. Hence $(x^{-1} \bigcirc_B h) \bigcirc_A h_1 \in P^{-1}$ for all x. Then by $P7^*$, $(f \bigcirc_B h) \bigcirc_A h_1 \in P^{-1} \cup I$. This is a contradiction. Hence we must have that $f \bigcirc_B h \sim_A g \bigcirc_B h$ for all h.

(B3) Suppose that B is not null w.r.t. $\succeq_A$. We note by Lemma 7 and Claim 1 that

$$
\begin{aligned}
x \succ_A y &\iff x \bigcirc_A h \in P \text{ and } y \bigcirc_A h \in P^{-1} \text{ for some simple } h \\
&\iff \phi(x) > \phi(y).
\end{aligned}
$$

If f is simple, then we have

$$
\begin{aligned}
& x \bigcirc_B f \succ_A y \bigcirc_B f \\
&\iff (x \bigcirc_B f) \bigcirc_A h \in P \text{ and } (y \bigcirc_B f) \bigcirc_A h \in P^{-1} \text{ for some simple } h \\
&\iff \phi(x) > \phi(y)
\end{aligned}
$$

Hence the desired result obtains.

Assume next that f is not simple. Then for any simple g, it follows from B2 that

$$
x \bigcirc_B f \succ_A y \bigcirc_B f \iff x \bigcirc_B g \succ_A y \bigcirc_B g.
$$

Hence the desired result follows from the preceding paragraph.

(B4) This follows from Claim 2 and Lemma 7.

(B5) By P5, $\boldsymbol{a} \in P$ for some $\boldsymbol{a} \in X^2$. Thus $\boldsymbol{a} \in P_A$ and $\boldsymbol{a}^{-1} \in P_A^{-1}$, so $\boldsymbol{a} \succ_A \boldsymbol{a}^{-1}$.

(B6) Suppose that $\boldsymbol{f}$ or $\boldsymbol{g}$ are simple and $\boldsymbol{f} \succ_A \boldsymbol{g}$. We assume that $\boldsymbol{f}$ is simple. When $\boldsymbol{g}$ is simple, the proof is similar. By Lemma 7, $\boldsymbol{f} \bigcirc_A \boldsymbol{h} \in P$ and $\boldsymbol{g} \bigcirc_A \boldsymbol{h} \in P^{-1}$ for some simple $\boldsymbol{h}$. We are to show that given $\boldsymbol{x} \in X^2$, there is a partition $\omega(A)$ such that for all $C \in \omega(A)$, $(\boldsymbol{x} \bigcirc_C \boldsymbol{f}) \bigcirc_A \boldsymbol{h} \in P$ and $(\boldsymbol{x} \bigcirc_C \boldsymbol{g}) \bigcirc_A \boldsymbol{h} \in P^{-1}$.

Since $\boldsymbol{f}$ is simple, there is a partition $\omega(A) = \{A_1, \ldots, A_n\}$ such that for $k = 1, \ldots, n$,

$$\boldsymbol{f}(s) = \boldsymbol{a}_k \text{ for all } s \in A_k \text{ and some } \boldsymbol{a}_k \in X^2.$$

It follows from Lemma 2 that for $k = 1, \ldots, n$, there is a partition $\omega(A_k) = \{A_{k1}, \ldots, A_{kn_k}\}$ such that for $i = 1, \ldots, n_k$,

$$((\boldsymbol{x} \bigcirc_{A_{ki}} \boldsymbol{a}_k) \bigcirc_{A_k} \boldsymbol{f}) \bigcirc_A \boldsymbol{h} \in P.$$

Given $\{A_{ki} : k = 1, \ldots, n \text{ and } i = 1, \ldots, n_k\}$, it follows from Lemma 2 that for each A_{ki}, there is a partition $\omega(A_{ki})$ such that for all $C \in \omega(A_{ki})$,

$$(\boldsymbol{x} \bigcirc_C \boldsymbol{g}) \bigcirc_A \boldsymbol{h} \in P^{-1}.$$

It remains to show that $(\boldsymbol{x} \bigcirc_C \boldsymbol{f}) \bigcirc_A \boldsymbol{h} \in P$. Therefore, letting $\omega(A) = \cup_{k,i}\omega(A_{ki})$, the desired result obtains.

Given A_{ki}, it suffices to show that for all $C \subseteq A_{ki}$, $(\boldsymbol{x} \bigcirc_C \boldsymbol{f}) \bigcirc_A \boldsymbol{h} \in P$. By Claim 1,

$$\begin{aligned} ((\boldsymbol{x} \bigcirc_{A_{ki}} \boldsymbol{a}_k) \bigcirc_{A_k} \boldsymbol{f}) \bigcirc_A \boldsymbol{h} \in P &\iff \pi(A_{ki})\phi(\boldsymbol{x}) > \alpha, \\ \boldsymbol{f} \bigcirc_A \boldsymbol{h} \in P &\iff \pi(A_{ki})\phi(\boldsymbol{a}_k) > \alpha, \end{aligned}$$

where

$$\alpha = \int_{A \setminus A_{ki}} \phi(\boldsymbol{f}^{-1}(s))d\pi(s) + \int_{A^c} \phi(\boldsymbol{h}^{-1}(s))d\pi(s).$$

Then we have

$$\pi(C)\phi(\boldsymbol{x}) + (\pi(A_{ki}) - \pi(C))\phi(\boldsymbol{a}_k) > \alpha,$$

so that $(\boldsymbol{x} \bigcirc_C \boldsymbol{f}) \bigcirc_A \boldsymbol{h} \in P$.

(B7) Suppose that $\boldsymbol{f} \bigcirc_B \boldsymbol{h} \succ_A \boldsymbol{g}(s) \bigcirc_B \boldsymbol{h}$ for all $s \in B$. A similar proof applies to another case. Then by Lemma 7, there is a simple $\boldsymbol{h}_1$ such that for all $s \in B$,

$$(\boldsymbol{f} \bigcirc_B \boldsymbol{h}) \bigcirc_A \boldsymbol{h}_1 \in P \text{ and } (\boldsymbol{g}(s) \bigcirc_B \boldsymbol{h}) \bigcirc_A \boldsymbol{h}_1 \in P^{-1}.$$

By $P7^*$, $(\boldsymbol{g} \bigcirc_B \boldsymbol{h}) \bigcirc_A \boldsymbol{h}_1 \in P^{-1} \cup I$. Hence $\boldsymbol{f} \bigcirc_B \boldsymbol{h} \succeq_A \boldsymbol{g} \bigcirc_B \boldsymbol{h}$.

□

Step 5. This step completes the proof of the extended SSA theorem. We need the following lemma.

Lemma 8 *If* $\pi(A) = \frac{1}{2}, f \sim_A g$, *and* $f \sim_{A^c} g$, *then* $f \in P \Longleftrightarrow g \in P$.

Proof. Suppose that $\pi(A) = \frac{1}{2}, f \sim_A g$, and $f \sim_{A^c} g$. Assume that $f \in P$. We have three cases to exmine.

Case 1. (f is normal both on A and A^c) We note that g is also normal both on A and A^c. By Lemma 5, there is a simple h such that $f \bigcirc_A h \in I$ and $g \bigcirc_A h \in I$. Since h is simple, it follows from Claim 1 that $h_1 \bigcirc_A h \in I$ for some simple h_1. Let $\omega(S) = \{A, A^c\}$ and

$$\begin{aligned} f^1 &= f \bigcirc_A f, \\ f^2 &= f^{-1} \bigcirc_A h^{-1}, \\ f^3 &= h_1 \bigcirc_A h, \\ f^4 &= h_1^{-1} \bigcirc_A f^{-1}, \end{aligned}$$

so $(f^1, f^2, f^3, f^4) \in E_\omega^4$. Since $f^1 \in P, f^2 \in I$, and $f^3 \in I$, $P2^{**}$ implies that $f^4 \in P^{-1}$. Thus by Lemma 7, $f \succ_{A^c} h$. Since $f \sim_{A^c} g$, and $\succ_{A^c}$ is a weak order, we have $g \succ_{A^c} h$. Thus by Lemma 7, $h_2 \bigcirc_A g \in P$ and $h_2 \bigcirc_A h \in P^{-1}$ for some simple h_2. Let $\omega(S) = \{A, A^c\}$ and

$$\begin{aligned} g^1 &= h_2 \bigcirc_A g, \\ g^2 &= h_2^{-1} \bigcirc_A h^{-1}, \\ g^3 &= g \bigcirc_A h, \\ g^4 &= g^{-1} \bigcirc_A g^{-1}, \end{aligned}$$

so $(g^1, g^2, g^3, g^4) \in E_\omega^4$. Since $g^1 \in P, g^2 \in P$, and $g^3 \in I$ are in I, $P2^{**}$ implies that $g^4 \in P^{-1}$, so $g \in P$.

Case 2. (f is either big both on A and A^c, or little both on A and A^c) Suppose that f is big both on A and A^c. The proof for the other case is similar. Then $f \in P_A$ and $f \in P_{A^c}$. By $P2^*$, $f \in P$. Since g is also big both on A and A^c, we have $g \in P$, so that the desired result obtains.

Case 3. (f is big on A and little on A^c) By Lemma 4, $f \in I$. Since g is also big on A and little on A^c, we have $g \in I$. Henc the desired result obtains. □

We now complete the proof of the extended SSA theorem. Suppose that $\pi(A) = \frac{1}{2}$. Then it follows from Lemma 7 and Claim 1 that for all simpe $f, g \in \mathcal{F}^2$,

$$\begin{aligned} f \succ_A g &\Longleftrightarrow f \bigcirc_A h \in P \text{ and } g \bigcirc_A h \in P^{-1} \text{ for some simple } h \\ &\Longleftrightarrow \int_A \phi(f(s))d\pi(s) > \int_A \phi(h^{-1}(s))d\pi(s) > \int_A \phi(g(s))d\pi(s) \text{ for some simple } h \\ &\Longleftrightarrow \int_A \phi(f(s))d\pi(s) > \int_A \phi(g(s))d\pi(s). \end{aligned}$$

Hence Claim 3 implies that for all $f, g \in \mathcal{F}^2$,

$$f \succ_A g \Longleftrightarrow \int_A \phi(f(s))d\pi(s) > \int_A \phi(g(s))d\pi(s).$$

In what follows, we shall assume $\sup \phi = 1$. Thus we note that

$$\begin{aligned} \int_A \phi(f(s))d\pi(s) &= \frac{1}{2} && \text{when } f \text{ is big on } A, \\ &= -\frac{1}{2} && \text{when } f \text{ is little on } A, \\ |\int_A \phi(f(s))d\pi(s)| &< \frac{1}{2} && \text{when } f \text{ is nomral on } A. \end{aligned}$$

First we assume that f is normal both on A and A^c. Then by Lemma 5, there is a simple g such that $f \sim_A g$ and $f \sim_{A^c} g$. Hence it follows from Lemma 8, Claim 1, and the preceding paragraph that

$$\begin{aligned} f \in P &\Longleftrightarrow g \in P \\ &\Longleftrightarrow \int_S \phi(g(s))d\pi(s) > 0 \\ &\Longleftrightarrow \int_A \phi(g(s))d\pi(s) + \int_{A^c} \phi(g(s))\pi(s) > 0 \\ &\Longleftrightarrow \int_A \phi(f(s))d\pi(s) + \int_{A^c} \phi(f(s))\pi(s) > 0 \\ &\Longleftrightarrow \int_S \phi(f(s))d\pi(s) > 0. \end{aligned}$$

Next we assume that f is big on A and not little on A^c. Then $f \in P$. Also we have

$$\int_A \phi(f(s))d\pi(s) = \frac{1}{2} \text{ and } \int_{A^c} \phi(f(s))d\pi(s) > -\frac{1}{2},$$

so that

$$\int_S \phi(f(s))d\pi(s) = \int_A \phi(f(s))d\pi(s) + \int_{A^c} \phi(f(s))d\pi(s) > 0.$$

When f is little on A and not big on A^c, the desired result similarly follows.

Last we assume that f is big on A and little on A^c. Then we have $f \in I$. Also we obtain

$$\int_A \phi(f(s))d\pi(s) = \frac{1}{2} \text{ and } \int_{A^c} \phi(f(s))d\pi(s) = -\frac{1}{2},$$

so that

$$\int_S \phi(f(s))d\pi(s) = 0:$$

This completes the proof.

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