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Game Logic and Its Applications II

by

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Abstract

This paper provides a Genzten style formulation of our game logic framework GL_m $(0 \leq m \leq \omega)$, and proves the cut-elimination theorem for GL_m. As its application, we prove the term existence theorem for GL_ω used in Part I.

1. Introduction

This is a sequel to our development of the game logic framework. In Part I, we presented the framework in the Hilbert style formulation and showed some applications of the framework to game theory - - the epistemic axiomatization of Nash equilibrium and the undecidability on the playability of a game. To obtain the undecidability results (Theorems 6.A and 6.B of Part I), we used the result called the *term existence theorem*. This is a metatheorem stating an evaluation of provability on an existential formula. The Hilbert style formulation is convenient in presentation, but is difficult in managing an evaluation of such provability. In general, it would be better to reformulate the Hilbert style formulation into a Gentzen style sequent calculus for the purpose of evaluating provability. This paper provides a Gentzen style formulation of the game logic framework, and proves the cut-elimination theorem. As its application, we prove the term existence theorem used in Part I.

First, we provide the sequent calculus corresponding to GL_ω, which is an infinitary predicate extension of the sequent calculus formulation of propositional *KD4* along the line of Ohnishi-Matsumoto [7], [8]. Then we present the cut-elimination theorem for sequent calculus GL_ω. We discuss also the infinitary predicate extensions of *K4*, *K* as well as *S4*. Section 4 provides a sequent calculus formulation of logic GL_m for a finite m.

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This provides also a different sequent calculus formulation of GL_ω. The corresponding cut-elimination theorem will be provided, which will be proved in Section 5.

In Section 3, we prove the term existence theorem in sequent calculus GL_ω, using the cut-elimination theorem. As steps to this result, we provide a more basic term existence theorem and another result which we call a *separation theorem*. These two results hold, in fact, in GL_m for any m.

We will find that the separation theorem manifests cognitive relativism in that the epistemic world of each player is separated from the others'. This enables us to prove the results of Section 3. These results fail to hold for the *S4*-type extension, in which knowledge must be always true relative to the thinker and ultimately relative to the investigator.

In our sequent calculus, the Barcan rule, $\bigwedge K_i(\Phi) \supset K_i(\bigwedge \Phi)$, for an allowable set Φ, is formulated as an inference rule, $(B\text{-}\bigwedge)$, which does not satisfy the subformula property. Hence the cut-elimination theorem for GL_m does not give a proof with the full subformula property. Nevertheless, this is not an obstacle in obtaining the term existence theorem. For other applications, unfortunately, this inference becomes an obstacle. A final remark is that since the Barcan rule is not needed in the finitary fragment of GL_ω, the cut-elimination theorem for the finitary fragment of GL_ω provides a cut-free proof with the full subformula property.

2. Sequent Calculus GL_ω and its Variations

2.1. Sequent Calculus GL_ω and the Cut-Elimination Theorem

We work on the same language $\mathcal{P}_\omega$ as in Part I, and prepare an auxiliary symbol $\rightarrow$. We call the expression $\Gamma \rightarrow \Theta$ a *sequent* iff Γ and Θ are finite sets of formulae. The sets Γ and Θ are called the *antecedent* and *succedent* of the sequent $\Gamma \rightarrow \Theta$. The expression $\Gamma, \Delta \rightarrow \Theta, \Lambda$ is used to denote $\Gamma \cup \Delta \rightarrow \Theta \cup \Lambda$. We omit the set-theoretical bracket $\{A\}$, for example, $\{A\}, \Gamma \rightarrow \Theta, \{B\}$ is denoted as $A, \Gamma \rightarrow \Theta, B$.

Our sequent calculus GL_ω is defined as follows:

Initial Sequents: An initial sequent is of the form $A \rightarrow A$ for any formula A.

Inference Rules: We have three kinds of inference rules: structural, operational and K-inference rules.

Structural Inferences:

$$\frac{\Gamma \rightarrow \Theta}{\Delta, \Gamma \rightarrow \Theta, \Lambda} \text{ (th)}$$

$$\frac{\Gamma \rightarrow \Theta, M \quad M, \Delta \rightarrow \Lambda}{\Gamma, \Delta \rightarrow \Theta, \Lambda} \ (M) \text{ (cut)},$$

where M is called the *cut-formula.*

Operational Inferences:

$$\frac{A, \Gamma \to \Theta}{\bigwedge \Phi, \Gamma \to \Theta} (\bigwedge \to) \; (A \in \Phi) \qquad \frac{\{\Gamma \to \Theta, A : A \in \Phi\}}{\Gamma \to \Theta, \bigwedge \Phi} (\to \bigwedge)$$

$$\frac{\{A, \Gamma \to \Theta : A \in \Phi\}}{\bigvee \Phi, \Gamma \to \Theta} (\bigvee \to) \qquad \frac{\Gamma \to \Theta, A}{\Gamma \to \Theta, \bigvee \Phi} (\to \bigvee) \; (A \in \Phi)$$

$$\frac{\Gamma \to \Theta, A \quad B, \Gamma \to \Theta}{A \supset B, \Gamma \to \Theta} (\supset\to) \qquad \frac{A, \Gamma \to \Theta, B}{\Gamma \to \Theta, A \supset B} (\to\supset)$$

$$\frac{\Gamma \to \Theta, A}{\neg A, \Gamma \to \Theta} (\neg \to) \qquad \frac{A, \Gamma \to \Theta}{\Gamma \to \Theta, \neg A} (\to \neg)$$

$$\frac{A(t), \Gamma \to \Theta}{\forall x A(x), \Gamma \to \Theta} (\forall \to) \qquad \frac{\Gamma \to \Theta, A(a)}{\Gamma \to \Theta, \forall x A(x)} (\to \forall)$$

$$\frac{A(a), \Gamma \to \Theta}{\exists x A(x), \Gamma \to \Theta} (\exists \to) \qquad \frac{\Gamma \to \Theta, A(t)}{\Gamma \to \Theta, \exists x A(x)} (\to \exists),$$

where Φ is an allowable set, t is a term, and a is a free variable which must not occur in the lower sequents of $(\to \forall)$ and $(\exists \to)$.

In an operational inference, the formulae to be changed in the upper sequents are called the *side formulae,* and the formula newly created in the lower sequent is called the *principal formula.* The free variable a in $(\to \forall)$ and $(\exists \to)$ is called an *eigenvariable.*

The following two inference rules are specific to our system. We need the first to guarantee that the Barcan sequent $\bigwedge K_i(\Phi) \to K_i(\bigwedge \Phi)$ is provable.

K-Inferences:

$$\frac{\{\Gamma \to \Theta, K_i(A) : A \in \Phi\} \qquad K_i(\bigwedge \Phi), \Gamma \to \Theta}{\Gamma \to \Theta} (B\text{-}\bigwedge),$$

where Φ is an allowable set. We call this the *Barcan inference.* The *side formulae* of this inference are $K_i(A)$ $(A \in \Phi)$ and $K_i(\bigwedge \Phi)$.

The last inference rule is as follows:

$$\frac{\Gamma, K_i(\Delta) \to \Theta}{K_i(\Gamma, \Delta) \to K_i(\Theta)} (K \to K),$$

where $|\Theta|$, the cardinality of Θ, is at most one. Recall that $K_i(\Gamma)$ denotes the set $\{K_i(A) : A \in \Gamma\}$ and $K_i(\Gamma, \Delta)$ is $K_i(\Gamma \cup \Delta)$.

In a similar manner to in Part I, a *proof* is defined to be a countable tree with the following properties: (i) every path from the root is finite; (ii) a sequent is associated with each node, and the sequent associated with each leaf is an initial sequent; and (iii) adjoining nodes together with the associated sequents form an instance of the above inference rules. A sequent $\Gamma \to \Theta$ is said to be *provable* in GL_ω, denoted by $\vdash_\omega \Gamma \to \Theta$, iff there is a proof such that $\Gamma \to \Theta$ is associated with its root. This $\Gamma \to \Theta$ is called the *endsequent* of the proof.

Without K-inferences, the above system is an infinitary extension of Gentzen's LK. If we restrict the system to the finitary propositional fragment, then it is the sequent calculus formulation of *KD4* with n knowledge operators. Ohnishi-Matsumoto [7], [8] first formulated some modal propositional logics including propositional *T*, *S4* and *S5*, in sequential calculi. Ours is an infinitary predicate extension of *KD4* with the Barcan rule along this line of research. We will discuss the extensions of some other systems later.

We use the same notations GL_ω and $\vdash_\omega$ as in Part I. This is due to the following theorem.

Theorem 2.A. Let Φ be an allowable set of closed formulae and A a formula. Then $\vdash_\omega \bigwedge \Phi \to A$ if and only if $\Phi \vdash_\omega A$ in the sense of Part I.

The following lemma together with the deduction theorem (Lemma 2.1) of Part I implies Theorem 2.A.

Lemma 2.1.1): If $\vdash_\omega A$ in the sense of Part I, then $\vdash_\omega \to A$.

2): If $\vdash_\omega \Gamma \to \Theta$, then $\vdash_\omega \bigwedge \Gamma \supset \bigvee \Theta$ in the sense of Part I, where Γ or Θ is empty, $\bigwedge \Gamma$ or $\bigvee \Theta$ is interpreted as $\neg A \bigvee A$ or $\neg A \bigwedge A$, respectively.

Assertion (1) is proved inductively on the structure of a proof of A in the Hilbert Style. Assertion (2) is proved inductively on the structure of a proof of $\Gamma \to \Theta$. For example, (PI_i): $K_i(A) \supset K_iK_i(A)$ is proved in the sequent calculus as follows:

$$\frac{K_i(A) \to K_i(A)}{K_i(A) \to K_iK_i(A)}\ (K \to K),$$

which implies $\vdash_\omega \;\to K_i(A) \supset K_iK_i(A)$ by $(\to\supset)$. Conversely, let us prove that $(K \to K)$ is allowed in the Hilbert style formulation: Suppose $\vdash_\omega (\bigwedge \Gamma) \bigwedge(\bigwedge K_i(\Delta)) \supset \bigvee \Theta$.

By Lemma 3.3.1) of Part I and (MP_i), $\vdash_\omega K_i((\bigwedge\Gamma)\bigwedge(\bigwedge K_i(\Delta))) \supset K_i(\bigvee\Theta)$. Using $\vdash_\omega \bigwedge K_i(\Sigma) \supset\subset K_i(\bigwedge\Sigma)$ for any finite set Σ and also (PI_i), we have $\vdash_\omega \bigwedge K_i(\Gamma,\Delta) \supset K_i(\bigvee\Theta)$. Since $|\Theta| \le 1$, $\vdash_\omega \bigwedge K_i(\Gamma,\Delta) \supset \bigvee K_i(\Theta)$. Note that the Barcan axiom ($\bigwedge$-B_i) is not used here.

In GL_ω, the sequent $\bigwedge K_i(\Phi) \to K_i(\bigwedge\Phi)$ is provable for any allowable set Φ, using (B-$\bigwedge$): For any $A \in \Phi$, $\vdash_\omega \bigwedge K_i(\Phi) \to K_i(A)$, which implies $\vdash_\omega \bigwedge K_i(\Phi) \to K_i(\bigwedge\Phi), K_i(A)$ by (th). Hence

$$\frac{\{\bigwedge K_i(\Phi) \to K_i(\bigwedge\Phi), K_i(A) : A \in \Phi\} \qquad K_i(\bigwedge\Phi) \to K_i(\bigwedge\Phi)}{\bigwedge K_i(\Phi) \to K_i(\bigwedge\Phi)} \ (B\text{-}\textstyle\bigwedge).$$

In the case of finite Φ, $\bigwedge K_i(\Phi) \to K_i(\bigwedge\Phi)$ is provable without (B-$\bigwedge$).

In Section 4, we will give a different formulation of GL_ω and prove the cut-elimination theorem for it in Section 5. This cut-elimination theorem implies the following.

Theorem 2.B (Cut-Elimination for GL_ω). If $\vdash_\omega \Gamma \to \Theta$, there is a cut-free proof of $\Gamma \to \Theta$ in GL_ω.

In GL_ω, if the Barcan inference (B-$\bigwedge$) occurs in a proof, some formulae of the form $K_i(A)$ ($A \in \Phi$) and $K_i(\bigwedge\Phi)$ are eliminated in the inference. Hence the above cut-elimination theorem does not imply the full subformula property that every formula occurring in a cut-free proof is a subformula of a formula occurring in its endsequent.[1] Nevertheless, the subformula property holds for the other kinds of formulae. This partial violation of the subformula property is sometimes an obstacle and sometimes not. Fortunately, it does not prevent us from proving the term existence theorem, which is the present primary purpose of Part II.

In logic GL_ω, we give a special attention to a part of a cut-free proof, where the violation of the subformula property is kept minimal. Consider a proof P. In the path from the root to a leaf, the lower sequent of the lowest occurrence of $(K \to K)$ is called a *boundary*. If the path does not have such an inference, the *boundary* is the initial sequent. The part of P from the endsequent to all boundaries is called the *trunk* of P. In the trunk of a cut-free proof, there is no occurrence of inference $(K \to K)$. Since the side formulae of inference (B-$\bigwedge$) are of the form $K_j(B)$ for some j and B, the following holds for a cut-free proof P:

$$\begin{array}{c}\text{Any formula occurring in the trunk of } P \text{ is}\\ \text{a subformula of some formula occurring in the endsequent}\\ \text{or has the outermost symbol } K_j \text{ for some } j.\end{array} \tag{2.1}$$

[1] A *subformula* of a given formula A is defined in the standard inductive manner, but note the step for quantification that if $\forall x B(x)$ and $\exists x B(x)$ are subformulae of A, then $B(t)$ is a subformula of A for any term t.

This property will be used in proving the term existence theorem in Section 3.

When we prohibit the use of $(K \rightarrow K)$ and $(B\text{-}\bigwedge)$, the system is essentially the same as Gentzen's LK with the infinitary modification, which corresponds to base logic GL_0 defined in Part I. We denote the provability in GL_0 by $\vdash_0$. Then it is shown as Proposition 4.1 of Part I that

$$\vdash_\omega \Gamma \rightarrow \Theta \text{ implies } \vdash_0 \epsilon\Gamma \rightarrow \epsilon\Theta. \tag{2.2}$$

Recall that $\epsilon\Gamma$ is obtained from Γ by eliminating all K_j $(j = 1, ..., n)$. Theorem 4.A of this paper states that cut-elimination holds for GL_0 and implies the full subformula property, since GL_0 does not allow $(B\text{-}\bigwedge)$. Hence logic GL_0 is contradiction-free, which together with (2.2) implies that logic GL_ω is also contradiction-free.

When we prohibit only the Barcan inference $(B\text{-}\bigwedge)$, the system corresponds to GL_ω^{-B} in the sense of Part I. The cut-elimination theorem holds for GL_ω^{-B}. Since $(B\text{-}\bigwedge)$ is prohibited in GL_ω^{-B}, a cut-free proof satisfies the full subformula property. Using this fact, we can prove that the assertion of Lemma 2.4 of Part I, i.e., $C(A) \rightarrow K_i(C(A))$, is not provable in GL_ω^{-B}. Hence the Barcan axiom is needed to have $C(A) \rightarrow K_i(C(A))$. This will be discussed more in a different paper.

When we restrict our attention to the finitary fragment of our logic, the Barcan inference $(B\text{-}\bigwedge)$ become unnecessary, as was already stated. Therefore a cut-free proof in the finitary fragment of GL_ω has the full subformula property.

2.2. Variations of GL_ω

The sequent calculus formulation of $GL_{\omega p}$ is obtained from GL_ω by substituting $(K \rightarrow K)_p$ for $(K \rightarrow K)$:

$$\frac{\Gamma \rightarrow \Theta}{K_i(\Gamma) \rightarrow K_i(\Theta)} \; (K \rightarrow K)_p,$$

where $|\Theta| \leq 1$. Logic $GL_{\omega p}$ is weaker than GL_ω, e.g., $K_i(A) \rightarrow K_iK_i(A)$ is not provable in $GL_{\omega p}$, but is provable in GL_ω. Cut-elimination as well as the other metatheorems of Section 3 hold for $GL_{\omega p}$. In this sense, $GL_{\omega p}$ has a status similar to GL_ω, except that the epistemic axiomatization of Nash equilibrium needs some modification in $GL_{\omega p}$, i.e., the common knowledge formula should be modified by using $\bigcup_{m<\omega} \mathsf{K}_p(m)$ instead of $\bigcup_{m<\omega} \mathsf{K}(m)$. Recall that $\mathsf{K}_p(m) = \{K_{i_1}...K_{i_m}$: each i_t is one of $K_1, ..., K_n\}$ and $\mathsf{K}(m) = \{K_{i_1}...K_{i_m}$: each i_t is one of $K_1, ..., K_n$ with $i_t \neq i_{t+1}$ for $t = 1, ..., m-1\}$.

Among others, the infinitary predicate extension of modal logic *S4* is of special interest. The sequent calculus of the infinitary predicate extension of *S4* is obtained from GL_ω by substituting the following two inferences for $(K \rightarrow K)$:

$$\frac{\Gamma, \Delta \rightarrow \Theta}{K_i(\Gamma), \Delta \rightarrow \Theta} (K \rightarrow) \qquad \frac{K_i(\Gamma) \rightarrow A}{K_i(\Gamma) \rightarrow K_i(A)} (\rightarrow K).$$

We denote this system by $GL_{\omega S4}$. The cut-elimination theorem for $GL_{\omega S4}$ is obtained by modifying the proof for GL_m given in Section 5. One important consequence is that a cut-free proof in the finitary fragment of $GL_{\omega S4}$ satisfies the full subformula property.

In the full predicate calculus $GL_{\omega S4}$, the Barcan inference (B-$\bigwedge$) is needed and becomes an obstacle in applying the cut-elimination theorem in that (2.1) does not hold. Therefore we do not obtain the results in Section 3 in $GL_{\omega S4}$.

3. Applications of the Cut-Elimination Theorem

Using the cut-elimination theorem (Theorem 2.B), we prove several theorems, from which we obtain the term existence theorem (Theorem 3.C) used for the undecidability results in Part I.

3.1. Term Existence Theorems and Separation Theorem

The first result is term existence for an individual player, which will be proved in Subsection 3.2.

Theorem 3.A (Term Existence I). If $\vdash_\omega K_i(\Gamma) \rightarrow \exists x_1 ... \exists x_\ell K_i(A(x_1, ..., x_\ell))$, then $\vdash_\omega K_i(\Gamma) \rightarrow K_i(A(t_1, ..., t_\ell))$ for some terms $t_1, ..., t_\ell$.

When the system has closed terms and no free variables occur in Γ and $\exists x_1 ... \exists x_\ell K_i(A(x_1, ..., x_\ell))$, we can assert that $t_1, ..., t_\ell$ are closed terms.

It can be proved, using Theorem 3.A, that $K_i(\exists x A(x)) \rightarrow \exists x K_i(A(x))$ is not provable for some A in GL_ω. Also, a parallel assertion holds for a disjunctive formula:

$$\text{If } \vdash_\omega K_i(\Gamma) \rightarrow \bigvee_{A \in \Phi} K_i(A), \text{ then } \vdash_\omega K_i(\Gamma) \rightarrow K_i(A) \text{ for some } A \in \Phi, \tag{3.1}$$

which is also proved in the same way as Theorem 3.A. Using this fact, we find that $K_i(A \bigvee B) \rightarrow K_i(A) \bigvee K_i(B)$ is not necessarily provable. Hence the statements of Proposition 3.1 in Part I are, indeed, unparalleled.

Theorem 3.A manifests a similarity between our logic GL_ω and intuitionistic logic. The distinction between $K_i(\Gamma) \rightarrow K_i(\exists x A(x))$ and $K_i(\Gamma) \rightarrow \exists x K_i(A(x))$ is parallel to that between existential claims in classical logic and intuitionistic logic. A similar result is found for intuitionistic logic, cf., Harrop [2], [3] (see also Van Dalen [9]). From the view point of formal systems, logic GL_ω has the restriction of the succedents of the upper and lower sequents of $(K \to K)$ to contain at most one formula, while intuitionistic logic has the same restriction on any sequent (cf., Gentzen [1]). In fact, we modify the Barcan inference so that this restriction holds for all sequents in the trunk of the cut-free proof of $K_i(\Gamma) \to \exists x_1 ... \exists x_\ell K_i(A(x_1, ..., x_\ell))$. Then the similarity in formalism becomes more apparent, which is stated in Lemma 3.1.

To state Lemma 3.1 and Theorem 3.A, we use the following, slightly different formulation of $(B\text{-}\bigwedge)$:

$$\frac{\{\Gamma \to \Theta, K_i(A) : A \in \Phi\} \qquad K_i(\bigwedge \Phi), \Delta \to \Lambda}{\Gamma, \Delta \to \Theta, \Lambda} \, (B - \bigwedge)^*,$$

where Φ is an allowable set. When we have (th), $(B\text{-}\bigwedge)$ is equivalent to $(B\text{-}\bigwedge)^*$ in that provability $\vdash_\omega$, as well as the cut-elimination theorem, is not affected by the use of $(B\text{-}\bigwedge)^*$ instead of $(B\text{-}\bigwedge)$. Nevertheless, $(B\text{-}\bigwedge)^*$ is more convenient in proving Theorem 3.A, and $(B\text{-}\bigwedge)$ is more for other purposes.

Lemma 3.1. Suppose $\vdash_\omega K_i(\Gamma) \to \exists x_1 ... \exists x_\ell K_i(A(x_1, ..., x_\ell))$. Then there is a cut-free proof P of $K_i(\Gamma) \to \exists x_1 ... \exists x_\ell K_i(A(x_1, ..., x_\ell))$ such that the succedent of each sequent in the trunk of P has at most one formula.

As was discussed, by modifying $(K \to K)$ slightly, we obtain an infinitary predicate extension of *S4*, where the cut-elimination theorem holds, too. Although this *S4*-type extension has also the same restriction as that of GL_ω, we have not succeeded in proving Theorem 3.A. In GL_ω, K_i is applied simultaneously to the antecedent and succedent, while in the *S4*-type extension, it is applied separately to them. This difference prevents from having Lemma 3.1 and becomes an obstacle against proving Theorem 3.A in the *S4*-type extension.

To prove the term existence theorem stated in Part I, we need one more theorem. A formula A is said to be *indecomposable* iff A is atomic or the outermost symbol of A is K_j for some $j = 1, ..., n$. We say that for $i = 1, ..., n$, a formula A is a *K_i-formula* iff the outermost symbol of every maximal indecomposable subformula of A is K_i; and that A is a *K_{-i}-formula* iff K_i occurs only in the scope of K_j for some $j \neq i$. These two notions are mutually exclusive. For example, $K_1(K_2(A) \supset B) \supset K_1(B)$ is a K_1-formula, $K_2(K_1(A) \supset B) \supset K_3(B)$ a K_{-1}-formula, and $K_1(K_2(A) \supset B) \supset K_2(B)$ is neither a K_1-formula nor K_{-1}-formula.

Theorem 3.B (Separation Theorem). Let Γ_i, Θ_i be finite sets of K_i-formulae, and Γ_{-i}, Θ_{-i} finite sets of K_{-i}-formulae ($i = 1, ..., n$). If $\vdash_\omega \Gamma_i, \Gamma_{-i} \to \Theta_i, \Theta_{-i}$, then $\vdash_\omega \Gamma_i \to \Theta_i$ or $\vdash_\omega \Gamma_{-i} \to \Theta_{-i}$.

This theorem is proved based on the following lemma.

Lemma 3.2. Let Γ_i, Θ_i be finite sets of K_i-formulae, and Γ_{-i}, Θ_{-i} finite sets of K_{-i}-formulae ($i = 1, ..., n$). Let P be a cut-free proof of $\Gamma_i, \Gamma_{-i} \to \Theta_i, \Theta_{-i}$. Then every formula occurring in the trunk of P is either a K_i-formula or a K_{-i}-formula.

Using Theorem 3.B repeatedly, we have a refinement: Let Γ_i, Θ_i be finite sets of K_i-formulae for $i = 0, 1, ..., n$, where Γ_0 and Θ_0 are finite sets of nonepistemic formulae. Then

$$\begin{array}{c}\text{If } \vdash_\omega \Gamma_0, \Gamma_1, ..., \Gamma_n \to \Theta_0, \Theta_1, ..., \Theta_n, \\ \text{then } \vdash_\omega \Gamma_i \to \Theta_i \text{ for some } i = 0, 1, ..., n,\end{array} \tag{3.2}$$

Now we can state the term existence theorem used in Part I.

Theorem 3.C (Term Existence II). Let Γ be a finite set of nonepistemic formulae, and A a nonepistemic formula. If $\vdash_\omega C(\Gamma) \to \exists x_1 ... \exists x_\ell C(A(x_1, ..., x_\ell))$, then $\vdash_\omega C(\Gamma) \to C(A(t_1, ..., t_\ell))$ for some terms $t_1, ..., t_\ell$.

Theorems 3.A and 3.B can be obtained for GL_m and GL_{mp} ($m \leq \omega$) as well as for their finitary fragments. However, they fail to hold for the *S4*-type extensions. For example, $\vdash_{\omega S4} K_1(\exists x K_2(P(x))) \to \exists x K_2(P(x))$, but Theorem 3.A does not hold for this sequent in $GL_{\omega S4}$, where $P(.)$ is a unary predicate. Also, $\vdash_{\omega S4} K_1(P(a)), K_2(\neg P(a)) \to$, but Theorem 3.B does not hold for this sequent in $GL_{\omega S4}$. Nevertheless, it still remains open whether Theorem 3.C holds in $GL_{\omega S4}$.

Theorem 3.B manifests cognitive relativism in GL_ω in that the epistemic world of each player (even in the mind of another player) is separated from the others'. In contrast, the *S4*-type extension $GL_{\omega S4}$ does not permit this separation as was mentioned above, and assumes that knowledge must be true relative to the thinker and ultimately relative to the investigator. In GL_ω, cognitive relativism manifested as the separation of epistemic worlds enables us to obtain our results.

3.2. Proofs of the Results of Section 3.1

Recall that we use $(B\text{-}\bigwedge)^*$ instead of $(B\text{-}\bigwedge)$ in the proofs of Lemma 3.1 and Theorem 3.A.

Proof of Lemma 3.1. Let P be a cut-free proof of $K_i(\Gamma) \to \exists x_1 ... \exists x_\ell K_i(A(x_1, ..., x_\ell))$ in GL_ω. Consider the trunk of P. By the form of the endsequent and (2.1), the trunk has only three types of inferences, (th), $(\to \exists)$ and $(B\text{-}\bigwedge)^*$, and each boundary is either an initial sequent or the lower sequent of inference $(K \to K)$.

We prove that for any sequent $\Lambda \to \Theta$ in the trunk of P, there is a sequent $\Lambda \to \Theta^*$ and a cut-free proof P' such that

(i): Θ^* is a subset of Θ;

(ii): the succedent of any sequent in the trunk of P' has at most one formula.

Of course, Θ^* has at most one formula. If this is done, we have a cut-free proof of $K_i(\Gamma) \to \exists x_1 ... \exists x_\ell K_i(A(x_1, ..., x_\ell))$ with the property (ii). We prove this assertion by induction on the tree structure of the trunk of P from each boundary.

First, consider a boundary of the trunk of P. Then $\Lambda \to \Theta$ is an initial sequent or the lower sequent of inference $(K \to K)$. Thus Θ has at most one formula. Therefore the subproof of $\Lambda \to \Theta$ in P is a cut-free proof with the properties (i) and (ii).

Next, consider a sequent $\Delta \to \Psi$ in the trunk of P which is not a boundary. We make the induction hypothesis that for any sequent $\Lambda \to \Theta$ immediately above $\Delta \to \Psi$, there is a cut-free proof of $\Lambda \to \Theta^*$ with the properties (i) and (ii). We consider the three possible cases, (th), $(\to \exists)$ and $(B\text{-}\bigwedge)^*$. That is, $\Lambda \to \Theta$ and $\Delta \to \Psi$ are an upper sequent of and the lower sequent of one of these inferences.

Consider (th). The upper sequent $\Lambda \to \Theta$ of (th) satisfies $\Lambda \subseteq \Delta$ and $\Theta \subseteq \Psi$. By the induction hypothesis, there is a cut-free proof of $\Lambda \to \Theta^*$ with (i) and (ii). Adding (th) to the proof of $\Lambda \to \Theta^*$, we have a cut-free proof of $\Delta \to \Theta^*$ with (i) and (ii).

Consider $(\to \exists)$. This $(\to \exists)$ is represented as

$$\frac{\Lambda \to \Theta', \exists y_1 ... \exists y_k K_i(A(t, y_1, ..., y_k))}{\Lambda \to \Theta', \exists y \exists y_1 ... \exists y_k K_i(A(y, y_1, ..., y_k))},$$

where $\Delta \to \Psi$ is $\Lambda \to \Theta', \exists y \exists y_1 ... \exists y_k K_i(A(y, y_1, ..., y_k))$. The induction hypothesis states that there is a cut-free proof P' of $\Lambda \to \Theta^*$ with the properties (i), i.e., $\Theta^* \subseteq \Theta' \cup \{\exists y_1 ... \exists y_k K_i(A(t, y_1, ..., y_k))\}$, and (ii). If Θ^* does not contain $\exists y_1 ... \exists y_k K_i(A(t, y_1, ..., y_k))$, we conclude that the proof P' of $\Lambda \to \Theta^*$ is the desired one. If Θ^* consists of $\exists y_1 ... \exists y_k K_i(A(t, y_1, ..., y_k))$, we add the following step to the proof P':

$$\frac{\Lambda \to \exists y_1 ... \exists y_k K_i(A(t, y_1, ..., y_k))}{\Lambda \to \exists y \exists y_1 ... \exists y_k K_i(A(y, y_1, ..., y_k))} \ (\to \exists).$$

Finally, consider $(B\text{-}\bigwedge)^*$:

$$\frac{\{\Sigma \to \Xi, K_j(B) : B \in \Phi\} \quad K_j(\bigwedge \Phi), \Pi \to \Upsilon}{\Sigma, \Pi \to \Xi, \Upsilon},$$

where $\Delta \to \Psi$ is $\Sigma, \Pi \to \Xi, \Upsilon$. By the induction hypothesis, for each $B \in \Phi$ we have a cut-free proof P_B of $\Sigma \to \Xi_B$ with the properties (i), i.e., $\Xi_B \subseteq \Xi \cup \{K_j(B)\}$, and (ii), and also we have a cut-free proof P' of $K_j(\bigwedge \Phi), \Pi \to \Upsilon^*$ with (i), i.e., $\Upsilon^* \subseteq \Upsilon$, and (ii).

If Ξ_B does not contain $K_j(B)$ for some $B \in \Phi$, then it is a subset of Ξ. Hence we obtain a proof of $\Sigma, \Pi \to \Xi_B$ by adding a (th) to the proof P_B, which has the properties

(i) and (ii). Now consider the case where Ξ_B consists of $K_j(B)$ for any $B \in \Phi$. Then we have a proof of $\Sigma, \Pi \to \Upsilon^*$ combining proofs P_B ($B \in \Phi$) and P' in the following way:

$$\frac{\{\Sigma \to K_j(B) : B \in \Phi\} \quad K_j(\bigwedge \Phi), \Pi \to \Upsilon^*}{\Sigma, \Pi \to \Upsilon^*} \ (B\text{-}\bigwedge)^*.$$

This cut-free proof of $\Sigma, \Pi \to \Upsilon^*$ satisfies (i) and (ii). □

Proof of Theorem 3.A. By Lemma 3.1, there is a cut-free proof P of $K_i(\Gamma) \to \exists x_1 ... \exists x_\ell K_i(A(x_1, ..., x_\ell))$ such that the succedent of any sequent in the trunk of P has at most one formula. By induction on the structure of the trunk of P, we prove that for any sequent $\Lambda \to \Theta$ in the trunk,

(*): if Θ is represented as $\{\exists x_k ... \exists x_\ell K_i(A(t_1, ..., t_{k-1}, x_k, ..., x_\ell))\}$ with $k \leq \ell$, then $\vdash_\omega \Lambda \to K_i(A(t_1, ..., t_\ell))$ for some terms $t_k, ..., t_\ell$.

From this, we have the conclusion that $\vdash_\omega K_i(\Gamma) \to K_i(A(t_1, ..., t_\ell))$ for some terms $t_1, ..., t_\ell$.

For a boundary $\Lambda \to \Theta$, the premise of (*) does not hold, since it is an initial sequent of the form $K_j(\cdot) \to K_j(\cdot)$ or the lower sequent of $(K \to K)$.

Consider a sequent $\Lambda \to \Theta$ which is not a boundary in the trunk. We make the induction hypothesis that for any sequent immediately above $\Lambda \to \Theta$, (*) holds. We assume that Θ is represented as $\{\exists x_k ... \exists x_\ell K_i(A(t_1, ..., t_{k-1}, x_k, ..., x_\ell))\}$. We have three cases (th), $(\to \exists)$ and $(B\text{-}\bigwedge)^*$.

Consider (th). Then its upper sequent has the form $\Lambda' \to \Theta'$ with $\Lambda' \subseteq \Lambda$ and $\Theta' \subseteq \Theta$. If Θ' consists of $\exists x_k ... \exists x_\ell K_i(A(t_1, ..., t_{k-1}, x_k, ..., x_\ell))$, then $\vdash_\omega \Lambda' \to K_i(A(t_1, ..., t_\ell))$ for some $t_k, ..., t_\ell$ by the induction hypothesis, which together with (th) implies $\vdash_\omega \Lambda \to K_i(A(t_1, ..., t_\ell))$. If Θ' is empty, then $\vdash_\omega \Lambda \to K_i(A(t_1, ..., t_\ell))$ for any terms $t_1, ..., t_\ell$ by (th).

Consider $(\to \exists)$. Then its upper sequent is $\Lambda \to \exists x_{k+1} ... \exists x_\ell K_i(A(t_1, ..., t_k, x_{k+1}, ..., x_\ell))$ for some terms $t_1, ..., t_k$. If $k = \ell$, the upper sequent is $\Lambda \to K_i(A(t_1, ..., t_\ell))$, which is already the assertion for the lower sequent. If $k < \ell$, then $\vdash_\omega \Lambda \to K_i(A(t_1, ..., t_\ell))$ for some terms $t_{k+1}, ..., t_\ell$ by the induction hypothesis, which is also the assertion.

Finally, consider $(B\text{-}\bigwedge)^*$:

$$\frac{\{\Sigma \to K_j(B) : B \in \Phi\} \quad K_j(\bigwedge \Phi), \Delta \to \exists x_k ... \exists x_\ell K_i(A(t_1, ..., t_{k-1}, x_k, ..., x_\ell))}{\Sigma, \Delta \to \exists x_k ... \exists x_\ell K_i(A(t_1, ..., t_{k-1}, x_k, ..., x_\ell))}.$$

By the induction hypothesis, $\vdash_\omega K_j(\bigwedge \Phi), \Delta \to K_i(A(t_1, ..., t_\ell))$ for some terms $t_k, ..., t_\ell$. Then we have a proof of $\Sigma, \Delta \to K_i(A(t_1, ..., t_\ell))$ by adding the following:

$$\frac{\{\Sigma \to K_j(B) : B \in \Phi\} \qquad K_j(\bigwedge\Phi), \Delta \to K_i(A(t_1, ..., t_\ell))}{\Sigma, \Delta \to K_i(A(t_1, ..., t_\ell))} \ (B\text{-}\bigwedge)^*.$$

□

In the following, we use the original $(B\text{-}\bigwedge)$.

Proof of Lemma 3.2. As was noted in (2.1), the violation of the subformula property is caused by in the occurrences of $(B\text{-}\bigwedge)$ in the trunk of P. The side formulae of $(B\text{-}\bigwedge)$ are of the form $K_j(\cdot)$ for some $j = 1, ..., n$; they are K_i-formulae if $j = i$ or K_{-i}-formulae if $j \neq i$. For any formula of other form, the subformula property holds in the trunk. Hence every formula occurring in the trunk must be a K_i-formula or a K_{-i}-formula, since the endsequent consists of K_i-formulae and K_{-i}-formulae. □

Proof of Theorem 3.B. Suppose $\vdash_\omega \Gamma_i, \Gamma_{-i} \to \Theta_i, \Theta_{-i}$, where Γ_i, Θ_i are finite sets of K_i-formulae and Γ_{-i}, Θ_{-i} are finite sets of K_{-i}-formulae. Then there is a cut-free proof P of $\Gamma_i, \Gamma_{-i} \to \Theta_i, \Theta_{-i}$ by the cut-elimination theorem. Consider the trunk of P. Lemma 3.2 states that any sequent in the trunk is represented as $\Delta_i, \Delta_{-i} \to \Lambda_i, \Lambda_{-i}$, where Δ_i, Λ_i and $\Delta_{-i}, \Lambda_{-i}$ consist of K_i-formulae and K_{-i}-formulae, respectively. We prove the following by the structure of the trunk from boundaries:

$$(*):\ \vdash_\omega \Delta_i \to \Lambda_i \text{ or } \vdash_\omega \Delta_{-i} \to \Lambda_{-i}.$$

We call $\Delta_i \to \Lambda_i$ and $\Delta_{-i} \to \Lambda_{-i}$ the *K_i-part* and *K_{-i}-part.*

A boundary is an initial sequent or the lower sequent of $(K \to K)$. In the first case, it is represented as $A \to A$. Then A is either a K_i-formula or K_{-i}-formula by Lemma 3.2. In the second case, it is of the form $K_j(\Delta) \to K_j(\Lambda)$, and hence the formulae are K_i-formulae if j is i, or are K_{-i}-formulae if j is not i. Thus the assertion $(*)$ holds.

Consider a sequent $\Delta \to \Lambda$ in the trunk which is not a boundary. We make the induction hypothesis that every sequent immediately above $\Delta \to \Lambda$ satisfies $(*)$. There are three cases we have to consider: $\Delta \to \Lambda$ is the lower sequent of (th), some operational inference or $(B\text{-}\bigwedge)$.

(th): In this case, it follows from Lemma 3.2 that its upper and lower sequents are described as $\Delta'_i, \Delta'_{-i} \to \Lambda'_i, \Lambda'_{-i}$ and $\Delta_i, \Delta_{-i} \to \Lambda_i, \Lambda_{-i}$, where $\Delta'_i \subseteq \Delta_i, \Delta'_{-i} \subseteq \Delta_{-i}, \Lambda'_i \subseteq \Lambda_i$ and $\Lambda'_{-i} \subseteq \Lambda_{-i}$. By the induction hypothesis, $\vdash_\omega \Delta'_i \to \Lambda'_i$ or $\vdash_\omega \Delta'_{-i} \to \Lambda'_{-i}$. Thus, by (th), $\vdash_\omega \Delta_i \to \Lambda_i$ or $\vdash_\omega \Delta_{-i} \to \Lambda_{-i}$.

(Operational Inferences): By Lemma 3.2, there are only two cases: (a) the side formulae are K_i-formulae; and (b) they are K_{-i}-formulae.

Consider case (a). If the K_{-i}-part of some upper sequent is provable, the K_{-i}-part of the lower sequent is provable, since the K_{-i}-parts of the upper and lower sequents are the same. If the K_{-i}-part is not provable for any upper sequent, then the K_i-part of every upper sequent is provable by the induction hypothesis, which implies that the K_i-part of the lower sequent is also provable.

Consider case (b). If the K_i-part of some upper sequent is provable, the K_i-part of the lower sequent is also provable, since the K_i-parts of the upper and lower sequents are the same. Suppose that the K_i-part of any upper sequent is not provable. Then the K_{-i}-part of every upper sequent is provable. In this case, the inference can be directly applied to the K_{-i}-parts of the upper sequents of the inference. Thus the K_{-i}-part of the lower sequent is also provable.

(B-$\bigwedge$): Suppose that $\Delta \rightarrow \Lambda$ is the lower sequent of $(B\text{-}\bigwedge)$. Then $(B\text{-}\bigwedge)$ has the following form:

$$\frac{\{\Delta_i, \Delta_{-i} \rightarrow \Lambda_i, \Lambda_{-i}, K_j(A) : A \in \Phi\} \qquad K_j(\bigwedge \Phi), \Delta_i, \Delta_{-i} \rightarrow \Lambda_i, \Lambda_{-i}}{\Delta_i, \Delta_{-i} \rightarrow \Lambda_i, \Lambda_{-i}}.$$

Let $j = i$. Then if the K_{-i}-part of one of the upper sequents is provable, then $\vdash_\omega \Delta_{-i} \rightarrow \Lambda_{-i}$. If the K_i-parts of all upper sequents are provable, then $\vdash_\omega \Delta_i \rightarrow \Lambda_i$, since

$$\frac{\{\Delta_i \rightarrow \Lambda_i, K_i(A) : A \in \Phi\} \qquad K_i(\bigwedge \Phi), \Delta_i \rightarrow \Lambda_i}{\Delta_i \rightarrow \Lambda_i} \ (B\text{-}\bigwedge).$$

Let $j \neq i$. If the K_i-part of one of the upper sequent is provable, then $\vdash_\omega \Delta_i \rightarrow \Lambda_i$. If the K_i-part of no upper sequent is provable, the K_{-i}-parts of all upper sequents are provable. Then $\vdash_\omega \Delta_{-i} \rightarrow \Lambda_{-i}$, since

$$\frac{\{\Delta_{-i} \rightarrow \Lambda_{-i}, K_j(A) : A \in \Phi\} \qquad K_j(\bigwedge \Phi), \Delta_{-i} \rightarrow \Lambda_{-i}}{\Delta_{-i} \rightarrow \Lambda_{-i}} \ (B\text{-}\bigwedge).$$

Thus the K_{-i}-part of the lower sequent is provable. □

Proof of Theorem 3.C. By Lemma 2.1, $\vdash_\omega C(\Gamma) \rightarrow \exists x_1 ... \exists x_\ell C(A(x_1, ..., x_\ell))$ is equivalent to $\vdash_\omega \bigwedge C(\Gamma) \rightarrow \exists x_1 ... \exists x_\ell C(A(x_1, ..., x_\ell))$. Since $\vdash_\omega \Gamma, K_1(C(\Gamma)), ..., K_n(C(\Gamma)) \rightarrow \bigwedge C(\Gamma)$, we have $\vdash_\omega \Gamma, K_1(C(\Gamma)), ..., K_n(C(\Gamma)) \rightarrow \exists x_1 ... \exists x_\ell C(A(x_1, ..., x_\ell))$. This implies $\vdash_\omega \Gamma, K_1(C(\Gamma)), ..., K_n(C(\Gamma)) \rightarrow \exists x_1 ... \exists x_\ell K_1(A(x_1, ..., x_\ell))$. Hence it follows from (3.2) that $\vdash_\omega \Gamma \rightarrow$, $\vdash_\omega K_j(C(\Gamma)) \rightarrow$ for $j \neq 1$ or $\vdash_\omega K_1(C(\Gamma)) \rightarrow \exists x_1 ... \exists x_\ell K_1(A(x_1, ..., x_\ell))$. In the first two cases, we have $\vdash_0 \Gamma \rightarrow$ by (2.2), which implies $\vdash_\omega C(\Gamma) \rightarrow$. Thus $\vdash_\omega C(\Gamma) \rightarrow C(A(t_1, ..., t_\ell))$ for any terms $t_1, ..., t_\ell$. In the third case, it follows from Theorem 3.A that $\vdash_\omega K_1(C(\Gamma)) \rightarrow K_1(A(t_1, ..., t_\ell))$ for some terms $t_1, ..., t_\ell$. This implies $\vdash_0 \Gamma \rightarrow A(t_1, ..., t_\ell)$ by (2.2). Hence $\vdash_\omega K(\Gamma) \rightarrow K(A(t_1, ..., t_\ell))$ for any $K \in \bigcup_{t<\omega} \mathsf{K}(t)$. It follows from this that $\vdash_\omega C(\Gamma) \rightarrow K(A(t_1, ..., t_\ell))$ for any $K \in \bigcup_{t<\omega} \mathsf{K}(t)$. Thus $\vdash_\omega C(\Gamma) \rightarrow C(A(t_1, ..., t_\ell))$. □

4. Sequent Calculus GL_m for m $(0 \leq m \leq \omega)$

Logic GL_m for a finite m needs a different formulation of a sequent calculus. For the future use as well as for the understanding of GL_ω, we present the sequent calculus formulation of GL_m. In fact, the new formulation works also for $m = \omega$, and cut-elimination holds for all m $(0 \leq m \leq \omega)$. In this sense, the new formulation is a generalization of sequent calculus GL_ω. We also present some other systems.

4.1. Sequent Calculus GL_m and Cut-Elimination

The main difference between the new formulation and the sequent calculus given in Section 2 is that each sequent has an outer $K \in \bigcup_{t<1+m} \mathsf{K}(t)$, that is, a sequent has the form $K[\Gamma \to \Theta]$, where Γ are Θ are finite sets of formulae. When m is finite, a sequent with an outer K of at most depth m is allowed, and when m is ω, a sequent with an outer K of any depth is allowed. When K is represented as $K_{i_1}K_{i_2}...K_{i_m}$, an infinitary extension of Gentzen's LK is given to the player i_m in the mind of player i_{m-1} in the mind of player i_{m-2} ... of player i_1. Sequents with different outer K and K' are connected by two inference rules corresponding to $(K \to K)$.

Specifically, sequent calculus GL_m $(0 \leq m \leq \omega)$ is formulated as follows:

Initial Sequents: An initial sequent is of the form $K[A \to A]$, where $K \in \bigcup_{t<1+m} \mathsf{K}(t)$ and A is a formula.

Inference Rules: The structural and operational inference rules are the same as those in Section 2, except the outer $K \in \bigcup_{t<1+m} \mathsf{K}(t)$ with each sequent in inferences, for example, (cut) and $(\bigwedge \to)$ are given as

$$\frac{K[\Gamma \to \Theta, M] \qquad K[M, \Delta \to \Lambda]}{K[\Gamma, \Delta \to \Theta, \Lambda]}\ (M)\ (\text{cut})$$

$$\frac{K[A, \Gamma \to \Theta]}{K[\bigwedge \Phi, \Gamma \to \Theta]}\ (\bigwedge \to)\ (A \in \Phi).$$

The Barcan inference takes the following form: for $K \in \bigcup_{t<m} \mathsf{K}(t)$ or $K = K'K_i \in \bigcup_{t<1+m} \mathsf{K}(t)$,

$$\frac{\{K[\Gamma \to \Theta, K_i(A)] : A \in \Phi\} \qquad K[K_i(\bigwedge \Phi), \Gamma \to \Theta]}{K[\Gamma \to \Theta]}\ (B\text{-}\bigwedge),$$

where Φ is an allowable set. When $m = 0$, this inference is not allowed, and when $m = 1$, the outer K of $(B\text{-}\bigwedge)$ is the null symbol or is the same as the outermost K_i of the side formulae of $(B\text{-}\bigwedge)$.

The inference $(K \to K)$ is modified as follows: for any $KK_i \in \bigcup_{t<1+m} \mathsf{K}(t)$,

$$\frac{KK_i[\Gamma, K_i(\Delta) \to \Theta]}{K[K_i(\Gamma, \Delta) \to K_i(\Theta)]} \; (K \to K)_C \qquad \frac{KK_i[\Gamma, K_i(\Delta) \to \Theta]}{KK_i[K_i(\Gamma, \Delta) \to K_i(\Theta)]} \; (K \to K)_U \,,$$

where $|\Theta| \leq 1$. When $m = 0$, these rules are not allowed, and when $m = 1$, the outer K is the null symbol.

These two rules are needed to describe the idea that each player has the inference ability described by $(K \to K)_C$ as well as he knows the ability. In the case of $(B\text{-}\bigwedge)$, he can use and knows the Barcan rule. The latter is described in the case where the innermost symbol of K coincides with the outermost symbol of the side formulae in $(B\text{-}\bigwedge)$.

When $m = 0$, no K-inference is allowed. Thus GL_0 is simply an infinitary extension of Gentzen's LK.

The following examples illustrate some difference between GL_1 and GL_2: for any i, j $(i \neq j)$,

$$\vdash_1 \;\; \to \bigwedge K_i(\Phi) \supset K_i(\bigwedge \Phi); \text{ and } \vdash_1 \;\; \to K_i(\bigwedge K_i(\Phi) \supset K_i(\bigwedge \Phi));$$

$$\vdash_2 \;\; \to K_j(\bigwedge K_i(\Phi) \supset K_i(\bigwedge \Phi)); \text{ and } \vdash_2 \;\; \to K_j K_i(\bigwedge K_i(\Phi) \supset K_i(\bigwedge \Phi)).$$

We use the same notations GL_m and $\vdash_m$ as in Part I. This is due to the following theorem.

Theorem 4.A. Let Φ be an allowable set of closed formulae and A a formula. Then, for any m $(0 \leq m \leq \omega)$, $\vdash_m \bigwedge \Phi \to A$ if and only if $\Phi \vdash_m A$ in the sense of Part I.

Cut-elimination holds for GL_m for any m with $0 \leq m \leq \omega$.

Theorem 4.B (Cut-Elimination in GL_m). If $K[\Gamma \to \Theta]$ is provable in GL_m, there is a cut-free proof of $K[\Gamma \to \Theta]$.

In logic GL_m for $m \geq 1$, when $(B\text{-}\bigwedge)$ occurs in a cut-free proof, the cut-free proof does not satisfy the full subformula property as in GL_ω of Section 2. In GL_0, however, since $(B\text{-}\bigwedge)$ is not allowed, the cut-elimination theorem ensures the full subformula property for a cut-free proof.

The relationship between the sequent calculus GL_ω in this section and that in Section 2 is given by the following proposition.

Proposition 4.1. A sequent $\Gamma \to \Theta$ is provable in the present formulation of GL_ω if and only if it is provable in the formulation of GL_ω in Section 2.

Proof. Suppose that P is a proof of $\Gamma \to \Theta$ in GL_ω in the sense of this section. Removing all outer K's from the sequents in the proof P, we obtain a proof of $\Gamma \to \Theta$ in GL_ω in the sense of Section 2.

Conversely, suppose that P is a proof of $\Gamma \to \Theta$ in GL_ω in the sense of Section 2. We associate an outer K with each sequent by induction from the endsequent as follows. We associate the null symbol with the endsequent. Consider an inference (η) in P, and make the induction hypothesis that K is associated with the lower sequent of (η). If (η) is not $(K \to K)$, we associate the same K with every upper sequent of (η). Suppose that (η) is $(K \to K)$ and introduces K_i. Then we associate K with the upper sequent of (η) if the innermost symbol of K is K_i, and associate KK_i with it otherwise. Then we have a proof P' of $\Gamma \to \Theta$ in the present formulation of GL_ω. □

The proof of Proposition 4.1 associates a proof in the present formulation of GL_ω with one in GL_ω in the sense of Section 2. The associated proofs have the same structures of inference rules. Therefore, the cut-elimination theorem for GL_ω in the present formulation provides a cut-free proof in GL_ω in the sense of Section 2, and *vice versa.* In Section 5, we prove the cut-elimination theorem (Theorem 4.B) for GL_m $(0 \leq m \leq \omega)$, which implies Theorem 2.B.

Theorems 3.A and 3.B for GL_ω in Section 3 are obtained also in GL_m for any finite m. The proofs of them given in Section 3 work almost directly. Theorem 3.C is specific to the case $m = \omega$.

4.2. Logics GL_{mp} $(1 \leq m \leq \omega)$

First, recall that logic GL_{mp} which is defined by the axiom set $\Delta_{mp} = \{K(A) : A \in \Delta_p$ and $K \in \bigcup_{t<m} \mathsf{K}_p(t)\}$. The sequent calculus formulation of logic GL_{mp} is obtained from GL_m with the following modifications: An outer K of each sequent in initial sequents, structural and operational inferences is taken from $\bigcup_{t<1+m} \mathsf{K}_p(t)$ instead of $\bigcup_{t<1+m} \mathsf{K}(t)$. We substitute the following inferences for $(B\text{-}\bigwedge), (K \to K)_C$ and $(K \to K)_U$: for any $K \in \bigcup_{t<m} \mathsf{K}_p(t)$,

$$\frac{\{K[\Gamma \to \Theta, K_i(A)] : A \in \Phi\} \qquad K[K_i(\forall x A(x)), \Gamma \to \Theta]}{K[\Gamma \to \Theta]} \; (B\text{-}\textstyle\bigwedge)_p$$

$$\frac{KK_i[\Gamma \to \Theta]}{K[K_i(\Gamma) \to K_i(\Theta)]} \; (K \to K)_p$$

where $|\Theta| \leq 1$. If the additional condition $|\Theta| \leq 1$ is replaced by $|\Theta| = 1$, then the system is denoted by GL_{mK}, which corresponds to the logic defined by Δ_0 without (PI_i) and $(\bot_i)$.

Cut-elimination is obtained for GL_{mp} as well as for GL_{mK} from the proof given in Section 5 with the desired modifications.

When $m = \omega$, again, we do not need outer K and KK_i in the inference rules as in GL_ω. Logics $GL_{\omega p}$ and $GL_{\omega K}$ are the infinitary predicate extensions of modal logics KD and K.

Proposition 2.2 (Faithful Representation) of Part I is a special case of the following proposition.

Proposition 4.2. (Faithful Representation): For a finite m, $\vdash_{(m+1)p} K_i(\Gamma) \to K_i(\Theta)$ if and only if $\vdash_{mp} \Gamma \to \Theta$.

Proof. We prove the *only-if* part. Let P be a proof of $K_i(\Gamma) \to K_i(\Theta)$ in $GL_{(m+1)p}$. We can define boundaries and the trunk of P in the same way as in GL_ω. In the trunk, by the form of the endsequent, every formula has the form $K_j(B)$ for some j and B, and only inferences (th) and $(B\text{-}\bigwedge)_p$ may occur. We prove by induction from boundaries in the trunk that $\Delta^\# \to \Lambda^\#$ is provable in GL_{mp} for any sequent $\Delta \to \Lambda$ in the trunk of P, where $\Delta^\#, \Lambda^\#$ are obtained from Δ, Λ by eliminating the outermost K_j $(j = 1, ..., n)$ of each formula in Δ, Λ.

A boundary $\Delta \to \Lambda$ is either an initial sequent or the lower sequent of $(K \to K)_p$. If $\Delta \to \Lambda$ is an initial sequent, it has the form $K_j(A) \to K_j(A)$. Then $A \to A$ is provable in GL_{mp}. Next suppose that a boundary is the lower sequent of $(K \to K)_p$, which is expressed as

$$\frac{K_j[\Sigma \to \Xi]}{K_j(\Sigma) \to K_j(\Xi)}\,.$$

All sequents above this inference $(K \to K)_p$ have the outermost symbol K_j, i.e., $K_jK'[\cdot \to \cdot]$, where K' may be the null symbol. We eliminate this outermost K_j of each sequent in the proof of $K_j[\Sigma \to \Xi]$ and obtain a proof of $\Sigma \to \Xi$ in GL_{mp}.

We make the induction hypothesis that for every upper sequent $\Delta \to \Lambda$ of (th) or $(B\text{-}\bigwedge)_p$, $\Delta^\# \to \Lambda^\#$ is provable in GL_{mp}. We have to consider only two cases: (th) and $(B\text{-}\bigwedge)_p$.

Consider (th):

$$\frac{\Delta \to \Lambda}{\Pi, \Delta \to \Lambda, \Psi}\,.$$

By the form of the endsequent $K_i(\Gamma) \to K_i(\Theta)$, each formula in Π and Ψ is the form $K_j(B)$ for some $j = 1, ..., n$ and B. Since $\Delta^\# \to \Lambda^\#$ is provable in GL_{mp} by the induction hypothesis, the sequent $\Pi^\#, \Delta^\# \to \Lambda^\#, \Psi^\#$ is provable by (th).

Consider $(B\text{-}\bigwedge)_p$:

$$\frac{\{\Sigma \to \Psi, K_j(A) : A \in \Phi\} \qquad K_j(\bigwedge \Phi), \Sigma \to \Psi}{\Sigma \to \Psi}\,.$$

By the induction hypothesis, the sequent $\Sigma^{\#} \to \Psi^{\#}, A$ is provable for any $A \in \Phi$ and $\bigwedge \Phi, \Sigma^{\#} \to \Psi^{\#}$ is also provable in GL_{mp}. Hence $\Sigma^{\#} \to \Psi^{\#}$ is provable in GL_{mp} since

$$\dfrac{\dfrac{\{\Sigma^{\#} \to \Psi^{\#}, A : A \in \Phi\}}{\Sigma^{\#} \to \Psi^{\#}, \bigwedge \Phi}\,(\to \bigwedge) \qquad \bigwedge \Phi, \Sigma^{\#} \to \Psi^{\#}}{\Sigma^{\#} \to \Psi^{\#}}\,(\text{cut}).$$

□

5. Proof of the Cut-Elimination Theorem for GL_m

5.1. Preliminaries

Our proof of the cut-elimination theorem GL_m $(0 \leq m \leq \omega)$ is based on the original proof of Gentzen [1]. There are several differences between Gentzen's LK and our GL_m. We have additional inference rules $(K \to K)_C$, $(K \to K)_U$, $(B\text{-}\bigwedge)$, and also our system is infinitary. We have to give careful attentions to these differences.

As in Gentzen [1], we focus our attention to a proof P having a (cut) only at the last inference:

$$\frac{K[\Gamma \to \Theta, M] \qquad K[M, \Delta \to \Lambda]}{K[\Gamma, \Delta \to \Theta, \Lambda]}\,(M)(\text{cut}). \tag{5.1}$$

We show that

(∗): for any proof with a (cut) only at the last inference, there is a cut-free proof with the same endsequent.

If this is done, we can eliminate every (cut) from an arbitrary proof by induction on the tree structure of a proof from initial sequents.

To prove the assertion (∗), we use triple induction. For this purpose, we define the "grade", the "left rank" and "right rank" of the (cut).

We assign to each formula A an ordinal number $\text{gr}(A)$ called the *grade* of formula A. The grade $\text{gr}(A)$ is defined by induction on the structure of a formula as follows:

$$\begin{gathered}\mathrm{gr}(A) = 0 \text{ for every atomic formula } A;\\ \mathrm{gr}(\neg A) = \mathrm{gr}(A)+1;\quad \mathrm{gr}(A \supset B) = \max(\mathrm{gr}(A),\mathrm{gr}(B))+1;\\ \mathrm{gr}(\forall x A(x)) = \mathrm{gr}(A)+1;\quad \mathrm{gr}(\exists x A(x)) = \mathrm{gr}(A)+1;\\ \mathrm{gr}(K_i(A)) = \mathrm{gr}(A)+2 \text{ for } i = 1,...,n;\\ \mathrm{gr}(\bigwedge \Phi) = \sup\{\mathrm{gr}(A) : A \in \Phi\}+1;\\ \mathrm{gr}(\bigvee \Phi) = \sup\{\mathrm{gr}(A) : A \in \Phi\}+1.\end{gathered} \tag{5.2}$$

Here $\alpha + \beta$ is the standard sum of two ordinal numbers α, β.[2] The *grade* of the (cut) of (5.1), denoted by γ, is defined by

$$\gamma = \begin{cases} \mathrm{gr}(M) + \ell & \text{if the outermost symbol of } M \text{ is } K_{i_\ell} \\ \mathrm{gr}(M) + \ell + 1 & \text{othwerwise,} \end{cases} \tag{5.3}$$

where $K = K_{i_1}...K_{i_\ell}$, i.e., ℓ is the depth of the outer K. The grade of the (cut) is the sum of the grade of the cut-formula and the depth of the outer K if the outermost symbol of M coincides with the innermost symbol of K. We count the depth of K as $\ell + 1$ if they do not coincide. The second case is applied if $\ell = 0$.

We also associate other two ordinal numbers, called the left and right ranks, with the cut-formula M in (5.1). The left rank is defined as follows. Let P be a proof of the form (5.1). For an initial sequent η, we define

$$\rho_\ell(\eta) = \begin{cases} 1 & \text{if } \eta \text{ has the form } K'[M \to M] \text{ for some outer } K'; \\ 0 & \text{otherwise.} \end{cases} \tag{5.4}$$

Now let η be the lower sequent of some occurrence (J) of an inference in P, and suppose that the left rank $\rho_\ell(\xi)$ of M at each upper sequent ξ of (J) is already defined. Then

$$\rho_\ell(\eta) = \begin{cases} \sup\{\rho_\ell(\xi) : \xi \text{ is an upper sequent of } (J)\} + 1 & \\ & \text{if the succedent of } \eta \text{ contains } M; \\ 0 & \text{otherwise,} \end{cases} \tag{5.5}$$

The *left rank* ρ_ℓ of the (cut) of the proof of (5.1) is the left rank of M at the left upper sequent $K[\Gamma \to \Theta, M]$ of (5.1). The *right rank* ρ_r of the proof of (5.1) is defined in the dual manner.

[2]The grade of any formula in $\mathcal{P}_\omega$ is smaller than ω^2. More precisely, it can be verified that $\mathrm{gr}(A) < \omega(t+1)$ for any $A \in \mathcal{P}_t$ and t $(0 \le t < \omega)$,which implies $\mathrm{gr}(A) < \omega^2$ for any $A \in \mathcal{P}_\omega$.

To prove the assertion $(*)$, we carry out three inductions:

Induction Step 1 (Subsection 5.2.1): Under the induction hypothesis that $(*)$ holds for any proof where the grade of the (cut) is less than γ, we prove $(*)$ for any proof where the grade is γ, the left rank is one and the right rank is also one.

Induction Step 2 (Subsection 5.2.2): Under the induction hypothesis that $(*)$ holds for any proof where the grade is γ, the left rank is one and the right rank is less than ρ_r, we prove $(*)$ for any proof where the grade is γ, the left rank is one and the right rank is ρ_r.

Induction Step 3 (Subsection 5.2.3): Under the induction hypothesis that $(*)$ holds for any proof where the grade is γ, the left rank is less than ρ_ℓ and the right rank is equal to ρ_r, we prove $(*)$ for any proof where the grade is γ, the left rank is ρ_ℓ and the right rank is ρ_r.

Before going to the main body of the proof, we mention certain lemmata about the substitution of free variables.

Lemma 5.1. Let $\vdash_m K[\Gamma \to \Theta]$. Then there is a proof P of $K[\Gamma \to \Theta]$ in GL_m such that there remain an infinite number of free variables not occurring in P.

Proof. Let P' be a proof of $K[\Gamma \to \Theta]$. Since Γ, Θ are finite sets and each formula has at most finite number of free variables as was noted in Part I, they contain only a finite number of free variables. Denote the set of free variables not occurring in $K[\Gamma \to \Theta]$ by $B = \{b_0, b_1, ...\}$. We substitute the free variable b_{2t} for each free variable a_t in P' but not in $K[\Gamma \to \Theta]$ $(t = 0, 1, ...)$. The new tree is denoted by P. Then we can prove, by induction on the proof P', that P is a proof of $K[\Gamma \to \Theta]$. There remain an infinite number of free variables not occurring in P. □

From this lemma, we can always assume that there remain an infinite number of free variables not occurring in a proof.

Lemma 5.2. Let P be a proof of $K[\Gamma(a) \to \Theta(a)]$ in GL_m.

(1): Let b be a free variable not occurring in P. The tree P' obtained from P by substituting b for every occurrence of a is a proof of $K[\Gamma(b) \to \Theta(b)]$.

(2): Let t be a term. Then there is a proof P' of $K[\Gamma(t) \to \Theta(t)]$ which is obtained from P by substituting free variables not occurring in P for some finite number of free variables in P and by substituting t for every occurrence of a.

Proof. (1): By induction on the proof P.

(2): This assertion can be proved in the same way as (1) under the *assumption* that every eigenvariable occurring in P is neither the free variable a nor is included in t. When P does not satisfy the assumption, we change the proof P into another one with the same endsequent so that it satisfies this assumption as follows. Let b be a free variable occurring as an eigenvariable in P, and suppose that b is either a itself or is included in the term t. The number of such free variables is at most finite, since t has at most finite number of free variables. We substitute a new free variable not occurring in P for every occurrence of b above the inference whose eigenvariable is b. This new tree P^* is also a proof of $K[\Gamma(a) \to \Theta(a)]$ and does not include b as an eigenvariable. Repeating this process finite times, we obtain a proof P^{**} of $K[\Gamma(a) \to \Theta(a)]$ which satisfies the assumption. Then we obtain a proof P' by substituting the term t for all occurrences of the free variable a. This is a proof of $K[\Gamma(t) \to \Theta(t)]$ desired. □

5.2. Reductions

In the following, we consider a proof P whose last inference is of the form (5.1).

[1] Suppose that M belongs to at least one of $\Gamma, \Delta, \Theta, \Lambda$.

[1.1] When $M \in \Gamma$, we change the last part of the proof into

$$\frac{K[M, \Delta \to \Lambda]}{K[\Gamma, \Delta \to \Theta, \Lambda]} \text{(th)}.$$

The upper sequent of this inference is the right upper sequent of (5.1). Thus we simply eliminate the (cut).

[1.2] When $M \in \Theta, \Delta$ or Λ, we can eliminate the (cut) in a similar way.

From [1.1] and [1.2], we can assume $M \notin \Gamma \cup \Theta \cup \Delta \cup \Lambda$. This assumption is made throughout the remaining part of Subsection 5.2. Then neither upper sequent of (5.1) is an initial sequent, i.e., either is the lower sequent of some inference. That is, the last part has the following form:

$$\frac{\dfrac{\circ \quad \circ \quad \circ}{K[\Gamma \to \Theta, M]}(I_1) \qquad \dfrac{\circ \quad \circ \quad \circ}{K[M, \Delta \to \Lambda]}(I_2)}{K[\Gamma, \Delta \to \Theta, \Lambda]} (M) \text{ (cut)}.$$

We consider every case according to inferences (I_1) and (I_2) for the upper sequents of the (cut).

5.2.1. Induction Step 1

Consider a proof of the form (5.1) with grade γ and $\rho_\ell = \rho_r = 1$. We show by induction that we can find a cut-free proof with the same endsequent. We make the induction hypothesis that

$$\begin{array}{c}\text{for any proof of the form (5.1) with the grade smaller than } \gamma, \\ \text{we can find a cut-free proof with the same endsequent.}\end{array} \tag{5.6}$$

[2] Suppose that at least one of (I_1) and (I_2) is (th).

[2.1] When (I_1) is (th), the last part of the proof is expressed as

$$\dfrac{\dfrac{K[\Gamma' \to \Theta']}{K[\Gamma \to \Theta, M]}\,(\mathrm{th}) \qquad K[M, \Delta \to \Lambda]}{K[\Gamma, \Delta \to \Theta, \Lambda]}\,(M)(\mathrm{cut}),$$

where $\Gamma' \subseteq \Gamma$ and $\Theta' \subseteq \Theta$. Then we can eliminate the (cut) as follows:

$$\dfrac{K[\Gamma' \to \Theta']}{K[\Gamma, \Delta \to \Theta, \Lambda]}\,(\mathrm{th}).$$

[2.2] When (I_2) is (th), we can eliminate the (cut) in the dual manner.

Therefore we assume in the remaining of Subsection 5.2.1 that neither (I_1) nor (I_2) is (th). Hence M is the principal formula of (I_1). Similarly, M is the principal formula of (I_2). Also, neither (I_1) nor (I_2) is $(B\text{-}\bigwedge)$, since $\rho_\ell = \rho_r = 1$. Thus we have the following cases.

(a): When M is of the form $\neg A$, (I_1) and (I_2) are $(\to \neg)$ and $(\neg \to)$.

(b): When M is of the form $A \supset B$, (I_1) and (I_2) are $(\to\supset)$ and $(\supset\to)$.

(c): When M is of the form $\forall x A(x)$, (I_1) and (I_2) are $(\to \forall)$ and $(\forall \to)$, respectively; and similarly when M is of the form $\exists x A(x)$, (I_1) and (I_2) are $(\to \exists)$ and $(\exists \to)$, respectively.

(d): When M is of the form $\bigwedge \Phi$, (I_1) and (I_2) are $(\to \bigwedge)$ and $(\bigwedge \to)$. When M is of the form $\bigvee \Phi$, (I_1) and (I_2) are $(\to \bigvee)$ and $(\bigvee \to)$.

(e): When M is of the form $K_i(A)$, there are two cases based on the innermost symbol of the outer K of the lower sequents of (I_1) and (I_2). If the innermost symbol is not K_i of K, both (I_1) and (I_2) are $(K \to K)_C$, and otherwise, both are $(K \to K)_U$.

[3] Suppose that (I_1) and (I_2) are operational inferences whose principal formulae are M.

[3.1] When the outermost symbol of M is $\bigwedge$, the last part of the proof is

$$\dfrac{\dfrac{\{K[\Gamma \to \Theta, B] : B \in \Phi\}}{K[\Gamma \to \Theta, \bigwedge \Phi]}\,(\to \bigwedge) \qquad \dfrac{K[A, \Delta \to \Lambda]}{K[\bigwedge \Phi, \Delta \to \Lambda]}\,(\bigwedge \to)}{K[\Gamma, \Delta \to \Theta, \Lambda]}\,(\bigwedge \Phi)\ (\mathrm{cut}),$$

where $A \in \Phi$ and Φ is an allowable set. This is reduced into

$$\dfrac{K[\Gamma \to \Theta, A] \qquad K[A, \Delta \to \Lambda]}{K[\Gamma, \Delta \to \Theta, \Lambda]}\,(A)\ (\mathrm{cut}).$$

If the outermost symbol of A is the innermost symbol K_{j_ℓ} of K, the grade of the new (cut) is $\mathrm{gr}(A)+\ell$, and otherwise, it is $\mathrm{gr}(A)+\ell+1$. In either case, the grade of this (cut) is smaller than $\gamma = \mathrm{gr}(\bigwedge \Phi)+\ell+1$. Hence we can eliminate this (cut) by the induction hypothesis.

[3.2] When the outermost logical connective of M is $\forall$, the last part is:

$$\dfrac{\dfrac{K[\Gamma \to \Theta, A(a)]}{K[\Gamma \to \Theta, \forall x A(x)]}\,(\to \forall) \qquad \dfrac{K[A(t), \Delta \to \Lambda]}{K[\forall x A(x), \Delta \to \Lambda]}\,(\forall \to)}{K[\Gamma, \Delta \to \Theta, \Lambda]}\,(\forall x A(x))\ (\text{cut}).$$

Let P' be the subproof of $K[\Gamma \to \Theta, A(a)]$ in P. Lemma 5.2.2 ensures that there is a proof P'' of $K[\Gamma \to \Theta, A(t)]$ which is obtained from P' by substituting new free variables for some free variables in P' and substituting t for a. Then we can reduce the last part into

$$\dfrac{K[\Gamma \to \Theta, A(t)] \qquad K[A(t), \Delta \to \Lambda]}{K[\Gamma, \Delta \to \Theta, \Lambda]}\,(A(t))\ (\text{cut}).$$

Since the grade of the new (cut) is $\mathrm{gr}(A(t))+\ell+1$ or $\mathrm{gr}(A(t))+\ell$, it is smaller than the grade of the original (cut), $\mathrm{gr}(\forall x A(x))+\ell+1$. Hence we can find a cut-free proof of $K[\Gamma, \Delta \to \Theta, \Lambda]$ by the induction hypothesis.

[3.3] When the outermost logical connective of M is $\neg, \bigvee, \supset$ or $\exists$, we can reduce the proof into one with a smaller grade in a similar manner (see Gentzen [1]), and then we can eliminate the (cut) by the induction hypothesis.

[4] Suppose that (I_1) and (I_2) are K-inferences. Recall that neither (I_1) nor (I_2) is $(B\text{-}\bigwedge)$. We have to consider the following two cases: both (I_1) and (I_2) are $(K \to K)_C$ or both are $(K \to K)_U$.

[4.1] When M is $K_i(A)$ for some A and K_i is not the innermost symbol of K, the last part of the proof is:

$$\dfrac{\dfrac{KK_i[\Gamma, K_i(\Xi) \to A]}{K[K_i(\Gamma, \Xi) \to K_i(A)]}\,(K \to K)_C \qquad \dfrac{KK_i[A, \Delta, K_i(\Pi) \to \Lambda]}{K[K_i(A), K_i(\Delta, \Pi) \to K_i(\Lambda)]}\,(K \to K)_C}{K[K_i(\Gamma, \Xi, \Delta, \Pi) \to K_i(\Lambda)]}\,(\text{cut}).$$

The grade of this (cut) is $\gamma = \mathrm{gr}(A)+\ell+3$. The last part is reduced into

$$\dfrac{\dfrac{KK_i[\Gamma, K_i(\Xi) \to A] \qquad KK_i[A, \Delta, K_i(\Pi) \to \Lambda]}{KK_i[\Gamma, \Delta, K_i(\Xi, \Pi) \to \Lambda]}\,(\text{cut})}{K[K_i(\Gamma, \Delta, \Xi, \Pi) \to K_i(\Lambda)]}\,(K \to K)_C\ .$$

The grade of this new (cut) is $\mathrm{gr}(A) + \ell + 2$ or $\mathrm{gr}(A) + \ell + 1$, which is smaller than the grade of the original (cut), $\mathrm{gr}(A) + \ell + 3$. Thus we can eliminate this (cut) by the induction hypothesis.

[4.2] Suppose that M is $K_i(A)$ for some A and K_i is the innermost symbol of K. The last part of the proof is:

$$\dfrac{\dfrac{K[\Gamma, K_i(\Xi) \to A]}{K[K_i(\Gamma, \Xi) \to K_i(A)]}(K \to K)_U \qquad \dfrac{K[A, \Delta, K_i(\Pi) \to \Lambda]}{K[K_i(A), K_i(\Delta, \Pi) \to K_i(\Lambda)]}(K \to K)_U}{K[K_i(\Gamma, \Xi, \Delta, \Pi) \to K_i(\Lambda)]}(\mathrm{cut}).$$

The grade of this (cut) is $\gamma = \mathrm{gr}(A) + \ell + 2$. The last part is reduced into

$$\dfrac{\dfrac{K[\Gamma, K_i(\Xi) \to A] \qquad K[A, \Delta, K_i(\Pi) \to \Lambda]}{K[\Gamma, \Delta, K_i(\Xi, \Pi) \to \Lambda]}(\mathrm{cut})}{K[K_i(\Gamma, \Delta, \Xi, \Pi) \to K_i(\Lambda)]}(K \to K)_U .$$

The grade of this new (cut) is $\mathrm{gr}(A) + \ell$ or $\mathrm{gr}(A) + \ell + 1$ by (5.3), and is smaller γ. Hence we can eliminate this (cut) by the induction hypothesis.

5.2.2. Induction Step 2

Consider a proof of the form (5.1) where the grade is γ, the left rank ρ_ℓ is 1 and the right rank is $\rho_r > 1$. We prove by induction that there is a cut-free proof with the same endsequent. We make the induction hypothesis that

for any proof of the form (5.1) with the grade γ,
the left rank equal to 1 and the right rank lower than ρ_r, (5.7)
we can find a cut-free proof with the same endsequent.

[5] When (I_2) is (th), we can change the proof into one with a lower right rank. Then we can eliminate the (cut) by the induction hypothesis.

When (I_1) is (th), the (cut) is eliminated in the same manner as in [2.1] since $\rho_\ell = 1$. Hence we can assume in the remaining of Subsection 5.2.2 that (I_1) is not (th).

In the following reduction steps except for [7.2], the outer K and the cut-formula M remain unchanged. Hence the grade of the (cut) remains the same, too.

[6] Suppose that (I_2) is an operational inference.

[6.1] When (I_2) is $(\supset\to)$, the last part of the proof is

$$\dfrac{K[\Gamma \to \Theta, M] \qquad \dfrac{K[M, \Delta' \to \Lambda, A] \qquad K[B, M, \Delta' \to \Lambda]}{K[A \supset B, M, \Delta' \to \Lambda]}(\supset\to)}{K[\Gamma, [A \supset B], \Delta' \to \Theta, \Lambda]}(\mathrm{cut}),$$

where $[A \supset B]$ is $A \supset B$ and Δ is $\Delta' \cup \{A \supset B\}$ if M is not $A \supset B$, and $[A \supset B]$ is empty and Δ is Δ' if M is $A \supset B$.

[6.1.1] When M is not $A \supset B$, the last part is reduced into

$$\dfrac{\dfrac{K[\Gamma \to \Theta, M] \quad K[M, \Delta' \to \Lambda, A]}{K[\Gamma, \Delta' \to \Theta, \Lambda, A]}\,(\text{cut}) \qquad \dfrac{K[\Gamma \to \Theta, M] \quad K[B, M, \Delta' \to \Lambda]}{K[B, \Gamma, \Delta' \to \Theta, \Lambda]}\,(\text{cut})}{K[A \supset B, \Gamma, \Delta' \to \Theta, \Lambda]}\,(\supset\to).$$

Since these cuts have lower right ranks than ρ_r, we can eliminate these (cut)'s by the induction hypothesis. Note that even when M is $A \supset B$, this reduction is possible.

[6.1.2] When M is $A \supset B$, we have a cut-free proof of $K[A \supset B, \Gamma, \Delta \to \Theta, \Lambda]$ by [6.1.1] and continue

$$\dfrac{K[\Gamma \to \Theta, M] \qquad K[A \supset B, \Gamma, \Delta \to \Theta, \Lambda]}{K[\Gamma, \Delta \to \Theta, \Lambda]}\,(\text{cut}).$$

Since the right rank of this (cut) is 1, we can eliminate this (cut) by the induction hypothesis.

[6.2] When (I_2) is $(\bigvee \to)$, the last part of the proof is

$$\dfrac{K[\Gamma \to \Theta, M] \qquad \dfrac{\{K[M, A, \Delta' \to \Lambda] : A \in \Phi\}}{K[M, \bigvee \Phi, \Delta' \to \Lambda]}\,(\bigvee \to)}{K[\Gamma, \bigvee \Phi, \Delta' \to \Theta, \Lambda]}\,(\text{cut}),$$

where Φ is an allowable set, and Δ is $\Delta' \cup \{\bigvee \Phi\}$. The right rank of M at each upper sequent of the $(\bigvee \to)$ is lower than the right rank ρ_r of the (cut) by (5.5). We combine the subproof rooted at each upper sequent of the $(\bigvee \to)$ with the subproof rooted at the right upper sequent of the (cut) as follows: for each $A \in \Phi$,

$$\dfrac{K[\Gamma \to \Theta, M] \qquad K[M, A, \Delta' \to \Lambda]}{K[\Gamma, A, \Delta' \to \Theta, \Lambda]}\,(\text{cut}).$$

Since the right rank of each of these (cut)'s is lower than the right rank ρ_r of the original (cut), we can find a cut-free proof of $K[\Gamma, A, \Delta' \to \Theta, \Lambda]$ for each $A \in \Phi$ by the induction hypothesis. Then we continue

$$\dfrac{\{K[\Gamma, A, \Delta' \to \Theta, \Lambda] : A \in \Phi\}}{K[\Gamma, \bigvee \Phi, \Delta' \to \Theta, \Lambda]}\,(\bigvee \to).$$

[6.3] When (I_2) is one of the other operational inferences, we can reduce the proof into one with a lower right rank (see Gentzen [1]). In the cases of $(\to \forall)$ and $(\exists \to)$, we need Lemma 5.2.1).

[7] Suppose that (I_2) is a K-inference.

[7.1] When (I_2) is $(B\text{-}\bigwedge)$, the last part of the proof is

$$\dfrac{K[\Gamma \to \Theta, M] \quad \dfrac{\{K[M, \Delta \to \Lambda, K_i(A)] : A \in \Phi\} \quad K[K_i(\bigwedge \Phi), M, \Delta \to \Lambda]}{K[M, \Delta \to \Lambda]}\,(B\text{-}\bigwedge)}{K[\Gamma, \Delta \to \Theta, \Lambda]}\,(\text{cut}).$$

where Φ is an allowable set. Then this is reduced into

$$\dfrac{\left\{\dfrac{K[\Gamma \to \Theta, M] \quad K[M, \Delta \to \Lambda, K_i(A)]}{K[\Gamma, \Delta \to \Theta, \Lambda, K_i(A)]}\right\}_{A \in \Phi} \quad \dfrac{K[\Gamma \to \Theta, M] \quad K[M, K_i(\bigwedge \Phi), \Delta \to \Lambda]}{K[K_i(\bigwedge \Phi)), \Gamma, \Delta \to \Theta, \Lambda]}}{K[\Gamma, \Delta \to \Theta, \Lambda]}\,(B\text{-}\bigwedge),$$

where the above inferences are (cut)'s with the cut-formula M. Since these (cut)'s have lower right ranks than ρ_r by (5.5), we can eliminate the (cut)'s by the induction hypothesis.

[7.2] When (I_2) is $(K \to K)_C$, the last part is, by the assumption $\rho_\ell = 1$,

$$\dfrac{\dfrac{KK_i[\Gamma, K_i(\Xi) \to A]}{K[K_i(\Gamma, \Xi) \to K_i(A)]}\,(K \to K)_C \quad \dfrac{KK_i[\Delta, K_i(A), K_i(\Pi) \to \Lambda]}{K[K_i(A), K_i(\Delta, \Pi) \to K_i(\Lambda)]}\,(K \to K)_C}{K[K_i(\Gamma, \Xi, \Delta, \Pi) \to K_i(\Lambda)]}\,(\text{cut}).$$

The grade of this (cut) is $\gamma = \mathrm{gr}(A) + \ell + 3$. We reduce the last part into

$$\dfrac{\dfrac{\dfrac{KK_i[\Gamma, K_i(\Xi) \to A]}{KK_i[K_i(\Gamma, \Xi) \to K_i(A)]}\,(K \to K)_U \quad KK_i[K_i(A), \Delta, K_i(\Pi) \to \Lambda]}{KK_i[\Delta, K_i(\Gamma, \Xi, \Pi) \to \Lambda]}\,(\text{cut})}{K[K_i(\Gamma, \Xi, \Delta, \Pi) \to K_i(\Lambda)]}\,(K \to K)_C\,.$$

The grade of this new (cut) is $\mathrm{gr}(A) + \ell + 3$, which is the same as γ. Since this new (cut) has a lower right rank than ρ_r, we can eliminate the (cut) by the induction hypothesis.

[7.3] When (I_2) is $(K \to K)_U$, the last part is, by the assumption $\rho_\ell = 1$,

$$\dfrac{\dfrac{K[\Gamma, K_i(\Xi) \to A]}{K[K_i(\Gamma, \Xi) \to K_i(A)]}\,(K \to K)_U \quad \dfrac{K[\Delta, K_i(A), K_i(\Pi) \to \Lambda]}{K[K_i(A), K_i(\Delta, \Pi) \to K_i(\Lambda)]}\,(K \to K)_U}{K[K_i(\Gamma, \Xi, \Delta, \Pi) \to K_i(\Lambda)]}\,(\text{cut}).$$

This is reduced into

$$\dfrac{\dfrac{\dfrac{K[\Gamma, K_i(\Xi) \to A]}{K[K_i(\Gamma, \Xi) \to K_i(A)]}\,(K \to K)_U \quad K[K_i(A), \Delta, K_i(\Pi) \to \Lambda]}{K[\Delta, K_i(\Gamma, \Xi, \Pi) \to \Lambda]}\,(\text{cut}).}{K[K_i(\Gamma, \Xi, \Delta, \Pi) \to K_i(\Lambda)]}\,(K \to K)_U$$

Again, the right rank of this (cut) is lower than ρ_r, so we can eliminate the (cut) by the induction hypothesis.

5.2.3. Induction Step 3

Now we consider a proof of the form (5.1) where the grade is γ, the left rank is $\rho_\ell > 1$ and the right rank is ρ_r. In this case, the succedent of at least one upper sequent of the inference (I_1) has M. We make the induction hypothesis that

$$\begin{array}{c}\text{for any proof of the form (5.1) with the grade } \gamma,\\ \text{the left rank lower than } \rho_\ell \text{ and the right rank equal to } \rho_r,\\ \text{we can find a cut-free proof with the same endsequent.}\end{array} \tag{5.8}$$

[8] When (I_1) is (th), it is easy to reduce the proof into one with a lower left rank. By the induction hypothesis, we can eliminate the (cut).

In the following reduction steps, the outer K and the cut-formula M remain unchanged. Hence the grade of the (cut) remains the same, too.

[9] Suppose that (I_1) is an operational inference.

[9.1] When (I_1) is $(\supset\rightarrow)$, the last part of the proof is

$$\dfrac{\dfrac{K[\Gamma' \rightarrow \Theta, M, A] \quad K[B, \Gamma' \rightarrow \Theta, M]}{K[A \supset B, \Gamma' \rightarrow \Theta, M]}(\supset\rightarrow) \qquad K[M, \Delta \rightarrow \Lambda]}{K[A \supset B, \Gamma', \Delta \rightarrow \Theta, \Lambda]}(\text{cut}).$$

where Γ is $\Gamma' \cup \{A \supset B\}$ and $A \supset B \notin \Gamma'$.

[9.1.1] When neither A nor B is M, the last part is reduced into

$$\dfrac{\dfrac{K[\Gamma' \rightarrow \Theta, M, A] \quad K[M, \Delta \rightarrow \Lambda]}{K[\Gamma', \Delta \rightarrow \Theta, \Lambda, A]}(\text{cut}) \qquad \dfrac{K[B, \Gamma' \rightarrow \Theta, M] \quad K[M, \Delta \rightarrow \Lambda]}{K[B, \Gamma', \Delta \rightarrow \Theta, \Lambda]}(\text{cut})}{K[A \supset B, \Gamma', \Delta \rightarrow \Theta, \Lambda]}(\supset\rightarrow).$$

Each (cut) has a lower left rank. Hence we can eliminate each (cut) by the induction hypothesis.

[9.1.2] When A is M, the last part is reduced into

$$\dfrac{\dfrac{K[\Gamma' \rightarrow \Theta, M] \quad K[M, \Delta \rightarrow \Lambda]}{K[\Gamma', \Delta \rightarrow \Theta, \Lambda]}(\text{cut})}{K[A \supset B, \Gamma' \rightarrow \Theta, \Lambda]}(\text{th})\,.$$

Since this (cut) has a lower left rank than ρ_ℓ, we can eliminate this (cut) by the induction hypothesis.

[9.1.3] When B is M, the last part is reduced into

$$\dfrac{\dfrac{K[\Gamma' \rightarrow \Theta, M, A] \quad K[M, \Delta \rightarrow \Lambda]}{K[\Gamma', \Delta \rightarrow \Theta, \Lambda, A]}(\text{cut}) \qquad \dfrac{K[B, \Gamma' \rightarrow \Theta, M] \quad K[M, \Delta \rightarrow \Lambda]}{K[B, \Gamma', \Delta \rightarrow \Theta, \Lambda]}(\text{cut})}{K[A \supset B, \Gamma', \Delta \rightarrow \Theta, \Lambda]}(\supset\rightarrow).$$

Again the left ranks of these (cut)'s are lower than ρ_ℓ by the definition of left ranks, so we can eliminate these (cut)'s by the induction hypothesis.

[9.2] We omit the other cases of operational inferences for (I_1). Note that Lemma 5.2.1) is needed for the cases of $(\to \forall)$ and $(\exists \to)$.

[10] Suppose that (I_1) is a K-Inference.

[10.1] The inference (I_1) can be neither $(K \to K)_C$ nor $(K \to K)_U$, since $\rho_\ell > 1$.

[10.2] When (I_1) is $(B\text{-}\bigwedge)$, the proof is

$$\frac{\dfrac{\{K[\Gamma \to \Theta, M, K_i(A)] : A \in \Phi\} \quad K[K_i(\bigwedge \Phi), \Gamma \to \Theta, M]}{K[\Gamma \to \Theta, M]}\,(B\text{-}\bigwedge) \qquad K[M, \Delta \to \Lambda]}{K[\Gamma, \Delta \to \Theta, \Lambda]}\,(\text{cut}),$$

where Φ is an allowable set. This is reduced into

$$\frac{\left\{\dfrac{K[\Gamma \to \Theta, K_i(A), M] \quad K[M, \Delta \to \Lambda]}{K[\Gamma, \Delta \to \Theta, \Lambda, K_i(A)]}\right\}_{A \in \Phi} \qquad \dfrac{K[K_i(\bigwedge \Phi), \Gamma \to \Theta, M] \quad K[M, \Delta \to \Lambda]}{K[K_i(\bigwedge \Phi), \Gamma, \Delta \to \Theta, \Lambda]}}{K[\Gamma, \Delta \to \Theta, \Lambda]}\,(B\text{-}\bigwedge),$$

where the above inferences are (cut)'s with the cut-formula M. Since each (cut) has a lower left rank than ρ_ℓ, we can eliminate these (cut)'s. □

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