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Global Optimization Problem with Several Reverse Convex Constraints and its Application to Out-of-Roundness Problem

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Abstract We consider a global minimization problem: $\min\{\ c^{\mathsf T}x + d^{\mathsf T}y \mid x \in X,\ y \in Y,\ (x,y) \in Z,\ y \in R^{n_2} \setminus \cup_{h=1}^{m} G_h\ \}$, where X and Y are polytopes in $R_+^{n_1}$ and $R_+^{n_2}$, respectively; Z is a closed convex set in $R^{n_1+n_2}$; and G_h $(h = 1, \ldots, m)$ is an open convex set in R^{n_2}. We propose an algorithm based on a combination of polyhedral outer approximation, branch-and-bound and cutting plane techniques. We also show that an out-of-roundness problem can be solved by the algorithm.

Keywords: global optimization, reverse convex constraint, polyhedral outer approximation, cutting plane, branch-and-bound, out-of-roundness problem

1 Introduction

In this paper we consider the following minimization problem:

$$
(Q) \qquad \left|\begin{array}{ll} \min & c^{\mathsf T}x + d^{\mathsf T}y \\ \text{s.t.} & x \in X,\ y \in Y, \\ & (x,y) \in Z,\ y \in R^{n_2} \setminus \displaystyle\bigcup_{h=1}^{m} G_h, \end{array}\right.
$$

where X and Y are polytopes in $R_+^{n_1}$ and $R_+^{n_2}$, respectively; Z is a closed convex set in $R^{n_1+n_2}$; and G_h $(h = 1, \ldots, m)$ is an open convex set in R^{n_2}; the vectors c and

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d are in R^{n_1} and R^{n_2}, respectively. As in many applications the constraints of (Q) are usually given as a system of inequalities, we assume in this paper that

$$
\begin{aligned}
\Omega &= \{\, (x,y) \mid x \in X,\ y \in Y,\ (x,y) \in Z \} \\
&= \{\, (x,y) \mid \phi_i(x,y) \le 0,\ i = 1,\dots,k \}, \\
G_h &= \{\, y \mid g_h(y) < 0 \,\},\ h = 1,\dots,m, \\
G &= \bigcup_{h=1}^{m} G_h, \\
D &= \{\, (x,y) \mid (x,y) \in \Omega,\ y \notin G \,\}.
\end{aligned} \tag{1.1}
$$

where ϕ_i $(i = 1,\dots,k)$ and g_h $(h = 1,\dots,m)$ are convex functions. The constraint $g_h \ge 0$ is often called *reverse convex constraint* (see Horst and Tuy [6]). By setting

$$\phi(x,y) = \max_{i=1,\dots,k} \phi_i(x,y),$$

Problem (Q) is equivalent to the following *noncanonical d.c. problem:*

$$
(Q) \qquad \left|\ \begin{array}{ll} \min & c^{\mathsf{T}}x + d^{\mathsf{T}}y \\ \text{s.t.} & \phi(x,y) \le 0, \\ & g_h(y) \ge 0,\ h = 1,\dots,m. \end{array}\right.
$$

Note that $\phi(\cdot)$ is a convex function.

Problem (Q) includes several important classes of global optimization problems, such as a special d.c. programming problem, a class of problems with multiplicative terms [4]. Moreover it certainly contains the *canonical d.c. programming.*

Recently several algorithms [2, 4, 7] are proposed for solving a special case of (Q) in which only one additional reverse convex constraint is considered. Since a set of reverse convex constraints can not be represented by a single reverse convex constraint, their algorithm is not applicable to Problem (Q). The general branch-and-bound algorithm is a sole method for solving Problem (Q) (section X.2 in [6]), which does not make use of the structure of the problem. Since Problem (Q) possesses a special structure that the reverse convex constraints are defined only on the y-space, we devise an algorithm which takes advantage of the specific structure. Moreover, from practical point of view, it is useful to combine several basic optimization techniques for problems of medium and large sizes [3, 5].

Our new algorithm is based mainly on a combination of polyhedral outer approximation method and conical branch-and-bound in which the partition is made only in the y-space. Since only linear programming problems are solved in each step, it should not be costly to determine a solution for subproblem even the feasible region is changed by adding a new cut. Therefore we can incorporate the cutting plane method whenever a cut is available. The algorithm can be regarded as a generalization of the first algorithm proposed in [4].

To use polyhedral outer approximation and conical subdivision we assume that

(A1) $\operatorname{int} \Omega \neq \emptyset$.

(A2) $\bigcap_{h=1}^{m} G_h \neq \emptyset$ and a point $y^0 \in \bigcap_{h=1}^{m} G_h$ is available.

Furthermore we require that the reverse convex constraints are essential, i.e.,

(A3) there exists a point $(x^\star, y^\star)$ such that $(x^\star, y^\star) \in \Omega$, $y^\star \in G$ and $c^\top y^\star + d^\top y^\star < c^\top x + d^\top y$ for any $(x, y) \in D$.

The remainder of this paper is organized as follows. Section 2 describes a partition of D based on a conical partition of R^{n_2}. We also show how to find the lower and upper bounds in Section 2. Section 3 gives the algorithm and proves its convergence. In Section 4 we show that the out-of-roundness problem [1, 8] can be formulated as Problem (Q) and describe the details of the method for obtaining the lower and upper bounds for this specific problem.

2 The branching and bounding operations

We establish a subdivision underlying the branch-and-bound algorithm in this paper. Due to the special structure of the problem we first subdivide the subspace R^{n_2} and then build the subdivision on the whole space $R^{n_1+n_2}$. We use a conical partition as the subdivision of the subspace R^{n_2}. The bounding operations are carried out by solving a series of linear programming problems.

2.1 A conical partition

Let y^0, z^i $(i = 1, ..., n_2)$ be $n_2 + 1$ affinely independent points of R^{n_2}. We call the convex polyhedral cone $\{ y \in R^{n_2} \,|\, y = \sum_{i=1}^{n_2} \lambda_i(z^i - y^0) + y^0, \lambda_i \geq 0 \}$ the cone generated by points $y^0, z^1, ..., z^{n_2}$. The cone has exactly n_2 edges emanating from point y^0. We denote it by C_{y^0}. Without loss of generality, we assume that all z^i are on the ball $B_1(y^0) = \{ z \in R^{n_2} \,|\, \| z - y^0 \| = 1 \}$.

A collection $\mathcal{D}$ of finitely many cones $\{ C_1, ..., C_t \}$ defined as above is called a *conical partition* of R^{n_2} if $\cup_{j=1}^{t} C_j = R^{n_2}$ and $\operatorname{int} C_i \cap \operatorname{int} C_j = \emptyset$ for $i \neq j$. We call a collection $\{ T_1, \ldots, T_t \}$, where $T_j = R^{n_1} \times C_j$, a C_{y^0}*-partition* of $R^{n_1+n_2}$. The set T_j is called a C_{y^0}*-partition set.*

Let $\mathcal{D}_k$ be a given conical partition of R^{n_2}. A conical partition $\mathcal{D}_{k+1}$ of R^{n_2} is said to be a *refinement* of $\mathcal{D}_k$ if for any $C_j \in \mathcal{D}_{k+1}$ there exists a cone $C_{j'} \in \mathcal{D}_k$ such that $C_j \subseteq C_{j'}$. For each cone $C \in \mathcal{D}_k$, the collection $\{ C_1, ..., C_r \}$ of cones of $\mathcal{D}_{k+1}$ is said to be a conical partition of C if $\cup_{j=1}^{r} C_j = C$. Similarly, we call the collection $\{ R^{n_1} \times C_1, \ldots, R^{n_1} \times C_r \}$ a C_{y^0}-partition of $T = R^{n_1} \times C$.

In our algorithm we repeatedly refine the conical partition of R^{n_2} to yield $\{ \mathcal{D}_k \}_{k=1,2,...}$, a sequence of conical partitions. The refinement process is called *exhaustive* if for every strictly nested sequence $\{ C_k \}$ satisfying $C_k \in \mathcal{D}_k$ and $C_{k+1} \subset C_k$ for every k, there exists a vector $\bar{z}$ on $B_1(y^0)$ such that

$$\lim_{k \to \infty} z_k^i = \bar{z} \quad \text{for all} \quad i = 1, ..., n_2.$$

2.2 The lower and upper bounds

Given a C_{y^0}-partition $\{ T_1, \ldots, T_t \}$ of $R^{n_1+n_2}$, we consider how to compute a lower bound L_j of $c^\top x + d^\top y$ over $(D \cap P) \cap T_j = (D \cap P) \cap (R^{n_1} \times C_j)$ for $j = 1, \ldots, n_2$, where D is the feasible region of Problem (Q) defined by (1.2), and P is a polytope containing the optimal solution of (Q). For the sake of brevity, we omit the subscript j and let $T = R^{n_1} \times C$ denote a C_{y^0}-partition set of $\{ T_1, ..., T_t \}$ throughout this and next subsections.

Assume that the polytope P is defined by the following system of inequalities:

$$Ax + By \leq b,$$

where A, B, and b are matrices and a vector of appropriate sizes. Note that P does not necessarily contain the whole set Ω. Denoting $D_P = D \cap P$, we propose a procedure for calculating the lower bound L of the objective function over the set $D_P \cap T$.

For the cone C, define for every G_h $(h = 1, \ldots, m)$ a set of n_2 points

$$v^{i,h} = y^0 + \theta_i^h(z^i - y^0) \quad \text{with} \quad \theta_i^h = \min\{\,\bar{\theta}_{big}, \bar{\theta}\,\},\ i = 1, ..., n_2, \tag{2.1}$$

where $\bar{\theta}_{big}$ is a given sufficient large positive number, and $\bar{\theta} = \sup\{\,\theta \,|\, y^0 + \theta(z^i - y^0) \in G_h\,\}$. We denote by θ^h the n_2-dimensional vector $(\theta_1^h, ..., \theta_{n_2}^h)^\top$, U^h the $n_2 \times n_2$-matrix $(v^{1,h} - y^0, ..., v^{n_2,h} - y^0)$. Define a half space H^h in R^{n_2} as

$$\begin{aligned} H^h &= \{\, y \in R^{n_2} \,|\, e^\top (U^h)^{-1}(y - y^0) \geq 1\,\} \\ &= \{\, y \in R^{n_2} \,|\, y = y^0 + U^h\lambda^h, e^\top\lambda^h \geq 1\,\}, \end{aligned}$$

where $e = (1, ..., 1)^\top$. Then the intersection I^h of H^h and the cone C is written as

$$\begin{aligned} I^h &= H^h \cap C \\ &= \{\, y \in R^{n_2} \,|\, y = y^0 + U^h\lambda^h, e^\top\lambda^h \geq 1, \lambda^h \geq 0\,\}. \end{aligned} \tag{2.2}$$

We see that for every point $(x, y) \in P \cap (\,R^{n_1} \times \cap_{h=1}^m I^h)$, there exist nonnegative vectors $\lambda^h, \lambda^h = (\lambda_1^h, ..., \lambda_{n_2}^h)^\top$ $(h = 1, ..., m)$, such that $e^\top\lambda^h \geq 1$ and

$$(x, y) = (0^{n_1}, y^0) + (x, U^h\lambda^h). \tag{2.3}$$

Lemma 2.1 *For every $(x, y) \in P \cap (\,R^{n_1} \times \cap_{h=1}^m I^h\,)$, λ^h in (2.3) is bounded for $h = 1, ..., m$.*

Proof. Let h be an arbitrary index of $\{1, ..., m\}$. Note that $v^{i,h} - y^0 \neq 0$ for all i since y^0 is in the open set G_h, and that they are linearly independent. From (2.3) point y with $(x, y) \in P \cap (\,R^{n_1} \times \cap_{h=1}^m I^h\,)$ is written as

$$y = y^0 + U^h\lambda^h,$$

that is

$$\lambda^h = (U^h)^{-1}(y - y^0).$$

Then we obtain

$$\| \lambda^h \| \leq \| (U^h)^{-1} \| \| y - y^0 \|,$$

which is bounded since y is in the polytope Y. □

Lemma 2.2 *For $h_1, h_2 \in \{1, \ldots, m\}$ if $\theta^{h_1} \geq \theta^{h_2}$, then $I^{h_1} \subseteq I^{h_2}$.*

Proof. For $y \in I^{h_1}$, there exists a vector $\lambda^{h_1} \geq 0$ such that $e^\top \lambda^{h_1} \geq 1$ and

$$\begin{aligned} y &= y^0 + U^{h_1}\lambda^{h_1} \\ &= y^0 + (v^{1,h_1} - y^0, \ldots, v^{n_2,h_1} - y^0)(\lambda_1^{h_1}, \ldots, \lambda_{n_2}^{h_1})^\top \\ &= y^0 + [(z^1 - y^0)\theta_1^{h_1}, \ldots, (z^{n_2} - y^0)\theta_{n_2}^{h_1}](\lambda_1^{h_1}, \ldots, \lambda_{n_2}^{h_1})^\top. \end{aligned}$$

Let $t_i = \theta_i^{h_1}/\theta_i^{h_2}$, which is well defined by $\theta_i^{h_2} > 0$, then $t_i \geq 1$ and we obtain

$$\begin{aligned} y &= y^0 + [(z^1 - y^0)\theta_1^{h_2}t_1, \ldots, (z^{n_2} - y^0)\theta_{n_2}^{h_2}t_{n_2}]\,(\lambda_1^{h_1}, \ldots, \lambda_{n_2}^{h_1})^\top \\ &= y^0 + [(z^1 - y^0)\theta_1^{h_2}, \ldots, (z^{n_2} - y^0)\theta_{n_2}^{h_2}](t_1\lambda_1^{h_1}, \ldots, t_{n_2}\lambda_{n_2}^{h_1})^\top. \end{aligned}$$

Note that $t_i\lambda_i^{h_1} \geq 0$ for all i and $\sum t_i\lambda_i^{h_1} \geq 1$. This means $y \in I^{h_2}$. □

Let

$$L = \min\{\, c^\top x + d^\top y \,|\, (x, y) \in P \cap (\, R^{n_1} \times \cap_{h=1}^m I^h \,)\,\}.$$

By the definition of I^h in (2.2), we have

$$\begin{aligned} L = \min\{\, c^\top x + d^\top y \,|\; & Ax + By \leq b, \\ & y = y^0 + U^h\lambda^h, \;\; h = 1, \ldots, m, \\ & e^\top\lambda^h \geq 1, \; \lambda^h \geq 0, \;\; h = 1, \ldots, m\,\}. \end{aligned} \tag{2.4}$$

The following lemma shows that L is a lower bound of $c^\top x + d^\top y$ over the set $D_P \cap T$.

Lemma 2.3

(i) If $P \cap (R^{n_1} \times \cap_{h=1}^m I^h)$ is empty, then the optimal solution of (Q) is not in $D_P \cap T$.

(ii) If $P \cap (R^{n_1} \times \cap_{h=1}^{m} I^h)$ is not empty, then

$$L \leq \min\{\, c^\top x + d^\top y \,|\, (x, y) \in D_P \cap T \,\}.$$

Proof. From the definitions of I^h and G, we see that $\{\, y \,|\, y \notin G \,\} \cap C \subseteq \cap_{h=1}^{m} I^h$, then $D_P \cap T \subseteq P \cap (R^{n_1} \times \cap_{h=1}^{m} I^h)$. □

Note that if $P \cap (R^{n_1} \times \cap_{h=1}^{m} I^h) = \emptyset$ then we define $L = +\infty$. From the above lemma, it seems that we have to solve a linear program with a lot of linear constraints when m is large. However from Lemma 2.2, it is likely that we can remove many of such constraints of I^{h_1} in computing (2.4).

The following lemma is derived from Assumption (A3).

Lemma 2.4 *Let (x^*, y^*) be a global optimal solution of (Q), then $y^* \in R^{n_2}$ is on the boundary of G.*

Proof. Suppose that the optimal solution (x^*, y^*) of (Q) is not on the boundary of G. Then $g_h(y^*) > 0$ for any $h \in \{1, \ldots, m\}$. Let $(x(\lambda), y(\lambda)) = (\lambda(x^\star, y^\star) + (1 - \lambda)(x^*, y^*))$ for $(x^\star, y^\star)$ of Assumption (A3). Then for any $\lambda \in (0, 1]$

$$c^\top x(\lambda) + d^\top y(\lambda) < c^\top x^* + d^\top y^*.$$

Therefore $c^\top x(\bar{\lambda}) + d^\top y(\bar{\lambda}) < c^\top x^* + d^\top y^*$ for some $\bar{\lambda} \in (0, 1]$ and $g_h(y(\bar{\lambda})) > 0$ for all h. By the convexity of Ω, we also see that $(x(\bar{\lambda}), y(\bar{\lambda})) \in \Omega$. It implies that $(x(\bar{\lambda}), y(\bar{\lambda}))$ is a feasible solution of (Q). It is a contradiction. □

After solving the linear programming problem of (2.4) we obtain an optimal solution $(\bar{x}, \bar{y})$ and the corresponding objective function value L. If the point $(\bar{x}, \bar{y})$ lies in the feasible region D_P, it is an optimal solution of $\min\{\, c^\top x + d^\top y \,|\, (x, y) \in D_P \cap T \,\}$. Then the currently considered C_{y^0}-partition set T is not necessary to be subdivided further. Moreover, L serves as an upper bound of the optimal value of Problem (Q).

If $(\bar{x}, \bar{y})$ is not in D_P, then we possibly find a feasible point of (Q) by moving from $(\bar{x}, \bar{y})$ along some specific direction. A possible choice of the direction is (c, d). Define a point $(\hat{x}, \hat{y})$ by

(2.5) $\quad (\hat{x}, \hat{y}) = \hat{\tau}(c, d) + (\bar{x}, \bar{y})$ with $\hat{\tau} = \min\{\, \bar{\theta}_{big},\ \sup\{\, \tau \,|\, \tau d + \bar{y} \in G \,\}\,\}.$

If $(\hat{x},\hat{y}) \in \Omega$, then the value $c^\top \hat{x} + d^\top \hat{y}$ is an upper bound of the optimal value of (Q). If $(\hat{x},\hat{y}) \notin \Omega$, we fix $\hat{y}$ and search a test point $(\tilde{x},\hat{y})$ defined by

$$(2.6) \qquad \tilde{x} = \hat{x} + \hat{\lambda}(\ddot{x} - \hat{x}) \text{ with } \hat{\lambda} = \min\{\, 1, \sup\{\, \lambda \,|\, (\hat{x} + \lambda(\ddot{x} - \hat{x}), \hat{y}) \notin \Omega \,\}\,\},$$

where

$$(2.7) \qquad \ddot{x} = \arg\max\{\, c^\top x \,|\, Ax \leq b - B\hat{y} \,\}.$$

Lemma 2.5 *If $\hat{\lambda} < 1$ then $(\tilde{x},\hat{y}) \in \Omega$.*

Proof. If any point $(x,\hat{y})$ on the line segment $(\,(\hat{x},\hat{y}),(\ddot{x},\hat{y})\,)$ is not in Ω, then $\sup\{\, \lambda \,|\, (\hat{x} + \lambda(\ddot{x} - \hat{x}), \hat{y}) \notin \Omega \,\} \geq 1$, i.e., $\hat{\lambda} = 1$. □

2.3 Polyhedral outer approximation and cutting plane

At the beginning of the algorithm, we take the polytope $X \times Y$ as an initial polytope P_1 containing Ω. The algorithm generates a sequence of polytopes $\{\, P_k \,|\, k = 1,2,...\}$ such that $P_1 \supset P_2 \cdots$ and each P_k contains an optimal solution of (Q).

At the kth iteration, we construct a C_{y^0}-partition over some set T chosen in the (k-1)st iteration. By solving linear programming problem (2.4) for all sets in the partition, we obtain a sequence of lower bounds. We also obtain several, possibly no, feasible points of (Q), which are generated by solving (2.4) or by (2.5)-(2.7). After bounding operations (see section 3 for the details) we choose a point with minimal lower bound to obtain a point $(\bar{x}_k, \bar{y}_k)$, which is an optimal solution of (2.4) for some C_{y^0}-partition set. If we find some feasible points, then choose one of them, say $(\dot{x},\dot{y})$ having the smallest objective function value. We can take the inequality

$$(2.8) \qquad c^\top x + d^\top y \leq c^\top \dot{x} + d^\top \dot{y}$$

as a cutting plane if the value $c^\top \dot{x} + d^\top \dot{y}$ is less than the current upper bound. Adding (2.8) to the constraints of P_k will not cut off the optimal solution of (Q). Moreover, if $(\bar{x}_k, \bar{y}_k) \notin \Omega$, compute a subgradient $s_\phi(\bar{x}_k, \bar{y}_k)$ of ϕ at $(\bar{x}_k, \bar{y}_k)$ and let

$$(2.9) \qquad l_k(x,y) = [\,(x,y) - (\bar{x}_k, \bar{y}_k)\,] s_\phi(\bar{x}_k, \bar{y}_k) + \phi(\bar{x}_k, \bar{y}_k).$$

Then the inequality

$$l_k(x,y) \le 0 \tag{2.10}$$

will cut off the point $(\bar{x}_k, \bar{y}_k)$ but no any feasible points of (Q) in P_k. Therefore we can define the polytope P_{k+1} for the next iteration by

$$P_{k+1} = P_k \cap \{\, (x,y) \,|\, l_k(x,y) \le 0,\ c^\top x + d^\top y \le c^\top \dot{x} + d^\top \dot{y} \,\},$$

However, on the situation that only one of the cutting planes (2.8) and (2.10) or no cutting planes can be constructed, P_{k+1} is defined by adding the corresponding cutting plane to P_k or keep $P_{k+1} = P_k$, respectively.

3 The Algorithm

Based on the above discussion we propose an algorithm for solving Problem (Q) as follows.

Algorithm $\mathcal{GO}$

begin

  Construct a polytope $P_1 : P_1 \supseteq \Omega$ and a C_{y^0}-conical partition $\mathcal{D}$ of R^{n_2};

  $\mathcal{M}_1 := \mathcal{D}$; $\gamma := +\infty$; $k := 1$;

  while $\mathcal{M}_k \neq \emptyset$ **do**

    begin

      for each $C \in \mathcal{M}_k$ **do**

        begin

          Solve linear program (2.4);

          $(\bar{x}(C), \bar{y}(C)) :=$ the optimal solution; $L(C) := c^\top \bar{x}(C) + d^\top \bar{y}(C)$;

          if $(\bar{x}(C), \bar{y}(C)) \in D$ **and** $\gamma > L(C)$ **then**

            begin

              $\gamma := L(C)$; $(\dot{x}, \dot{y}) := (\bar{x}(C), \bar{y}(C))$

            end

          else

            begin

Compute $(\hat{x}, \hat{y})$ by (2.5);
if $(\hat{x}, \hat{y}) \in D$ **and** $\gamma > c^\top \hat{x} + d^\top \hat{y}$ **then**
  begin
    $\gamma := c^\top \hat{x} + d^\top \hat{y}$; $(\dot{x}, \dot{y}) := (\hat{x}, \hat{y})$
  end
else
  begin
    Compute $(\tilde{x}, \hat{y})$ by (2.6) with (2.7);
    if $(\tilde{x}, \hat{y}) \in D$ **and** $\gamma > c^\top \tilde{x} + d^\top \hat{y}$ **then**
    begin
      $\gamma := c^\top \tilde{x} + d^\top \hat{y}$; $(\dot{x}, \dot{y}) := (\tilde{x}, \hat{y})$
    end
  end
end
end;
$\mathcal{M}_{k+1} := \{ C \in \mathcal{M}_k \,|\, L(C) < \gamma \}$;
$L := \min\{ L(C) \,|\, C \in \mathcal{M}_{k+1} \}$;
Choose a set $C \in \mathcal{M}_{k+1}$ satisfying $c^\top \bar{x}(C) + d^\top \bar{y}(C) = L$;
$C_k := C$; $(\bar{x}_k, \bar{y}_k) := (\bar{x}(C_k), \bar{y}(C_k))$;
if γ is updated **then** $P_{k+1} := P_k \bigcap \{ (x, y) \,|\, c^\top x + d^\top y \le \gamma \}$
else $P_{k+1} := P_k$;
if $(\bar{x}_k, \bar{y}_k) \notin \Omega$ **then**
  begin
    $l_k := [\,(x, y) - (\bar{x}_k, \bar{y}_k)\,] s_\phi(\bar{x}_k, \bar{y}_k) + \phi(\bar{x}_k, \bar{y}_k) \le 0$;
    $P_{k+1} := P_{k+1} \bigcap \{ (x, y) \,|\, l_k(x, y) \le 0 \}$
  end
Construct a C_{y^0}-conical partition $\mathcal{C}$ of C_k;
$\mathcal{M}_{k+1} := \mathcal{M}_{k+1} \setminus \{ C_k \} \bigcup \mathcal{C}$; $k := k + 1$
end;
if $\gamma := +\infty$ **then** **writeln**(' The problem is infeasible ')

else **writeln**(' The solution is ', $(\dot{x}, \dot{y})$)

end.

3.1 The proof of the convergence of the algorithm

If the algorithm terminates within a finite number of iterations, then clearly we obtain an optimal solution of Problem (Q). Otherwise an infinite sequence $\{ (\bar{x}_k, \bar{y}_k) \}$ is generated. Let (x^*, y^*) be a cluster point of the sequence. There exists a subsequence of $\{ (\bar{x}_k, \bar{y}_k) \}$ converging to (x^*, y^*). Since the conical partition of a cone consists of finitely many cones, there exists at least one cone containing infinitely many points $(\bar{x}_k, \bar{y}_k)$. Therefore we can always choose a subsequence $\{ (\bar{x}_{k_q}, \bar{y}_{k_q}) \}$ of the above subsequence such that $(\bar{x}_{k_q}, \bar{y}_{k_q}) \in C_{k_q}$ and $\{ C_{k_q} \}$ is a nested sequence of cones.

The following two cases can happen.

Case(1) there exists a $\bar{q}$ such that for all $q > \bar{q}$, $(\bar{x}_{k_q}, \bar{y}_{k_q}) \in \Omega$;

Case(2) for any $\bar{q}$ there exists $q > \bar{q}$ such that $(\bar{x}_{k_q}, \bar{y}_{k_q}) \notin \Omega$.

In order to prove the convergence of the algorithm we first prove $(x^*, y^*) \in \Omega$. If case (1) happens then clearly $(x^*, y^*) \in \Omega$. Therefore we only consider case (2). For simplicity we assume that the points $(\bar{x}_{k_q}, \bar{y}_{k_q})$ does not belong to Ω for every q by taking a suitable subsequence of $\{ (\bar{x}_{k_q}, \bar{y}_{k_q}) \}$ if necessary. For a positive ε let us introduce a closed ε-neighborhood $P(\varepsilon)$ of P_1, i.e.,

$$P(\varepsilon) = \{ (x, y) \,|\, \exists (x_0, y_0) \in P_1, \ \| (x, y) - (x_0, y_0) \| \leq \varepsilon \}.$$

Recall P_1 is the initial polytope of the algorithm satisfying $\Omega \subset P_1$. Then $P_1 \subset P(\varepsilon)$ and any $(x, y) \in P_1$ implies $(x, y) \in \mathrm{int}P(\varepsilon)$.

Lemma 3.1 *The cutting plane functions $\{ l_k \}$ are uniformly equicontinuous on P_1.*

Proof. From the compactness of $P(\varepsilon)$ we see that the convex function $\phi(x, y)$ is bounded on $P(\varepsilon)$. By the definition of subgradient $\phi(x, y) \geq [\,(x, y) - (\bar{x}_k, \bar{y}_k)\,] s_\phi(\bar{x}_k, \bar{y}_k)$

$+ \phi(\bar{x}_k, \bar{y}_k)$ for all $(x, y) \in P(\varepsilon)$, and consequently for a sufficiently large number M

$$[(x,y) - (\bar{x}_k, \bar{y}_k)]s_\phi(\bar{x}_k, \bar{y}_k) \le M \tag{3.1}$$

holds for all $(x, y) \in P(\varepsilon)$ and all k. Suppose that $\{ s_\phi(\bar{x}_k, \bar{y}_k) \mid k = 1, 2, \dots \}$ is unbounded, then there exists at least one unbounded component of $s_\phi(\bar{x}_k, \bar{y}_k)$. We assume without loss of generality that the first component of $s_\phi(\bar{x}_k, \bar{y}_k)$ is unbounded. By $(\bar{x}_k, \bar{y}_k) \in P_1$, we can take a point $(x, y) \in P(\varepsilon)$ such that $(x, y) - (\bar{x}_k, \bar{y}_k) = (\pm\varepsilon, 0, \dots, 0)$. By choosing an appropriate sign of ε, we have $[(x, y) - (\bar{x}_k, \bar{y}_k)]s_\phi(\bar{x}_k, \bar{y}_k) > M$ for the sufficiently large k, a contradiction to (3.1). Therefore $\{ s_\phi(\bar{x}_k, \bar{y}_k) \mid k = 1, 2, \dots \}$ is bounded.

Let M_1, M_2 and M_3 be sufficiently large numbers such that $\| (x, y) - (\bar{x}_k, \bar{y}_k) \| \le M_1$, $\| s_\phi(\bar{x}_k, \bar{y}_k) \| \le M_2$ and $|\phi(x, y)| \le M_3$ for all $(x, y) \in P_1$ and for all k. Then $|l_k(x, y)| \le M_1M_2 + M_3$ for all $(x, y) \in P_1$ and for all k, which means that $\{ l_k(x, y) \mid (x, y) \in P_1,\ k = 1, 2, \dots \}$ is bounded. Therefore, both $\sup\{ l_k(x, y) \mid (x, y) \in P_1,\ k = 1, 2, \dots \}$ and $\inf\{ l_k(x, y) \mid (x, y) \in P_1,\ k = 1, 2, \dots \}$ are finite. The desired result follows from Theorem 10.6 of [9]. $\square$

Since $\{ l_k(x, y) \mid (x, y) \in P_1,\ k = 1, 2, \dots \}$ is bounded, by Theorem 10.8 of [9], there exists a subsequence $\{ l_{k_q} \}$ of $\{ l_k \}$ converging uniformly to a continuous function l, i.e.,

$$\lim_{q\to\infty} \sup_{(x,y)\in P_1} |l_{k_q}(x, y) - l(x, y)| = 0.$$

We have

Lemma 3.2 $\lim_{q\to\infty} l_{k_q}(\bar{x}_{k_q}, \bar{y}_{k_q}) = l(x^*, y^*)$.

Proof. Since l_{k_q} converges uniformly to l, we have

$$\forall\varepsilon > 0, \exists\, q_1 \text{ such that } \forall q \ge q_1, \sup_{(x,y)\in P_1} |l_{k_q}(x, y) - l(x, y)| \le \varepsilon/2.$$

From $\lim_{q\to\infty}(\bar{x}_{k_q}, \bar{y}_{k_q}) = (x^*, y^*)$ and the continuity of l, we have $\lim_{q\to\infty} l(\bar{x}_{k_q}, \bar{y}_{k_q}) = l(x^*, y^*)$, i.e.,

$$\forall\varepsilon > 0, \exists\, q_2 \text{ such that } \forall q \ge q_2, |l(\bar{x}_{k_q}, \bar{y}_{k_q}) - l(x^*, y^*)| \le \varepsilon/2.$$

Then we have for all $q \geq \max\{ q_1, q_2 \}$,

$$\begin{aligned} & |\, l_{k_q}(\bar{x}_{k_q}, \bar{y}_{k_q}) - l(x^*, y^*)\,| \\ \leq\; & |\, l_{k_q}(\bar{x}_{k_q}, \bar{y}_{k_q}) - l(\bar{x}_{k_q}, \bar{y}_{k_q})\,| + |\, l(\bar{x}_{k_q}, \bar{y}_{k_q}) - l(x^*, y^*)\,| \\ \leq\; & \sup_{(x,y)\in P_1} |l_{k_q}(x, y) - l(x, y)\,| + |\, l(\bar{x}_{k_q}, \bar{y}_{k_q}) - l(x^*, y^*)\,| \\ \leq\; & \varepsilon/2 + \varepsilon/2 \\ =\; & \varepsilon. \end{aligned}$$

□

Lemma 3.3 *If $l(x^*, y^*) \leq 0$ then $(x^*, y^*) \in \Omega$.*

Proof. Note that $l_{k_q}(x, y) = [\,(x, y) - (\bar{x}_{k_q}, \bar{y}_{k_q})]s_\phi(\bar{x}_{k_q}, \bar{y}_{k_q}) + \phi(\bar{x}_{k_q}, \bar{y}_{k_q})$,

$$\lim_{q\to\infty} l_{k_q}(x, y) = l(x, y) \quad \text{and} \quad \lim_{q\to\infty} \phi(\bar{x}_{k_q}, \bar{y}_{k_q}) = \phi(x^*, y^*).$$

By the boundedness of $s_\phi(\bar{x}_{k_q}, \bar{y}_{k_q})$, we can find a suitable subsequence of $s_\phi(\bar{x}_{k_q}, \bar{y}_{k_q})$ converging to a vector $\bar{s}_\phi$. Therefore

$$l(x, y) = [\,(x, y) - (x^*, y^*)]\bar{s}_\phi + \phi(x^*, y^*).$$

Then we obtain $\phi(x^*, y^*) = l(x^*, y^*)$. By the definition of Ω, we have the lemma. □

Lemma 3.4 $\lim_{q\to\infty} l_{k_q}(\bar{x}_{k_{q+1}}, \bar{y}_{k_{q+1}}) = l(x^*, y^*)$.

Proof. By $\lim_{q\to\infty} l_{k_q}(\bar{x}_{k_q}, \bar{y}_{k_q}) = l(x^*, y^*)$ and $\lim_{q\to\infty} (\bar{x}_{k_q}, \bar{y}_{k_q}) = (x^*, y^*)$, we see that

$$\forall\varepsilon > 0, \exists q_1 \text{ such that } \forall q \geq q_1, |\, l(x^*, y^*) - l_{k_q}(\bar{x}_{k_q}, \bar{y}_{k_q})\,| < \varepsilon/3,$$

and

$$\forall\delta > 0, \exists q_2 \text{ such that } \forall q \geq q_2, \|\, (x^*, y^*) - (\bar{x}_{k_q}, \bar{y}_{k_q})\,\| < \delta.$$

From Lemma 3.1, we see that $\{\, l_{k_q}(x, y)\,\}$ is equicontinous at (x^*, y^*), i.e.,

$$\forall\varepsilon > 0, \exists\delta > 0 \;\text{ such that if }\; \|\, (x, y) - (x^*, y^*)\,\| \leq \delta,$$

then

$$|\, l_{k_q}(x^*, y^*) - l_{k_q}(x, y)\,| < \varepsilon/3 \text{ for all } q.$$

Therefore for all $\varepsilon > 0$ we have that for $q \geq \max\{ q_1, q_2 \}$

$$\begin{aligned}
& | l(x^*, y^*) - l_{k_q}(\bar{x}_{k_{q+1}}, \bar{y}_{k_{q+1}}) | \\
\leq & | l(x^*, y^*) - l_{k_q}(\bar{x}_{k_q}, \bar{y}_{k_q}) | + | l_{k_q}(x^*, y^*) - l_{k_q}(\bar{x}_{k_q}, \bar{y}_{k_q}) | \\
& + | l_{k_q}(x^*, y^*) - l_{k_q}(\bar{x}_{k_{q+1}}, \bar{y}_{k_{q+1}}) | \\
\leq & \varepsilon/3 + \varepsilon/3 + \varepsilon/3 \\
= & \varepsilon.
\end{aligned}$$

□

Theorem 3.5 $(x^*, y^*) \in \Omega$.

Proof. Since $(\bar{x}_{k_{q+1}}, \bar{y}_{k_{q+1}}) \in \{ (x, y) \,|\, l_{k_q}(x, y) \leq 0 \}$, $l(x^*, y^*) \leq 0$ by Lemma 3.4. By Lemma 3.3, it implies that $(x^*, y^*) \in \Omega$. □

We have proved that every cluster point of the sequence $\{ \bar{x}_k, \bar{y}_k \}$ generated by the algorithm belongs to Ω.

Theorem 3.6 *If the conical partitions generated by the algorithm are exhaustive then every cluster point of the sequence $\{ (\bar{x}_k, \bar{y}_k) \}$ is an optimal solution of Problem (Q).*

Proof. Let (x^*, y^*) be a cluster point of $\{ (\bar{x}_k, \bar{y}_k) \}$. Assume that $\{ (\bar{x}_{k_q}, \bar{y}_{k_q}) \}$ is a subsequence of $\{ (\bar{x}_k, \bar{y}_k) \}$ such that $(\bar{x}_{k_q}, \bar{y}_{k_q}) \in T_{k_q} = R^{n_1} \times C_{k_q}$ and $\lim_{q \to \infty} (\bar{x}_{k_q}, \bar{y}_{k_q}) = (x^*, y^*)$ and that $C_{k_{q+1}} \subset C_{k_q}$ for all q. By the definition of $(\bar{x}_{k_q}, \bar{y}_{k_q})$ and C_{k_q},

$$\text{(3.2)} \quad \bar{y}_{k_q} = y^0 + U^h_{k_q} \lambda^h_{k_q} = y^0 + (v^{1,h}_{k_q} - y^0, ..., v^{n_2,h}_{k_q} - y^0)(\lambda^h_{1,k_q}, ..., \lambda^h_{n_2,k_q})^\top,$$

where $e^\top \lambda^h_{k_q} \geq 1$ and $\lambda^h_{k_q} \geq 0$. From the definition of $v^{i,h}_{k_q}$ in (2.1) (where index k_q is omitted) we see that $\{ v^{i,h}_{k_q} \,|\, i = 1, ..., n,\ h = 1, ..., m,\ q = 1, 2, ... \}$ is bounded. By Lemma 2.1, we have that $\{ \lambda^h_{k_q} \,|\, h = 1, ..., m, q = 1, 2, ... \}$ is also bounded. Taking a subsequence if necessary, we obtain that

$$\lim_{q \to \infty} v^{i,h}_{k_q} = \vartheta^{i,h}, \quad \lim_{q \to \infty} \lambda^h_{i,k_q} = \gamma^h_i \ \text{ for } \ i = 1, ..., n_2 \ \text{ and } \ \sum \gamma^h_i \geq 1, \gamma^h_i \geq 0.$$

Therefore

$$\text{(3.3)} \qquad y^* = \lim_{q \to \infty} \bar{y}_{k_q} = y^0 + (\vartheta^{1,h} - y^0, ..., \vartheta^{n_2,h} - y^0)(\gamma^h_1, ..., \gamma^h_{n_2})^\top.$$

Note that $v_{k_q}^{i,h} = y^0 + \theta_{i,k_q}^h (z_{k_q}^i - y^0)$, and θ_{i,k_q}^h is taken as $\bar{\theta}_{big}$ or such that $v_{k_q}^{i,h} \in \partial G_h$. We see that $\vartheta^{i,h} \in \partial G_h$ or $\vartheta^{i,h}$ is a sufficiently large vector. By the compactness of P_1 and $(\bar{x}_{k_q}, \bar{y}_{k_q}) \in P_1$, we obtain that $y^* \in P_1$. Therefore there exists at least one vector $\vartheta^{i,h}$ in (3.3) such that $\vartheta^{i,h} \in \partial G_h$ for every h. From the assumption that $\{ C_{k_q} \}$ is exhaustive, i.e., there exists a vector $\bar{z} \in R^{n_2}$ such that

$$\lim_{q \to \infty} z_{k_q}^i = \bar{z} \quad \text{for all} \quad i,$$

we see that $\vartheta^{i,h}$ is on the ray $\{ y \,|\, y = y^0 + \theta(\bar{z} - y^0), \theta \geq 0 \}$. Therefore we obtain

$$\vartheta^{i,h} = y^0 + \theta_i^h (\bar{z} - y^0), \;\; i = 1, ..., n_2, \;\; h = 1, ..., m. \tag{3.4}$$

Moreover,

$$\begin{aligned} y^* = \lim_{q \to \infty} \bar{y}_{k_q} &= y^0 + [\,(\theta_1^h(\bar{z} - y^0), ..., \theta_{n_2}^h(\bar{z} - y^0)\,](\gamma_1^h, ..., \gamma_{n_2}^h)^\top \\ &= y^0 + [\,(\bar{z} - y^0), ..., (\bar{z} - y^0)\,](\theta_1^h \gamma_1^h, ..., \theta_{n_2}^h \gamma_{n_2}^h)^\top \\ &= y^0 + (\bar{z} - y^0) \sum_{i=1}^{n_2} \theta_i^h \gamma_i^h \qquad \text{for all } h. \end{aligned} \tag{3.5}$$

Suppose $y^* \in G$, then there exists at least one h_0 such that $y^* \in G_{h_0}$. Taking $\hat{\theta}^h = \min_i \theta_i^h$, we see that the point $\vartheta^{i,h}(\hat{\theta}^h) = y^0 + \hat{\theta}^h(\bar{z} - y^0) \in \partial G_h$ for every h. Therefore $\sum \theta_i^{h_0} \gamma_i^{h_0} < \hat{\theta}^{h_0} = \min_i \theta_i^{h_0}$. On the other hand, $\sum \theta_i^{h_0} \gamma_i^{h_0} \geq \sum \hat{\theta}^{h_0} \gamma_i^{h_0} \geq \hat{\theta}^{h_0}$, a contradiction. It implies that $y^* \notin G$.

Combining the above result with Theorem 3.5 we see that (x^*, y^*) is a feasible solution of Problem (Q), i.e., $(x^*, y^*) \in D$.

Let V^* be the optimal value of (Q). Note that $c^\top \bar{x}_{k_q} + d^\top \bar{y}_{k_q}$ is a lower bound of V^*, therefore we see that

$$c^\top x^* + d^\top y^* = \lim_{q \to \infty} c^\top \bar{x}_{k_q} + d^\top \bar{y}_{k_q} \leq V^*.$$

It implies that (x^*, y^*) is an optimal solution of Problem (Q). □

4 The out-of-roundness problem

Let S be a set of finitely many points $s^1, \ldots, s^m$ in R^n. The *out-of-roundness* problem is formulated as follows.

$$
(R) \qquad \left| \begin{array}{ll} \min & t - r \\ \text{s.t.} & \| s - s^h \| \le t, \; h = 1, \ldots, m, \\ & \| s - s^h \| \ge r, \; h = 1, \ldots, m, \\ & s \in C(S), \end{array} \right.
$$

where $C(S)$ is the convex hull of the set S. The problem is to find a pair of cocentric balls one of which contains all the points $s^1, \ldots, s^m$ and the other contains none of the points such that the difference of two radii is minimized. If the objective function $t - r$ is small enough, we can conclude that the given points $s^1, \ldots, s^m$ lie on the surface of a ball.

There are several algorithms [1, 8] dealing with the problems. However they can only solve problems of two or three dimension. To authors' knowledge there are no practical algorithms for solving problems with dimension higher than three. In the remaining part of this section we show that the out-of-roundness problem can be formulated as Problem (Q). We also show how the algorithm proposed in the previous section can be applied efficiently to the problem.

We consider the problem (R) with the last constraint $s \in C(S)$ dropped, i.e.,

$$
(R1) \qquad \left| \begin{array}{ll} \min & t - r \\ \text{s.t.} & \| s - s^h \| \le t, \; h = 1, \ldots, m, \\ & \| s - s^h \| \ge r, \; h = 1, \ldots, m. \end{array} \right.
$$

Assumption 4.1 *The optimal solution of $(R1)$ is in the convex hull $C(S)$.*

Obviously the out-of-roundness problem is equivalent to the problem $(R1)$ under Assumption (4.1). In practice the given points $s^1, \ldots, s^m$ represent the location of sample points on the surface of an almost round object. Therefore it is very likely that the solution of $(R1)$ lies somewhere in the convex hull of S. Let

$$
\begin{aligned}
s^0 &= \frac{1}{m} \sum_{h=1}^{m} s^h, \\
\alpha_j^0 &= \min\{ e_j^\top s^h \mid h = 1, \ldots, m \}, \; j = 1, \ldots, n, \\
\beta_j^0 &= \max\{ e_j^\top s^h \mid h = 1, \ldots, m \}, \; j = 1, \ldots, n,
\end{aligned}
$$

where e^j is a jth unit vector in R^n. Further we define

$$\begin{aligned}
Z &= \{ (t,r,s) \mid \| s - s^h \| \le t,\ h = 1,\dots,m \}, \\
G_h &= \{ (r,s) \mid \| s - s^h \| < r \},\ h = 1,\dots,m, \\
G &= \bigcup_{h=1}^{m} G_h, \\
\rho^0 &= \max_h \| s^0 - s^h \|, \\
X &= \{ t \mid 0 \le t \le 2\rho^0 \}, \\
Y &= \{ (r,s) \mid 0 \le r \le 2\rho^0,\ \alpha_j^0 \le s_j \le \beta_j^0,\ j = 1,\dots,n \}.
\end{aligned}$$

and consider

$$(R2) \qquad \left| \begin{array}{ll} \min & t - r \\ \text{s.t.} & t \in X,\ (r,s) \in Y, \\ & (t,r,s) \in Z,\ (r,s) \in R^{n+1} \setminus G. \end{array} \right.$$

Let $g_h(r,s) = \| s - s^h \| - r$ $(h = 1,\dots,m)$, then

$$G_h = \{ (r,s) \mid g_h(r,s) < 0 \},\ h = 1,\dots,m.$$

Note that in this problem the ratio of m/n is usually very large. The polytope X is just an interval and the polytope Y is a hypercube of dimension $n+1$.

Theorem 4.2 *Under Assumption 4.1 the problems (R1) and (R2) are equivalent.*

Proof. Let $(t^*(1), r^*(1), s^*(1))$ and $(t^*(2), r^*(2), s^*(2))$ be optimal solutions of the problem $(R1)$ and the problem $(R2)$, respectively. Note $r^*(1) \le t^*(1)$, $r^*(2) \le t^*(2)$. Since $(t^*(2), r^*(2), s^*(2)) \in Z$ and $(r^*(2), s^*(2)) \notin G$, $(t^*(2), r^*(2), s^*(2))$ is also a feasible point of the problem $(R1)$. Therefore

$$(4.6) \qquad t^*(1) - r^*(1) \le t^*(2) - r^*(2).$$

On the other hand, from Assumption 4.1, $s^*(1) = \sum_{h=1}^{m} \lambda_h^* s^h$ for some nonnegative λ_h^* such that $\sum_{h=1}^{m} \lambda_h^* = 1$. Then for some $h_1 \in \{1,\dots,m\}$

$$\begin{aligned}
t^*(1) &= \| s^*(1) - s^{h_1} \| = \| s^*(1) - s^0 + s^0 - s^{h_1} \| \\
&\le \| s^*(1) - s^0 \| + \| s^0 - s^{h_1} \|
\end{aligned}$$

$$\begin{aligned}
&= \|\sum_{h=1}^{m}\lambda_h^* s^h - s^0 \| + \| s^0 - s^{h_1} \| \\
&\le \sum_{h=1}^{m}\lambda_h^* \| s^h - s^0 \| + \| s^0 - s^{h_1} \| \\
&\le 2\rho^0.
\end{aligned}$$

Furthermore, $s^*(1) \in C(S)$ implies $s^*(1) \in \{ s \mid \alpha_j^0 \le s_j \le \beta_j^0, j = 1, \ldots, n \}$. Therefore $(t^*(1), r^*(1), s^*(1))$ is a feasible point of the problem $(R2)$. We have

$$t^*(2) - r^*(2) \le t^*(1) - r^*(1). \tag{4.7}$$

By (4.6) and (4.7), $t^*(1) - r^*(1) = t^*(2) - r^*(2)$, which proves the theorem. □

From Theorem 4.1 and Assumption 4.1, the out-of-roundness problem is equivalent to the problem $(R2)$, which is solvable by the algorithm proposed in section 3.

To start the algorithm we choose $X \times Y$ as an initial polytope P_1 containing Ω, where Ω is defined as before. Take r^0 to be any value greater than $\rho^0 = \max_h \| \bar{s} - s^h \|$. Then the point (r^0, s^0) belongs to $\cap_{h=1}^m G_h$, and can serve as point y^0 of the algorithm.

Suppose that we are at the kth iteration of the algorithm, let the polytope P_k be defined by

$$P_k = \{ (t, r, s) \mid A_t^k t + A_r^k r + A_s^k s \le b^k \},$$

where A_t^k, A_r^k and b^k are m^k-dimensional vectors, and A_s^k is an $m^k \times n$-matrix. In order to obtain a lower bound over a set $D \cap P_k \cap T$, we need first to calculate for every G_h $(h = 1 \ldots, m)$, a set of $n+1$ points which are on the intersection of ∂G_h.

Let $(r^1, z^1), \ldots, (r^{n+1}, z^{n+1})$ be points generating the cone C. Since each constraint $g_h(r, s) < 0$ defining the set G_h is very simple, it is not necessary to solve a maximization problem $\sup\{ \theta \mid (r^0, s^0) + \theta(r^i - r^0, z^i - s^0) \in G_h \}$ to obtain the value θ_i^h in (2.1). Solving the equation

$$\| s^0 + \theta_i^h(z^i - s^0) - s^h \| - (r^0 + \theta_i^h(r^i - r^0)) = 0$$

yields the value of θ_i^h, for which $(r^{i,h}, s^{i,h}) = (r^0, s^0) + \theta_i^h(r^i - r^0, z^i - s^0)$ lies on the intersection of ∂G_h and the ith ray of the cone C if $\theta_i^h < \bar{\theta}_{big}$. After computing the set of $n+1$ points $(r^{1,h}, s^{1,h}), \ldots, (r^{n+1,h}, s^{n+1,h})$ for every h $(h = 1, \ldots, m)$, we have to solve a linear program (2.4) to obtain a lower bound and possibly an upper bound. The linear program (2.4) can be written as

$$\text{(4.8)} \qquad \left| \begin{array}{ll} \min & t - r \\ \text{s.t.} & A_t^k t + A_r^k r + A_s^k s \leq b^k, \\ & (r, s) = (r^0, s^0) + U^h \lambda^h, \ \forall h, \\ & e^\top \lambda^h \geq 1, \ \forall h, \\ & \lambda^h, \ t, \ r \geq 0, \ \forall h. \end{array} \right.$$

Recall $U^h = [(r^{1,h} - r^0, s^{1,h} - s^0), \ldots, (r^{n+1,h} - r^0, s^{n+1,h} - s^0)]$. Let

$$U^h = \begin{pmatrix} U_r^h \\ U_s^h \end{pmatrix},$$

then

$$\text{(4.9)} \qquad r = r^0 + U_r^h \lambda^h, \ \forall h,$$

$$\text{(4.10)} \qquad s = s^0 + U_s^h \lambda^h, \ \forall h.$$

Take an arbitrary number of $1, 2, \ldots, m$, for instance 1 and substitute (4.9) and (4.10) with $h = 1$ for r and s of (4.8), respectively. Then the problem (4.8) reduces to

$$\text{(4.11)} \qquad \left| \begin{array}{llcl} \min & t - U_r^1 \lambda^1 - r^0 & & \\ \text{s.t.} & \tilde{A}^k t + \tilde{B}^k \lambda^1 & \geq & \tilde{b}^k, \\ & U^h \lambda^h - U^1 \lambda^1 & = & 0, \ h = 2, \ldots, m, \\ & e^\top \lambda^h & \geq & 1, \ \forall h, \\ & \lambda^h, \ t & \geq & 0, \ \forall h, \end{array} \right.$$

where

$$\begin{aligned} \tilde{A}^k &= -A_t^k, \\ \tilde{B}^k &= -A_r^k U_r^1 - A_s^k U_s^1, \\ \tilde{b}^k &= A_r^k r^0 + A_s^k s^0 - b^k. \end{aligned}$$

The above problem has $m^k + (n+1)(m-1) + m$ constraints, and $(n+1)m + 1$ variables, Moreover, the number m^k will grow at each iteration due to adding of cutting planes. Therefore it is time consuming to solve this problem directly. We deal with this shortcoming as follows.

There are a lots of redundant constraints in (4.11). Using Lemma 2.2 we can remove h from the set $\{1,..,m\}$ if there exists an h' such that $\theta^{h'} < \theta^h$. Let I be the remaining subset of $\{1,..,m\}$ after removing all those h, relabel the elements in I as $1,\ldots,|I|$, and relabel also correspondingly U^h and λ^h. We consider the dual problem of (4.11). Let ζ, $\eta^1,\ldots,\eta^{|I|-1}$, ξ be dual variables of the reduced and relabeled problem (4.11), where ζ is a vector of m^k-dimension, $\eta^1,\ldots,\eta^{|I|-1}$ are vectors of $(n+1)$-dimension, and ξ is a vector of $|I|$-dimension. The dual problem is

$$(4.12)\qquad \left|\begin{array}{lll} \max & (\tilde{b}^k)^\top\zeta + e^\top\xi & \\ \text{s.t.} & (\tilde{A}^k)^\top\zeta & \leq\ 1, \\ & (\tilde{B}^k)^\top\zeta - \sum\limits_{h=1}^{|I|-1}(U^1)^\top\eta^h + \xi e & \leq\ -(U_r^1)^\top, \\ & (U^h)^\top\eta^{h-1} + \xi e & \leq\ 0,\ h=2,\ldots,|I|, \\ & \zeta,\ \xi & \geq\ 0. \end{array}\right.$$

Note that the above problem has $(n+1)|I|+1$ constraints. It is obvious that solving (4.12) is less time consuming in comparison with solving (4.11) directly.

The other thing we like to point out is that the methods of finding possible feasible points in (2.5) and (2.6)-(2.7) are extremely simple. By solving (4.12) we obtain a point $(\bar{t},\bar{r},\bar{s})$. Suppose $(\bar{t},\bar{r},\bar{s}) \notin \Omega$. Then

$$\|\ \bar{s} - s^h\ \| - \bar{r} < 0 \quad \text{for} \quad h = 1,\ldots,m.$$

The value of $\hat{\tau} = \sup\{\,\tau \mid \tau d + \bar{y} \in G\,\}$ in (2.5), where $d = (-1,0)$, $\bar{y} = (\bar{r},\bar{s})$, can be determined by

$$\hat{\tau} = \max\{\bar{r} - \|\ \bar{s} - s^h\ \| \mid h = 1,\ldots,m\,\}.$$

In fact the point $(\hat{\tau}d + \bar{y}) = (\bar{r} - \hat{\tau}, \bar{s})$ belongs to G, since

$$\|\ \bar{s} - s^h\ \| - (\bar{r} - \hat{\tau}) \geq 0,\ h = 1,\ldots,m,$$

and at least one equation holds. Then the point $(\hat{x},\hat{y}) = (\hat{t},\hat{r},\hat{s})$ in (2.5) can be determined by $\hat{\tau}(c,d) + (\bar{x},\bar{y}) = \hat{\tau}(1,-1,0) + (\bar{t},\bar{r},\bar{s})$.

If it is necessary to obtain a point $(\tilde{x},\hat{y}) = (\tilde{t},\hat{r},\hat{s})$ in (2.6), we have to determine first $\ddot{x} = \ddot{t}$ satisfying (2.7), i.e., to solve the following maximization problem

$$\max\{\,t \mid A_t^k t + A_r^k\hat{r} + A_s^k\hat{s} \leq b^k\,\}.$$

But the optimal solution of the problem is simply given by

$$\ddot{t} = \min\{ \frac{b_i}{a_i} \mid i = 1, \ldots, m_k \},$$

where a_i and b_i are the ith components of the vectors A_t^k and $b^k - A_{\eta}^k \hat{r} - A_s^k \hat{s}$.

5 The conclusions

We have proposed an algorithm $\mathcal{GO}$ for solving a global optimization problem with a set of reverse convex constraints by means of cutting plane techniques and branch-and-bound method. The out-of-roundness problem has been discussed as a special case of the problem considered in this paper. Techniques proposed to find a possible feasible point in (2.5)-(2.7) become very simple when applied to the out-of-roundness problem.

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