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Utility Theories in Cooperative Games

by

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Abstract

The utility theories for the definitions of cooperative games and associated solution concepts are discussed. A game with side payments needs the assumptions of transferable utility and side payments. We discuss the axioms for the transferable utility assumption and also the role of side payments for various solution concepts. We also discuss games without side payments, which do not require the assumptions of transferable utility and side payments. We see how some solution concepts depend upon these assumptions.

1 Introduction

In principle there are no utility theories exclusive to cooperative game theory. In practice, however, cooperative game theory sometimes uses some forms of utility theories in particular ways. Special consideration of utility theories is required because of the feature that, to achieve some agreement, payments between players may be made "on the side". A *game with side payments* requires both the assumption that side payments are possible and an assumption of transferability of utility. A *game without side payments* requires neither of these assumptions. In this chapter, we discuss the axioms for the transferable utility assumption and also the role of side payments in various solution concepts for both games with and without side payments.

The assumption of transferable utility, sometimes called "quasi-linearity", is that the utility function of each player is linearly separable with respect to some perfectly divisible commodity called money. The assumption required to define a game with side payments is that, in addition to the strategic choice of an alternative, players can choose to make transfers of money. A game with side payments consists of a set of players and a function which assigns a set of total utilities, summarized by a real number, to each coalition. This framework provides a convenient tool and has an appealingly simple formulation. If we drop either the assumption of side payments or of transferable utility, however, the mathematically heavier concept of a game without side payments is required. The subtleties of games with side payments arise in interpretation of solution concepts.

Solution concepts for games with side payments depend upon the assumptions of transferable utility and side payments. Some solution concepts apparently deviate from the original intention of a game with side payments.

We consider to what extent such deviations from the original intention exist and to what extent they may be justified. Since neither transferable utility or side payments are required for games without side payments, solution concepts developed for games without side payments might avoid subtle difficulties in utility theories. Many solution concepts have been developed first, however, for games with side payments and then extended to games without side payments. Thus, the difficulties present in interpretation of solution concepts for games with side payments also may arise in extensions of these solution concepts to games without side payments.

In this chapter, we first focus on the basic assumptions, with respect to utility and side payments, of a game with side payments. We will see how a game with side payments is used in the definitions of several solution concepts. To illustrate the status of a game with side payments and solution concepts for such a game, in the context of general utility theory and cooperative game theory we discuss market games and voting games. Then we consider solution concepts for a game without side payments. Some solution concepts involve no difficulty in utility theories but some inherit difficulties from games with side payments.

In Section 2 we review the concept of a game with side payments and several examples from the literature. In Section 3 we consider the assumptions of transferable utility and of side payments. We give axiomatic characterizations of the transferable utility assumption in the cases of no uncertainty and of uncertainty. In Section 4 we discuss some solution concepts for games with side payments, specifically, the core, the von Neumann-Morgenstern stable set, the Shapley value and the nucleolus. In Section 5 we discuss games with side payments and see how those solution concepts depend upon the assumptions of transferable utility and of side payments.

2 A Game with Side Payments

A *game with side payments* consists of a pair (N, v), where $N = \{1, ..., n\}$ is the *player set* and $v : 2^N \to R$ is the *characteristic function* with $v(\emptyset) = 0$. The function v assigns to each coalition S in 2^N the maximum total payoff that can be obtained by collective activities of the players in S. A game (N, v) describes a social situation in terms of the payoffs achievable by the collective activities of groups of players.

The game is *superadditive* iff

$$v(S) + v(T) \leq v(S \cup T) \text{ for all } S, T \in 2^N \text{ with } S \cap T = \emptyset.$$

In the modelling of social situations as games with side payments, superadditivity is typically satisfied. For example, a game is superadditive if the opportunities open to a coalition include dividing into smaller coalitions, each of which can operate independently of the other. All examples to be discussed in the present Chapter satisfy superadditivity.

Two considerations are particularly important for our discussion. The first is that the assumptions of transferable utility and side payments are required for the description of a game (N, v). Much of our discussion concerns the interpretations and roles of these assumptions. The remainder of our discussion concerns the distribution of payoffs among the players, addressed by solution theories. Some solution concepts require no additional assumptions beyond those of transferable utility and side payments, while other solution concepts make additional implicit assumptions.

In this Section, we illustrate the derivation of a game with side payments from a market game, a majority voting game, and a strategic game. We discuss the roles of transferable utility and side payments in these examples.

2.1 Transferable utility and side payments

Consider an individual player i with utility function $U^i : X \times R \to R$. The space $X \times R$ is called the *outcome space for player* i and the space $X \times R^N$ is the *outcome space*. The space X may represent a commodity space, a set of social alternatives or the outcome space of a noncooperative game. The space R of real numbers is typically interpreted as representing a perfectly divisible composite commodity called "money." This commodity represents purchasing power for other commodities outside the model. The value $U^i(x, \xi)$ represents the utility from the outcome x and the increment (or decrement) ξ of money from some initial level. The interpretation of the unbounded domain of money is that any individually rational outcome can be achieved, without meeting any boundary conditions, by monetary transfers. That is, relative to individually rational payoffs that might arise from the game, incomes are sufficiently large to avoid the need for boundary conditions.

The *transferable utility assumption*, also called "quasi-linearity" in the economics literature, is that U^i is linearly separable with respect to ξ, that is, there is a function $u^i : X \to R$ such that

$$U^i(x, \xi) = u^i(x) + \xi \text{ for all } (x, \xi) \in X \times R. \tag{1}$$

The utility function in (1) is interpreted as uniquely determined up to a parallel transformation. That is, as will be clarified in Section 3.1, if $V^i(x, \xi) = v^i(x) + \xi$ and $v^i(x) = u^i(x) + c$ for some constant c, then $V^i(x, \xi)$ can be regarded as equivalent to $U^i(x, \xi)$.

The term "transferable utility" is motivated by the following observation. When two players have utility functions of form (1), since the utility level of each player changes by the amount of the transfer, a transfer of money between the players appears as a transfer of utility.

Let $x_0 \in X$ be an arbitrarily chosen outcome, interpreted as an initial situation or the "status quo". For a utility function $U^i(x, \xi)$ of form (1), it holds that for any $x \in X$

$$U^i(x, \xi) = U^i(x_0, u^i(x) - u^i(x_0) + \xi).$$

The above formula implies that $u^i(x) - u^i(x_0)$ represents the monetary equivalent of the change in utility brought about by the change from x_0 to x. In other words, $u^i(x) - u^i(x_0)$ is the amount of "willingness to pay" for the transition of the outcome from x_0 to x (cf., Hicks (1956)).

In the terminology of economics, (1) implies that there are no income effects on the choice behavior of player i. "No income effects" means the preferences over X are independent of money holdings, that is, $u^i(x) - u^i(x_0)$ does not depend on ξ. This guarantees the well-definedness of consumer surplus. When player i compares paying the amounts of money p and p_0 for x and x_0 respectively, his "surplus" due to the change from x_0 to x is $(u^i(x) - p) - (u^i(x_0) - p_0)$. When $u^i(x_0) - p_0$ is normalized to equal zero by a parallel transformation of u^i, $u^i(x) - p$ is defined as the player's *consumer surplus* due to the change.

The no income effects condition is justified as a local approximation to a situation where the initial income of the consumer is large relative to any money transfers that may arise in the game. This also provides a justification for the assumption of the unbounded domain of money. We should always keep these justifications in mind: some applications or extensions of games with side payments are not consistent with these justifications.

The assumption of side payments is independent of the assumption of transferable utility. "Side payments" permits transfers of money in addition to any sort of transfer embodied in the outcome x. We consider the role of side payments in the contexts of market games, majority voting games and games derived from strategic games.

2.2 A market game

Consider an exchange economy with players $1, ..., n$ and commodities $1, ..., m$, $m+1$. The space X, called the *consumption space* of the first m commodities, is taken as the non-negative orthant R^m_+ of R^m. Each player i has an *endowment* of commodities $\omega^i \in X$, describing his initial holding of the first m commodities. Each player also has an endowment of the $m+1^{th}$ commodity but this is assumed to be sufficiently large so that it is not binding. Thus we do not need to specify the endowment of money: only increments or decrements from the initial level are considered. The value $U^i(x, \xi)$ represents the utility from consuming commodity bundle x and the initial money holdings plus the increment or decrement in ξ.

Under the transferable utility assumption for all players, a game with side payments is defined as follows: for each coalition S,[1]

$$v(S) = \max \sum_{i \in S} u^i(x^i) \text{ subject to} \sum_{i \in S} x^i = \sum_{i \in S} \omega^i \text{ and } x^i \in X \text{ for all } i \in S. \tag{2}$$

The characteristic function assigns to each coalition the maximum total payoff achievable by exchanges of commodities among the members of the coalition.

The characteristic function (2) may *appear* to suggest that players in a coalition maximize total utility and players make interpersonal comparisons of utilities. As we discuss below, some solution concepts based on the characteristic function may indeed make interpersonal comparisons of utilities. These may violate the original intention of von-Neumann-Morgenstern (1944) that the value $v(S)$ simply describes the Pareto frontier and the feasible payoffs for S.Definition (2) by itself does not involve any behavioral assumptions or interpersonal utility comparisons.

To describe the Pareto frontier, define *an allocation* $(x^i, \xi^i)_{i \in S}$ *for* S by

$$(x^i, \xi^i) \in X \times R \text{ for all } i \in S, \sum_{i \in S} x^i = \sum_{i \in S} \omega^i \text{ and } \sum_{i \in S} \xi^i = 0. \tag{3}$$

An allocation $(x^i, \xi^i)_{i \in S}$ is said to be *Pareto-optimal for* S iff there is no

[1]When $u^i(x^i)$ is continuous, the following maximization problem is well defined. In the sequel, when we use "max" we are assuming that it is well defined.

other allocation $(y^i, \eta^i)_{i\in S}$ for S such that

$$\begin{aligned} U^i(y^i,\eta^i) &\geq U^i(x^i,\xi^i) \text{ for all } i \in S; \text{ and} \\ U^i(y^i,\eta^i) &> U^i(x^i,\xi^i) \text{ for some } i \in S. \end{aligned} \tag{4}$$

The value $v(S)$ describes Pareto-optimal allocations in the following sense:

Proposition 2.1. An allocation $(x^i, \xi^i)_{i\in S}$ is Pareto-optimal for S if and only if $v(S) = \sum_{i\in S} u^i(x^i)$.

Proof: If $(x^i, \xi^i)_{i\in S}$ is not Pareto-optimal for S, then (4) holds for some allocation $(y^i, \eta^i)_{i\in S}$ for S. This, together with $\Sigma_{i\in S}\, \eta^i = \Sigma_{i\in S}\xi^i = 0$, implies that $\Sigma_{i\in S}\, u^i(x^i) < \Sigma_{i\in S}\, u^i(y^i) \leq v(S)$. Conversely, if $\Sigma_{i\in S}\, u^i(x^i) < v(S)$, there is a $(y^i)_{i\in S}$ with $\Sigma_{i\in S}\, u^i(x^i) < \Sigma_{i\in S}\, u^i(y^i)$. This implies that (4) holds for an appropriate choice of $(\eta^i)_{i\in S}$. □

The characteristic function delineates feasible payoffs. In the market, the feasibility of payoffs $(a_i)_{i\in S}$ for coalition S is described as:

$$\text{for some allocation } (x^i,\xi^i)_{i\in S},\ a_i \leq u^i(x^i) + \xi^i \text{ for all } i \in S. \tag{5}$$

This feasibility is summarized by the characteristic function v, since $(a_i)_{i\in S}$ is feasible for S if and only if

$$\Sigma_{i\in S}\, a_i \leq v(S). \tag{6}$$

In the terminology of economics, the value $v(S)$ is the maximum sum of the consumer surpluses over the players in S. From the no income effects condition, this sum $v(S)$ is independent of the distribution of the money holdings among the members of S.

The definition of Pareto optimality (4) is unaffected by monotone increasing transformations of utility functions. That is, the Pareto-optimality of an allocation for a coalition S is unaffected by such transformations of the utility functions of its members. On the other hand, the definition of the characteristic function requires particular (transformations of) utility functions. Nevertheless, Proposition 2.1 guarantees that the value $v(S)$ determines the Pareto frontier for coalition S.

In the context of markets, the side payments assumption simply means that side payments – transfers of the last commodity – are possible. In other contexts such as voting games, discussed in the next subsection, side payments have a more significant role.

2.3 Majority voting game with side payments

Consider a voting situation with n players where one alternative is chosen by majority voting from the set X of social alternatives. Suppose that the utility function of each player i, $U^i(x,\xi^i)$: $X \times R \to R$, is of form (1). Define a characteristic function $v : 2^N \to R$ by

$$v(S) = \begin{matrix} \max_{x\in X} \sum_{i\in S} u^i(x) & \text{if } |S| > \frac{n}{2} \\ \min_{x\in X} \sum_{i\in S} u^i(x) & \text{otherwise,} \end{matrix} \tag{7}$$

where $|S|$ is the number of members in S. A majority coalition S, $|S| > \frac{n}{2}$, can choose any social alternative x from X. Therefore the members of a majority coalition can maximize the total payoff $\Sigma_{i\in S}\ u^i(x)$. A minority coalition S, $|S| \leq \frac{n}{2}$, cannot make an effective choice. Thus $v(S)$ is defined as the value the members of S can certainly guarantee for themselves.

In a voting game, as in a market game, for a majority coalition S the value $v(S)$ determines the Pareto frontier for S. For a minority coalition S, $v(S)$ also determines the Pareto frontier among all feasible outcomes that the members of S can guarantee for themselves.

The main issue of the majority voting game is the choice of a social alternative $x \in X$. Besides the choice of x, the players are able to make transfers of money, that is, side payments. This allows the possibility of obtaining the consent of other players to a particular alternative by purchasing their votes.

In the market game of the above subsection, side payments have only a trivial meaning in the sense that transfers of money are parts of the main issue of the market; if such transfers are prohibited, the situation is no longer a market. On the other hand, in voting situations, such side payments are sometimes difficult or regarded as impossible. In such a case, the formulation (7) is inappropriate: we need the formulation of a game without side payments, which will be discussed in Section 5.

2.4 The cooperative game derived from a strategic form game

Let $G = (N, \{\Sigma_i\}_{i\in N}, \{h_i\}_{i\in N})$ be an n-person finite strategic form game, that is, $N = \{1, ..., n\}$ is the player set, Σ_i is a finite strategy space for player $i \in N$, and $h_i : \Sigma_1 \times \cdots \times \Sigma_n \to R$ is the payoff function of player i. The *space of mixed strategies* of player i is the set of all probability distributions over Σ_i, denoted by $M(\Sigma_i)$. Note that $M(\Sigma_i)$ is the $\mid \Sigma_i \mid -1$ dimensional unit simplex. When the players in a coalition S cooperate, they can coordinate their strategies to play a *joint mixed strategy*, a probability distribution over $\Sigma_S = \Pi_{i \in S}\, \Sigma_i$. We denote the set of all joint strategies for S by $M(\Sigma_S)$,which is also a unit simplex. The payoff function $h_i(\cdot)$ is extended to $M(\Sigma_N)$ as the expectation of $h_i(s)$ over Σ_N. In fact, $h_i(\cdot)$ corresponds to u_i in the expression (1) so that $U^i(s, \xi) = h_i(s) + \xi$. Thus the whole utility function U^i is defined on $M(\Sigma_N) \times R$, and the space X of Section 2.1 is now $M(\Sigma_N)$.

In the derivation of a cooperative game with side payments from a strategic game, transfers of money between the players in a coalition are permitted. When transferable utility in the sense of Subsection 2.1 is assumed, the total utility

$$\sum_{i \in S} h_i(\sigma_S, \sigma_{-S}), \text{ where } \sigma_S \in M(\Sigma_S) \text{ and } \sigma_{-S} \in M(\Sigma_{-S}),$$

is independent of the monetary transfers. Thus the total utility $\sum\limits_{i \in S} h_i(\sigma_S, \sigma_{-S})$ can be freely distributed among the players in S by the players via side payments $(\xi^i)_{i \in S}$ with $\Sigma_{i \in S} \xi^i = 0$. Each player evaluates an outcome (σ_S, σ_{-S}) by the expected value of $h_i(\cdot)$ and may make transfers to other players in return for the agreements to play the joint mixed strategy.

Von Neumann and Morgenstern (1944) defined the characteristic function v,

$$v(S) = \max_{\sigma_S \in M(\Sigma_S)} \min_{\sigma_{-S} \in M(\Sigma_{N-S})} \Sigma_{i \in S}\, h_i(\sigma_S\, , \sigma_{-S}) \text{ for all } S \in 2^N. \qquad (8)$$

That is, the value $v(S)$ is defined by regarding the game situation as a two-person zero-sum game with one player taken as S and the other as the complementary coalition $N - S$.

The game involves uncertainty in that the players can choose joint mixed strategies. It is assumed that when side payments are permitted, even though they might play mixed strategies, players can make monetary transfers without uncertainty.

In the standard treatment of a strategic game $G = (N, \{\Sigma_i\}_{i \in N}, \{h_i\}_{i \in N})$, the game G is a closed world in the sense that no additional structure is assumed. In the treatment here, side payments can be made by making transfers of money. Money represents purchasing power in the world outside the game. In this sense, the game is not a closed world.

3 Axiomatic Characterization of Transferable Utility

It may be helpful in understanding the assumption of transferable utility to look at an axiomatic characterization of preferences having transferable utility representations. We will discuss axioms for both preferences over outcomes with and without uncertainty. The derivation with no uncertainty is close to the classical utility theory (cf., Debreu (1957)). With uncertainty, the derivation is a special case of the von Neumann-Morgenstern utility theory.

3.1 Transferable utility with no uncertainty

In the absence of uncertainty, a preference relation $\succeq_i$ is defined on $X \times R$. Consider the following four conditions on $\succeq_i$:

(T1) $\succeq_i$ is a complete preordering on $X \times R$;

(T2) $\succeq_i$ is strictly monotone on R;

(T3) for any (x,ξ), $(y,\eta) \in X \times R$ with $(x,\xi) \succeq_i (y,\eta)$, there is an $\epsilon \in R$ such that $(x,\xi) \sim_i (y,\eta+\epsilon)$; and

(T4) $(x,\xi) \sim_i (y,\eta)$ and $\epsilon \in R$ imply $(x,\xi+\epsilon) \sim_i (y,\eta+\epsilon)$,

where $\sim_i$ is the indifference part of the relation $\succeq_i$. Conditions (T1) and (T2) are standard. Condition (T3) means that some amount of money substitutes for a change in outcome. Condition (T4), the most essential, means that the player's choice behavior on X does not depend on his money holdings.

The following result holds (Aumann (1960), Kaneko (1976)):

Proposition 3.1. A preference relation $\succeq_i$ satisfies (T1)-(T4) if and only if there is a function $u^i : X \to R$ such that $(x,\xi) \succeq_i (y,\eta) \Leftrightarrow u^i(x) + \xi \geq u^i(y) + \eta$.

A utility function $U^i(x,\xi) = u^i(x) + \xi$ is one representation of a preference relation $\succeq_i$ satisfying (T1)-(T4). Note that any monotone transformation $\varphi(u^i(x) + \xi)$ of U^i is also a representation of the preference relation $\succeq_i$. Nevertheless, as already seen in Section 2, the representation $u^i(x) + \xi$ has a special status in defining a game with side payments.

Proof of Proposition 3.1. If there is a utility function U^i of form (1), then $\succeq_i$ determined by U^i satisfies (T1)-(T4). Suppose, conversely, that

$\succeq_i$ satisfies (T1)-(T4). Choose an arbitrary x_0 in X. For each x in X, define $u^i(x)$ by

$$u^i(x) = \eta - \xi, \text{ where } (x, \xi) \sim_i (x_0, \eta). \tag{9}$$

The existence of such numbers ξ and η is ensured by (T3) and the difference $\eta - \xi$ is uniquely determined by (T2) and (T4). Note that (9) and (T2) imply $u^i(x_0) = 0$ and $(x, \xi) \sim_i (x_0, u^i(x) + \xi)$, i.e., $u^i(x)$ is the amount of willingness-to-pay for the transition from x_0 to x. The function $u^i(x)$ represents the preference relation $\succeq_i$. Indeed, $(x, \xi) \succeq_i (y, \eta) \iff (x_0, u^i(x) + \xi) \sim_i (x, \xi) \succeq_i (y, \eta) \sim_i (x_0, u^i(y) + \eta) \iff u^i(x) + \xi \geq u^i(y) + \eta$. □

The following facts hold (Kaneko (1976)):

$$U^i(x, \xi) = u^i(x) + \xi \text{ is quasi-concave iff } u^i(x) \text{ is concave;} \tag{10}$$

$$U^i(x, \xi) = u^i(x) + \xi \text{ is continuous iff } u^i(x) \text{ is continuous.} \tag{11}$$

In (10) and (11) some convex and topological structures on X are assumed. It follows from (10) and (11) that a condition for $\succeq_i$ to be convex or to be continuous is the concavity or continuity of u^i respectively.

3.2 Transferable utility with uncertainty

When the game situation involves uncertainty, as in Section 2.4, for example, $U^i(x, \xi) = u^i(x) + \xi$ is a von Neuman-Morgenstern utility representation. In this case, the domain of a preference relation $\succeq_i$ is the set of probability distributions on $X \times R$. We describe conditions on preference relations $\succeq_i$ with this domain to have a utility function representation of form (1).

A *probability distribution on $X \times R$ with finite support* is a function $p : X \times R \to [0, 1]$ satisfying the property that for some finite subset S of $X \times R$, $\Sigma_{t \in S}\, p(t) = 1$ and $p(t) > 0$ implies $t \in S$. We extend $X \times R$ to the set $M(X \times R)$ of all probability distributions on $X \times R$ with finite supports. Regarding a one-point distribution $f_{(x,\xi)}$ (i.e., $f_{(x,\xi)}(x, \xi) = 1$) as (x, ξ) itself, the space $X \times R$ becomes a subset of $M(X \times R)$. Also, $M(X) \times R$ is a subset of $M(X \times R)$; this is relevant in Section 2.4 (where we take X as Σ_N). For $p, q \in M(X \times R)$ and $\lambda \in [0, 1]$, we define a convex combination $\lambda p * (1 - \lambda) q$ by

$$(\lambda p * (1 - \lambda) q)(x, \xi) = \lambda p(x, \xi) + (1 - \lambda) q(x, \xi) \text{ for all } (x, \xi) \in X \times R. \tag{12}$$

With this operation, $M(X \times R)$ is a convex set. Usually, $\lambda p * (1-\lambda)q$ is regarded as a compound lottery in the sense that p and q occur with probabilities λ and $(1-\lambda)$ respectively and then the random choice according to p or q is made. Condition (12) requires that the compound lottery be composed into one lottery.

We impose the following four axioms on $\succsim_i$:

(NM1) $\succsim_i$ is a complete preordering on $M(X \times R)$;

(NM2) $p \succsim_i q \succsim_i r$ implies $\alpha p * (1-\alpha)r \sim_i q$ for some $\alpha \in [0,1]$;

(NM3) $p \succsim_i q$, $r \in M(X \times R)$ and $\alpha \in [0,1]$ imply $\alpha p * (1-\alpha)r \succsim_i \alpha q * (1-\alpha)r$;

(NM4) $p \succ_i q$ and $\alpha > \beta$ imply $\alpha p * (1-\alpha)q \succ_i \beta p * (1-\beta)q$,

where $\succ_i$ is the asymmetric part of $\succsim_i$, i.e., $p \succ_i q$ means that player i strictly prefers p to q. Condition (NM1) is the same as (T1) except that condition (NM1) is applied to the larger domain $M(X \times R)$; thus (NM1) implies (T1). Condition (NM2) states that for any lottery q between two other lotteries p and r, there is a compound lottery $\alpha p*(1-\alpha)r$ indifferent to q. Condition (NM2), as condition (T2), is a continuity property. Conditions (NM3) and (NM4) mean that the comparison of compound lotteries is based on the outcomes of these lotteries, which implies that the evaluation of a lottery depends eventually upon the sure outcomes of the lottery, as is shown in (15) below. Thus conditions (NM3) and (NM4) are called the "Sure-Thing" Principle.[2]

The following is known as the Expected Utility Theorem (cf., von Neumann-Morgenstern (1944), Herstein-Milnor (1953)):

Proposition 3.2. A preference relation $\succsim_i$ satisfies (NM1)-(NM4) if and only if there is a function $V^i : M(X \times R) \to R$ such that for any $p, q \in M(X \times R)$ and $\lambda \in [0,1]$,

$$p \succsim_i q \Longleftrightarrow V^i(p) \geq V^i(q); \text{and} \tag{13}$$

$$V^i(\lambda p * (1-\lambda)q) = \lambda V^i(p) + (1-\lambda)V^i(q). \tag{14}$$

The function V^i is called a *von Neuman-Morgenstern utility function.* In contrast to the representability in Section 3.1, $V^i(x,\xi)$ allows only a positive linear transformation, not necessarily an arbitrary monotonic transformation, i.e., if U^i also satisfies (13) and (14), there is a positive real number a and a real number b such that $U^i(p) = aV^i(p) + b$ for all $p \in M(X \times R)$.

[2]Using some additional topological conditions, (NM4) is derived from (NM3) (cf., Herstein-Milnor (1953)).

Since $X \times R$ is a subset of $M(X \times R)$, V^i assigns a value $V^i(x, \xi)$ to each (x, ξ) in $X \times R$. For each $p \in M(X \times R)$, the value $V^i(p)$ is represented as the expected value of $V^i(x, \xi)$ with $p(x, \xi) > 0$. Indeed, since each $p \in M(X \times R)$ has finite support S, by repeated applications of (14), we obtain

$$V^i(p) = \sum_{(x,\xi) \in S} p(x, \xi) V^i(x, \xi). \tag{15}$$

That is, the utility from the probability distribution p is given as the expected utility value with respect to the distribution p. This fact motivates the term "expected utility theory".

Proof of Proposition 3.2. The "if" part is straightforward. Consider the "only-if" part. Suppose that $a \succ_i b$ for some $a, b \in M(X \times R)$. If such distributions a and b do not exist, the claim is shown by assigning zero to every p. Now we define $V^i_{ab}(p)$ for any p with $a \succeq_i p \succeq_i b$ by

$$V^i_{ab}(p) = \lambda, \text{ where } p \sim_i \lambda a * (1 - \lambda) b. \tag{16}$$

The unique existence of such λ is ensured by (NM2) and (NM4). Then it follows from (NM1) and (NM4) that $V^i_{ab}(p) \geq V^i_{ab}(q) \Leftrightarrow p \succeq_i q$, which is (13). Finally, $\mu := V^i_{ab}(\lambda p * (1 - \lambda) q)$ satisfies

$$\begin{aligned}
&\mu a * (1 - \mu) b \sim_i \lambda p * (1 - \lambda) q \quad \text{(by (16))} \\
\sim_i\ &\lambda [V^i_{ab}(p) a * (1 - V^i_{ab}(p)) b] * (1 - \lambda)[V^i_{ab}(q) a * (1 - V^i_{ab}(q) b] \quad \text{(by (NM3))} \\
\sim_i\ &[\lambda V^i_{ab}(p) + (1 - \lambda) V^i_{ab}(q)] a * (1 - [\lambda V^i_{ab}(p) + (1 - \lambda) V^i_{ab}(q)]) b.
\end{aligned}$$

The coefficients for a in the first and last terms must be the same by (NM1) and (NM4), that is, $\mu = V^i_{ab}(\lambda p * (1 - \lambda) q) = \lambda V^i_{ab}(p) + (1 - \lambda) V^i_{ab}(q)$. Thus (14) holds.

It remains to extend the function V^i_{ab} to the entire space $M(X \times R)$. We give a sketch of how this extension is made (cf., Herstein-Milnor (1953) for a more detailed proof). Let c, d, e, f be arbitrary elements in $M(X \times R)$ with $e \succeq_i c \succeq_i a$ and $b \succeq_i d \succeq_i f$. Applying the above proof, we obtain utility functions V^i_{cd} and V^i_{ef} satisfying (13) and (14) with domains $\{p : c \succeq_i p \succeq_i d\}$ and $\{p : e \succeq_i p \succeq_i f\}$. Then $V^i_{cd}(c) = V^i_{ef}(e) = 1$ and $V^i_{cd}(d) = V^i_{ef}(f) = 0$. We define new utility functions U^i_{cd} and U^i_{ef} by the following positive linear transformations:

$$U^i_{cd}(p) = (V^i_{cd}(p) - V^i_{cd}(a)) / (V^i_{cd}(b) - V^i_{cd}(a)) \text{ for all } p \text{ with } c \succeq_i p \succeq_i d;$$

$$U^i_{ef}(p) = (V^i_{ef}(p) - V^i_{ef}(a)) / (V^i_{ef}(b) - V^i_{ef}(a)) \text{ for all } p \text{ with } e \succeq_i p \succeq_i f.$$

Then it can be shown that these functions U^i_{cd} and U^i_{ef} coincide on $\{p : c \succeq_i p \succeq_i d\}$. This fact ensures that we can define $V^i(p) = U^i_{cd}(p)$ for any $p \in M(X \times R)$, where c, d are chosen so that $c \succeq_i p \succeq_i d$ and $c \succeq_i a \succ_i b \succeq_i d$. Since $U^i_{cd}(p)$ satisfies (13) and (14), so does the function V^i. □

When $\succeq_i$ satisfies (T2)-(T4) on the domain $X \times R$ in addition to (NM1)-(NM4) on $M(X \times R)$, it holds that there is a monotone function $\varphi : R \to R$ satisfying

$$V^i(x, \xi) = \varphi(u^i(x) + \xi) \text{ for all } (x, \xi) \in X \times R.$$

Indeed, since the preference $\succeq_i$ over $X \times R$ is represented by $u^i(x) + \xi$ and is also represented by the restriction of V^i to $X \times R$, the functions $u^i(x) + \xi$ and $V^i(x, \xi)$ are related by a monotone transformation φ. The function φ expresses the risk attitude of player i.

For $u^i(x) + \xi$ to be a von Neuman-Morgenstern utility function, we need one more assumption:

(RN) $\frac{1}{2}(x, \xi) * \frac{1}{2}(x, \eta) \sim_i (x, \frac{1}{2}\xi + \frac{1}{2}\eta)$ for all $(x, \xi), (x, \eta) \in X \times R$.

This assumption describes risk neutrality with respect to money; given x, player i is indifferent between ξ and η with equal probabilities and the average of ξ and η.

From (RN) and (14) it follows that

$$\frac{1}{2}\varphi(u^i(x) + \xi) + \frac{1}{2}\varphi(u^i(x) + \eta) = \varphi(u^i(x) + \frac{1}{2}\xi + \frac{1}{2}h). \tag{17}$$

Indeed since ξ, η are arbitrary elements of R, $u^i(x) + \xi$ and $u^i(x) + \eta$ can take arbitrary values. Thus (17) can be regarded as a functional equation: for each α and β in R,

$$\frac{1}{2}\varphi(\alpha) + \frac{1}{2}\varphi(\beta) = \varphi(\frac{1}{2}\alpha + \frac{1}{2}\beta).$$

This, together with the monotonicity of φ, implies that φ can be represented as $\varphi(\alpha) = a\alpha + \beta$ for all α, where $a > 0$ and b are given constants. We can normalize a and b to be $a = 1$ and $b = 0$. Thus we have the following:

Proposition 3.3. A preference relation $\succeq_i$ satisfies (NM-1)-(NM4), (T2)-(T4) and (RN) if and only if $\succeq_i$ is represented by utility function of form $V^i(x, \xi) = u^i(x) + \xi$ in the sense of (13), (14) and (15).

4 Solution Concepts for Games with Side Payments

A game in characteristic function form describes what each coalition S can obtain by the cooperation of the members of S. Given a game, solution theory addresses the question of how payoffs are distributed. Each solution concept explicitly or implicitly describes the behavior of coalitions and makes some prediction on the occurrence of distributions of payoffs. Some solution concepts are faithful to the basic objective of the definition of the characteristic function discussed in Section 2, but some depend critically upon the numerical expression of the characteristic function. In this section, we discuss four solution concepts, namely, the core, the von Neumann-Morgenstern stable set, the nucleolus, and the Shapley value.

We prepare some notions before discussing solution concepts. Let a game (N, v) with side payments be given. An *imputation* is a payoff vector $(a_1, ..., a_n)$ satisfying

$$\begin{array}{ll} (\textit{Individual rationality}): & a_i \geq v(\{i\}) \text{ for all } i \in N; \\ (\textit{Feasibility}): & \Sigma_{i \in N}\, a_i = v(N). \end{array}$$

Individual rationality means that a possible candidate for the distribution of payoffs gives to each player not less than what the player can certainly obtain by himself. Feasibility means that the maximum payoff $v(N)$ that the total coalition N can obtain is distributed among the players. We denote the set of imputations by $I(N, v)$.

For imputations a and b in $I(N, v)$, we say that *a dominates b via a coalition S*, denoted by a dom_S b, iff

$$a_i > b_i \text{ for all } i \in S \tag{18}$$

and

$$v(S) \geq \Sigma_{i \in S}\, a_i. \tag{19}$$

Condition (18) means that every player in S prefers a to b and (19), called *effectiveness* by von Neumann-Morgenstern (1944), means that the imputation a is feasible for coalition S (cf.,(5), (6)). Here the characteristic function is used to describe the feasibility of $(a_i)_{i \in S}$ for coalition S. We denote a dom_S b for some S by a dom b.

4.1 The core

The *core* is defined to be the set of all undominated imputations, that is,

$$\{a \in I(N,v) : \text{not } b \textit{ dom } a \text{ for any } b \in I(N,v)\}.$$

Although the core is defined to be a set, the stability property of the core is an attribute of each imputation in the core. The core can alternatively be defined to be the set of all imputations satisfying *coalitional rationality:*

$$\Sigma_{i\in S}\, a_i \geq v(S) \text{ for all } S \in 2^N. \tag{20}$$

In the market game of Section 2.2, if $v(S) > \Sigma_{i\in S}\, a_i$ for some coalition S then there is an allocation $(x^i, \xi^i)_{i\in S}$ for S such that $u^i(x^i) + \xi^i > a^i$ for all $i \in S$, that is, the players in S can be better off by their own exchanges of commodities. The coalitional rationality of the core rules out such possibilities. This definition simply depends upon individual preferences and the feasibility described by the characteristic function. No interpersonal comparisons are involved in the definition of the core.[3,4]

For two-player games, the core is simply the imputation space $I(N,v)$. For more than two players, games may have empty cores. In the following we consider the role of side payments in some examples of games with empty cores and some with nonempty cores.

Example 3.1. Consider two different three-person voting games with total player set $N = \{1,2,3\}$ and $X = \{x,y\}$. For one game the utility functions of the players are given by the column below on the left and for the other game, by the column on the right.

[3]In the literature on market games, the nonemptiness of the core and the relationship between the core and the competitive equilibria has been extensively studied. The reader can find a comprehensive list of references in Shubik (1984).

[4]Since side payments permit unbounded transfers of the commodity "money," the competitive equilibrium concept requires some modification. A *competitive equilibrium* is a pair $(p, (x^i, p(\omega^i - x^i))_{i\in N})$ consisting of a *price vector* p and an *allocation* $(x^i, p(\omega^i - x^i)))_{i\in N}$ with the following properties:

$$u^i(x^i) + p(\omega^i - x^i) \geq u^i(y^i) + p(\omega^i - y) \text{ for all } y \in X.$$

Since money can be traded in any amount, positive or negative, the budget constraint is non-binding. Under the assumptions of concavity and continuity on the utility functions and the assumption that $\Sigma_{i\in N}\, \omega^i > 0$, the existence of a competitive equilibrium is proven by using the Kuhn-Tucker Theorem (cf., Uzawa (1958) and Negishi (1960)).

$u^1(x) = u^2(x) = 10,\ u^3(x) = 0$ $\qquad$ $\overline{u}^1(x) = \overline{u}^2(x) = 10,\ \overline{u}^3(x) = 0$
$u^1(y) = u^2(y) = 0,\ \ u^3(y) = 15$ $\qquad$ $\overline{u}^1(y) = \overline{u}^2(y) = 0,\ \overline{u}^3(y) = 0.$

The characteristic functions, defined by (7), are given as follows:

$$v(N) = 20, \qquad \overline{v}(N) = 20,$$
$$v(\{1,2\}) = 20, \qquad \overline{v}(\{1,2\}) = 20,$$
$$v(\{2,3\}) = v(\{1,3\}) = 15, \text{ and} \qquad \overline{v}(\{2,3\}) = \overline{v}(\{1,3\}) = 10, \text{ and}$$
$$v(\{1\}) = v(\{2\}) = v(\{3\}) = 0. \qquad \overline{v}(\{1\}) = \overline{v}(\{2\}) = \overline{v}(\{3\}) = 0.$$

The core of the first game (N, v) is empty. Indeed, consider the regular triangle with height 20, as in Figure 4.1.

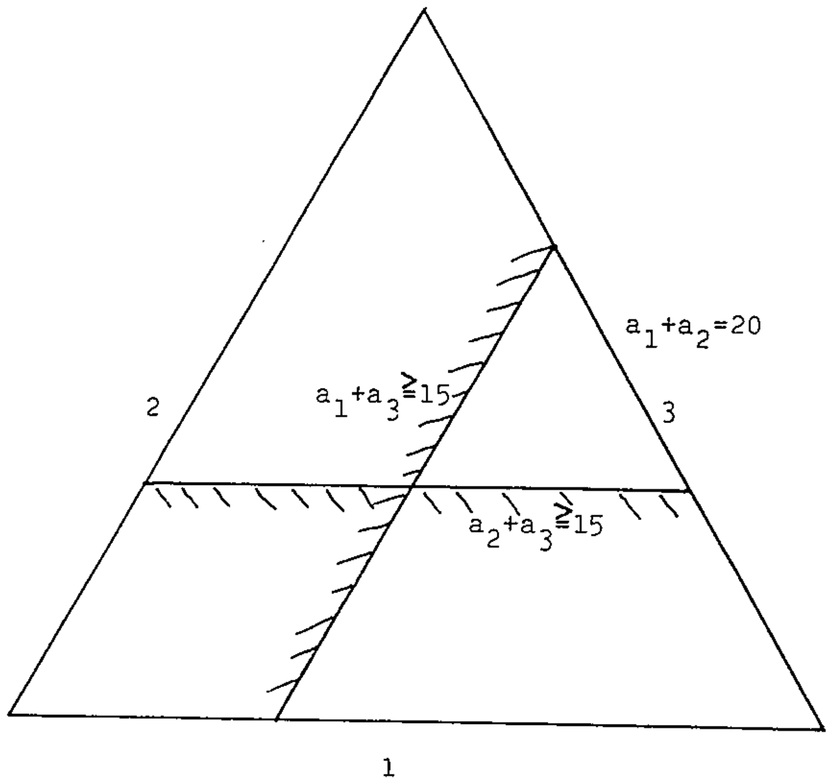

Figure 4.1.

Each point in the triangle in Figure 4.1 corresponds to a vector (a_1, a_2, a_3), where a_i is the height of the perpendicular to the base i. The inequalities

$$a_1 + a_2 \geq 20 = v(\{1,2\}),\ a_2 + a_3 \geq 15 = v(\{2,3\}),$$
$$\text{and } a_1 + a_3 \geq 15 = v(\{1,3\})$$

determine the areas that the corresponding coalitions can guarantee. The core is the intersection of those three areas. In this example, the core is empty.

For the second game $(N, \overline{v})$, the core consists of a single imputation, (10,10,0), designated by A in Figure 4.2. Since $(10,10,0) = (\overline{u}^1(x), \overline{u}^2(x), \overline{u}^3(x))$ this imputation is obtained by choosing alternative x and making no side payments. Any other imputation is dominated. For domination, side payments may be required. For example, the imputation (14,6,0) is dominated by (10,8,2). Players 2 and 3 can choose x and make a side payment of 2 units of money from player 2 to player 3, so as to ensure the payoffs of 8 and 2 for themselves.

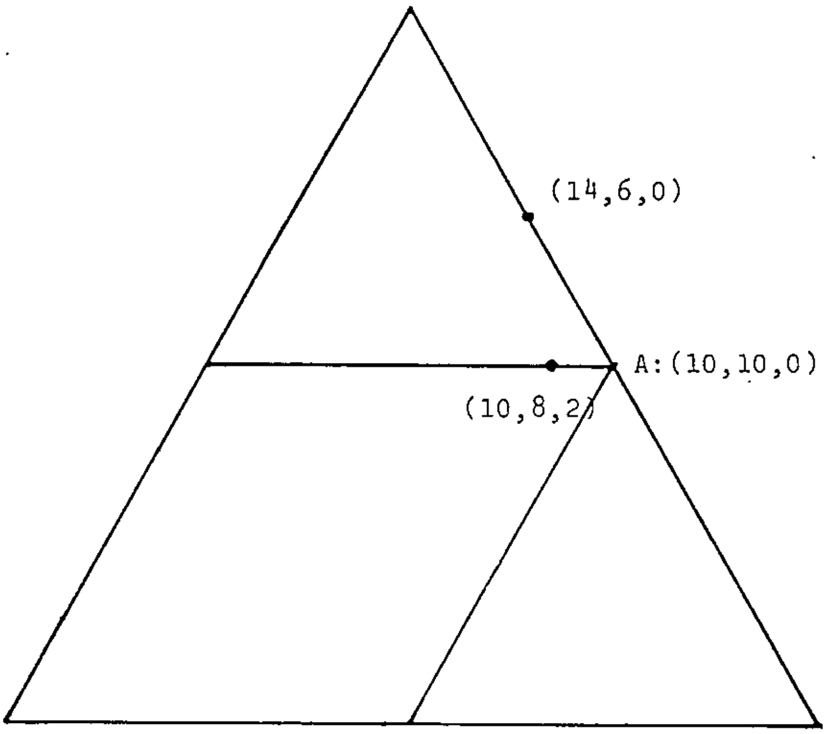

Figure 4.2

A necessary and sufficient condition for the nonemptiness of the core of the voting game in Section 2.3 was given in Kaneko (1975). This condition states that every majority coalition has the same most preferred social alternative x^*, i.e., $v(S) = \sum_{i \in S} u^i(x^*)$ for all $S \in 2^N$ with $|S| > \frac{n}{2}$. In this case the core consists of the unique payoff vector $(u^1(x^*), ..., u^n(x^*))$; the common alternative x^* is chosen and no side payments are made.

Now we will see how much side payments are required for the core. For this purpose, we consider briefly the Shapley-Shubik (1971) assignment game and its core. In the assignment game model, only pairs of players from two groups L and M ($L \cup M = N$ and $L \cap M = \emptyset$) play essential roles, i.e., an essential coalition T is $T = \{i, j\}$, $i \in L$ and $j \in M$. We denote the set of all such essential pairs by $\mathcal{P}$. Now $\Pi(S)$ denotes the set of all partitions of S into essential pairs or singleton coalitions. The value $v(S)$ of an arbitrary coalition S is obtained by partitioning coalition S into pairs and singletons, that is, a game (N, v) with side payments is called an *assignment game* iff

$$v(S) = \max_{\pi \in \Pi} \sum_{T \in \pi} v(T) \text{ for all } S \in 2^N. \tag{21}$$

The assignment game has interesting applications to markets with indivisible goods (cf., Shapley-Shubik (1971)).

For the core of the assignment game (N, v), side payments are effectively required only for essential pairs. Indeed, define a *pairwise feasible payoff vector* a by

$$a_i \geq v(\{i\}) \text{ for all } i \in N; \text{ and}$$

$$\text{for some partition } \pi \in \Pi(N),\ a_i + a_j \leq v(\{i, j\}) \text{ if } \{i, j\} \in \pi \text{ and}$$

$$a_i = v(\{i\}) \text{ if } \{i\} \in \pi.$$

That is, a pairwise feasible payoff vector is obtained by cooperation of essential pairs in some partition π. We denote the set of all pairwise feasible payoff vectors by $P(N, v)$. This set is typically much smaller than the entire imputation space $I(N, v)$. One can prove that the core of the assignment game (N, v) coincides with the set $\{a \in P(N, v) : a_i + a_j \geq v(\{i, j\})$ for all $\{i, j\} \in \mathcal{P}\}$. In the definition of a pairwise feasible payoff vector and in coalitional rationality for essential pairs $\{i, j\} \in \mathcal{P}$, side payments are allowed only between two players in each essential coalition. Thus, for the consideration of the core of an assignment game, side payments are only required within essential coalitions. In different game models, we cannot make exactly the same assertion, but often the similar tendency can be found.

4.2 The von Neumann-Morgenstern stable set

Now consider the von Neumann-Morgenstern stable set. Let (N, v) be a game with side payments. A subset K of $I(N, v)$ is called a *stable set* iff it satisfies the following two properties:

(*Internal stability*): for any $a, b \in K$, neither a *dom* b nor b *dom* a;

(*External stability*): for any $a \in I(N, v) - K$, there is $b \in K$ such that b *dom* a.

Von Neumann-Morgenstern described the stability property of a stable set as follows: each stable set is a candidate for a stable standard of behavior in recurrent situations of the game. Once a stable set has become socially acceptable, each imputation in the stable set is a possible stable (stationary) outcome. The stability of each outcome in the stable set is supported by the entire structure of the stable set. In general each game also has a great multiplicity of stable sets. Two of these stable sets for the above three person game examples are depicted in Figures 4.3 and 4.4.

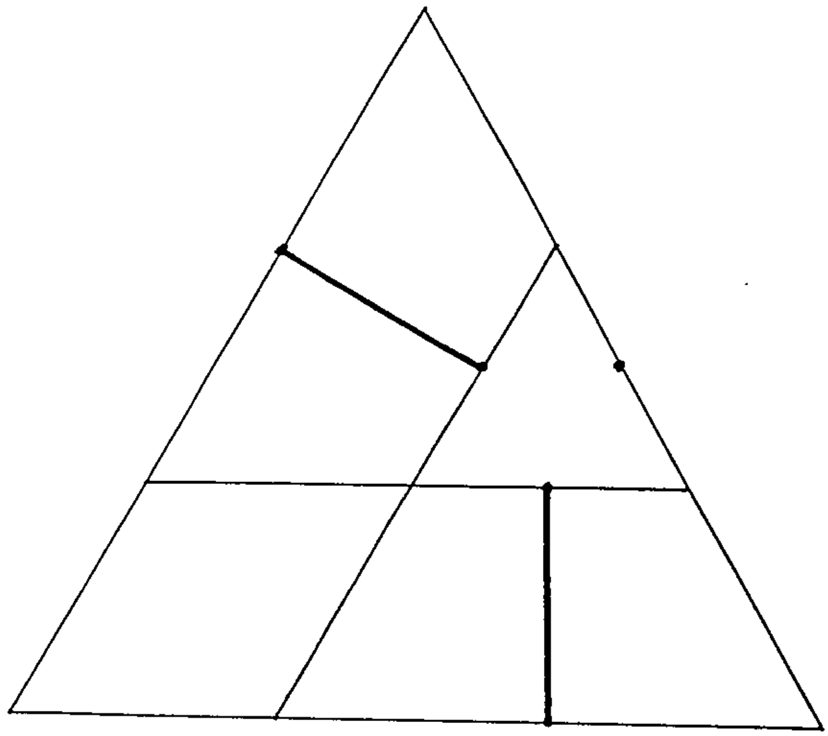

Figure 4.3

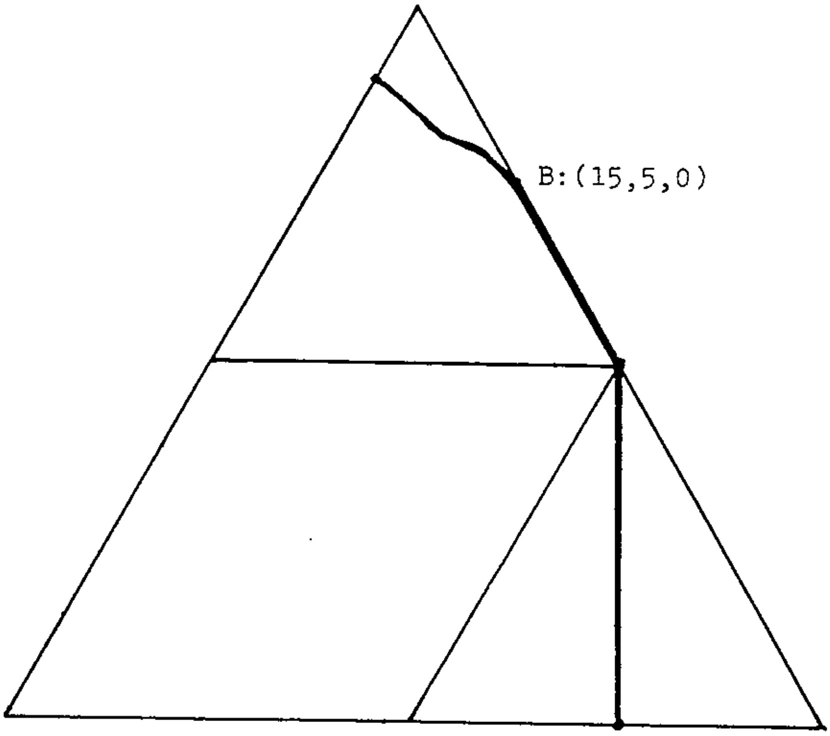

Figure 4.4

The stable sets consist of the points in the bold lines. Which outcome in a stable set and which stable set arises is determined by the history of the society. For a full explanation, see von Neumann-Morgenstern (1944).

The definition of a stable set is based on dominance relations. Thus, like the core, the definition depends only upon individual preferences and the feasibility described by the characteristic function. Nevertheless, the definition of a stable set depends crucially upon the entire imputation space $I(N, v)$, in contrast to the core. Some imputations in $I(N, v)$ need large transfers among all the players. For example, the point $B = (15, 0, 5)$ in Figure 4.4 is in the stable set and is obtained by choosing alternative x and making the transfer of 5 each to players 1 and 3 from player 2. The point $C = (0, 20, 0)$ is not in the stable set but needs to be taken into account for a stable set.

When the game involves a large number of players, the dependence of a stable set upon the entire imputation space becomes problematic. Imputations where a few players get all the surplus and the others only receive their individually rational payoffs cannot be ignored. It may require cooperation and agreement among a large number of players to make large amounts of side payments to obtain such imputations. In this case, the justification for the framework of games with side payments and no boundary conditions for money, discussed in Section 2, becomes problematic.

In the Shapley-Shubik assignment game described above, for example, the core can be defined by coalitional rationality for essential pairs in $\mathcal{P}$ and the pairwise feasible payoff space $P(N,v)$; it does not need the entire imputation space $I(N,v)$. In contrast to the core, a stable set crucially depends upon the specification of the entire feasible payoff set. If we adopt a different set of feasible payoff vectors, a stable set would change drastically, and also there remains an arbitrariness in the choice of such a space.

Here we do not intend to suggest the superiority of the core to the stable set. Apparently the stable set has a richer underlying interpretation than the core, and may give some good hints for applications of game theory to new and different models of social problems. Our intent is to suggest that simplistic applications or extensions of the stable set may violate the original justification and motivation for the framework of games with side payments.

4.3 The nucleolus

Some solution concepts make apparently intrinsic use of the monetary representation of $v(S)$. In this and the following subsections we discuss two such solution concepts, the nucleolus and the Shapley value. It is often claimed that these concepts involve interpersonal utility comparisons. We consider how we might interpret these interpersonal comparisons.

Let (N,v) be a game and let a be an imputation. Define the "dissatisfaction" of coalition $S \in 2^N$ by

$$e(a,S) = v(S) - \Sigma_{i \in S}\, a_i. \tag{22}$$

Let $\theta(a)$ be the 2^n-vectors whose components are $e(a,S), S \in 2^N$ and are ordered in a descending way, i.e., $\theta_t(a) \geq \theta_s(a)$ for all s and t from 1 to 2^n with $t \leq s$. The lexicographic ordering $\succ_l$ is defined as follows:

$$
\begin{aligned}
&a \succ_I b \text{ iff there is an } s\ (s = 1, \cdots, 2^n) \text{ such that} \\
&\theta_t(a) = \theta_t(b) \text{ for all } t = 1, ..., s-1 \text{ and } \theta_s(a) > \theta_s(b).
\end{aligned} \tag{23}
$$

The relation $\succ_I$ is a complete ordering on I. The *nucleolus* is defined to be the minimal element in I with respect to the ordering $\succ_I$. Schmeidler (1969) showed that the nucleolus exists and is unique.

The nucleolus has various technical merits. The apparent merit is the unique existence. Also, when the core is nonempty, the nucleolus belongs to the core and, for any $\epsilon \geq 0$, when the ϵ-core is nonempty, the nucleolus belongs to the ϵ-core. In the examples of Section 4.1, the nuclei are $(\frac{25}{3}, \frac{25}{3}, \frac{10}{3})$ and (10,10,0) respectively. In the first case the nucleolus is in the ϵ-core, and in the second case the nucleolus coincides with the core. The nucleolus is related to other solution concepts – the bargaining set M_1^i of Aumann-Maschler (1964) and the kernel of Davis-Maschler (1966).

The nucleolus is frequently regarded as a possible candidate for a normative outcome of a game, meaning that the nucleolus expresses some equity or fairness.[5] Sometimes, it is regarded as a descriptive concept since it always belongs to the core or the ϵ-core. Either interpretation, normative or descriptive, presents, however, some difficulties related to the treatments of transferable utility and side payments. The first difficulty is in the question of how to interpret comparisons of dissatisfactions $v(S) - \sum_{i \in S} a_i$ and $v(T) - \sum_{i \in T} b_i$ for different coalitions S, T and different imputations a, b. If the dissatisfactions are compared for a single coalition, the minimization of dissatisfaction is equivalent to the original role of $v(S)$ described by (5) and (6), but we need to make comparisons over different coalitions. The second difficulty is the lack of motivation for the criterion of lexicographic minimization of dissatisfactions.

The first difficulty consists of two parts: (A) individual utilities (gains or losses) are compared over players; and (B) sums of utilities (gains or losses) for some players are compared for different coalitions. In either case, making such comparisons already deviates from the initial intention of the characteristic function discussed in Section 2.

If the nucleolus is regarded as normative, the second difficulty is less problematic since the normative observer may be motivated to minimize dissatisfactions. The question here is the basis for the criterion of minimizing

[5]The normative aspect attributed to the nucleolus is derived chiefly by its similarity to Rawles' (1970) minmax principle or the leximin welfare function as the interpretation of the maximin principle given by economists.

dissatisfactions in the lexicographic manner. Thus the first question is more relevant form the normative viewpoint.

The intuitive appeal of the nucleolus to some researchers may be based on the feature that dissatisfactions are compared using monetary units, perhaps because monetary comparisons are familiar from our everyday life. As discussed in Section 2, the assumption of transferable utility prohibits income effects, but for distributional normative issues, income effects are central.

4.4 The Shapley value

The value, introduced by Shapley (1953), resembles the nucleolus as a game theoretical concept; it exists uniquely for any game (N, v). From the viewpoint of utility theory, the Shapley value also needs the intrinsic use of the particular definition of a characteristic function. Nevertheless, it is less problematic than the nucleolus. First, we give a brief review of the Shapley value.

Shapley (1953) derived his value originally from four axioms on a solution function. A *solution function* ψ is a function on the set Γ of all n-person superadditive characteristic function games (N, v), with fixed player set N, which assigns a payoff vector to each game. Since the player set N is fixed, the game is identified with a characteristic function v. Thus, a value function $\psi : \Gamma \to R^n$ is denoted by $\psi(v) = (\psi_1(v), ..., \psi_n(v))$. Shapley gave the following four axioms on ψ:

S1 (Pareto Optimality): for any game $(N, v) \in \Gamma$, $\sum_{i \in N} \psi_i(v) = v(N)$;

S2 (Symmetry): for any permutation π of N, $\psi(\pi v) \in (\psi_{\pi(1)}(v), ..., \psi_{\pi(n)}(v))$, where πv is defined by $\pi v(S) = v(\{\pi(i) : i \in S\})$ for all $S \in 2^N$; ←

S3 (Additivity): for any two games, $v, w \in \Gamma$, $\psi(v + w) = \psi(v) + \psi(w)$ where $v + w$ is defined by $(v + w)(S) = v(S) + w(S)$ for all $S \in 2^N$;

S4 (Dummy Axiom): for any game $v \in \Gamma$ and $i \in N$, if $v(S \cup \{i\}) = v(S) + v(\{i\})$ for all $S \in 2^N$ with $i \notin S$, then $\psi_i(v) = v(\{i\})$.

In general, the solution function ψ depends upon the game described by a characteristic function, but Axiom S2 means that ψ should not depend on the names of players given by the index numbers $1, 2, ..., n$. The mathematical meanings of the other axioms are also clear, but they have not been given a meaning beyond their mathematical expressions.

Shapley (1953) proved the following: if a solution function ψ satisfies Axioms S1 through S4, then ψ is uniquely determined as

$$\psi_i(v) = \sum_{\substack{S \subseteq N \\ S \ni i}} \frac{|S|!(n-|S|-1)!}{n!} [v(S \cup \{i\}) - v(S)] \text{ for all } i \in N. \quad (24)$$

Although each of the above axioms and Shapley's result are mathematically clear, they do not indicate the utility theory underlying the concept of the Shapley value. Formula (24), however, does provide some utility theoretic interpretation of the Shapley value.

Suppose that the players come to participate in the game in random order and that each player i gets his marginal contribution $v(S \cup \{i\}) - v(S)$ when the players S are already in the game and then player i enters the game. Before the game is played, it is equally probable for player i that he comes to the game at any place in the ordering of $1, 2, ..., n$. The probability that player i comes after the players S is given by the coefficients in formula (24). Thus player i's expected utility from this process is given as formula (24).

In the above interpretation, the utility theory underlying the Shapley value is relatively clear. The marginal contribution $v(S \cup \{i\}) - v(S)$ is the monetary payoff to player i and the expectation of these marginal payoffs is taken: the risk neutral von Neuman-Morgenstern utility function suffices. In this interpretation, however, the game is assumed to be played in a different manner than that intended by the motivation initially given for a game in characteristic function form.

Similarly to the nucleolus, the Shapley value is also interpreted as a normative (fair or equitable) outcome, mainly because of the symmetry axiom. As already mentioned, the Symmetry axiom simply states that a solution function does not depend upon the names of the players, a necessary but not sufficient condition for an equitable outcome, since the game itself may be inequitable.

5 Games Without Side Payments and Some Solution Concepts

Although a game with side payments is a convenient tool, it needs the assumption of transferable utility and side payments. The transferable utility assumption may be inappropriate for some situations in that it ignores income effects. Side payments may be prohibited or impossible. When either the assumption of transferable utility or of side payments is eliminated, we need games without side payments. In this section we discuss a game without side payments together with some solution concepts from the viewpoint of utility theories.

The term "a game without side payments" is slightly misleading in the sense that the framework allows games with side payments and transferable utility as special cases. However, we follow this standard terminology.

5.1 Games without side payments

A game without side payments is given as a pair (N, V) consisting of the player set N and a characteristic function V on 2^N. For each coalition S, the set $V(S)$ is a subset of R^S, where R^S is $|S|$ –dimensional Euclidean space with coordinates labelled by the members of S.[6] The set $V(S)$ describes the set of all payoff vectors for coalition S that are attainable by the members in S themselves. We assume the following technical conditions: for all $S \in 2^N$,

$$V(S) \text{ is a closed subset of } R^S; \tag{25}$$

$$a^S \in V(S) \text{ and } b^S \leq a^S \text{ imply } b^S \in V(S); \tag{26}$$

$$\{a^S \in V(S) : a_i^S \geq \max\ V(\{i\}) \text{ for all } i \in S\} \text{ is nonempty and bounded.}$$

For games without side payments, *superadditivity* is given by

$$V(S) \times V(T) \subseteq V(S \cup T) \text{ for all } S, T \in 2^N \text{ with } S \cap T = \emptyset. \tag{27}$$

Within the framework of games without side payments, a game with side payments is described as

$$V(S) = \{a^S : \Sigma_{i \in S}\, a_i^S \leq v(S)\} \text{ for all } S \in 2^N. \tag{28}$$

Thus $V(S)$ describes directly the set of attainable payoffs for S. The three examples of games with side payments in Section 2 are directly described by (28). It will be seen below that using the framework without side payments, the assumptions of transferable utility and side payments are not needed.

A game without side payments is a heavy mathematical tool. It is suitable to discuss general problems such as the nonemptinesss of the core (cf.,

[6] $R^{\emptyset} = \{0\}$.

Scarf (1967)) but when a specific game situation is given, it is often more convenient to work on the situation directly instead of describing it as a game without side payments. Nevertheless, in order to see general principles underlying cooperative games, it is useful to formulate game situations in terms of games without side payments. In the following, we will see the descriptions of the examples given in Section 2 in terms of games without side payments.

5.2 Examples

5.2.1 Market games

Consider a market game with n players and $m+1$ commodities. In contrast to the previous formulation of market games with quasi-linear utility functions, we now assume that the continuous utility function U^i is defined on R_+^{m+1} and the endowment of player i is given as a vector in R_+^{m+1}. The $m+1^{th}$ commodity is treated in the same way as the first m commodities. An *S-allocation* $(x^i)_{i\in S}$ is defined by $\Sigma_{i\in S}\, x^i = \Sigma_{i\in S}\, \omega^i$ and $x^i \in R_+^{m+1}$ for all $i \in S$. The characteristic function V is defined by

$$V(S) = \{a^S \in R^S : a_i^S \leq U^i(x^i) \text{ for some } S\text{-allocation } (x^i)_{i\in S}\} \quad \text{for all } S \in 2^N. \tag{29}$$

Then this characteristic function V satisfies conditions (25)- (27). For this definition, only the existence of a utility function U^i representing a preference relation $\succeq_i$ is required (see Debreu (1957) for conditions ensuring the existence of a continuous utility function).

For the definition (29), we can assume that the utility function U^i satisfies the transferable utility assumption, i.e., the linear separability. If, however, the endowments of ω_{m+1}^i of the $m+1^{th}$ commodity are small, then side payments may not be freely permitted. If the endowments ω_{m+1}^i are sufficiently large to avoid the relevant constraints, then side payments are effectively unbounded. This is the case of a market game in Section 2.2. Nevertheless, side payments are part of the problem.

5.2.2 Voting games

Consider a voting game where the assumption of transferable utility is satisfied but no side payments are permitted. In such a case, the characteristic function is given by

$$\begin{aligned} V(S) &= \{a^S \in R^S : \text{for some } x \in X,\ a_i \leq u^i(x) \text{ for all } i \in S\} \text{ if } |S| > \tfrac{n}{2} \\ &= \{a^S \in R^S : \text{for all } x \in X,\ a_i \leq u^i(x) \text{ for all } i \in S\} \text{ if } |S| \leq \tfrac{n}{2}. \end{aligned} \tag{30}$$

This majority voting game has been extensively discussed in the social choice literature (Nakamura (1975), Moulin (1988), for example.)

Observe that in the above voting game, we have the independence of the two assumptions of transferrable utility and side payments. A game makes the transferable utility assumption, which is not necessarily required, but also makes the assumption that side payments are not allowed.

5.2.3 Cooperative games derived from strategic games

Suppose that side payments are not allowed in the normal form game $G = (N, \{\Sigma_i\}_{i\in N}, \{h_i\}_{i\in N})$. This means that either the economy including the game G has money but money transfers are prohibited, or that G is a full description of the game in question and nothing other than in the game is available in playing the game. In either case, the relevant utility functions of players, given by $\{h_i\}_{i\in N}$, are von Neumann-Morgenstern utility functions over the domain $M(\Sigma_N)$.

Corresponding to definition (8), the characteristic function V_α is defined by

$$\begin{gathered} V_\alpha(S) = \{a \in R^S : \text{ there is some } \sigma \in M(\Sigma_S) \text{ such that} \\ \text{for any } \sigma_{-S} \in M(\textstyle\sum_{N-S}), a_i \leq h_i(\sigma_S, \sigma_{-S}) \text{ for all } i \in S\} \text{ for all } S \in 2^N. \end{gathered} \tag{31}$$

The value $V_\alpha(S)$ of the characteristic function V_α is the set of all payoff vectors for the members of the coalition S that can be obtained with certainty by the cooperation of the members of S. This is a faithful extension of definition (8) in the absence of side payments. In (8), in fact, the min-max value, which is obtained by changing the order of the max and min operators, coincides with the value of (8) because of the von Neumann mini-max Theorem. This suggests another definition of a characteristic function;

$$\begin{gathered} V_\beta(S) = \{a \in R^S : \text{ for any } \sigma_{-S} \in M(\Sigma_{N-S}) \text{ there is some } \sigma_S \in M(\Sigma_S) \\ \text{such that } a_i \leq h_i(\sigma_S, \sigma_{-S}) \text{ for all } i \in S\} \text{ for all } S \in 2^N. \end{gathered} \tag{32}$$

Unlike games with side payments, these two definitions may give different sets (cf., Aumann (1961)). The first and second are often called the α and β – *characteristic functions.* A general nonemptiness result for the α-core, defined using the $\alpha-$ characteristic function, is obtained in Scarf (1971). The β-core, defined by the $\beta-$characteristic function, is closely related to the folk theorem for repeated games (cf., Aumann (1959,1981)).

5.3 Solution concepts

The characteristic function V of a game without side payments describes, for each coalition S, the set of payoff vectors attainable by the members of S. Once this is given, the imputation space and dominance relations are extended to a game without side payments in a straightforward manner. The imputation space $I(N,V)$ is simply the set

$$\{a \in V(N) : a_i \geq \max V(\{i\}) \text{ for all } i \in N\}.$$

The dominance relation a *dom* b is defined by:

$$\text{for some } S \in 2^N,\ a_i > b_i \text{ for all } i \in S \text{ and and } (a_i)_{i \in S} \in V(S).$$

The core is defined to be the set of all undominated imputations. The von Neumann-Morgenstern stable set is also defined with the internal and external stability requirements in the same way as in a game with side payments.

Consider the core and stable set for a voting game without side payments for Example 3.1. Since no transfer of money is allowed, the problem is which alternative x or y to choose. In both examples, players 1 and 2 prefer x to y and thus x is chosen. Actually, x constitutes the core and also the unique stable set. In the first example, when side payments are involved, Player 3 can compensate for Player 1 or 2 to obtain his cooperation for the alternative y. This causes the core to be empty. Our point is that the possibility of side payments may drastically change the nature of the game. But this is almost independent of the assumption of the quasi-linear form of the utility function.

The nucleolus and Shapley value are based intrinsically on the numerical expression of the characteristic function with side payments. Nevertheless, some authors modify the definitions of these concepts. Here we discuss only one example – the λ-transfer value introduced by Shapley (1969).

Shapley transformed a game (N, V) without side payments into a game (N, v_λ) with side payments by using "utility transfer weights" $\lambda = (\lambda_1, ..., \lambda_n) > 0$ by defining

$$v_\lambda(S) = \max\{\sum_{i \in S} \lambda_i a_i : a \in V(S)\} \text{ for all } S \in 2^N. \tag{33}$$

The $\lambda-$transfer value is defined as follows: a payoff vector $a = (a_1, ..., a_n)$ is a *λ-transfer value* iff there are transfer weights $\lambda = (\lambda_1, ..., \lambda_n) > 0$ such that a is the Shapley value of the game (N, v_λ) and a is feasible in (N, V) i.e., $a \in V(N)$.

Shapley (1969) proved the existence of a λ-transfer value for a game without side payments, but the uniqueness property does not hold. Aumann (1985) provided an axiomatization of the $\lambda-$transfer value.

From the viewpoint of utility theory, it is difficult to evaluate the $\lambda-$transfer value. First, the meaning of the transformation from (N, V) to (N, v_λ) is unclear: some authors claim the utility units are compared with these weights, but this does not imply that the game can be transformed into a game with side payments. Second, the axiomatization does not help us evaluate the status. A question of a meaningful interpretation of the $\lambda-$transfer value remains open.

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