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Abstract: This paper addresses ranked voting systems to determine an ordering of candidates in terms of the aggregate vote by rank for each candidate. It is shown that specifying nothing arbitrary, we can obtain a total ordering of candidates by using DEA/AR (Data Envelopment Analysis/ Assurance Region) exclusion model. Explaining the evaluation criterion used to rank candidates, it is concluded that we may consider the system proposed at least as an alternative to determine an ordering of candidates.

Keywords: Data envelopment analysis; Ranked voting system; DEA/ assurance region analysis; DEA exclusion model

1. Ranked voting systems

This paper considers ranked voting systems in which each voter selects and ranks the top t candidates. It is assumed that there are no ties in each voter's ranking. The problem is to determine an ordering of all n candidates by obtaining a total score $s_j = \sum_{r=1}^{t} u_r y_{rj}$ for each candidate $j = 1, ..., n$, where y_{rj} is the number of rth place votes candidate j receives; u_r, $r = 1, ..., t$, is the sequence of weights given to rth place vote. Here, we assume only knowledge of the aggregate vote by rank for each candidate. Because of no established ways to determine the weights, many arbitrary choices of the sequence of weights can exist. The well known Borda method, $u_r = t - r + 1$, $r = 1, ..., t$, is an example.

Since any sequence of weights to be used in ranked voting systems cannot but be somewhat arbitrary, it is desirable to obtain an ordering of candidates without specifying the weights. Stein *et al.* (1994) showed that the stochastic dominance can give a *partial* ordering of candidates not specifying the weights but assuming the condition of *decreasing and convex* sequence of weights:

$$u_r > u_{r+1} > 0, \quad r = 1, ..., t-1, \tag{1.1}$$

$$u_r - u_{r+1} \geq u_{r+1} - u_{r+2}, \quad r = 1, ..., t-2. \tag{1.2}$$

However, in order to obtain a *total* ordering of candidates, we need to specify the sequence of weights satisfying this condition arbitrarily. [Note that the Borda method satisfies the decreasing and convex condition (1.1)–(1.2).]

Cook and Kress (1990) considered an alternative method not specifying the sequence of weights by using *DEA (Data Envelopment Analysis)* (Charnes *et al.*, 1978; Boussofiane *et al.*, 1991). Here, the candidates in ranked voting systems are regarded as *DMUs (Decision Making Units)* in DEA, and each DMU is considered to have t *outputs* (ranked votes) and only one *input* with amount unity (Hashimoto, 1993) [i.e., the *pure output* DEA model (Adolphson *et al.*, 1991)]. This method uses the *DEA/AR (DEA/ Assurance Region)* model (Thompson *et al.*, 1986; Hashimoto and Ishikawa, 1993) as follows:

$$\text{Maximize} \quad h_{j_0} = \sum_{r=1}^{t} u_r y_{rj_0} \tag{1.3}$$

$$\text{subject to} \quad \sum_{r=1}^{t} u_r y_{rj} \leq 1, \quad j = 1, ..., n, \tag{1.4}$$

$$u_r - u_{r+1} \geq d(r, \delta), \quad r = 1, ..., t-1, \tag{1.5}$$

$$u_t \geq d(t, \delta), \tag{1.6}$$

where $d(r, \delta)$ is a positive function implying the minimum gap between successively ranked weights, the discrimination intensity function, and nondecreasing in $\delta > 0$. This problem is solved for each candidate $j_0 = 1, ..., n$ based on the consideration that it is fair to evaluate each candidate in terms of the weights optimal to himself/herself [see also Hashimoto (1993)].

Specifying the $d(r, \delta)$ function and solving problem (1.3)–(1.6), we obtain a total score, the maximum h_{j0}^*, for each candidate. However, several candidates are tied for the first place as *DEA/AR efficient* DMUs ($h_{j0}^* = 1$). This method seeks to discriminate these top candidates by maximizing the $d(r, \delta)$ function subject to the condition that they remain DEA/AR efficient. The ordering of candidates can vary by the $d(r, \delta)$ function. Therefore, in order to determine a total ordering of candidates, we also need to specify the $d(r, \delta)$ function arbitrarily. In the case that $d(r, \delta) = \delta$, this method is equivalent to the Borda method. [See also Cook *et al.* (1992).]

Based on the considerations above, this paper proposes a method to determine a total ordering of candidates specifying nothing arbitrary but assuming only the condition of decreasing and convex sequence of weights. Because the decreasing and convex sequence of weights is considered as one that most people would regard fair, and ranked voting systems virtually always use this type of weights (Stein *et al.*, 1994). It might be considered that the Borda method has had its share of the method to determine an ordering of candidates in ranked voting systems because it uses a decreasing and convex sequence of weights. [If we assume the condition of decreasing sequence of weights (not necessarily convex), we may consider neglecting inequality (1.2).]

2. Proposing the use of DEA/AR exclusion model

This paper proposes to use DEA agreeing with Cook and Kress (1990) that ranked voting systems may allow for flexibility in the assignment of weights from one candidate to another. We incorporate the condition of decreasing and convex sequence of weights into DEA as the assurance region. However, we do not use the DEA/AR model like Cook and Kress (1990) but here use the DEA/AR *exclusion* model.

Andersen and Petersen (1993) proposed the DEA exclusion model as one that discriminates DEA efficient DMUs. In this model, the DMU being evaluated is excluded from the comparison set (Adolphson *et al.*, 1991). Applying this to ranked voting systems as a DEA/AR exclusion model, the formulation is as follows:

$$\text{Maximize} \quad g_{j_0} = \sum_{r=1}^{t} u_r y_{rj_0} \tag{2.1}$$

$$\text{subject to} \quad \sum_{r=1}^{t} u_r y_{rj} \le 1, \quad j = 1, \ldots, n,\ j \ne j_0, \tag{2.2}$$

$$u_r - u_{r+1} \ge \varepsilon, \quad r = 1, \ldots, t-1, \tag{2.3}$$

$$u_t \ge \varepsilon, \tag{2.4}$$

$$u_r - 2u_{r+1} + u_{r+2} \ge 0, \quad r = 1, \ldots, t-2, \tag{2.5}$$

where ε is a positive non-Archimedean infinitesimal.

Constraint set (2.3)–(2.5), the assurance region, is equivalent to condition (1.1)–(1.2), and constraint (2.2) is different from (1.4) by that candidate j_0 being evaluated is not included in (2.2). Note that unlike the standard DEA model, the exclusion model allows DEA efficiency scores to exceed unity. Solving problem (2.1)–(2.5) for each candidate $j_0 = 1, \ldots, n$, the ties in the first place in the DEA/AR model are broken because the DEA/AR exclusion model discriminates DEA/AR efficient DMUs, so that we can obtain a total ordering of candidates in accordance with the DEA/AR exclusion scores g_{j0}^*. Therefore, we need not specify the $d(r, \delta)$ function in model (1.3)–(1.6) nor the sequence of weights.

We can certainly break the ties in the first place only by using the DEA/AR exclusion model instead of the DEA/AR model. However, in order to employ this method as one to determine a total ordering of candidates, we should explain the criterion based on which we evaluate and discriminate DEA/AR efficient candidates.

For interpretations of the DEA exclusion model, Andersen and Petersen (1993) state for the dual problem of one corresponding to model (2.1)–(2.5) as follows: DEA inefficient DMUs are assigned the same efficiency scores as the standard DEA model. These scores may be interpreted as the reciprocals of the minimum proportional increase in outputs yielding DEA efficiency. DEA efficient DMUs are assigned the DEA exclusion scores greater than or equal to unity. These scores may be interpreted as the reciprocals of the maximum proportional decrease in outputs preserving DEA efficiency. [These statements are converted in accordance with the current pure output model since Andersen and Petersen (1993) explain for the *input oriented* model.]

In the case of the ranked voting system using DEA/AR exclusion model, we can likewise state for the dual of model (2.1)–(2.5) that the DEA/AR exclusion scores greater than or equal to unity may be interpreted as the reciprocals of the maximum proportional decrease in votes preserving the top rank. This seems to be one criterion to evaluate the candidates tied for the first place as DEA/AR efficient DMUs.

Additionally, for the primal problem (2.1)–(2.5) itself, we can interpret the ranked voting system using DEA/AR exclusion model as follows: The DEA/AR exclusion score less than unity is of course the index how far the candidate being evaluated can approach to the top when the score of the top is supposed to be unity. On the other hand, the DEA/AR exclusion score greater than or equal to unity is the index how far the candidate being evaluated can have a lead on the candidate in the second place when the score of the second place is supposed to be unity. This interpretation seems to be more appropriate to an evaluation criterion in the ranked voting system.

Andersen and Petersen (1993) state the shortcoming of the DEA exclusion model that specialized DMUs are ranked too high because the maximum proportional decrease in the output vector preserving efficiency in some instances is determined in a subspace of the output space. This can be resolved in the DEA/AR exclusion model because the weights assigned for the outputs take some *a priori* conditions into consideration as the assurance region. Rather, we should use the exclusion model not as the DEA exclusion model but as the DEA/AR exclusion model like the ranked voting system considered here. This strengthens appropriateness of the ranked voting system using DEA/AR exclusion model.

Table 1 shows a DEA/AR exclusion ordering of candidates. Since candidates A and B have the DEA/AR exclusion scores greater than unity, they would have been tied for the first place if we use the DEA/AR model. We should note that a total ordering of candidates different from the Borda one is obtained specifying nothing arbitrary.

3. Summary and conclusions

This paper considered a ranked voting system using DEA/AR exclusion model. In this system, we could obtain a total ordering of candidates specifying nothing arbitrary but only assuming the condition of decreasing and convex sequence of weights. The evaluation criterion used to rank candidates is how far to approach to the top or how far to have a lead on the second place. Although we cannot say this is the best, it seems to be appropriate to one evaluation criterion. Therefore, it is considered that this ranked voting system may at least have its share of the method to determine an ordering of candidates.

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Table 1

An example of the DEA/AR exclusion ordering[†]

	Rank					Borda		DEA/AR exclusion	
Candidate	1	2	3	4	5	Score	Order	Score[‡]	Order
A	27	38	15	7	4	350	1	1.0750	2
B	38	15	16	14	7	333	2	1.4074	1
C	21	25	30	8	5	316	3	0.9780	3
D	2	8	10	23	19	137	4	0.6813	4
E	1	1	11	22	24	110	5	0.6484	5
F	2	4	4	7	13	65	6	0.3297	6
G	1	-	4	7	13	44	7	0.2747	7
H	-	1	1	-	-	7	8	0.0220	11
I	-	-	1	1	1	6	9	0.0330 ($-$ 5.5385ε)	8
J	-	-	-	1	2	4	10	0.0330 ($-$ 7.5385ε)	9
K	-	-	-	1	2	4	10	0.0330 ($-$ 7.5385ε)	9
L	-	-	-	1	-	2	12	0.0110 ($-$ 1.8462ε)	12
M	-	-	-	-	1	1	13	0.0110 ($-$ 2.8462ε)	13
N	-	-	-	-	1	1	13	0.0110 ($-$ 2.8462ε)	13

† Data of the aggregate votes are quoted from Stein *et al.* (1994).

‡ ε = a positive non-Archimedean infinitesimal. In terms of the two-phased method to solve the dual of model (2.1)-(2.5). The values of infinitesimal term are not computed unless necessary.