

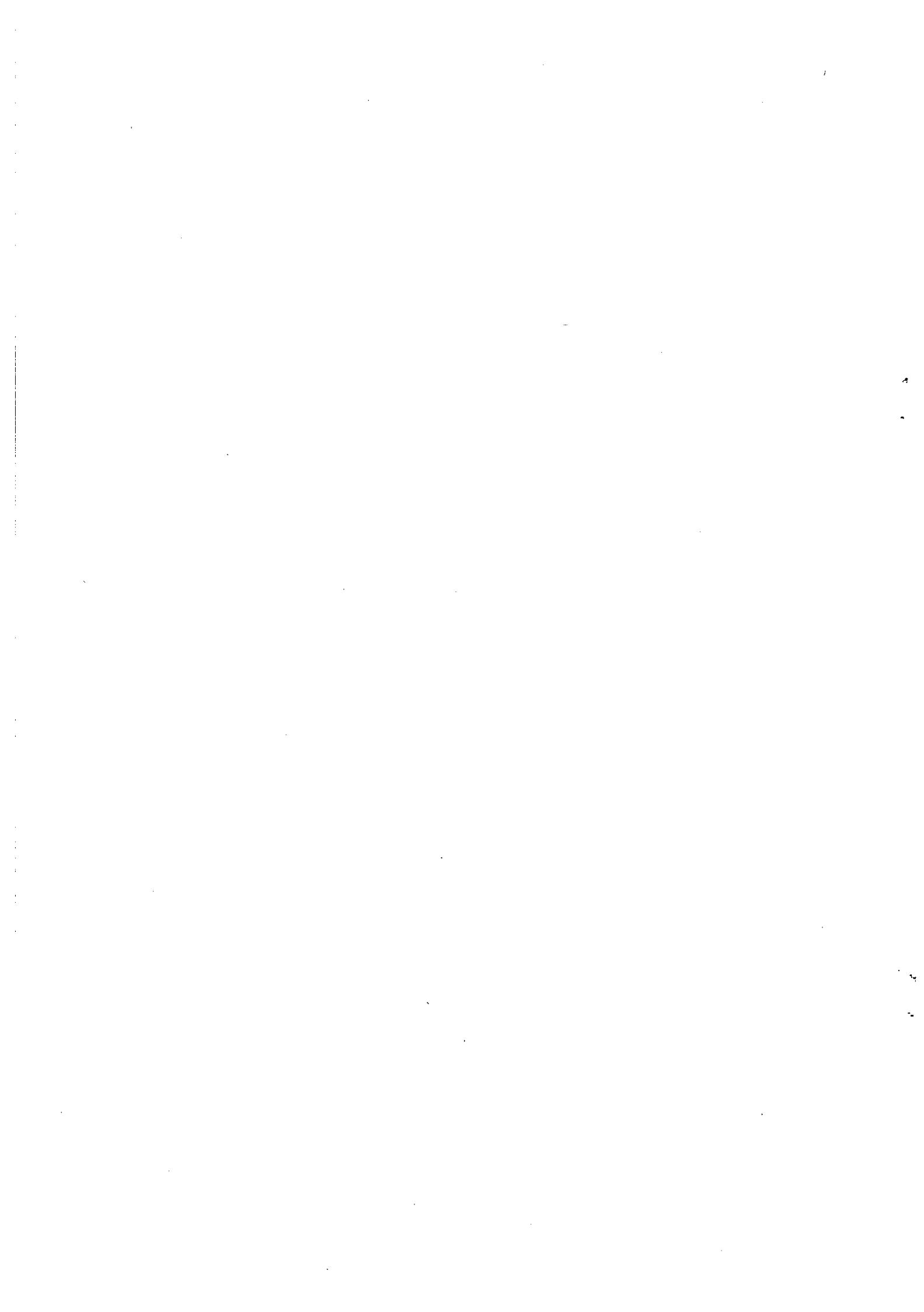
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Symbolic Moment Calculation
for an M/G/1 Queue

by

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Symbolic Moment Calculation for an M/G/1 Queue

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We show a sequence of several commands of *Mathematica*, a software with functions for symbolic formula manipulation, that calculates algebraically the distribution moment of any order from the Pollaczek-Khinchine formula for the waiting time of an M/G/1 queue with first-come first-served discipline. An additional command generates the LATEX source statements for formatting the resultant equations. Similar programs are also provided for the queue size, the length of a busy period, and the waiting time in the last-come first-served system. The moments of up to the 10th order for these quantities are explicitly displayed.

Key Words: Queue, M/G/1, symbolic calculation, waiting time, queue size, busy period.

1. Introduction

The waiting time, the queue size, and the length of a busy period are basic characteristics for an M/G/1 queueing system. Therefore, every standard textbook on queueing theory, for example, Cooper[1981] and Kleinrock [1975], shows the equations for their distributions in the form of transforms, and derives the expressions for their mean values. Calculating the higher-order moments is straightforward in principle, because it only involves the evaluation of the derivatives of a given function or a given functional equation. However, the expressions for these derivatives soon become so complicated as the order goes up that only the first few moments are explicitly available in the literature. Such limitation of manual calculation can be broken through by using a software package for symbolic formula manipulation that prevails today.

In this paper, we show that a set of several functions of *Mathematica* [Wolfram 1991] can calculate the moments of up to any order (as far as the computational time and the memory space allow) for the waiting time, the queue size, and the length of a busy period. The results are given in terms of the arrival rate and the moments of the service time. An additional command can be used to generate the LATEX source statements for formatting the resultant equations. This method is in contrast with the numerical evaluation of higher-order moments for the length of a busy period by Klimko and Neuts [1973]. Our symbolic moment calculation also complements the use of *Mathematica* for the numerical evaluation of formulas and the discrete event simulation for computer performance analysis shown by Allen [1990, 1994].

Specifically, we deal with the waiting time and the response time (the waiting time plus the service time) in the first-come first-served (FCFS) system in Section 2. In Section 3, we consider the queue size in the system as well as that in the waiting line. In Section 4, we treat the length of a busy period. In Section 5, we handle the waiting time in the last-come first-served (LCFS) system.

In each of these sections, we start with the equation for the quantity whose moments we are to calculate. We comment on the available results. We then provide a *Mathematica* program that symbolically calculates the moments of any order, rearranges the terms for better appearance, and finally generates the *LATEX* source statements for formatting.

As examples, we obtained up to the 10th moments of each quantity. Minor cosmetic changes (such as line breaking and global symbol replacement) were made manually on the *LATEX* source for the final expression. The results are shown in the Appendices. Some comments on extension and limitation of the present technique are mentioned in Section 6. All the programs in this paper were written and executed using *Mathematica* Version 2.2 for SPARC from Wolfram Research, Inc. [Wolfram 1991].

Let us introduce the notation for the M/G/1 queueing system considered in this paper. The arrival rate is denoted by λ . The Laplace-Stieltjes transform (LST) of the distribution function (DF), the mean, and the n th moment of the service time are denoted by $B^*(s)$, b , and $b^{(n)}$ ($n = 2, 3, \dots$), respectively. The traffic intensity is defined as $\rho := \lambda b$, which is assumed to be less than unity for the stability of the queue. The LST of the DF for the waiting time W of a customer is denoted by $W^*(s)$. Similarly, that for the response time T is denoted by $T^*(s)$. The probability generating function (PGF) for the queue size L in the system at an arbitrary time (also immediately after a service completion) is denoted by $\Pi(z)$. Similarly, the PGF for the queue size ℓ in the waiting line at an arbitrary time (also immediately after a service start) is denoted by $\pi(z)$. Finally, the LST of the DF for the length Θ of a busy period is denoted by $\Theta^*(s)$.

2. Waiting Time and Response Time in an FCFS System

The waiting time W of a customer is a time interval from the arrival to the service start. The LST $W^*(s)$ of the DF for W in an FCFS system is given by the *Pollaczek-Khinchine formula* [Cooper 1981 (sec. 5.8); Kleinrock 1975 (sec. 5.7)]:

$$W^*(s) = \frac{(1 - \rho)s}{s - \lambda + \lambda B^*(s)} \quad (2.1)$$

The mean waiting time can be obtained by evaluating the first derivative of (2.1) at $s = 0$, which involves the application of L'Hôpital's rule twice. The result is given by

$$E[W] = -W^{*(1)}(0) = \frac{\lambda b^{(2)}}{2(1 - \rho)} \quad (2.2)$$

In order to obtain the n th moment $E[W^n] = (-1)^n W^{*(n)}(0)$, we can differentiate the r.h.s. of (2.1) n times and evaluate at $s = 0$; but this would be very cumbersome even for $n = 3$. Instead, we write (2.1) as

$$W^*(s) \left[1 - \lambda \frac{1 - B^*(s)}{s} \right] = 1 - \rho \quad (2.3)$$

and differentiate both sides n times and let $s = 0$. We then get the recursive relation that expresses $E[W^n]$ in terms of $E[W^m]$, $1 \leq m \leq n - 1$ [Takács 1962]

$$E[W^n] = \frac{\lambda}{1 - \rho} \sum_{m=1}^n \binom{n}{m} \frac{b^{(m+1)}}{m+1} E[W^{n-m}] \quad n = 1, 2, \dots \quad (2.4)$$

```

f[s] = (1 - lambda b) s / (s - lambda + lambda B[s])
B[0] = 1
B'[0] = -b
Derivative[n_][B][0] = Derivative[n][b]*(-1)^n
taylor = Series[f[s], {s, 0, 10}]
coeff[n_] := Coefficient[taylor, s^n]*n!*(-1)^n
ew[n_] := Expand[Apart[coeff[n], b]]
ew2tex[n_] := Do[TeXForm[ew[i]] >>> waiting_time.tex, {i, 1, n}]

```

Figure 1. Symbolic calculation of the moments of the waiting time for an FCFS M/G/1 system: Taylor series expansion of the Pollaczek-Khinchine formula.

Takács [1963] also shows how to calculate $E[W^n]$ by evaluating the n th derivative of (2.1) at $s = 0$ using the Faà di Bruno formula, which is a formula for the n th derivative of a composite function. He thus gives the expressions for $E[W^2]$ and $E[W^3]$.

A *Mathematica* program that calculates $E[W^n]$ by evaluating $W^{*(n)}(0)$ from the Taylor series expansion of (2.1) is given in Figure 1. Let us annotate each line. Line 1 defines the function $W^*(s)$, that corresponds to (2.1). Lines 2 through 4 simply specifies the replacement of $B^*(0)$ by 1, $B^{*(1)}(0)$ by $-b$, and $B^{*(n)}(0)$ by $(-1)^n b^{(n)}$ whenever these expressions appear in the subsequent calculation. Line 5 expands $W^*(s)$ in Taylor series around $s = 0$ up to $o(s^{10})$. Line 6 sets the coefficient of s^n in the Taylor series, multiplied by $(-1)^n n!$, into $\text{coeff}[n]$, which is already $E[W^n]$. Line 7 rearranges the terms with respect to the power of b . Line 8 requests to generate the *LATEX* source statements for $E[W^i]$, $i = 1-n$, in the *waiting_time.tex* file. In this file we make several global changes of symbols manually. The results of formatting are shown in Appendix A.

Note that we have used the Taylor series expansion in order to obtain the derivatives of (2.1) at $s = 0$. This is because the *Limit* function of *Mathematica* does not handle the indefinite form 0/0.

The LST $T^*(s)$ of the DF for the response time T is given by

$$T^*(s) = W^*(s)B^*(s) = \frac{(1 - \rho)sB^*(s)}{s - \lambda + \lambda B^*(s)} \quad (2.5)$$

The mean response time is then given by

$$E[T] = -T^{*(1)}(0) = \frac{\lambda b^{(2)}}{2(1 - \rho)} + b \quad (2.6)$$

For the calculation of the moment $E[T^n]$, we can use the same *Mathematica* program as in Figure 1 with the only replacement of Line 1 by

```
f[s] = (1 - lambda b) s B[s] / (s - lambda + lambda B[s])
```

The moments $E[T^n]$ for $n = 1-10$ thus obtained are shown in Appendix B.

From (2.5), we easily get [Allen 1990 (sec. 5.3.1); Kleinrock 1975 (sec. 5.7)]

$$E[T^n] = \sum_{m=0}^n \binom{n}{m} b^{(m)} E[W^{n-m}] \quad n = 0, 1, 2, \dots \quad (2.7)$$

Comparing (2.4) and (2.7), we get the relation [Takács 1963]

$$E[T^n] = E[W^n] + \frac{n}{\lambda} E[W^{n-1}] \quad n = 2, 3, \dots \quad (2.8)$$

3. Queue Sizes in the System and in the Waiting Line

We next consider the number of customers (queue size) L present in the system at an arbitrary time. The PGF $\Pi(z)$ for L is also given by the *Pollaczek-Khinchine formula* [Cooper 1981 (sec. 5.8); Kleinrock 1975 (sec. 5.6)]:

$$\Pi(z) = \frac{(1-\rho)(1-z)B^*(\lambda - \lambda z)}{B^*(\lambda - \lambda z) - z} \quad (3.1)$$

The mean queue size can be obtained by evaluating the first derivative of (3.1) at $z = 1$, which needs to use L'Hôpital's rule twice. The result is given by

$$E[L] = \Pi^{(1)}(1) = \frac{\lambda^2 b^{(2)}}{2(1-\rho)} + \rho \quad (3.2)$$

Allen [1990 (sec. 5.7, prob. 40)] gives $\text{Var}[L]$.

If we differentiate the r.h.s. of (3.1) m times and evaluate at $z = 1$, we get the m th factorial moment of L :

$$E[L(L-1)(L-2)\cdots(L-m+1)] = \Pi^{(m)}(1) \quad m = 1, 2, \dots \quad (3.3)$$

The n th moment $E[L^n]$ is then obtained by

$$E[L^n] = \sum_{m=1}^n \begin{Bmatrix} n \\ m \end{Bmatrix} \Pi^{(m)}(1) \quad n = 1, 2, \dots \quad (3.4)$$

where $\begin{Bmatrix} n \\ m \end{Bmatrix}$ is a *Stirling number of the second kind* generated by [Knuth 1973 (sec. 1.2.6)]:

$$\begin{Bmatrix} n \\ 0 \end{Bmatrix} = 0, \quad \begin{Bmatrix} n \\ n \end{Bmatrix} = 1, \quad \begin{Bmatrix} n \\ m \end{Bmatrix} = m \begin{Bmatrix} n-1 \\ m \end{Bmatrix} + \begin{Bmatrix} n-1 \\ m-1 \end{Bmatrix}, \quad n > m \geq 1 \quad (3.5)$$

Figure 2 shows a *Mathematica* program that calculates $E[L^n]$ by evaluating $\Pi^{(m)}(1)$ ($m = 1, 2, \dots, n$) from the Taylor series expansion of (3.1) around $z = 1$, and then using (3.4). Line 1 defines the function $\Pi(z)$ as in (3.1). Line 5 expands $\Pi(z)$ in Taylor series around $z = 1$ up to $o((z-1)^{10})$. Line 6 sets the coefficient of $(z-1)^n$ in the Taylor series, multiplied by $n!$, into $\text{coeff}[n]$, which is $\Pi^{(n)}(1)$. Line 7 calculates $E[L^n]$ and rearranges the terms, using the Stirling number of the second kind $\begin{Bmatrix} n \\ m \end{Bmatrix}$ given by $\text{StirlingS2}[n, m]$. Line 8 generates the *LATEX* source statements for $E[L^n]$, $n = 1-10$, in the *queue_size.tex* file. The results of formatting are given in Appendix C.

Note that the PGF $\Pi(z)$ in (3.1) is also the PGF for the queue size in the M/G/1 system immediately after a service completion (departure of a customer). Therefore, we have the relation

$$\Pi(z) = T^*(\lambda - \lambda z) \quad (3.6)$$

```

g[z] = (1 - lambda b)(1-z) B[lambda - lambda z]/(B[lambda - lambda z]-z)
B[0] = 1
B'[0] = - b
Derivative[n_][B][0] = Derivative[n][b]*(-1)^n
taylor = Series[g[z],{z,1,10}]
coeff[n_] := Coefficient[taylor,(-1+z)^n] * n!
eL[n_] := Expand[Apart[Sum[StirlingS2[n,m]*coeff[m],{m,1,n}],b]]
eL2tex[n_] := Do[TeXForm[eL[i]] >>> queue_size.tex,{i,1,n}]

```

Figure 2. Symbolic calculation of the moments of the queue size in an M/G/1 system: Taylor series expansion of the Pollaczek-Khinchine formula.

where $T^*(\lambda - \lambda z)$ is the PGF for the number of customers that arrive while a departing customer was in the system.

From (3.6), we get the relation [Allen 1990 (sec. 5.3.1), Kleinrock 1975 (prob. 5.25), Marshall and Wolff 1971]

$$E[L(L-1)(L-2)\cdots(L-n+1)] = \lambda^n E[T^n] \quad n = 1, 2, \dots \quad (3.7)$$

Therefore, we have the following relationship for the moments of the queue size and response time [Keilson and Servi 1990]:

$$E[L^n] = \sum_{m=1}^n \begin{Bmatrix} n \\ m \end{Bmatrix} \lambda^m E[T^m] \quad n = 1, 2, \dots \quad (3.8)$$

$$\lambda^n E[T^n] = \sum_{m=1}^n (-1)^{n-m} \begin{Bmatrix} n \\ m \end{Bmatrix} E[L^m] \quad n = 1, 2, \dots \quad (3.9)$$

where $\begin{Bmatrix} n \\ m \end{Bmatrix}$ is a *Stirling number of the first kind* generated by [Knuth 1973 (sec. 1.2.6)]:

$$\begin{Bmatrix} n \\ 0 \end{Bmatrix} = 0, \quad \begin{Bmatrix} n \\ n \end{Bmatrix} = 1, \quad \begin{Bmatrix} n \\ m \end{Bmatrix} = (n-1) \begin{Bmatrix} n-1 \\ m \end{Bmatrix} + \begin{Bmatrix} n-1 \\ m-1 \end{Bmatrix}, \quad n > m \geq 1 \quad (3.10)$$

The PGF $\pi(z)$ for the number ℓ of customers present in the waiting line at an arbitrary time (also immediately after a service start) is given by

$$\pi(z) = \frac{(1-\rho)(1-z)}{B^*(\lambda - \lambda z) - z} \quad (3.11)$$

The moments of ℓ are related with the moments of L by

$$E[L^n] = \rho + \sum_{m=1}^n \begin{Bmatrix} n \\ m \end{Bmatrix} E[\ell^m] \quad n = 1, 2, \dots \quad (3.12)$$

$$E[\ell^n] = (-1)^n \rho + \sum_{m=1}^n \begin{Bmatrix} n \\ m \end{Bmatrix} (-1)^{n-m} E[L^m] \quad n = 1, 2, \dots \quad (3.13)$$

From the relation

$$\pi(z) = W^*(\lambda - \lambda z) \quad (3.14)$$

```

(* Initial Setup *)
ew[0] = 1
Derivative[0][B][0] = 1
Derivative[1][B][0] = -rho/lambda
Derivative[n_][B][0] := Derivative[n][b]*(-1)^n
(* Finding E[W] *)
ew[1] = lambda Derivative[2][B][0]/(2(1-rho))
f[n_,m_] := Binomial[n,m] * Derivative[m+1][B][0]*(-1)^(m+1)/(m+1)*ew[n-m]
ew[n_] := ew[n] = Expand[lambda/(1-rho) * Sum[f[n,m],{m,1,n}]]
Do[TeXForm[ew[i]] >>> waiting_time.tex,{i,1,10}]
(* Finding E[T] *)
et[1] = rho/lambda + lambda Derivative[2][B][0]/(2(1-rho))
et[n_] := et[n] = Expand[ew[n]+n/lambda ew[n-1]]
Do[TeXForm[et[i]] >>> response_time.tex,{i,1,10}]
(* Finding E[L] *)
g[n_,m_] := StirlingS2[n,m]*lambda^m et[m]
eL[n_] := eL[n] = Expand[Sum[g[n,m],{m,1,n}]]
Do[TeXForm[eL[i]] >>> system_size.tex,{i,1,10}]
(* Finding E[l] *)
h[n_,m_] := Binomial[n,m]*(-1)^(n-m)*eL[m]
el[n_] := el[n] = Expand[(-1)^n*rho + Sum[h[n,m],{m,1,n}]]
Do[TeXForm[el[i]] >>> waiting_line.tex,{i,1,10}]

```

Figure 3. Recursive calculation of the moments for an FCFS M/G/1 system.

which can be obtained from the same reasoning as for (3.6), we get

$$E[\ell(\ell-1)(\ell-2)\cdots(\ell-n+1)] = \lambda^n E[W^n] \quad n = 1, 2, \dots \quad (3.15)$$

For the calculation of $E[\ell^n]$, we can use the same *Mathematica* program as in Figure 2 with the only replacement of Line 1 by

$$g[z] = (1 - \text{lambda } b)(1-z) / (\text{B}[\text{lambda} - \text{lambda } z] - z)$$

Appendix D shows the moments $E[\ell^n]$ for $n = 1-10$.

Capitalizing on the relationships shown above for the moments of W , T , L , and ℓ , we can also calculate them recursively starting with $E[W]$ given in (2.2). Namely, we obtain $E[W^n]$ from (2.4), $E[T^n]$ from (2.8), $E[L^n]$ from (3.8), and $E[\ell^n]$ from (3.13). This is done in a *Mathematica* program shown in Figure 3. We have confirmed that the moments calculated by this program agree with those by the previous programs.

4. Length of a Busy Period

The LST $\Theta^*(s)$ of the DF for the length Θ of a busy period in a stable M/G/1 system is given as the unique solution to the *Takács equation* [Cooper 1981 (sec. 5.8), Kleinrock 1975 (sec. 5.8)]:

$$\Theta^*(s) = B^*[s + \lambda - \lambda\Theta^*(s)] \quad (4.1)$$

```

eq = t[s] == B[s + lambda - lambda t[s]]
t[0] = 1
B[0] = 1
B'[0] = -rho/lambda
Derivative[n_][B][0] = Derivative[n][b]*(-1)^n
solve[n_] := Derivative[n][t][0] = Derivative[n][t][0] /.
    Solve[D[eq,{s,n}], s->0,Derivative[n][t][0]][[1]]
busy[n_] := Expand[Apart[solve[n]*(-1)^n,lambda]]
busy2tex[n_] := Do[TeXForm[busy[i]] >>> busy_period.tex,{i,1,n}]

```

Figure 4. Symbolic calculation of the moments of the length of a busy period in an M/G/1 system: Differentiation of the Takács equation.

```

f[n_] = (s/(s-lambda+lambda B[s]))^n
B[0] = 1
B'[0] = -b
Derivative[n_][B][0] = Derivative[n][b]*(-1)^n
busy[1] = b/(1 - lambda b )
busy[n_] := Expand[(-1)^(n-1)/lambda*(n-1) !
    *Coefficient[Series[f[n],{s,0,n-1}],s^(n-1)]]
busy2tex[n_] := Do[TeXForm[busy[i]] >>> busy_period.tex,{i,1,n}]

```

Figure 5. Symbolic calculation of the moments of the length of a busy period in an M/G/1 system: Evaluation by the Bürmann theorem.

If we differentiate both sides of (4.1) and let $s = 0$, we get an equation for $\Theta^{*(1)}(0)$, which solves to give

$$E[\Theta] = -\Theta^{*(0)}(0) = \frac{b}{1-\rho} \quad (4.2)$$

Higher-order moments of Θ can also be found by solving the recursive equations obtained by successively differentiating (4.1) and letting $s = 0$. Riordan [1962 (sec. 4.8)] gives $E[\Theta^n]$ for $n = 2, 3$, and 4 in this manner.

Figure 4 shows a *Mathematica* program that calculates $E[\Theta^n]$ using this algorithm. Line 1 defines the equation in (4.1). Line 6 solves the equation for $\Theta^{*(n)}(0)$ obtained by differentiating (4.1) n times and letting $s = 0$, and places the solution into `solve[n]`. Line 7 rearranges the terms.

Alternatively, Takács [1963] gives the explicit formula

$$E[\Theta^n] = \frac{(-1)^{n-1}}{\lambda} \left[\frac{d^{n-1}}{ds^{n-1}} \left(\frac{s}{s - \lambda + \lambda B^*(s)} \right)^n \right]_{s=0} \quad n = 2, 3, \dots \quad (4.3)$$

This is derived by using the Lagrange's theorem or Bürmann's theorem on the Taylor series expansion of an implicit function. Using (4.3), he also gives $E[\Theta^n]$ for $n = 2, 3$, and 4.

Figure 5 shows a *Mathematica* program for calculating $E[\Theta^n]$ by using (4.3). Note that Line 6 calculates the derivative and rearranges the resultant terms at once. The moments $E[\Theta^n]$ for $n = 1-10$ by both programs are given in Appendix E.

In passing, we note that the LST $D^*(s)$ of the DF for the depletion time D in an M/G/1

system at an arbitrary time is given by

$$D^*(s) = \frac{(1 - \rho)[s + \lambda - \lambda\Theta^*(s)]}{s} \quad (4.4)$$

The depletion time is defined as the time that the server requires to serve all customers present in the system as well as those that arrive subsequently until the system becomes empty. The depletion time is equivalent to the waiting time of a *polite customer* who is always served last in a busy period [Cooper 1981 (prob. 5.19); Takács 1963]. From (4.4), we get the n th moment of D as

$$E[D^n] = \frac{\lambda(1 - \rho)}{n + 1} E[\Theta^{n+1}] \quad n = 1, 2, \dots \quad (4.5)$$

Hence the expression for $E[D^n]$ can be obtained immediately from that for $E[\Theta^{n+1}]$ given in Appendix E.

5. Waiting Time in an LCFS System

The LST $W^*(s)$ of the DF for the waiting time W of a customer in an LCFS system is given by [Cooper 1981 (prob. 5.20), Kleinrock 1976 (sec. 3.5)]

$$W^*(s) = 1 - \rho + \frac{\lambda[1 - \Theta^*(s)]}{s + \lambda - \lambda\Theta^*(s)} \quad (5.1)$$

It is well known that the mean waiting times in the FCFS and LCFS systems are identical, and that the second moment of the waiting time in the LCFS system equals $(1 - \rho)^{-1}$ times that of the waiting time in the FCFS system.

From (5.1), the moments for the waiting time in the LCFS system can be obtained from the moments for the length of a busy period. This is done in a *Mathematica* program shown in Figure 6. This program must be executed after the program in Figure 4 has been executed. However, the execution time of this program increases rapidly as n grows mainly due to many substitution and rearrangement operations.

Takács [1963] shows the direct calculation

$$E[W^n] = \frac{(-1)^n}{n - 1} \left[\frac{d^n}{ds^n} \left(\frac{s}{s - \lambda + \lambda B^*(s)} \right)^{n-1} \right]_{s=0} \quad n = 2, 3, \dots \quad (5.2)$$

(The factor $(-1)^n$ on the r.h.s. of (5.2) is corrected from $(-1)^{n-1}$ in Takács [1963].) This is also obtained by the Bürmann theorem. Figure 7 shows a *Mathematica* program that calculates $E[W^n]$ by using (5.2). Again Line 6 calculates the derivative and rearranges the resultant terms. The execution of this program is very fast because it does not involve substitution and rearrangement. The moments of $E[W^n]$ for $n = 1-10$ for the LCFS system are shown in Appendix F.

6. Concluding Remarks

We have shown several *Mathematica* programs for the symbolic calculation of the moments for the waiting time, the queue size, the length of a busy period for an M/G/1 system. These

```
w[s] = 1 - rho + lambda (1 - t[s])/(s + lambda ( 1 - t[s]))
taylor = Series[w[s],{s,0,10}]
coeff[n_] := Coefficient[taylor,s^n]
ew[n_] := Expand[Apart[coeff[n] * n!*(-1)^n,lambda]]
ew2tex[n_] := Do[TeXForm[ew[i]] >>> waiting_time.tex,{i,1,n}]
```

Figure 6. Symbolic calculation of the moments of the waiting time for an LCFS M/G/1 system: Substitution of the busy period moments.

```
f[n_] = (s/( s - lambda + lambda B[s]))^n
B [0] = 1
B'[0] = -b
Derivative[n_][B][0] = Derivative[n][b]*(-1)^n
ew[1] = lambda b''/(1 - lambda b )
ew[n_] := Expand[(-1)^(n)/(n-1)*n!*Coefficient[Series[f[n-1],{s,0,n}],s^n]]
ew2tex[n_] := Do[TeXForm[ew[i]] >>> waiting_time.tex,{i,1,n}]
```

Figure 7. Symbolic calculation of the moments of the waiting time for an LCFS M/G/1 system: Evaluation by the Bürmann theorem.

programs only simulate manual calculation, but their speed, accuracy, and indefatigability go by far beyond the human ability. In fact, the calculation part is rather straightforward, and usually takes a small portion of the total computation time. It is the functions for rearrangement of the resultant terms in order to gain simplified appearance, such as *Expand* and *Apart*, that need careful programming. These functions also take a major portion of the total computation time.

We note that all the expressions for the moments shown in this paper have a special characteristic that simplifies the rearrangement of the terms. Namely, the numerator of each term does not contain ρ . The moments for the number of customers served in a busy period and the moments for the response time in an LCFS system do not have this property. Therefore, it is not easy to present the expressions for these moments in a naturally looking way, although their calculation is quite straightforward.

We can easily write down similar programs for the well-known variations of an M/G/1 system, such as a system with the server vacations, a system with batch arrivals, and so forth. On the other hand, it is not clear if *Mathematica* can be used to calculate the higher-order moments of the waiting time in an M/G/1 system with random order of service for which an explicit transform equation is not available.

The following implementation of *Mathematica* functions would be useful for not only queueing theory but also calculus and discrete mathematics: (1) L'Hôpital's rule in the *Limit* function when symbols are involved, and (2) *Apart* function with more than one reference variables.

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Appendix A. Moments of the Waiting Time in an FCFS M/G/1 System

$$\begin{aligned}
E[W] &= \frac{\lambda b^{(2)}}{2(1-\rho)} \\
E[W^2] &= \frac{\lambda^2 b^{(2)2}}{2(1-\rho)^2} + \frac{\lambda b^{(3)}}{3(1-\rho)} \\
E[W^3] &= \frac{3\lambda^3 b^{(2)3}}{4(1-\rho)^3} + \frac{\lambda^2 b^{(2)} b^{(3)}}{(1-\rho)^2} + \frac{\lambda b^{(4)}}{4(1-\rho)} \\
E[W^4] &= \frac{3\lambda^4 b^{(2)4}}{2(1-\rho)^4} + \frac{3\lambda^3 b^{(2)2} b^{(3)}}{(1-\rho)^3} + \frac{2\lambda^2 b^{(3)2}}{3(1-\rho)^2} + \frac{\lambda^2 b^{(2)} b^{(4)}}{(1-\rho)^2} + \frac{\lambda b^{(5)}}{5(1-\rho)} \\
E[W^5] &= \frac{15\lambda^5 b^{(2)5}}{4(1-\rho)^5} + \frac{10\lambda^4 b^{(2)3} b^{(3)}}{(1-\rho)^4} + \frac{5\lambda^3 b^{(2)} b^{(3)2}}{(1-\rho)^3} + \frac{15\lambda^3 b^{(2)2} b^{(4)}}{4(1-\rho)^3} + \frac{5\lambda^2 b^{(3)2} b^{(4)}}{3(1-\rho)^2} \\
&\quad + \frac{\lambda^2 b^{(2)} b^{(5)}}{(1-\rho)^2} + \frac{\lambda b^{(6)}}{6(1-\rho)} \\
E[W^6] &= \frac{45\lambda^6 b^{(2)6}}{4(1-\rho)^6} + \frac{75\lambda^5 b^{(2)4} b^{(3)}}{2(1-\rho)^5} + \frac{30\lambda^4 b^{(2)2} b^{(3)2}}{(1-\rho)^4} + \frac{10\lambda^3 b^{(3)3}}{3(1-\rho)^3} + \frac{15\lambda^4 b^{(2)3} b^{(4)}}{(1-\rho)^4} \\
&\quad + \frac{15\lambda^3 b^{(2)} b^{(3)} b^{(4)}}{(1-\rho)^3} + \frac{5\lambda^2 b^{(4)2}}{4(1-\rho)^2} + \frac{9\lambda^3 b^{(2)2} b^{(5)}}{2(1-\rho)^3} + \frac{2\lambda^2 b^{(3)} b^{(5)}}{(1-\rho)^2} + \frac{\lambda^2 b^{(2)} b^{(6)}}{(1-\rho)^2} + \frac{\lambda b^{(7)}}{7(1-\rho)} \\
E[W^7] &= \frac{315\lambda^7 b^{(2)7}}{8(1-\rho)^7} + \frac{315\lambda^6 b^{(2)5} b^{(3)}}{2(1-\rho)^6} + \frac{175\lambda^5 b^{(2)3} b^{(3)2}}{(1-\rho)^5} + \frac{140\lambda^4 b^{(2)} b^{(3)3}}{3(1-\rho)^4} + \frac{525\lambda^5 b^{(2)4} b^{(4)}}{8(1-\rho)^5} \\
&\quad + \frac{105\lambda^4 b^{(2)2} b^{(3)} b^{(4)}}{(1-\rho)^4} + \frac{35\lambda^3 b^{(3)2} b^{(4)}}{2(1-\rho)^3} + \frac{105\lambda^3 b^{(2)} b^{(4)2}}{8(1-\rho)^3} + \frac{21\lambda^4 b^{(2)3} b^{(5)}}{(1-\rho)^4} + \frac{21\lambda^3 b^{(2)} b^{(3)} b^{(5)}}{(1-\rho)^3} \\
&\quad + \frac{7\lambda^2 b^{(4)} b^{(5)}}{2(1-\rho)^2} + \frac{21\lambda^3 b^{(2)2} b^{(6)}}{4(1-\rho)^3} + \frac{7\lambda^2 b^{(3)} b^{(6)}}{3(1-\rho)^2} + \frac{\lambda^2 b^{(2)} b^{(7)}}{(1-\rho)^2} + \frac{\lambda b^{(8)}}{8(1-\rho)} \\
E[W^8] &= \frac{315\lambda^8 b^{(2)8}}{2(1-\rho)^8} + \frac{735\lambda^7 b^{(2)6} b^{(3)}}{(1-\rho)^7} + \frac{1050\lambda^6 b^{(2)4} b^{(3)2}}{(1-\rho)^6} + \frac{1400\lambda^5 b^{(2)2} b^{(3)3}}{3(1-\rho)^5} + \frac{280\lambda^4 b^{(3)4}}{9(1-\rho)^4} \\
&\quad + \frac{315\lambda^6 b^{(2)5} b^{(4)}}{(1-\rho)^6} + \frac{700\lambda^5 b^{(2)3} b^{(3)} b^{(4)}}{(1-\rho)^5} + \frac{280\lambda^4 b^{(2)} b^{(3)2} b^{(4)}}{(1-\rho)^4} + \frac{105\lambda^4 b^{(2)2} b^{(4)2}}{(1-\rho)^4} + \frac{35\lambda^3 b^{(3)} b^{(4)2}}{(1-\rho)^3} \\
&\quad + \frac{105\lambda^5 b^{(2)4} b^{(5)}}{(1-\rho)^5} + \frac{168\lambda^4 b^{(2)2} b^{(3)} b^{(5)}}{(1-\rho)^4} + \frac{28\lambda^3 b^{(3)2} b^{(5)}}{(1-\rho)^3} + \frac{42\lambda^3 b^{(2)} b^{(4)} b^{(5)}}{(1-\rho)^3} + \frac{14\lambda^2 b^{(5)2}}{5(1-\rho)^2} \\
&\quad + \frac{28\lambda^4 b^{(2)3} b^{(6)}}{(1-\rho)^4} + \frac{28\lambda^3 b^{(2)} b^{(3)} b^{(6)}}{(1-\rho)^3} + \frac{14\lambda^2 b^{(4)} b^{(6)}}{3(1-\rho)^2} + \frac{6\lambda^3 b^{(2)2} b^{(7)}}{(1-\rho)^3} + \frac{8\lambda^2 b^{(3)} b^{(7)}}{3(1-\rho)^2} + \frac{\lambda^2 b^{(2)} b^{(8)}}{(1-\rho)^2} + \frac{\lambda b^{(9)}}{9(1-\rho)} \\
E[W^9] &= \frac{2835\lambda^9 b^{(2)9}}{4(1-\rho)^9} + \frac{3780\lambda^8 b^{(2)7} b^{(3)}}{(1-\rho)^8} + \frac{6615\lambda^7 b^{(2)5} b^{(3)2}}{(1-\rho)^7} + \frac{4200\lambda^6 b^{(2)3} b^{(3)3}}{(1-\rho)^6} + \frac{700\lambda^5 b^{(2)} b^{(3)4}}{(1-\rho)^5} \\
&\quad + \frac{6615\lambda^7 b^{(2)6} b^{(4)}}{4(1-\rho)^7} + \frac{4725\lambda^6 b^{(2)4} b^{(3)} b^{(4)}}{(1-\rho)^6} + \frac{3150\lambda^5 b^{(2)2} b^{(3)2} b^{(4)}}{(1-\rho)^5} + \frac{280\lambda^4 b^{(3)3} b^{(4)}}{(1-\rho)^4} + \frac{1575\lambda^5 b^{(2)3} b^{(4)2}}{2(1-\rho)^5} \\
&\quad + \frac{630\lambda^4 b^{(2)} b^{(3)} b^{(4)2}}{(1-\rho)^4} + \frac{105\lambda^3 b^{(4)3}}{4(1-\rho)^3} + \frac{567\lambda^6 b^{(2)5} b^{(5)}}{(1-\rho)^6} + \frac{1260\lambda^5 b^{(2)3} b^{(3)} b^{(5)}}{(1-\rho)^5} + \frac{504\lambda^4 b^{(2)} b^{(3)2} b^{(5)}}{(1-\rho)^4} \\
&\quad + \frac{378\lambda^4 b^{(2)2} b^{(4)} b^{(5)}}{(1-\rho)^4} + \frac{126\lambda^3 b^{(3)} b^{(4)} b^{(5)}}{(1-\rho)^3} + \frac{189\lambda^3 b^{(2)} b^{(5)2}}{5(1-\rho)^3} + \frac{315\lambda^5 b^{(2)4} b^{(6)}}{2(1-\rho)^5} + \frac{252\lambda^4 b^{(2)2} b^{(3)} b^{(6)}}{(1-\rho)^4} \\
&\quad + \frac{42\lambda^3 b^{(3)2} b^{(6)}}{(1-\rho)^3} + \frac{63\lambda^3 b^{(2)} b^{(4)} b^{(6)}}{(1-\rho)^3} + \frac{42\lambda^2 b^{(5)} b^{(6)}}{5(1-\rho)^2} + \frac{36\lambda^4 b^{(2)} b^{(7)}}{(1-\rho)^4} + \frac{36\lambda^3 b^{(2)} b^{(3)} b^{(7)}}{(1-\rho)^3} \\
&\quad + \frac{6\lambda^2 b^{(4)} b^{(7)}}{(1-\rho)^2} + \frac{27\lambda^3 b^{(2)2} b^{(8)}}{4(1-\rho)^3} + \frac{3\lambda^2 b^{(3)} b^{(8)}}{(1-\rho)^2} + \frac{\lambda^2 b^{(2)} b^{(9)}}{(1-\rho)^2} + \frac{\lambda b^{(10)}}{10(1-\rho)}
\end{aligned}$$

$$\begin{aligned}
E[W^{10}] = & \frac{14175 \lambda^{10} b(2)^{10}}{4(1-\rho)^{10}} + \frac{42525 \lambda^9 b(2)^8 b(3)}{2(1-\rho)^9} + \frac{44100 \lambda^8 b(2)^6 b(3)^2}{(1-\rho)^8} + \frac{36750 \lambda^7 b(2)^4 b(3)^3}{(1-\rho)^7} + \frac{10500 \lambda^6 b(2)^2 b(3)^4}{(1-\rho)^6} \\
& + \frac{1400 \lambda^5 b(3)^5}{3(1-\rho)^5} + \frac{9450 \lambda^8 b(2)^7 b(4)}{(1-\rho)^8} + \frac{33075 \lambda^7 b(2)^5 b(3) b(4)}{(1-\rho)^7} + \frac{31500 \lambda^6 b(2)^3 b(3)^2 b(4)}{(1-\rho)^6} + \frac{7000 \lambda^5 b(2) b(3)^3 b(4)}{(1-\rho)^5} \\
& + \frac{23625 \lambda^6 b(2)^4 b(4)^2}{4(1-\rho)^6} + \frac{7875 \lambda^5 b(2)^2 b(3) b(4)^2}{(1-\rho)^5} + \frac{1050 \lambda^4 b(3)^2 b(4)^2}{(1-\rho)^4} + \frac{525 \lambda^4 b(2) b(4)^3}{(1-\rho)^4} + \frac{6615 \lambda^7 b(2)^6 b(5)}{2(1-\rho)^7} \\
& + \frac{9450 \lambda^6 b(2)^4 b(3) b(5)}{(1-\rho)^5} + \frac{6300 \lambda^5 b(2)^2 b(3)^2 b(5)}{(1-\rho)^5} + \frac{560 \lambda^4 b(3)^3 b(5)}{(1-\rho)^4} + \frac{3150 \lambda^5 b(2)^3 b(4) b(5)}{(1-\rho)^5} + \frac{2520 \lambda^4 b(2) b(3) b(4) b(5)}{(1-\rho)^4} \\
& + \frac{315 \lambda^3 b(4)^2 b(5)}{2(1-\rho)^3} + \frac{378 \lambda^4 b(2)^2 b(5)^2}{(1-\rho)^4} + \frac{126 \lambda^3 b(3) b(5)^2}{(1-\rho)^3} + \frac{945 \lambda^6 b(2)^5 b(6)}{(1-\rho)^6} + \frac{2100 \lambda^5 b(2)^3 b(3) b(6)}{(1-\rho)^5} \\
& + \frac{840 \lambda^4 b(2) b(3)^2 b(6)}{(1-\rho)^4} + \frac{630 \lambda^4 b(2)^2 b(4) b(6)}{(1-\rho)^4} + \frac{210 \lambda^3 b(3) b(4) b(6)}{(1-\rho)^3} + \frac{126 \lambda^3 b(2) b(5) b(6)}{(1-\rho)^3} + \frac{7 \lambda^2 b(6)^2}{(1-\rho)^2} \\
& + \frac{225 \lambda^5 b(2)^4 b(7)}{(1-\rho)^5} + \frac{360 \lambda^4 b(2)^2 b(3) b(7)}{(1-\rho)^4} + \frac{60 \lambda^3 b(3)^2 b(7)}{(1-\rho)^3} + \frac{90 \lambda^3 b(2) b(4) b(7)}{(1-\rho)^3} + \frac{12 \lambda^2 b(5) b(7)}{(1-\rho)^2} \\
& + \frac{45 \lambda^4 b(2)^3 b(8)}{(1-\rho)^4} + \frac{45 \lambda^3 b(2) b(3) b(8)}{(1-\rho)^3} + \frac{15 \lambda^2 b(4) b(8)}{2(1-\rho)^2} + \frac{15 \lambda^3 b(2)^2 b(9)}{2(1-\rho)^3} + \frac{10 \lambda^2 b(3) b(9)}{3(1-\rho)^2} \\
& + \frac{\lambda^2 b(2) b(10)}{(1-\rho)^2} + \frac{\lambda b(11)}{11(1-\rho)}
\end{aligned}$$

Appendix B. Moments of the Response Time in an FCFS M/G/1 System

$$\begin{aligned}
E[T] &= b + \frac{\lambda b^{(2)}}{2(1-\rho)} \\
E[T^2] &= \frac{b^{(2)}}{1-\rho} + \frac{\lambda^2 b^{(2)^2}}{2(1-\rho)^2} + \frac{\lambda b^{(3)}}{3(1-\rho)} \\
E[T^3] &= \frac{3\lambda b^{(2)^2}}{2(1-\rho)^2} + \frac{3\lambda^3 b^{(2)^3}}{4(1-\rho)^3} + \frac{b^{(3)}}{1-\rho} + \frac{\lambda^2 b^{(2)} b^{(3)}}{(1-\rho)^2} + \frac{\lambda b^{(4)}}{4(1-\rho)} \\
E[T^4] &= \frac{3\lambda^2 b^{(2)^3}}{(1-\rho)^3} + \frac{3\lambda^4 b^{(2)^4}}{2(1-\rho)^4} + \frac{4\lambda b^{(2)} b^{(3)}}{(1-\rho)^2} + \frac{3\lambda^3 b^{(2)^2} b^{(3)}}{(1-\rho)^3} + \frac{2\lambda^2 b^{(3)^2}}{3(1-\rho)^2} \\
&\quad + \frac{b^{(4)}}{1-\rho} + \frac{\lambda^2 b^{(2)} b^{(4)}}{(1-\rho)^2} + \frac{\lambda b^{(5)}}{5(1-\rho)} \\
E[T^5] &= \frac{15\lambda^3 b^{(2)^4}}{2(1-\rho)^4} + \frac{15\lambda^5 b^{(2)^5}}{4(1-\rho)^5} + \frac{15\lambda^2 b^{(2)^2} b^{(3)}}{(1-\rho)^3} + \frac{10\lambda^4 b^{(2)^3} b^{(3)}}{(1-\rho)^4} + \frac{10\lambda b^{(3)^2}}{3(1-\rho)^2} \\
&\quad + \frac{5\lambda^3 b^{(2)} b^{(3)^2}}{(1-\rho)^3} + \frac{5\lambda b^{(2)} b^{(4)}}{(1-\rho)^2} + \frac{15\lambda^3 b^{(2)^2} b^{(4)}}{4(1-\rho)^3} + \frac{5\lambda^2 b^{(3)} b^{(4)}}{3(1-\rho)^2} + \frac{b^{(5)}}{1-\rho} \\
&\quad + \frac{\lambda^2 b^{(2)} b^{(5)}}{(1-\rho)^2} + \frac{\lambda b^{(6)}}{6(1-\rho)} \\
E[T^6] &= \frac{45\lambda^4 b^{(2)^5}}{2(1-\rho)^5} + \frac{45\lambda^6 b^{(2)^6}}{4(1-\rho)^6} + \frac{60\lambda^3 b^{(2)^3} b^{(3)}}{(1-\rho)^4} + \frac{75\lambda^5 b^{(2)^4} b^{(3)}}{2(1-\rho)^5} + \frac{30\lambda^2 b^{(2)} b^{(3)^2}}{(1-\rho)^3} \\
&\quad + \frac{30\lambda^4 b^{(2)^2} b^{(3)^2}}{(1-\rho)^4} + \frac{10\lambda^3 b^{(3)^3}}{3(1-\rho)^3} + \frac{45\lambda^2 b^{(2)^2} b^{(4)}}{2(1-\rho)^3} + \frac{15\lambda^4 b^{(2)^3} b^{(4)}}{(1-\rho)^4} + \frac{10\lambda b^{(3)} b^{(4)}}{(1-\rho)^2} \\
&\quad + \frac{15\lambda^3 b^{(2)} b^{(3)} b^{(4)}}{(1-\rho)^3} + \frac{5\lambda^2 b^{(4)^2}}{4(1-\rho)^2} + \frac{6\lambda b^{(2)} b^{(5)}}{(1-\rho)^2} + \frac{9\lambda^3 b^{(2)^2} b^{(5)}}{2(1-\rho)^3} + \frac{2\lambda^2 b^{(3)} b^{(5)}}{(1-\rho)^2} \\
&\quad + \frac{b^{(6)}}{1-\rho} + \frac{\lambda^2 b^{(2)} b^{(6)}}{(1-\rho)^2} + \frac{\lambda b^{(7)}}{7(1-\rho)} \\
E[T^7] &= \frac{315\lambda^5 b^{(2)^6}}{4(1-\rho)^6} + \frac{315\lambda^7 b^{(2)^7}}{8(1-\rho)^7} + \frac{525\lambda^4 b^{(2)^4} b^{(3)}}{2(1-\rho)^5} + \frac{315\lambda^6 b^{(2)^5} b^{(3)}}{2(1-\rho)^6} + \frac{210\lambda^3 b^{(2)^2} b^{(3)^2}}{(1-\rho)^4} \\
&\quad + \frac{175\lambda^5 b^{(2)^3} b^{(3)^2}}{(1-\rho)^5} + \frac{70\lambda^2 b^{(3)^3}}{3(1-\rho)^3} + \frac{140\lambda^4 b^{(2)} b^{(3)^3}}{3(1-\rho)^4} + \frac{105\lambda^3 b^{(2)^3} b^{(4)}}{(1-\rho)^4} + \frac{525\lambda^5 b^{(2)^4} b^{(4)}}{8(1-\rho)^5} \\
&\quad + \frac{105\lambda^2 b^{(2)} b^{(3)} b^{(4)}}{(1-\rho)^3} + \frac{105\lambda^4 b^{(2)^2} b^{(3)} b^{(4)}}{(1-\rho)^4} + \frac{35\lambda^3 b^{(3)^2} b^{(4)}}{2(1-\rho)^3} + \frac{35\lambda b^{(4)^2}}{4(1-\rho)^2} + \frac{105\lambda^3 b^{(2)} b^{(4)^2}}{8(1-\rho)^3} \\
&\quad + \frac{63\lambda^2 b^{(2)^2} b^{(5)}}{2(1-\rho)^3} + \frac{21\lambda^4 b^{(2)^3} b^{(5)}}{(1-\rho)^4} + \frac{14\lambda b^{(3)} b^{(5)}}{(1-\rho)^2} + \frac{21\lambda^3 b^{(2)} b^{(3)} b^{(5)}}{(1-\rho)^3} + \frac{7\lambda^2 b^{(4)} b^{(5)}}{2(1-\rho)^2} \\
&\quad + \frac{7\lambda b^{(2)} b^{(6)}}{(1-\rho)^2} + \frac{21\lambda^3 b^{(2)^2} b^{(6)}}{4(1-\rho)^3} + \frac{7\lambda^2 b^{(3)} b^{(6)}}{3(1-\rho)^2} + \frac{b^{(7)}}{1-\rho} + \frac{\lambda^2 b^{(2)} b^{(7)}}{(1-\rho)^2} + \frac{\lambda b^{(8)}}{8(1-\rho)} \\
E[T^8] &= \frac{315\lambda^6 b^{(2)^7}}{(1-\rho)^7} + \frac{315\lambda^8 b^{(2)^8}}{2(1-\rho)^8} + \frac{1260\lambda^5 b^{(2)^5} b^{(3)}}{(1-\rho)^6} + \frac{735\lambda^7 b^{(2)^6} b^{(3)}}{(1-\rho)^7} + \frac{1400\lambda^4 b^{(2)^3} b^{(3)^2}}{(1-\rho)^5} \\
&\quad + \frac{1050\lambda^6 b^{(2)^4} b^{(3)^2}}{(1-\rho)^6} + \frac{1120\lambda^3 b^{(2)} b^{(3)^3}}{3(1-\rho)^4} + \frac{1400\lambda^5 b^{(2)^2} b^{(3)^3}}{3(1-\rho)^5} + \frac{280\lambda^4 b^{(3)^4}}{9(1-\rho)^4} + \frac{525\lambda^4 b^{(2)^4} b^{(4)}}{(1-\rho)^5} \\
&\quad + \frac{315\lambda^6 b^{(2)^5} b^{(4)}}{(1-\rho)^6} + \frac{840\lambda^3 b^{(2)^2} b^{(3)} b^{(4)}}{(1-\rho)^4} + \frac{700\lambda^5 b^{(2)^3} b^{(3)} b^{(4)}}{(1-\rho)^5} + \frac{140\lambda^2 b^{(3)^2} b^{(4)}}{(1-\rho)^3} + \frac{280\lambda^4 b^{(2)} b^{(3)^2} b^{(4)}}{(1-\rho)^4} \\
&\quad + \frac{105\lambda^2 b^{(2)} b^{(4)^2}}{(1-\rho)^3} + \frac{105\lambda^4 b^{(2)^2} b^{(4)^2}}{(1-\rho)^4} + \frac{35\lambda^3 b^{(3)} b^{(4)^2}}{(1-\rho)^3} + \frac{168\lambda^3 b^{(2)^3} b^{(5)}}{(1-\rho)^4} + \frac{105\lambda^5 b^{(2)^4} b^{(5)}}{(1-\rho)^5} \\
&\quad + \frac{168\lambda^2 b^{(2)} b^{(3)} b^{(5)}}{(1-\rho)^3} + \frac{168\lambda^4 b^{(2)^2} b^{(3)} b^{(5)}}{(1-\rho)^4} + \frac{28\lambda^3 b^{(3)^2} b^{(5)}}{(1-\rho)^3} + \frac{28\lambda b^{(4)} b^{(5)}}{(1-\rho)^2} + \frac{42\lambda^3 b^{(2)} b^{(4)} b^{(5)}}{(1-\rho)^3} \\
&\quad + \frac{14\lambda^2 b^{(5)^2}}{5(1-\rho)^2} + \frac{42\lambda^2 b^{(2)^2} b^{(6)}}{(1-\rho)^3} + \frac{28\lambda^4 b^{(2)^3} b^{(6)}}{(1-\rho)^4} + \frac{56\lambda b^{(3)} b^{(6)}}{3(1-\rho)^2} + \frac{28\lambda^3 b^{(2)} b^{(3)} b^{(6)}}{(1-\rho)^3} \\
&\quad + \frac{14\lambda^2 b^{(4)} b^{(6)}}{3(1-\rho)^2} + \frac{8\lambda b^{(2)} b^{(7)}}{(1-\rho)^2} + \frac{6\lambda^3 b^{(2)^2} b^{(7)}}{(1-\rho)^3} + \frac{8\lambda^2 b^{(3)} b^{(7)}}{3(1-\rho)^2} + \frac{b^{(8)}}{1-\rho} + \frac{\lambda^2 b^{(2)} b^{(8)}}{(1-\rho)^2} + \frac{\lambda b^{(9)}}{9(1-\rho)}
\end{aligned}$$

$$\begin{aligned}
E[T^9] &= \frac{2835 \lambda^7 b(2)^8}{2(1-\rho)^8} + \frac{2835 \lambda^9 b(2)^9}{4(1-\rho)^9} + \frac{6615 \lambda^6 b(2)^6 b(3)}{(1-\rho)^7} + \frac{3780 \lambda^8 b(2)^7 b(3)}{(1-\rho)^8} + \frac{9450 \lambda^5 b(2)^4 b(3)^2}{(1-\rho)^6} \\
&+ \frac{6615 \lambda^7 b(2)^5 b(3)^2}{(1-\rho)^7} + \frac{4200 \lambda^4 b(2)^2 b(3)^3}{(1-\rho)^5} + \frac{4200 \lambda^6 b(2)^3 b(3)^3}{(1-\rho)^6} + \frac{280 \lambda^3 b(3)^4}{(1-\rho)^4} + \frac{700 \lambda^5 b(2) b(3)^4}{(1-\rho)^5} \\
&+ \frac{2835 \lambda^5 b(2)^5 b(4)}{(1-\rho)^6} + \frac{6615 \lambda^7 b(2)^6 b(4)}{4(1-\rho)^7} + \frac{6300 \lambda^4 b(2)^3 b(3) b(4)}{(1-\rho)^5} + \frac{4725 \lambda^6 b(2)^4 b(3) b(4)}{(1-\rho)^6} + \frac{2520 \lambda^3 b(2) b(3)^2 b(4)}{(1-\rho)^4} \\
&+ \frac{3150 \lambda^5 b(2)^2 b(3)^2 b(4)}{(1-\rho)^5} + \frac{280 \lambda^4 b(3)^3 b(4)}{(1-\rho)^4} + \frac{945 \lambda^3 b(2)^2 b(4)^2}{(1-\rho)^4} + \frac{1575 \lambda^5 b(2)^3 b(4)^2}{2(1-\rho)^5} + \frac{315 \lambda^2 b(3) b(4)^2}{(1-\rho)^3} \\
&+ \frac{630 \lambda^4 b(2) b(3) b(4)^2}{(1-\rho)^4} + \frac{105 \lambda^3 b(4)^3}{4(1-\rho)^3} + \frac{945 \lambda^4 b(2)^4 b(5)}{(1-\rho)^5} + \frac{567 \lambda^6 b(2)^5 b(5)}{(1-\rho)^6} + \frac{1512 \lambda^3 b(2)^2 b(3) b(5)}{(1-\rho)^4} \\
&+ \frac{1260 \lambda^5 b(2)^3 b(3) b(5)}{(1-\rho)^5} + \frac{252 \lambda^2 b(3)^2 b(5)}{(1-\rho)^3} + \frac{504 \lambda^4 b(2) b(3)^2 b(5)}{(1-\rho)^4} + \frac{378 \lambda^2 b(2) b(4) b(5)}{(1-\rho)^3} + \frac{378 \lambda^4 b(2)^2 b(4) b(5)}{(1-\rho)^4} \\
&+ \frac{126 \lambda^3 b(3) b(4) b(5)}{(1-\rho)^3} + \frac{126 \lambda b(5)^2}{5(1-\rho)^2} + \frac{189 \lambda^3 b(2) b(5)^2}{5(1-\rho)^3} + \frac{252 \lambda^3 b(2)^3 b(6)}{(1-\rho)^4} + \frac{315 \lambda^5 b(2)^4 b(6)}{2(1-\rho)^5} \\
&+ \frac{252 \lambda^2 b(2) b(3) b(6)}{(1-\rho)^3} + \frac{252 \lambda^4 b(2)^2 b(3) b(6)}{(1-\rho)^4} + \frac{42 \lambda^3 b(3)^2 b(6)}{(1-\rho)^3} + \frac{42 \lambda b(4) b(6)}{(1-\rho)^2} + \frac{63 \lambda^3 b(2) b(4) b(6)}{(1-\rho)^3} \\
&+ \frac{42 \lambda^2 b(5) b(6)}{5(1-\rho)^2} + \frac{54 \lambda^2 b(2)^2 b(7)}{(1-\rho)^3} + \frac{36 \lambda^4 b(2)^3 b(7)}{(1-\rho)^4} + \frac{24 \lambda b(3) b(7)}{(1-\rho)^2} + \frac{36 \lambda^3 b(2) b(3) b(7)}{(1-\rho)^3} \\
&+ \frac{6 \lambda^2 b(4) b(7)}{(1-\rho)^2} + \frac{9 \lambda b(2) b(8)}{(1-\rho)^2} + \frac{27 \lambda^3 b(2)^2 b(8)}{4(1-\rho)^3} + \frac{3 \lambda^2 b(3) b(8)}{(1-\rho)^2} + \frac{b(9)}{1-\rho} + \frac{\lambda^2 b(2) b(9)}{(1-\rho)^2} + \frac{\lambda b(10)}{10(1-\rho)} \\
E[T^{10}] &= \frac{14175 \lambda^8 b(2)^9}{2(1-\rho)^9} + \frac{14175 \lambda^{10} b(2)^{10}}{4(1-\rho)^{10}} + \frac{37800 \lambda^7 b(2)^7 b(3)}{(1-\rho)^8} + \frac{42525 \lambda^9 b(2)^8 b(3)}{2(1-\rho)^9} + \frac{66150 \lambda^6 b(2)^5 b(3)^2}{(1-\rho)^7} \\
&+ \frac{44100 \lambda^8 b(2)^6 b(3)^2}{(1-\rho)^8} + \frac{42000 \lambda^5 b(2)^3 b(3)^3}{(1-\rho)^6} + \frac{36750 \lambda^7 b(2)^4 b(3)^3}{(1-\rho)^7} + \frac{7000 \lambda^4 b(2) b(3)^4}{(1-\rho)^5} + \frac{10500 \lambda^6 b(2)^2 b(3)^4}{(1-\rho)^6} \\
&+ \frac{1400 \lambda^5 b(3)^5}{3(1-\rho)^5} + \frac{33075 \lambda^6 b(2)^6 b(4)}{2(1-\rho)^7} + \frac{9450 \lambda^8 b(2)^7 b(4)}{(1-\rho)^8} + \frac{47250 \lambda^5 b(2)^4 b(3) b(4)}{(1-\rho)^6} + \frac{33075 \lambda^7 b(2)^5 b(3) b(4)}{(1-\rho)^7} \\
&+ \frac{31500 \lambda^4 b(2)^2 b(3)^2 b(4)}{(1-\rho)^5} + \frac{31500 \lambda^6 b(2)^3 b(3)^2 b(4)}{(1-\rho)^6} + \frac{2800 \lambda^3 b(3)^3 b(4)}{(1-\rho)^4} + \frac{7000 \lambda^5 b(2) b(3)^3 b(4)}{(1-\rho)^5} + \frac{7875 \lambda^4 b(2)^3 b(4)^2}{(1-\rho)^5} \\
&+ \frac{23625 \lambda^6 b(2)^4 b(4)^2}{4(1-\rho)^6} + \frac{6300 \lambda^3 b(2) b(3) b(4)^2}{(1-\rho)^4} + \frac{7875 \lambda^5 b(2)^2 b(3) b(4)^2}{(1-\rho)^5} + \frac{1050 \lambda^4 b(3)^2 b(4)^2}{(1-\rho)^4} + \frac{525 \lambda^2 b(4)^3}{2(1-\rho)^3} \\
&+ \frac{525 \lambda^4 b(2) b(4)^3}{(1-\rho)^4} + \frac{5670 \lambda^5 b(2)^5 b(5)}{(1-\rho)^6} + \frac{6615 \lambda^7 b(2)^6 b(5)}{2(1-\rho)^7} + \frac{12600 \lambda^4 b(2)^3 b(3) b(5)}{(1-\rho)^5} + \frac{9450 \lambda^6 b(2)^4 b(3) b(5)}{(1-\rho)^6} \\
&+ \frac{5040 \lambda^3 b(2) b(3)^2 b(5)}{(1-\rho)^4} + \frac{6300 \lambda^5 b(2)^2 b(3)^2 b(5)}{(1-\rho)^5} + \frac{560 \lambda^4 b(3)^3 b(5)}{(1-\rho)^4} + \frac{3780 \lambda^3 b(2)^2 b(4) b(5)}{(1-\rho)^4} + \frac{3150 \lambda^5 b(2)^3 b(4) b(5)}{(1-\rho)^5} \\
&+ \frac{1260 \lambda^2 b(3) b(4) b(5)}{(1-\rho)^3} + \frac{2520 \lambda^4 b(2) b(3) b(4) b(5)}{(1-\rho)^4} + \frac{315 \lambda^3 b(4)^2 b(5)}{2(1-\rho)^3} + \frac{378 \lambda^2 b(2) b(5)^2}{(1-\rho)^3} + \frac{378 \lambda^4 b(2)^2 b(5)^2}{(1-\rho)^4} \\
&+ \frac{126 \lambda^3 b(3) b(5)^2}{(1-\rho)^3} + \frac{1575 \lambda^4 b(2)^4 b(6)}{(1-\rho)^5} + \frac{945 \lambda^6 b(2)^5 b(6)}{(1-\rho)^6} + \frac{2520 \lambda^3 b(2)^2 b(3) b(6)}{(1-\rho)^4} + \frac{2100 \lambda^5 b(2)^3 b(3) b(6)}{(1-\rho)^5} \\
&+ \frac{420 \lambda^2 b(3)^2 b(6)}{(1-\rho)^3} + \frac{840 \lambda^4 b(2) b(3)^2 b(6)}{(1-\rho)^4} + \frac{630 \lambda^2 b(2) b(4) b(6)}{(1-\rho)^3} + \frac{630 \lambda^4 b(2)^2 b(4) b(6)}{(1-\rho)^4} + \frac{210 \lambda^3 b(3) b(4) b(6)}{(1-\rho)^3} \\
&+ \frac{84 \lambda b(5) b(6)}{(1-\rho)^2} + \frac{126 \lambda^3 b(2) b(5) b(6)}{(1-\rho)^3} + \frac{7 \lambda^2 b(6)^2}{(1-\rho)^2} + \frac{360 \lambda^3 b(2)^3 b(7)}{(1-\rho)^4} + \frac{225 \lambda^5 b(2)^4 b(7)}{(1-\rho)^5} \\
&+ \frac{360 \lambda^2 b(2) b(3) b(7)}{(1-\rho)^3} + \frac{360 \lambda^4 b(2)^2 b(3) b(7)}{(1-\rho)^4} + \frac{60 \lambda^3 b(3)^2 b(7)}{(1-\rho)^3} + \frac{60 \lambda b(4) b(7)}{(1-\rho)^2} + \frac{90 \lambda^3 b(2) b(4) b(7)}{(1-\rho)^3} \\
&+ \frac{12 \lambda^2 b(5) b(7)}{(1-\rho)^2} + \frac{135 \lambda^2 b(2)^2 b(8)}{2(1-\rho)^3} + \frac{45 \lambda^4 b(2)^3 b(8)}{(1-\rho)^4} + \frac{30 \lambda b(3) b(8)}{(1-\rho)^2} + \frac{45 \lambda^3 b(2) b(3) b(8)}{(1-\rho)^3} \\
&+ \frac{15 \lambda^2 b(4) b(8)}{2(1-\rho)^2} + \frac{10 \lambda b(2) b(9)}{(1-\rho)^2} + \frac{15 \lambda^3 b(2)^2 b(9)}{2(1-\rho)^3} + \frac{10 \lambda^2 b(3) b(9)}{3(1-\rho)^2} + \frac{b(10)}{1-\rho} + \frac{\lambda^2 b(2) b(10)}{(1-\rho)^2} + \frac{\lambda b(11)}{11(1-\rho)}
\end{aligned}$$

Appendix C. Moments of the Queue Size in an M/G/1 System

$$\begin{aligned}
E[L] &= \rho + \frac{\lambda^2 b(2)}{2(1-\rho)} \\
E[L^2] &= \rho + \frac{3\lambda^2 b(2)}{2(1-\rho)} + \frac{\lambda^4 b(2)^2}{2(1-\rho)^2} + \frac{\lambda^3 b(3)}{3(1-\rho)} \\
E[L^3] &= \rho + \frac{7\lambda^2 b(2)}{2(1-\rho)} + \frac{3\lambda^4 b(2)^2}{(1-\rho)^2} + \frac{3\lambda^6 b(2)^3}{4(1-\rho)^3} + \frac{2\lambda^3 b(3)}{1-\rho} + \frac{\lambda^5 b(2) b(3)}{(1-\rho)^2} + \frac{\lambda^4 b(4)}{4(1-\rho)} \\
E[L^4] &= \rho + \frac{15\lambda^2 b(2)}{2(1-\rho)} + \frac{25\lambda^4 b(2)^2}{2(1-\rho)^2} + \frac{15\lambda^6 b(2)^3}{2(1-\rho)^3} + \frac{3\lambda^8 b(2)^4}{2(1-\rho)^4} + \frac{25\lambda^3 b(3)}{3(1-\rho)} + \frac{10\lambda^5 b(2) b(3)}{(1-\rho)^2} \\
&\quad + \frac{3\lambda^7 b(2)^2 b(3)}{(1-\rho)^3} + \frac{2\lambda^6 b(3)^2}{3(1-\rho)^2} + \frac{5\lambda^4 b(4)}{2(1-\rho)} + \frac{\lambda^6 b(2) b(4)}{(1-\rho)^2} + \frac{\lambda^5 b(5)}{5(1-\rho)} \\
E[L^5] &= \rho + \frac{31\lambda^2 b(2)}{2(1-\rho)} + \frac{45\lambda^4 b(2)^2}{(1-\rho)^2} + \frac{195\lambda^6 b(2)^3}{4(1-\rho)^3} + \frac{45\lambda^8 b(2)^4}{2(1-\rho)^4} + \frac{15\lambda^{10} b(2)^5}{4(1-\rho)^5} \\
&\quad + \frac{30\lambda^3 b(3)}{1-\rho} + \frac{65\lambda^5 b(2) b(3)}{(1-\rho)^2} + \frac{45\lambda^7 b(2)^2 b(3)}{(1-\rho)^3} + \frac{10\lambda^9 b(2)^3 b(3)}{(1-\rho)^4} + \frac{10\lambda^6 b(3)^2}{(1-\rho)^2} \\
&\quad + \frac{5\lambda^8 b(2) b(3)^2}{(1-\rho)^3} + \frac{65\lambda^4 b(4)}{4(1-\rho)} + \frac{15\lambda^6 b(2) b(4)}{(1-\rho)^2} + \frac{15\lambda^8 b(2)^2 b(4)}{4(1-\rho)^3} + \frac{5\lambda^7 b(3) b(4)}{3(1-\rho)^2} \\
&\quad + \frac{3\lambda^5 b(5)}{1-\rho} + \frac{\lambda^7 b(2) b(5)}{(1-\rho)^2} + \frac{\lambda^6 b(6)}{6(1-\rho)} \\
E[L^6] &= \rho + \frac{63\lambda^2 b(2)}{2(1-\rho)} + \frac{301\lambda^4 b(2)^2}{2(1-\rho)^2} + \frac{525\lambda^6 b(2)^3}{2(1-\rho)^3} + \frac{210\lambda^8 b(2)^4}{(1-\rho)^4} + \frac{315\lambda^{10} b(2)^5}{4(1-\rho)^5} \\
&\quad + \frac{45\lambda^{12} b(2)^6}{4(1-\rho)^6} + \frac{301\lambda^3 b(3)}{3(1-\rho)} + \frac{350\lambda^5 b(2) b(3)}{(1-\rho)^2} + \frac{420\lambda^7 b(2)^2 b(3)}{(1-\rho)^3} + \frac{210\lambda^9 b(2)^3 b(3)}{(1-\rho)^4} \\
&\quad + \frac{75\lambda^{11} b(2)^4 b(3)}{2(1-\rho)^5} + \frac{280\lambda^6 b(3)^2}{3(1-\rho)^2} + \frac{105\lambda^8 b(2) b(3)^2}{(1-\rho)^3} + \frac{30\lambda^{10} b(2)^2 b(3)^2}{(1-\rho)^4} + \frac{10\lambda^9 b(3)^3}{3(1-\rho)^3} \\
&\quad + \frac{175\lambda^4 b(4)}{2(1-\rho)} + \frac{140\lambda^6 b(2) b(4)}{(1-\rho)^2} + \frac{315\lambda^8 b(2)^2 b(4)}{4(1-\rho)^3} + \frac{15\lambda^{10} b(2)^3 b(4)}{(1-\rho)^4} + \frac{35\lambda^7 b(3) b(4)}{(1-\rho)^2} \\
&\quad + \frac{15\lambda^9 b(2) b(3) b(4)}{(1-\rho)^3} + \frac{5\lambda^8 b(4)^2}{4(1-\rho)^2} + \frac{28\lambda^5 b(5)}{1-\rho} + \frac{21\lambda^7 b(2) b(5)}{(1-\rho)^2} + \frac{9\lambda^9 b(2)^2 b(5)}{2(1-\rho)^3} \\
&\quad + \frac{2\lambda^8 b(3) b(5)}{(1-\rho)^2} + \frac{7\lambda^6 b(6)}{2(1-\rho)} + \frac{\lambda^8 b(2) b(6)}{(1-\rho)^2} + \frac{\lambda^7 b(7)}{7(1-\rho)} \\
E[L^7] &= \rho + \frac{127\lambda^2 b(2)}{2(1-\rho)} + \frac{483\lambda^4 b(2)^2}{(1-\rho)^2} + \frac{5103\lambda^6 b(2)^3}{4(1-\rho)^3} + \frac{1575\lambda^8 b(2)^4}{(1-\rho)^4} + \frac{1995\lambda^{10} b(2)^5}{2(1-\rho)^5} \\
&\quad + \frac{315\lambda^{12} b(2)^6}{(1-\rho)^6} + \frac{315\lambda^{14} b(2)^7}{8(1-\rho)^7} + \frac{322\lambda^3 b(3)}{1-\rho} + \frac{1701\lambda^5 b(2) b(3)}{(1-\rho)^2} + \frac{3150\lambda^7 b(2)^2 b(3)}{(1-\rho)^3} \\
&\quad + \frac{2660\lambda^9 b(2)^3 b(3)}{(1-\rho)^4} + \frac{1050\lambda^{11} b(2)^4 b(3)}{(1-\rho)^5} + \frac{315\lambda^{13} b(2)^5 b(3)}{2(1-\rho)^6} + \frac{700\lambda^6 b(3)^2}{(1-\rho)^2} + \frac{1330\lambda^8 b(2) b(3)^2}{(1-\rho)^3} \\
&\quad + \frac{840\lambda^{10} b(2)^2 b(3)^2}{(1-\rho)^4} + \frac{175\lambda^{12} b(2)^3 b(3)^2}{(1-\rho)^5} + \frac{280\lambda^9 b(3)^3}{3(1-\rho)^3} + \frac{140\lambda^{11} b(2) b(3)^3}{3(1-\rho)^4} + \frac{1701\lambda^4 b(4)}{4(1-\rho)} \\
&\quad + \frac{1050\lambda^6 b(2) b(4)}{(1-\rho)^2} + \frac{1995\lambda^8 b(2)^2 b(4)}{2(1-\rho)^3} + \frac{420\lambda^{10} b(2)^3 b(4)}{(1-\rho)^4} + \frac{525\lambda^{12} b(2)^4 b(4)}{8(1-\rho)^5} + \frac{1330\lambda^7 b(3) b(4)}{3(1-\rho)^2} \\
&\quad + \frac{420\lambda^9 b(2) b(3) b(4)}{(1-\rho)^3} + \frac{105\lambda^{11} b(2)^2 b(3) b(4)}{(1-\rho)^4} + \frac{35\lambda^{10} b(3)^2 b(4)}{2(1-\rho)^3} + \frac{35\lambda^8 b(4)^2}{(1-\rho)^2} + \frac{105\lambda^{10} b(2) b(4)^2}{8(1-\rho)^3} \\
&\quad + \frac{210\lambda^5 b(5)}{1-\rho} + \frac{266\lambda^7 b(2) b(5)}{(1-\rho)^2} + \frac{126\lambda^9 b(2)^2 b(5)}{(1-\rho)^3} + \frac{21\lambda^{11} b(2)^3 b(5)}{(1-\rho)^4} + \frac{56\lambda^8 b(3) b(5)}{(1-\rho)^2} \\
&\quad + \frac{21\lambda^{10} b(2) b(3) b(5)}{(1-\rho)^3} + \frac{7\lambda^9 b(4) b(5)}{2(1-\rho)^2} + \frac{133\lambda^6 b(6)}{3(1-\rho)} + \frac{28\lambda^8 b(2) b(6)}{(1-\rho)^2} + \frac{21\lambda^{10} b(2)^2 b(6)}{4(1-\rho)^3} \\
&\quad + \frac{7\lambda^9 b(3) b(6)}{3(1-\rho)^2} + \frac{4\lambda^7 b(7)}{1-\rho} + \frac{\lambda^9 b(2) b(7)}{(1-\rho)^2} + \frac{\lambda^8 b(8)}{8(1-\rho)}
\end{aligned}$$

$$\begin{aligned}
E[L^8] = & \rho + \frac{255 \lambda^2 b(2)}{2(1-\rho)} + \frac{3025 \lambda^4 b(2)^2}{2(1-\rho)^2} + \frac{11655 \lambda^6 b(2)^3}{2(1-\rho)^3} + \frac{20853 \lambda^8 b(2)^4}{2(1-\rho)^4} + \frac{19845 \lambda^{10} b(2)^5}{2(1-\rho)^5} \\
& + \frac{10395 \lambda^{12} b(2)^6}{2(1-\rho)^6} + \frac{2835 \lambda^{14} b(2)^7}{2(1-\rho)^7} + \frac{315 \lambda^{16} b(2)^8}{2(1-\rho)^8} + \frac{3025 \lambda^3 b(3)}{3(1-\rho)} + \frac{7770 \lambda^5 b(2) b(3)}{(1-\rho)^2} \\
& + \frac{20853 \lambda^7 b(2)^2 b(3)}{(1-\rho)^3} + \frac{26450 \lambda^9 b(2)^3 b(3)}{(1-\rho)^4} + \frac{17325 \lambda^{11} b(2)^4 b(3)}{(1-\rho)^5} + \frac{5670 \lambda^{13} b(2)^5 b(3)}{(1-\rho)^6} + \frac{735 \lambda^{15} b(2)^6 b(3)}{(1-\rho)^7} \\
& + \frac{4634 \lambda^6 b(3)^2}{(1-\rho)^2} + \frac{13230 \lambda^8 b(2) b(3)^2}{(1-\rho)^3} + \frac{13860 \lambda^{10} b(2)^2 b(3)^2}{(1-\rho)^4} + \frac{6300 \lambda^{12} b(2)^3 b(3)^2}{(1-\rho)^5} + \frac{1050 \lambda^{14} b(2)^4 b(3)^2}{(1-\rho)^6} \\
& + \frac{1540 \lambda^9 b(3)^3}{(1-\rho)^3} + \frac{1680 \lambda^{11} b(2) b(3)^3}{(1-\rho)^4} + \frac{1400 \lambda^{13} b(2)^2 b(3)^3}{3(1-\rho)^5} + \frac{280 \lambda^{12} b(3)^4}{9(1-\rho)^4} + \frac{3885 \lambda^4 b(4)}{2(1-\rho)} \\
& + \frac{6951 \lambda^6 b(2) b(4)}{(1-\rho)^2} + \frac{19845 \lambda^8 b(2)^2 b(4)}{2(1-\rho)^3} + \frac{6930 \lambda^{10} b(2)^3 b(4)}{(1-\rho)^4} + \frac{4725 \lambda^{12} b(2)^4 b(4)}{2(1-\rho)^5} + \frac{315 \lambda^{14} b(2)^5 b(4)}{(1-\rho)^6} \\
& + \frac{4410 \lambda^7 b(3) b(4)}{(1-\rho)^2} + \frac{6930 \lambda^9 b(2) b(3) b(4)}{(1-\rho)^3} + \frac{3780 \lambda^{11} b(2)^2 b(3) b(4)}{(1-\rho)^4} + \frac{700 \lambda^{13} b(2)^3 b(3) b(4)}{(1-\rho)^5} + \frac{630 \lambda^{10} b(3)^2 b(4)}{(1-\rho)^3} \\
& + \frac{280 \lambda^{12} b(2) b(3)^2 b(4)}{(1-\rho)^4} + \frac{1155 \lambda^8 b(4)^2}{2(1-\rho)^2} + \frac{945 \lambda^{10} b(2) b(4)^2}{2(1-\rho)^3} + \frac{105 \lambda^{12} b(2)^2 b(4)^2}{(1-\rho)^4} + \frac{35 \lambda^{11} b(3) b(4)^2}{(1-\rho)^3} \\
& + \frac{6951 \lambda^5 b(5)}{5(1-\rho)} + \frac{2646 \lambda^7 b(2) b(5)}{(1-\rho)^2} + \frac{2079 \lambda^9 b(2)^2 b(5)}{(1-\rho)^3} + \frac{756 \lambda^{11} b(2)^3 b(5)}{(1-\rho)^4} + \frac{105 \lambda^{13} b(2)^4 b(5)}{(1-\rho)^5} \\
& + \frac{924 \lambda^8 b(3) b(5)}{(1-\rho)^2} + \frac{756 \lambda^{10} b(2) b(3) b(5)}{(1-\rho)^3} + \frac{168 \lambda^{12} b(2)^2 b(3) b(5)}{(1-\rho)^4} + \frac{28 \lambda^{11} b(3)^2 b(5)}{(1-\rho)^3} + \frac{126 \lambda^9 b(4) b(5)}{(1-\rho)^2} \\
& + \frac{42 \lambda^{11} b(2) b(4) b(5)}{(1-\rho)^3} + \frac{14 \lambda^{10} b(5)^2}{5(1-\rho)^2} + \frac{441 \lambda^6 b(6)}{1-\rho} + \frac{462 \lambda^8 b(2) b(6)}{(1-\rho)^2} + \frac{189 \lambda^{10} b(2)^2 b(6)}{(1-\rho)^3} \\
& + \frac{28 \lambda^{12} b(2)^3 b(6)}{(1-\rho)^4} + \frac{84 \lambda^9 b(3) b(6)}{(1-\rho)^2} + \frac{28 \lambda^{11} b(2) b(3) b(6)}{(1-\rho)^3} + \frac{14 \lambda^{10} b(4) b(6)}{3(1-\rho)^2} + \frac{66 \lambda^7 b(7)}{1-\rho} \\
& + \frac{36 \lambda^9 b(2) b(7)}{(1-\rho)^2} + \frac{6 \lambda^{11} b(2)^2 b(7)}{(1-\rho)^3} + \frac{8 \lambda^{10} b(3) b(7)}{3(1-\rho)^2} + \frac{9 \lambda^8 b(8)}{2(1-\rho)} + \frac{\lambda^{10} b(2) b(8)}{(1-\rho)^2} + \frac{\lambda^9 b(9)}{9(1-\rho)}
\end{aligned}$$

$$\begin{aligned}
E[L^9] = & \rho + \frac{511 \lambda^2 b(2)}{2(1-\rho)} + \frac{4665 \lambda^4 b(2)^2}{(1-\rho)^2} + \frac{102315 \lambda^6 b(2)^3}{4(1-\rho)^3} + \frac{127575 \lambda^8 b(2)^4}{2(1-\rho)^4} + \frac{342405 \lambda^{10} b(2)^5}{4(1-\rho)^5} \\
& + \frac{66150 \lambda^{12} b(2)^6}{(1-\rho)^6} + \frac{118125 \lambda^{14} b(2)^7}{4(1-\rho)^7} + \frac{14175 \lambda^{16} b(2)^8}{2(1-\rho)^8} + \frac{2835 \lambda^{18} b(2)^9}{4(1-\rho)^9} + \frac{3110 \lambda^3 b(3)}{1-\rho} \\
& + \frac{34105 \lambda^5 b(2) b(3)}{(1-\rho)^2} + \frac{127575 \lambda^7 b(2)^2 b(3)}{(1-\rho)^3} + \frac{228270 \lambda^9 b(2)^3 b(3)}{(1-\rho)^4} + \frac{220500 \lambda^{11} b(2)^4 b(3)}{(1-\rho)^5} + \frac{118125 \lambda^{13} b(2)^5 b(3)}{(1-\rho)^6} \\
& + \frac{33075 \lambda^{15} b(2)^6 b(3)}{(1-\rho)^7} + \frac{3780 \lambda^{17} b(2)^7 b(3)}{(1-\rho)^8} + \frac{28350 \lambda^6 b(3)^2}{(1-\rho)^2} + \frac{114135 \lambda^8 b(2) b(3)^2}{(1-\rho)^3} + \frac{176400 \lambda^{10} b(2)^2 b(3)^2}{(1-\rho)^4} \\
& + \frac{131250 \lambda^{12} b(2)^3 b(3)^2}{(1-\rho)^5} + \frac{47250 \lambda^{14} b(2)^4 b(3)^2}{(1-\rho)^6} + \frac{6615 \lambda^{16} b(2)^5 b(3)^2}{(1-\rho)^7} + \frac{19600 \lambda^9 b(3)^3}{(1-\rho)^3} + \frac{35000 \lambda^{11} b(2) b(3)^3}{(1-\rho)^4} \\
& + \frac{21000 \lambda^{13} b(2)^2 b(3)^3}{(1-\rho)^5} + \frac{4200 \lambda^{15} b(2)^3 b(3)^3}{(1-\rho)^6} + \frac{1400 \lambda^{12} b(3)^4}{(1-\rho)^4} + \frac{700 \lambda^{14} b(2) b(3)^4}{(1-\rho)^5} + \frac{34105 \lambda^4 b(4)}{4(1-\rho)} \\
& + \frac{42525 \lambda^6 b(2) b(4)}{(1-\rho)^2} + \frac{342405 \lambda^8 b(2)^2 b(4)}{4(1-\rho)^3} + \frac{88200 \lambda^{10} b(2)^3 b(4)}{(1-\rho)^4} + \frac{196675 \lambda^{12} b(2)^4 b(4)}{4(1-\rho)^5} + \frac{14175 \lambda^{14} b(2)^5 b(4)}{(1-\rho)^6} \\
& + \frac{6615 \lambda^{16} b(2)^6 b(4)}{4(1-\rho)^7} + \frac{38045 \lambda^7 b(3) b(4)}{(1-\rho)^2} + \frac{88200 \lambda^9 b(2) b(3) b(4)}{(1-\rho)^3} + \frac{78750 \lambda^{11} b(2)^2 b(3) b(4)}{(1-\rho)^4} + \frac{31500 \lambda^{13} b(2)^3 b(3) b(4)}{(1-\rho)^5} \\
& + \frac{4725 \lambda^{15} b(2)^4 b(3) b(4)}{(1-\rho)^6} + \frac{13125 \lambda^{10} b(3)^2 b(4)}{(1-\rho)^3} + \frac{12600 \lambda^{12} b(2) b(3)^2 b(4)}{(1-\rho)^4} + \frac{3150 \lambda^{14} b(2)^2 b(3)^2 b(4)}{(1-\rho)^5} \\
& + \frac{280 \lambda^{13} b(3)^3 b(4)}{(1-\rho)^4} + \frac{7350 \lambda^8 b(4)^2}{(1-\rho)^2} + \frac{39375 \lambda^{10} b(2) b(4)^2}{4(1-\rho)^3} + \frac{4725 \lambda^{12} b(2)^2 b(4)^2}{(1-\rho)^4} + \frac{1575 \lambda^{14} b(2)^3 b(4)^2}{2(1-\rho)^5} \\
& + \frac{1575 \lambda^{11} b(3) b(4)^2}{(1-\rho)^3} + \frac{630 \lambda^{13} b(2) b(3) b(4)^2}{(1-\rho)^4} + \frac{105 \lambda^{12} b(4)^3}{4(1-\rho)^3} + \frac{8505 \lambda^5 b(5)}{1-\rho} + \frac{22827 \lambda^7 b(2) b(5)}{(1-\rho)^2} + \frac{26460 \lambda^9 b(2)^2 b(5)}{(1-\rho)^3} \\
& + \frac{15750 \lambda^{11} b(2)^3 b(5)}{(1-\rho)^4} + \frac{4725 \lambda^{13} b(2)^4 b(5)}{(1-\rho)^5} + \frac{567 \lambda^{15} b(2)^5 b(5)}{(1-\rho)^6} + \frac{11760 \lambda^8 b(3) b(5)}{(1-\rho)^2} + \frac{15750 \lambda^{10} b(2) b(3) b(5)}{(1-\rho)^3} \\
& + \frac{7560 \lambda^{12} b(2)^2 b(3) b(5)}{(1-\rho)^4} + \frac{1260 \lambda^{14} b(2)^3 b(3) b(5)}{(1-\rho)^5} + \frac{1260 \lambda^{11} b(3)^2 b(5)}{(1-\rho)^3} + \frac{504 \lambda^{13} b(2) b(3)^2 b(5)}{(1-\rho)^4} + \frac{2625 \lambda^9 b(4) b(5)}{(1-\rho)^2} \\
& + \frac{1890 \lambda^{11} b(2) b(4) b(5)}{(1-\rho)^3} + \frac{378 \lambda^{13} b(2)^2 b(4) b(5)}{(1-\rho)^4} + \frac{126 \lambda^{12} b(3) b(4) b(5)}{(1-\rho)^3} + \frac{126 \lambda^{10} b(5)^2}{(1-\rho)^2} + \frac{189 \lambda^{12} b(2) b(5)^2}{5(1-\rho)^3} \\
& + \frac{7609 \lambda^6 b(6)}{2(1-\rho)} + \frac{5880 \lambda^8 b(2) b(6)}{(1-\rho)^2} + \frac{7875 \lambda^{10} b(2)^2 b(6)}{2(1-\rho)^3} + \frac{1260 \lambda^{12} b(2)^3 b(6)}{(1-\rho)^4} + \frac{315 \lambda^{14} b(2)^4 b(6)}{2(1-\rho)^5} \\
& + \frac{1750 \lambda^9 b(3) b(6)}{(1-\rho)^2} + \frac{1260 \lambda^{11} b(2) b(3) b(6)}{(1-\rho)^3} + \frac{252 \lambda^{13} b(2)^2 b(3) b(6)}{(1-\rho)^4} + \frac{42 \lambda^{12} b(3)^2 b(6)}{(1-\rho)^3} + \frac{210 \lambda^{10} b(4) b(6)}{(1-\rho)^2} \\
& + \frac{63 \lambda^{12} b(2) b(4) b(6)}{(1-\rho)^3} + \frac{42 \lambda^{11} b(5) b(6)}{5(1-\rho)^2} + \frac{840 \lambda^7 b(7)}{1-\rho} + \frac{750 \lambda^9 b(2) b(7)}{(1-\rho)^2} + \frac{270 \lambda^{11} b(2)^2 b(7)}{(1-\rho)^3} \\
& + \frac{36 \lambda^{13} b(2)^3 b(7)}{(1-\rho)^4} + \frac{120 \lambda^{10} b(3) b(7)}{(1-\rho)^2} + \frac{36 \lambda^{12} b(2) b(3) b(7)}{(1-\rho)^3} + \frac{6 \lambda^{11} b(4) b(7)}{(1-\rho)^2} + \frac{375 \lambda^8 b(8)}{4(1-\rho)} \\
& + \frac{45 \lambda^{10} b(2) b(8)}{(1-\rho)^2} + \frac{27 \lambda^{12} b(2)^2 b(8)}{4(1-\rho)^3} + \frac{3 \lambda^{11} b(3) b(8)}{(1-\rho)^2} + \frac{5 \lambda^9 b(9)}{1-\rho} + \frac{\lambda^{11} b(2) b(9)}{(1-\rho)^2} + \frac{\lambda^{10} b(10)}{10(1-\rho)}
\end{aligned}$$

$$\begin{aligned}
E[L^{10}] = & \rho + \frac{1023 \lambda^2 b(2)}{2(1-\rho)} + \frac{28501 \lambda^4 b(2)^2}{2(1-\rho)^2} + \frac{218625 \lambda^6 b(2)^3}{2(1-\rho)^3} + \frac{370095 \lambda^8 b(2)^4}{(1-\rho)^4} + \frac{2692305 \lambda^{10} b(2)^5}{4(1-\rho)^5} + \frac{2879415 \lambda^{12} b(2)^6}{4(1-\rho)^6} \\
& + \frac{467775 \lambda^{14} b(2)^7}{(1-\rho)^7} + \frac{363825 \lambda^{16} b(2)^8}{2(1-\rho)^8} + \frac{155925 \lambda^{18} b(2)^9}{4(1-\rho)^9} + \frac{14175 \lambda^{20} b(2)^{10}}{4(1-\rho)^{10}} + \frac{28501 \lambda^3 b(3)}{3(1-\rho)} \\
& + \frac{145750 \lambda^5 b(2) b(3)}{(1-\rho)^2} + \frac{740190 \lambda^7 b(2)^2 b(3)}{(1-\rho)^3} + \frac{1794870 \lambda^9 b(2)^3 b(3)}{(1-\rho)^4} + \frac{4799025 \lambda^{11} b(2)^4 b(3)}{2(1-\rho)^5} + \frac{1871100 \lambda^{13} b(2)^5 b(3)}{(1-\rho)^6} \\
& + \frac{848925 \lambda^{15} b(2)^6 b(3)}{(1-\rho)^7} + \frac{207900 \lambda^{17} b(2)^7 b(3)}{(1-\rho)^8} + \frac{42525 \lambda^{19} b(2)^8 b(3)}{2(1-\rho)^9} + \frac{493460 \lambda^6 b(3)^2}{3(1-\rho)^2} + \frac{897435 \lambda^8 b(2) b(3)^2}{(1-\rho)^3} \\
& + \frac{1919610 \lambda^{10} b(2)^2 b(3)^2}{(1-\rho)^4} + \frac{2079000 \lambda^{12} b(2)^3 b(3)^2}{(1-\rho)^5} + \frac{1212750 \lambda^{14} b(2)^4 b(3)^2}{(1-\rho)^6} + \frac{363825 \lambda^{16} b(2)^5 b(3)^2}{(1-\rho)^7} \\
& + \frac{44100 \lambda^{18} b(2)^6 b(3)^2}{(1-\rho)^8} + \frac{213290 \lambda^9 b(3)^3}{(1-\rho)^3} + \frac{554400 \lambda^{11} b(2) b(3)^3}{(1-\rho)^4} + \frac{538000 \lambda^{13} b(2)^2 b(3)^3}{(1-\rho)^5} + \frac{231000 \lambda^{15} b(2)^3 b(3)^3}{(1-\rho)^6} \\
& + \frac{36750 \lambda^{17} b(2)^4 b(3)^3}{(1-\rho)^7} + \frac{107800 \lambda^{12} b(3)^4}{3(1-\rho)^4} + \frac{38500 \lambda^{14} b(2) b(3)^4}{(1-\rho)^5} + \frac{10500 \lambda^{16} b(2)^2 b(3)^4}{(1-\rho)^6} + \frac{1400 \lambda^{15} b(3)^5}{3(1-\rho)^5} \\
& + \frac{72875 \lambda^4 b(4)}{2(1-\rho)} + \frac{246730 \lambda^6 b(2) b(4)}{(1-\rho)^2} + \frac{2692305 \lambda^8 b(2)^2 b(4)}{4(1-\rho)^3} + \frac{959805 \lambda^{10} b(2)^3 b(4)}{(1-\rho)^4} + \frac{779625 \lambda^{12} b(2)^4 b(4)}{(1-\rho)^5} \\
& + \frac{363825 \lambda^{14} b(2)^5 b(4)}{(1-\rho)^6} + \frac{363825 \lambda^{16} b(2)^6 b(4)}{4(1-\rho)^7} + \frac{9450 \lambda^{18} b(2)^7 b(4)}{(1-\rho)^8} + \frac{299145 \lambda^7 b(3) b(4)}{(1-\rho)^2} + \frac{959805 \lambda^9 b(2) b(3) b(4)}{(1-\rho)^3} \\
& + \frac{1247400 \lambda^{11} b(2)^2 b(3) b(4)}{(1-\rho)^4} + \frac{808500 \lambda^{13} b(2)^3 b(3) b(4)}{(1-\rho)^5} + \frac{259875 \lambda^{15} b(2)^4 b(3) b(4)}{(1-\rho)^6} + \frac{33075 \lambda^{17} b(2)^5 b(3) b(4)}{(1-\rho)^7} \\
& + \frac{207900 \lambda^{10} b(3)^2 b(4)}{(1-\rho)^3} + \frac{323400 \lambda^{12} b(2) b(3)^2 b(4)}{(1-\rho)^4} + \frac{173250 \lambda^{14} b(2)^2 b(3)^2 b(4)}{(1-\rho)^5} + \frac{31500 \lambda^{16} b(2)^3 b(3)^2 b(4)}{(1-\rho)^6} \\
& + \frac{15400 \lambda^{13} b(3)^3 b(4)}{(1-\rho)^4} + \frac{7000 \lambda^{15} b(2) b(3)^3 b(4)}{(1-\rho)^5} + \frac{319935 \lambda^8 b(4)^2}{4(1-\rho)^2} + \frac{155925 \lambda^{10} b(2) b(4)^2}{(1-\rho)^3} + \frac{121275 \lambda^{12} b(2)^2 b(4)^2}{(1-\rho)^4} \\
& + \frac{86625 \lambda^{14} b(2)^3 b(4)^2}{2(1-\rho)^5} + \frac{23625 \lambda^{16} b(2)^4 b(4)^2}{4(1-\rho)^6} + \frac{40425 \lambda^{11} b(3) b(4)^2}{(1-\rho)^3} + \frac{34650 \lambda^{13} b(2) b(3) b(4)^2}{(1-\rho)^4} + \frac{7875 \lambda^{15} b(2)^2 b(3) b(4)^2}{(1-\rho)^5} \\
& + \frac{1050 \lambda^{14} b(3)^2 b(4)^2}{(1-\rho)^4} + \frac{5775 \lambda^{12} b(4)^3}{4(1-\rho)^3} + \frac{525 \lambda^{14} b(2) b(4)^3}{(1-\rho)^4} + \frac{49346 \lambda^5 b(5)}{1-\rho} + \frac{179487 \lambda^7 b(2) b(5)}{(1-\rho)^2} + \frac{575883 \lambda^9 b(2)^2 b(5)}{2(1-\rho)^3} \\
& + \frac{249480 \lambda^{11} b(2)^3 b(5)}{(1-\rho)^4} + \frac{121275 \lambda^{13} b(2)^4 b(5)}{(1-\rho)^5} + \frac{31185 \lambda^{15} b(2)^5 b(5)}{(1-\rho)^6} + \frac{6615 \lambda^{17} b(2)^6 b(5)}{2(1-\rho)^7} + \frac{127974 \lambda^8 b(3) b(5)}{(1-\rho)^2} \\
& + \frac{249480 \lambda^{10} b(2) b(3) b(5)}{(1-\rho)^3} + \frac{194040 \lambda^{12} b(2)^2 b(3) b(5)}{(1-\rho)^4} + \frac{69300 \lambda^{14} b(2)^3 b(3) b(5)}{(1-\rho)^5} + \frac{9450 \lambda^{16} b(2)^4 b(3) b(5)}{(1-\rho)^6} \\
& + \frac{32340 \lambda^{11} b(3)^2 b(5)}{(1-\rho)^3} + \frac{27720 \lambda^{13} b(2) b(3)^2 b(5)}{(1-\rho)^4} + \frac{6300 \lambda^{15} b(2)^2 b(3)^2 b(5)}{(1-\rho)^5} + \frac{560 \lambda^{14} b(3)^3 b(5)}{(1-\rho)^4} + \frac{41580 \lambda^9 b(4) b(5)}{(1-\rho)^2} \\
& + \frac{48510 \lambda^{11} b(2) b(4) b(5)}{(1-\rho)^3} + \frac{20790 \lambda^{13} b(2)^2 b(4) b(5)}{(1-\rho)^4} + \frac{3150 \lambda^{15} b(2)^3 b(4) b(5)}{(1-\rho)^5} + \frac{6930 \lambda^{12} b(3) b(4) b(5)}{(1-\rho)^3} + \frac{2520 \lambda^{14} b(2) b(3) b(4) b(5)}{(1-\rho)^4} \\
& + \frac{315 \lambda^{13} b(4)^2 b(5)}{2(1-\rho)^3} + \frac{3234 \lambda^{10} b(5)^2}{(1-\rho)^2} + \frac{2079 \lambda^{12} b(2) b(5)^2}{(1-\rho)^3} + \frac{378 \lambda^{14} b(2)^2 b(5)^2}{(1-\rho)^4} + \frac{126 \lambda^{13} b(3) b(5)^2}{(1-\rho)^3} + \frac{59829 \lambda^6 b(6)}{2(1-\rho)} \\
& + \frac{63987 \lambda^8 b(2) b(6)}{(1-\rho)^2} + \frac{62370 \lambda^{10} b(2)^2 b(6)}{(1-\rho)^3} + \frac{32340 \lambda^{12} b(2)^3 b(6)}{(1-\rho)^4} + \frac{17325 \lambda^{14} b(2)^4 b(6)}{2(1-\rho)^5} + \frac{945 \lambda^{16} b(2)^5 b(6)}{(1-\rho)^6} \\
& + \frac{27720 \lambda^9 b(3) b(6)}{(1-\rho)^2} + \frac{32340 \lambda^{11} b(2) b(3) b(6)}{(1-\rho)^3} + \frac{13860 \lambda^{13} b(2)^2 b(3) b(6)}{(1-\rho)^4} + \frac{2100 \lambda^{15} b(2)^3 b(3) b(6)}{(1-\rho)^5} + \frac{2310 \lambda^{12} b(3)^2 b(6)}{(1-\rho)^3} \\
& + \frac{840 \lambda^{14} b(2) b(3)^2 b(6)}{(1-\rho)^4} + \frac{5390 \lambda^{10} b(4) b(6)}{(1-\rho)^2} + \frac{3465 \lambda^{12} b(2) b(4) b(6)}{(1-\rho)^3} + \frac{630 \lambda^{14} b(2)^2 b(4) b(6)}{(1-\rho)^4} + \frac{210 \lambda^{13} b(3) b(4) b(6)}{(1-\rho)^3} \\
& + \frac{162 \lambda^{11} b(5) b(6)}{(1-\rho)^2} + \frac{126 \lambda^{13} b(2) b(5) b(6)}{(1-\rho)^3} + \frac{7 \lambda^{12} b(6)^2}{(1-\rho)^2} + \frac{9141 \lambda^7 b(7)}{1-\rho} + \frac{11880 \lambda^9 b(2) b(7)}{(1-\rho)^2} + \frac{6930 \lambda^{11} b(2)^2 b(7)}{(1-\rho)^3}
\end{aligned}$$

to be continued

$E[L^{10}]$ (continuation)

$$\begin{aligned}
& + \frac{1980 \lambda^{13} b(2)^3 b(7)}{(1-\rho)^4} + \frac{225 \lambda^{15} b(2)^4 b(7)}{(1-\rho)^5} + \frac{3080 \lambda^{10} b(3) b(7)}{(1-\rho)^2} + \frac{1980 \lambda^{12} b(2) b(3) b(7)}{(1-\rho)^3} + \frac{360 \lambda^{14} b(2)^2 b(3) b(7)}{(1-\rho)^4} \\
& + \frac{60 \lambda^{13} b(3)^2 b(7)}{(1-\rho)^3} + \frac{330 \lambda^{11} b(4) b(7)}{(1-\rho)^2} + \frac{90 \lambda^{13} b(2) b(4) b(7)}{(1-\rho)^3} + \frac{12 \lambda^{12} b(5) b(7)}{(1-\rho)^2} + \frac{1485 \lambda^8 b(8)}{1-\rho} \\
& + \frac{1155 \lambda^{10} b(2) b(8)}{(1-\rho)^2} + \frac{1485 \lambda^{12} b(2)^2 b(8)}{4(1-\rho)^3} + \frac{45 \lambda^{14} b(2)^3 b(8)}{(1-\rho)^4} + \frac{165 \lambda^{11} b(3) b(8)}{(1-\rho)^2} + \frac{45 \lambda^{13} b(2) b(3) b(8)}{(1-\rho)^3} \\
& + \frac{15 \lambda^{12} b(4) b(8)}{2(1-\rho)^2} + \frac{385 \lambda^9 b(9)}{3(1-\rho)} + \frac{55 \lambda^{11} b(2) b(9)}{(1-\rho)^2} + \frac{15 \lambda^{13} b(2)^2 b(9)}{2(1-\rho)^3} + \frac{10 \lambda^{12} b(3) b(9)}{3(1-\rho)^2} \\
& + \frac{11 \lambda^{10} b(10)}{2(1-\rho)} + \frac{\lambda^{12} b(2) b(10)}{(1-\rho)^2} + \frac{\lambda^{11} b(11)}{11(1-\rho)}
\end{aligned}$$

Appendix D. Moments of the Queue Size in the Waiting Line of an M/G/1 System

$$\begin{aligned}
E[\ell] &= \frac{\lambda^2 b(2)}{2(1-\rho)} \\
E[\ell^2] &= \frac{\lambda^2 b(2)}{2(1-\rho)} + \frac{\lambda^4 b(2)^2}{2(1-\rho)^2} + \frac{\lambda^3 b(3)}{3(1-\rho)} \\
E[\ell^3] &= \frac{\lambda^2 b(2)}{2(1-\rho)} + \frac{3\lambda^4 b(2)^2}{2(1-\rho)^2} + \frac{3\lambda^6 b(2)^3}{4(1-\rho)^3} + \frac{\lambda^3 b(3)}{1-\rho} + \frac{\lambda^5 b(2) b(3)}{(1-\rho)^2} + \frac{\lambda^4 b(4)}{4(1-\rho)} \\
E[\ell^4] &= \frac{\lambda^2 b(2)}{2(1-\rho)} + \frac{7\lambda^4 b(2)^2}{2(1-\rho)^2} + \frac{9\lambda^6 b(2)^3}{2(1-\rho)^3} + \frac{3\lambda^8 b(2)^4}{2(1-\rho)^4} + \frac{7\lambda^3 b(3)}{3(1-\rho)} + \frac{6\lambda^5 b(2) b(3)}{(1-\rho)^2} \\
&\quad + \frac{3\lambda^7 b(2)^2 b(3)}{(1-\rho)^3} + \frac{2\lambda^6 b(3)^2}{3(1-\rho)^2} + \frac{3\lambda^4 b(4)}{2(1-\rho)} + \frac{\lambda^6 b(2) b(4)}{(1-\rho)^2} + \frac{\lambda^5 b(5)}{3(1-\rho)} \\
E[\ell^5] &= \frac{\lambda^2 b(2)}{2(1-\rho)} + \frac{15\lambda^4 b(2)^2}{2(1-\rho)^2} + \frac{75\lambda^6 b(2)^3}{4(1-\rho)^3} + \frac{15\lambda^8 b(2)^4}{(1-\rho)^4} + \frac{15\lambda^{10} b(2)^5}{4(1-\rho)^5} \\
&\quad + \frac{5\lambda^3 b(3)}{1-\rho} + \frac{25\lambda^5 b(2) b(3)}{(1-\rho)^2} + \frac{30\lambda^7 b(2)^2 b(3)}{(1-\rho)^3} + \frac{10\lambda^9 b(2)^3 b(3)}{(1-\rho)^4} + \frac{20\lambda^6 b(3)^2}{3(1-\rho)^2} \\
&\quad + \frac{5\lambda^8 b(2) b(3)^2}{(1-\rho)^3} + \frac{25\lambda^4 b(4)}{4(1-\rho)} + \frac{10\lambda^6 b(2) b(4)}{(1-\rho)^2} + \frac{15\lambda^8 b(2)^2 b(4)}{4(1-\rho)^3} + \frac{5\lambda^7 b(3) b(4)}{3(1-\rho)^2} \\
&\quad + \frac{2\lambda^5 b(5)}{1-\rho} + \frac{\lambda^7 b(2) b(5)}{(1-\rho)^2} + \frac{\lambda^6 b(6)}{6(1-\rho)} \\
E[\ell^6] &= \frac{\lambda^2 b(2)}{2(1-\rho)} + \frac{31\lambda^4 b(2)^2}{2(1-\rho)^2} + \frac{135\lambda^6 b(2)^3}{2(1-\rho)^3} + \frac{195\lambda^8 b(2)^4}{2(1-\rho)^4} + \frac{225\lambda^{10} b(2)^5}{4(1-\rho)^5} \\
&\quad + \frac{45\lambda^{12} b(2)^6}{4(1-\rho)^6} + \frac{31\lambda^3 b(3)}{3(1-\rho)} + \frac{90\lambda^5 b(2) b(3)}{(1-\rho)^2} + \frac{195\lambda^7 b(2)^2 b(3)}{(1-\rho)^3} + \frac{150\lambda^9 b(2)^3 b(3)}{(1-\rho)^4} \\
&\quad + \frac{75\lambda^{11} b(2)^4 b(3)}{2(1-\rho)^5} + \frac{130\lambda^6 b(3)^2}{3(1-\rho)^2} + \frac{75\lambda^8 b(2) b(3)^2}{(1-\rho)^3} + \frac{30\lambda^{10} b(2)^2 b(3)^2}{(1-\rho)^4} + \frac{10\lambda^9 b(3)^3}{3(1-\rho)^3} \\
&\quad + \frac{45\lambda^4 b(4)}{2(1-\rho)} + \frac{65\lambda^6 b(2) b(4)}{(1-\rho)^2} + \frac{225\lambda^8 b(2)^2 b(4)}{4(1-\rho)^3} + \frac{15\lambda^{10} b(2)^3 b(4)}{(1-\rho)^4} + \frac{25\lambda^7 b(3) b(4)}{(1-\rho)^2} \\
&\quad + \frac{15\lambda^9 b(2) b(3) b(4)}{(1-\rho)^3} + \frac{5\lambda^8 b(4)^2}{4(1-\rho)^2} + \frac{13\lambda^5 b(5)}{1-\rho} + \frac{15\lambda^7 b(2) b(5)}{(1-\rho)^2} + \frac{9\lambda^9 b(2)^2 b(5)}{2(1-\rho)^3} \\
&\quad + \frac{2\lambda^8 b(3) b(5)}{(1-\rho)^2} + \frac{5\lambda^6 b(6)}{2(1-\rho)} + \frac{\lambda^8 b(2) b(6)}{(1-\rho)^2} + \frac{\lambda^7 b(7)}{7(1-\rho)} \\
E[\ell^7] &= \frac{\lambda^2 b(2)}{2(1-\rho)} + \frac{63\lambda^4 b(2)^2}{2(1-\rho)^2} + \frac{903\lambda^6 b(2)^3}{4(1-\rho)^3} + \frac{525\lambda^8 b(2)^4}{(1-\rho)^4} + \frac{525\lambda^{10} b(2)^5}{(1-\rho)^5} \\
&\quad + \frac{945\lambda^{12} b(2)^6}{4(1-\rho)^6} + \frac{315\lambda^{14} b(2)^7}{8(1-\rho)^7} + \frac{21\lambda^3 b(3)}{1-\rho} + \frac{301\lambda^5 b(2) b(3)}{(1-\rho)^2} + \frac{1050\lambda^7 b(2)^2 b(3)}{(1-\rho)^3} \\
&\quad + \frac{1400\lambda^9 b(2)^3 b(3)}{(1-\rho)^4} + \frac{1575\lambda^{11} b(2)^4 b(3)}{2(1-\rho)^5} + \frac{315\lambda^{13} b(2)^5 b(3)}{2(1-\rho)^6} + \frac{700\lambda^6 b(3)^2}{3(1-\rho)^2} + \frac{700\lambda^8 b(2) b(3)^2}{(1-\rho)^3} \\
&\quad + \frac{630\lambda^{10} b(2)^2 b(3)^2}{(1-\rho)^4} + \frac{175\lambda^{12} b(2)^3 b(3)^2}{(1-\rho)^5} + \frac{70\lambda^9 b(3)^3}{(1-\rho)^3} + \frac{140\lambda^{11} b(2) b(3)^3}{3(1-\rho)^4} + \frac{301\lambda^4 b(4)}{4(1-\rho)} \\
&\quad + \frac{350\lambda^6 b(2) b(4)}{(1-\rho)^2} + \frac{525\lambda^8 b(2)^2 b(4)}{(1-\rho)^3} + \frac{315\lambda^{10} b(2)^3 b(4)}{(1-\rho)^4} + \frac{525\lambda^{12} b(2)^4 b(4)}{8(1-\rho)^5} + \frac{700\lambda^7 b(3) b(4)}{3(1-\rho)^2} \\
&\quad + \frac{315\lambda^9 b(2) b(3) b(4)}{(1-\rho)^3} + \frac{105\lambda^{11} b(2)^2 b(3) b(4)}{(1-\rho)^4} + \frac{35\lambda^{10} b(3)^2 b(4)}{2(1-\rho)^3} + \frac{105\lambda^8 b(4)^2}{4(1-\rho)^2} + \frac{105\lambda^{10} b(2) b(4)^2}{8(1-\rho)^3} \\
&\quad + \frac{70\lambda^5 b(5)}{1-\rho} + \frac{140\lambda^7 b(2) b(5)}{(1-\rho)^2} + \frac{189\lambda^9 b(2)^2 b(5)}{2(1-\rho)^3} + \frac{21\lambda^{11} b(2)^3 b(5)}{(1-\rho)^4} + \frac{42\lambda^8 b(3) b(5)}{(1-\rho)^2} \\
&\quad + \frac{21\lambda^{10} b(2) b(3) b(5)}{(1-\rho)^3} + \frac{7\lambda^9 b(4) b(5)}{2(1-\rho)^2} + \frac{70\lambda^6 b(6)}{3(1-\rho)} + \frac{21\lambda^8 b(2) b(6)}{(1-\rho)^2} + \frac{21\lambda^{10} b(2)^2 b(6)}{4(1-\rho)^3} \\
&\quad + \frac{7\lambda^9 b(3) b(6)}{3(1-\rho)^2} + \frac{3\lambda^7 b(7)}{1-\rho} + \frac{\lambda^9 b(2) b(7)}{(1-\rho)^2} + \frac{\lambda^8 b(8)}{8(1-\rho)}
\end{aligned}$$

$$\begin{aligned}
E[t^8] = & \frac{\lambda^2 b(2)}{2(1-\rho)} + \frac{127 \lambda^4 b(2)^2}{2(1-\rho)^2} + \frac{1449 \lambda^6 b(2)^3}{2(1-\rho)^3} + \frac{5103 \lambda^8 b(2)^4}{2(1-\rho)^4} + \frac{7875 \lambda^{10} b(2)^5}{2(1-\rho)^5} \\
& + \frac{5985 \lambda^{12} b(2)^6}{2(1-\rho)^6} + \frac{2205 \lambda^{14} b(2)^7}{2(1-\rho)^7} + \frac{315 \lambda^{16} b(2)^8}{2(1-\rho)^8} + \frac{127 \lambda^3 b(3)}{3(1-\rho)} + \frac{966 \lambda^5 b(2) b(3)}{(1-\rho)^2} \\
& + \frac{5103 \lambda^7 b(2)^2 b(3)}{(1-\rho)^3} + \frac{10500 \lambda^9 b(2)^3 b(3)}{(1-\rho)^4} + \frac{9975 \lambda^{11} b(2)^4 b(3)}{(1-\rho)^5} + \frac{4410 \lambda^{13} b(2)^5 b(3)}{(1-\rho)^6} + \frac{735 \lambda^{15} b(2)^6 b(3)}{(1-\rho)^7} \\
& + \frac{1134 \lambda^6 b(3)^2}{(1-\rho)^2} + \frac{5250 \lambda^8 b(2) b(3)^2}{(1-\rho)^3} + \frac{7980 \lambda^{10} b(2)^2 b(3)^2}{(1-\rho)^4} + \frac{4900 \lambda^{12} b(2)^3 b(3)^2}{(1-\rho)^5} + \frac{1050 \lambda^{14} b(2)^4 b(3)^2}{(1-\rho)^6} \\
& + \frac{2860 \lambda^9 b(3)^3}{3(1-\rho)^3} + \frac{3920 \lambda^{11} b(2) b(3)^3}{3(1-\rho)^4} + \frac{1400 \lambda^{13} b(2)^2 b(3)^3}{3(1-\rho)^5} + \frac{280 \lambda^{12} b(3)^4}{9(1-\rho)^4} + \frac{483 \lambda^4 b(4)}{2(1-\rho)} \\
& + \frac{1701 \lambda^6 b(2) b(4)}{(1-\rho)^2} + \frac{7875 \lambda^8 b(2)^2 b(4)}{2(1-\rho)^3} + \frac{3990 \lambda^{10} b(2)^3 b(4)}{(1-\rho)^4} + \frac{3675 \lambda^{12} b(2)^4 b(4)}{2(1-\rho)^5} + \frac{315 \lambda^{14} b(2)^5 b(4)}{(1-\rho)^6} \\
& + \frac{1750 \lambda^7 b(3) b(4)}{(1-\rho)^2} + \frac{3990 \lambda^9 b(2) b(3) b(4)}{(1-\rho)^3} + \frac{2940 \lambda^{11} b(2)^2 b(3) b(4)}{(1-\rho)^4} + \frac{700 \lambda^{13} b(2)^3 b(3) b(4)}{(1-\rho)^5} + \frac{490 \lambda^{10} b(3)^2 b(4)}{(1-\rho)^3} \\
& + \frac{280 \lambda^{12} b(2) b(3)^2 b(4)}{(1-\rho)^4} + \frac{665 \lambda^8 b(4)^2}{2(1-\rho)^2} + \frac{735 \lambda^{10} b(2) b(4)^2}{2(1-\rho)^3} + \frac{105 \lambda^{12} b(2)^2 b(4)^2}{(1-\rho)^4} + \frac{35 \lambda^{11} b(3) b(4)^2}{(1-\rho)^3} \\
& + \frac{1701 \lambda^5 b(5)}{5(1-\rho)} + \frac{1050 \lambda^7 b(2) b(5)}{(1-\rho)^2} + \frac{1197 \lambda^9 b(2)^2 b(5)}{(1-\rho)^3} + \frac{588 \lambda^{11} b(2)^3 b(5)}{(1-\rho)^4} + \frac{105 \lambda^{13} b(2)^4 b(5)}{(1-\rho)^5} \\
& + \frac{532 \lambda^8 b(3) b(5)}{(1-\rho)^2} + \frac{588 \lambda^{10} b(2) b(3) b(5)}{(1-\rho)^3} + \frac{168 \lambda^{12} b(2)^2 b(3) b(5)}{(1-\rho)^4} + \frac{28 \lambda^{11} b(3)^2 b(5)}{(1-\rho)^3} + \frac{98 \lambda^9 b(4) b(5)}{(1-\rho)^2} \\
& + \frac{42 \lambda^{11} b(2) b(4) b(5)}{(1-\rho)^3} + \frac{14 \lambda^{10} b(5)^2}{5(1-\rho)^2} + \frac{175 \lambda^6 b(6)}{1-\rho} + \frac{266 \lambda^8 b(2) b(6)}{(1-\rho)^2} + \frac{147 \lambda^{10} b(2)^2 b(6)}{(1-\rho)^3} \\
& + \frac{28 \lambda^{12} b(2)^3 b(6)}{(1-\rho)^4} + \frac{196 \lambda^9 b(3) b(6)}{3(1-\rho)^2} + \frac{28 \lambda^{11} b(2) b(3) b(6)}{(1-\rho)^3} + \frac{14 \lambda^{10} b(4) b(6)}{3(1-\rho)^2} + \frac{38 \lambda^7 b(7)}{1-\rho} \\
& + \frac{28 \lambda^9 b(2) b(7)}{(1-\rho)^2} + \frac{6 \lambda^{11} b(2)^2 b(7)}{(1-\rho)^3} + \frac{8 \lambda^{10} b(3) b(7)}{3(1-\rho)^2} + \frac{7 \lambda^8 b(8)}{2(1-\rho)} + \frac{\lambda^{10} b(2) b(8)}{(1-\rho)^2} + \frac{\lambda^9 b(9)}{9(1-\rho)}
\end{aligned}$$

$$\begin{aligned}
E[\ell^9] = & \frac{\lambda^2 b(2)}{2(1-\rho)} + \frac{255 \lambda^4 b(2)^2}{2(1-\rho)^2} + \frac{9075 \lambda^6 b(2)^3}{4(1-\rho)^3} + \frac{11655 \lambda^8 b(2)^4}{(1-\rho)^4} + \frac{104265 \lambda^{10} b(2)^5}{4(1-\rho)^5} \\
& + \frac{59535 \lambda^{12} b(2)^6}{2(1-\rho)^6} + \frac{72765 \lambda^{14} b(2)^7}{4(1-\rho)^7} + \frac{5670 \lambda^{16} b(2)^8}{(1-\rho)^8} + \frac{2835 \lambda^{18} b(2)^9}{4(1-\rho)^9} + \frac{85 \lambda^3 b(3)}{1-\rho} \\
& + \frac{3025 \lambda^5 b(2) b(3)}{(1-\rho)^2} + \frac{23310 \lambda^7 b(2)^2 b(3)}{(1-\rho)^3} + \frac{69510 \lambda^9 b(2)^3 b(3)}{(1-\rho)^4} + \frac{99225 \lambda^{11} b(2)^4 b(3)}{(1-\rho)^5} + \frac{72765 \lambda^{13} b(2)^5 b(3)}{(1-\rho)^6} \\
& + \frac{26460 \lambda^{15} b(2)^6 b(3)}{(1-\rho)^7} + \frac{3780 \lambda^{17} b(2)^7 b(3)}{(1-\rho)^8} + \frac{5180 \lambda^6 b(3)^2}{(1-\rho)^2} + \frac{34755 \lambda^8 b(2) b(3)^2}{(1-\rho)^3} + \frac{79380 \lambda^{10} b(2)^2 b(3)^2}{(1-\rho)^4} \\
& + \frac{80850 \lambda^{12} b(2)^3 b(3)^2}{(1-\rho)^5} + \frac{37800 \lambda^{14} b(2)^4 b(3)^2}{(1-\rho)^6} + \frac{6615 \lambda^{16} b(2)^5 b(3)^2}{(1-\rho)^7} + \frac{8820 \lambda^9 b(3)^3}{(1-\rho)^3} + \frac{21560 \lambda^{11} b(2) b(3)^3}{(1-\rho)^4} \\
& + \frac{16800 \lambda^{13} b(2)^2 b(3)^3}{(1-\rho)^5} + \frac{4200 \lambda^{15} b(2)^3 b(3)^3}{(1-\rho)^6} + \frac{1120 \lambda^{12} b(3)^4}{(1-\rho)^4} + \frac{700 \lambda^{14} b(2) b(3)^4}{(1-\rho)^5} + \frac{3025 \lambda^4 b(4)}{4(1-\rho)} \\
& + \frac{7770 \lambda^6 b(2) b(4)}{(1-\rho)^2} + \frac{104265 \lambda^8 b(2)^2 b(4)}{4(1-\rho)^3} + \frac{39690 \lambda^{10} b(2)^3 b(4)}{(1-\rho)^4} + \frac{121275 \lambda^{12} b(2)^4 b(4)}{4(1-\rho)^5} + \frac{11340 \lambda^{14} b(2)^5 b(4)}{(1-\rho)^6} \\
& + \frac{6615 \lambda^{16} b(2)^6 b(4)}{4(1-\rho)^7} + \frac{11585 \lambda^7 b(3) b(4)}{(1-\rho)^2} + \frac{39690 \lambda^9 b(2) b(3) b(4)}{(1-\rho)^3} + \frac{48510 \lambda^{11} b(2)^2 b(3) b(4)}{(1-\rho)^4} + \frac{25200 \lambda^{13} b(2)^3 b(3) b(4)}{(1-\rho)^5} \\
& + \frac{4725 \lambda^{15} b(2)^4 b(3) b(4)}{(1-\rho)^6} + \frac{8085 \lambda^{10} b(3)^2 b(4)}{(1-\rho)^3} + \frac{10080 \lambda^{12} b(2) b(3)^2 b(4)}{(1-\rho)^4} + \frac{3150 \lambda^{14} b(2)^2 b(3)^2 b(4)}{(1-\rho)^5} + \frac{280 \lambda^{13} b(3)^3 b(4)}{(1-\rho)^4} \\
& + \frac{6615 \lambda^8 b(4)^2}{2(1-\rho)^2} + \frac{24255 \lambda^{10} b(2) b(4)^2}{4(1-\rho)^3} + \frac{3780 \lambda^{12} b(2)^2 b(4)^2}{(1-\rho)^4} + \frac{1575 \lambda^{14} b(2)^3 b(4)^2}{2(1-\rho)^5} + \frac{1260 \lambda^{11} b(3) b(4)^2}{(1-\rho)^3} \\
& + \frac{630 \lambda^{13} b(2) b(3) b(4)^2}{(1-\rho)^4} + \frac{105 \lambda^{12} b(4)^3}{4(1-\rho)^3} + \frac{1554 \lambda^5 b(5)}{1-\rho} + \frac{6951 \lambda^7 b(2) b(5)}{(1-\rho)^2} + \frac{11907 \lambda^9 b(2)^2 b(5)}{(1-\rho)^3} \\
& + \frac{9702 \lambda^{11} b(2)^3 b(5)}{(1-\rho)^4} + \frac{3780 \lambda^{13} b(2)^4 b(5)}{(1-\rho)^5} + \frac{567 \lambda^{15} b(2)^5 b(5)}{(1-\rho)^6} + \frac{5292 \lambda^8 b(3) b(5)}{(1-\rho)^2} + \frac{9702 \lambda^{10} b(2) b(3) b(5)}{(1-\rho)^3} \\
& + \frac{6048 \lambda^{12} b(2)^2 b(3) b(5)}{(1-\rho)^4} + \frac{1260 \lambda^{14} b(2)^3 b(3) b(5)}{(1-\rho)^5} + \frac{1008 \lambda^{11} b(3)^2 b(5)}{(1-\rho)^3} + \frac{504 \lambda^{13} b(2) b(3)^2 b(5)}{(1-\rho)^4} + \frac{1617 \lambda^9 b(4) b(5)}{(1-\rho)^2} \\
& + \frac{1512 \lambda^{11} b(2) b(4) b(5)}{(1-\rho)^3} + \frac{378 \lambda^{13} b(2)^2 b(4) b(5)}{(1-\rho)^4} + \frac{126 \lambda^{12} b(3) b(4) b(5)}{(1-\rho)^3} + \frac{504 \lambda^{10} b(5)^2}{5(1-\rho)^2} + \frac{189 \lambda^{12} b(2) b(5)^2}{5(1-\rho)^3} \\
& + \frac{2317 \lambda^6 b(6)}{2(1-\rho)} + \frac{2646 \lambda^8 b(2) b(6)}{(1-\rho)^2} + \frac{4851 \lambda^{10} b(2)^2 b(6)}{2(1-\rho)^3} + \frac{1008 \lambda^{12} b(2)^3 b(6)}{(1-\rho)^4} + \frac{315 \lambda^{14} b(2)^4 b(6)}{2(1-\rho)^5} \\
& + \frac{1078 \lambda^9 b(3) b(6)}{(1-\rho)^2} + \frac{1008 \lambda^{11} b(2) b(3) b(6)}{(1-\rho)^3} + \frac{252 \lambda^{13} b(2)^2 b(3) b(6)}{(1-\rho)^4} + \frac{42 \lambda^{12} b(3)^2 b(6)}{(1-\rho)^3} + \frac{168 \lambda^{10} b(4) b(6)}{(1-\rho)^2} \\
& + \frac{63 \lambda^{12} b(2) b(4) b(6)}{(1-\rho)^3} + \frac{42 \lambda^{11} b(5) b(6)}{5(1-\rho)^2} + \frac{378 \lambda^7 b(7)}{1-\rho} + \frac{462 \lambda^9 b(2) b(7)}{(1-\rho)^2} + \frac{216 \lambda^{11} b(2)^2 b(7)}{(1-\rho)^3} \\
& + \frac{36 \lambda^{13} b(2)^3 b(7)}{(1-\rho)^4} + \frac{96 \lambda^{10} b(3) b(7)}{(1-\rho)^2} + \frac{36 \lambda^{12} b(2) b(3) b(7)}{(1-\rho)^3} + \frac{6 \lambda^{11} b(4) b(7)}{(1-\rho)^2} + \frac{231 \lambda^8 b(8)}{4(1-\rho)} \\
& + \frac{36 \lambda^{10} b(2) b(8)}{(1-\rho)^2} + \frac{27 \lambda^{12} b(2)^2 b(8)}{4(1-\rho)^3} + \frac{3 \lambda^{11} b(3) b(8)}{(1-\rho)^2} + \frac{4 \lambda^9 b(9)}{1-\rho} + \frac{\lambda^{11} b(2) b(9)}{(1-\rho)^2} + \frac{\lambda^{10} b(10)}{10(1-\rho)}
\end{aligned}$$

$$\begin{aligned}
E[\epsilon^{10}] = & \frac{\lambda^2 b(2)}{2(1-\rho)} + \frac{511\lambda^4 b(2)^2}{2(1-\rho)^2} + \frac{13995\lambda^6 b(2)^3}{2(1-\rho)^3} + \frac{102315\lambda^8 b(2)^4}{2(1-\rho)^4} + \frac{637875\lambda^{10} b(2)^5}{4(1-\rho)^5} \\
& + \frac{1027215\lambda^{12} b(2)^6}{4(1-\rho)^6} + \frac{231525\lambda^{14} b(2)^7}{(1-\rho)^7} + \frac{118125\lambda^{16} b(2)^8}{(1-\rho)^8} + \frac{127575\lambda^{18} b(2)^9}{4(1-\rho)^9} + \frac{14175\lambda^{20} b(2)^{10}}{4(1-\rho)^{10}} \\
& + \frac{511\lambda^3 b(3)}{3(1-\rho)} + \frac{9330\lambda^5 b(2) b(3)}{(1-\rho)^2} + \frac{102315\lambda^7 b(2)^2 b(3)}{(1-\rho)^3} + \frac{425250\lambda^9 b(2)^3 b(3)}{(1-\rho)^4} + \frac{1712025\lambda^{11} b(2)^4 b(3)}{2(1-\rho)^5} \\
& + \frac{926100\lambda^{13} b(2)^5 b(3)}{(1-\rho)^6} + \frac{551250\lambda^{15} b(2)^6 b(3)}{(1-\rho)^7} + \frac{170100\lambda^{17} b(2)^7 b(3)}{(1-\rho)^8} + \frac{42525\lambda^{19} b(2)^8 b(3)}{2(1-\rho)^9} + \frac{68210\lambda^6 b(3)^2}{3(1-\rho)^2} \\
& + \frac{212625\lambda^8 b(2) b(3)^2}{(1-\rho)^3} + \frac{684810\lambda^{10} b(2)^2 b(3)^2}{(1-\rho)^4} + \frac{1029000\lambda^{12} b(2)^3 b(3)^2}{(1-\rho)^5} + \frac{787500\lambda^{14} b(2)^4 b(3)^2}{(1-\rho)^6} + \frac{297675\lambda^{16} b(2)^5 b(3)^2}{(1-\rho)^7} \\
& + \frac{44100\lambda^{18} b(2)^6 b(3)^2}{(1-\rho)^8} + \frac{76090\lambda^9 b(3)^3}{(1-\rho)^3} + \frac{274400\lambda^{11} b(2) b(3)^3}{(1-\rho)^4} + \frac{350000\lambda^{13} b(2)^2 b(3)^3}{(1-\rho)^5} + \frac{189000\lambda^{15} b(2)^3 b(3)^3}{(1-\rho)^6} \\
& + \frac{36750\lambda^{17} b(2)^4 b(3)^3}{(1-\rho)^7} + \frac{70000\lambda^{12} b(2)^4}{3(1-\rho)^4} + \frac{31500\lambda^{14} b(2) b(3)^4}{(1-\rho)^5} + \frac{10500\lambda^{16} b(2)^2 b(3)^4}{(1-\rho)^6} + \frac{1400\lambda^{15} b(3)^5}{3(1-\rho)^5} \\
& + \frac{4665\lambda^4 b(4)}{2(1-\rho)} + \frac{34105\lambda^6 b(2) b(4)}{(1-\rho)^2} + \frac{637875\lambda^8 b(2)^2 b(4)}{4(1-\rho)^3} + \frac{342405\lambda^{10} b(2)^3 b(4)}{(1-\rho)^4} + \frac{385875\lambda^{12} b(2)^4 b(4)}{(1-\rho)^5} \\
& + \frac{236250\lambda^{14} b(2)^5 b(4)}{(1-\rho)^6} + \frac{297675\lambda^{16} b(2)^6 b(4)}{4(1-\rho)^7} + \frac{9450\lambda^{18} b(2)^7 b(4)}{(1-\rho)^8} + \frac{70875\lambda^7 b(3) b(4)}{(1-\rho)^2} + \frac{342405\lambda^9 b(2) b(3) b(4)}{(1-\rho)^3} \\
& + \frac{617400\lambda^{11} b(2)^2 b(3) b(4)}{(1-\rho)^4} + \frac{525000\lambda^{13} b(2)^3 b(3) b(4)}{(1-\rho)^5} + \frac{212625\lambda^{15} b(2)^4 b(3) b(4)}{(1-\rho)^6} + \frac{33075\lambda^{17} b(2)^5 b(3) b(4)}{(1-\rho)^7} \\
& + \frac{102900\lambda^{10} b(3)^2 b(4)}{(1-\rho)^3} + \frac{210000\lambda^{12} b(2) b(3)^2 b(4)}{(1-\rho)^4} + \frac{141750\lambda^{14} b(2)^2 b(3)^2 b(4)}{(1-\rho)^5} + \frac{31500\lambda^{16} b(2)^3 b(3)^2 b(4)}{(1-\rho)^6} \\
& + \frac{12600\lambda^{13} b(3)^3 b(4)}{(1-\rho)^4} + \frac{7000\lambda^{15} b(2) b(3)^3 b(4)}{(1-\rho)^5} + \frac{114135\lambda^8 b(4)^2}{4(1-\rho)^2} + \frac{77175\lambda^{10} b(2) b(4)^2}{(1-\rho)^3} + \frac{78750\lambda^{12} b(2)^2 b(4)^2}{(1-\rho)^4} \\
& + \frac{70875\lambda^{14} b(2)^3 b(4)^2}{2(1-\rho)^5} + \frac{23625\lambda^{16} b(2)^4 b(4)^2}{4(1-\rho)^6} + \frac{26250\lambda^{11} b(3) b(4)^2}{(1-\rho)^3} + \frac{28350\lambda^{13} b(2) b(3) b(4)^2}{(1-\rho)^4} + \frac{7875\lambda^{15} b(2)^2 b(3) b(4)^2}{(1-\rho)^5} \\
& + \frac{1050\lambda^{14} b(3)^2 b(4)^2}{(1-\rho)^4} + \frac{4725\lambda^{12} b(4)^3}{4(1-\rho)^3} + \frac{525\lambda^{14} b(2) b(4)^3}{(1-\rho)^4} + \frac{6821\lambda^5 b(5)}{1-\rho} + \frac{42525\lambda^7 b(2) b(5)}{(1-\rho)^2} + \frac{205443\lambda^9 b(2)^2 b(5)}{2(1-\rho)^3} \\
& + \frac{123480\lambda^{11} b(2)^3 b(5)}{(1-\rho)^4} + \frac{78750\lambda^{13} b(2)^4 b(5)}{(1-\rho)^5} + \frac{25515\lambda^{15} b(2)^5 b(5)}{(1-\rho)^6} + \frac{6615\lambda^{17} b(2)^6 b(5)}{2(1-\rho)^7} + \frac{45654\lambda^8 b(3) b(5)}{(1-\rho)^2} \\
& + \frac{123480\lambda^{10} b(2) b(3) b(5)}{(1-\rho)^3} + \frac{126000\lambda^{12} b(2)^2 b(3) b(5)}{(1-\rho)^4} + \frac{56700\lambda^{14} b(2)^3 b(3) b(5)}{(1-\rho)^5} + \frac{9450\lambda^{16} b(2)^4 b(3) b(5)}{(1-\rho)^6} \\
& + \frac{21000\lambda^{11} b(3)^2 b(5)}{(1-\rho)^3} + \frac{22680\lambda^{13} b(2) b(3)^2 b(5)}{(1-\rho)^4} + \frac{6300\lambda^{15} b(2)^2 b(3)^2 b(5)}{(1-\rho)^5} + \frac{560\lambda^{14} b(3)^3 b(5)}{(1-\rho)^4} + \frac{20580\lambda^9 b(4) b(5)}{(1-\rho)^2} \\
& + \frac{31500\lambda^{11} b(2) b(5)}{(1-\rho)^3} + \frac{17010\lambda^{13} b(2)^2 b(4) b(5)}{(1-\rho)^4} + \frac{3150\lambda^{15} b(2)^3 b(4) b(5)}{(1-\rho)^5} + \frac{5670\lambda^{12} b(3) b(4) b(5)}{(1-\rho)^3} + \frac{2520\lambda^{14} b(2) b(3) b(4) b(5)}{(1-\rho)^4} \\
& + \frac{315\lambda^{13} b(4)^2 b(5)}{2(1-\rho)^3} + \frac{2100\lambda^{10} b(5)^2}{(1-\rho)^2} + \frac{1701\lambda^{12} b(2) b(5)^2}{(1-\rho)^3} + \frac{378\lambda^{14} b(2)^2 b(5)^2}{(1-\rho)^4} + \frac{126\lambda^{13} b(3) b(5)^2}{(1-\rho)^3} + \frac{14175\lambda^6 b(6)}{2(1-\rho)} \\
& + \frac{22827\lambda^8 b(2) b(6)}{(1-\rho)^2} + \frac{30570\lambda^{10} b(2)^2 b(6)}{(1-\rho)^3} + \frac{21000\lambda^{12} b(2)^3 b(6)}{(1-\rho)^4} + \frac{14175\lambda^{14} b(2)^4 b(6)}{2(1-\rho)^5} + \frac{945\lambda^{16} b(2)^5 b(6)}{(1-\rho)^6} \\
& + \frac{13720\lambda^9 b(3) b(6)}{(1-\rho)^2} + \frac{21000\lambda^{11} b(2) b(3) b(6)}{(1-\rho)^3} + \frac{11340\lambda^{13} b(2)^2 b(3) b(6)}{(1-\rho)^4} + \frac{2100\lambda^{15} b(2)^3 b(3) b(6)}{(1-\rho)^5} + \frac{1890\lambda^{12} b(3)^2 b(6)}{(1-\rho)^3} \\
& + \frac{840\lambda^{14} b(2) b(3)^2 b(6)}{(1-\rho)^4} + \frac{3500\lambda^{10} b(4) b(6)}{(1-\rho)^2} + \frac{2835\lambda^{12} b(2) b(4) b(6)}{(1-\rho)^3} + \frac{630\lambda^{14} b(2)^2 b(4) b(6)}{(1-\rho)^4} + \frac{210\lambda^{13} b(3) b(4) b(6)}{(1-\rho)^3} \\
& + \frac{378\lambda^{11} b(5) b(6)}{(1-\rho)^2} + \frac{126\lambda^{13} b(2) b(5) b(6)}{(1-\rho)^3} + \frac{7\lambda^{12} b(6)^2}{(1-\rho)^2} + \frac{3261\lambda^7 b(7)}{1-\rho} + \frac{5880\lambda^9 b(2) b(7)}{(1-\rho)^2} + \frac{4500\lambda^{11} b(2)^2 b(7)}{(1-\rho)^3}
\end{aligned}$$

to be continued

$E[\ell^{10}]$ (continuation)

$$\begin{aligned}
& + \frac{1620 \lambda^{13} b(2)^3 b(7)}{(1-\rho)^4} + \frac{225 \lambda^{15} b(2)^4 b(7)}{(1-\rho)^5} + \frac{2000 \lambda^{10} b(3) b(7)}{(1-\rho)^2} + \frac{1620 \lambda^{12} b(2) b(3) b(7)}{(1-\rho)^3} + \frac{360 \lambda^{14} b(2)^2 b(3) b(7)}{(1-\rho)^4} \\
& + \frac{60 \lambda^{13} b(3)^2 b(7)}{(1-\rho)^3} + \frac{270 \lambda^{11} b(4) b(7)}{(1-\rho)^2} + \frac{90 \lambda^{13} b(2) b(4) b(7)}{(1-\rho)^3} + \frac{12 \lambda^{12} b(5) b(7)}{(1-\rho)^2} + \frac{735 \lambda^8 b(8)}{1-\rho} \\
& + \frac{750 \lambda^{10} b(2) b(8)}{(1-\rho)^2} + \frac{1215 \lambda^{12} b(2)^2 b(8)}{4(1-\rho)^3} + \frac{45 \lambda^{14} b(2)^3 b(8)}{(1-\rho)^4} + \frac{135 \lambda^{11} b(3) b(8)}{(1-\rho)^2} + \frac{45 \lambda^{13} b(2) b(3) b(8)}{(1-\rho)^3} \\
& + \frac{15 \lambda^{12} b(4) b(8)}{2(1-\rho)^2} + \frac{250 \lambda^9 b(9)}{3(1-\rho)} + \frac{45 \lambda^{11} b(2) b(9)}{(1-\rho)^2} + \frac{15 \lambda^{13} b(2)^2 b(9)}{2(1-\rho)^3} + \frac{10 \lambda^{12} b(3) b(9)}{3(1-\rho)^2} \\
& + \frac{9 \lambda^{10} b(10)}{2(1-\rho)} + \frac{\lambda^{12} b(2) b(10)}{(1-\rho)^2} + \frac{\lambda^{11} b(11)}{11(1-\rho)}
\end{aligned}$$

Appendix E. Moments of the Length of a Busy Period in an M/G/1 System

$$\begin{aligned}
E[\Theta] &= \frac{b}{1-\rho} \\
E[\Theta^2] &= \frac{b(2)}{(1-\rho)^3} \\
E[\Theta^3] &= \frac{3\lambda b(2)^2}{(1-\rho)^5} + \frac{b(3)}{(1-\rho)^4} \\
E[\Theta^4] &= \frac{15\lambda^2 b(2)^3}{(1-\rho)^7} + \frac{10\lambda b(2)b(3)}{(1-\rho)^6} + \frac{b(4)}{(1-\rho)^5} \\
E[\Theta^5] &= \frac{105\lambda^3 b(2)^4}{(1-\rho)^9} + \frac{105\lambda^2 b(2)^2 b(3)}{(1-\rho)^8} + \frac{10\lambda b(3)^2}{(1-\rho)^7} + \frac{15\lambda b(2)b(4)}{(1-\rho)^7} + \frac{b(5)}{(1-\rho)^6} \\
E[\Theta^6] &= \frac{945\lambda^4 b(2)^5}{(1-\rho)^{11}} + \frac{1260\lambda^3 b(2)^3 b(3)}{(1-\rho)^{10}} + \frac{280\lambda^2 b(2)b(3)^2}{(1-\rho)^9} + \frac{210\lambda^2 b(2)^2 b(4)}{(1-\rho)^9} + \frac{35\lambda b(3)b(4)}{(1-\rho)^8} \\
&\quad + \frac{21\lambda b(2)b(5)}{(1-\rho)^8} + \frac{b(6)}{(1-\rho)^7} \\
E[\Theta^7] &= \frac{10395\lambda^5 b(2)^6}{(1-\rho)^{13}} + \frac{17325\lambda^4 b(2)^4 b(3)}{(1-\rho)^{12}} + \frac{6300\lambda^3 b(2)^2 b(3)^2}{(1-\rho)^{11}} + \frac{280\lambda^2 b(3)^3}{(1-\rho)^{10}} + \frac{3150\lambda^3 b(2)^3 b(4)}{(1-\rho)^{11}} \\
&\quad + \frac{1280\lambda^2 b(2)b(3)b(4)}{(1-\rho)^{10}} + \frac{35\lambda b(4)^2}{(1-\rho)^9} + \frac{378\lambda^2 b(2)^2 b(5)}{(1-\rho)^{10}} + \frac{56\lambda b(3)b(5)}{(1-\rho)^9} + \frac{28\lambda b(2)b(6)}{(1-\rho)^9} + \frac{b(7)}{(1-\rho)^8} \\
E[\Theta^8] &= \frac{135135\lambda^6 b(2)^7}{(1-\rho)^{15}} + \frac{270270\lambda^5 b(2)^5 b(3)}{(1-\rho)^{14}} + \frac{138600\lambda^4 b(2)^3 b(3)^2}{(1-\rho)^{13}} + \frac{15400\lambda^3 b(2)b(3)^3}{(1-\rho)^{12}} + \frac{51975\lambda^4 b(2)^4 b(4)}{(1-\rho)^{13}} \\
&\quad + \frac{34650\lambda^3 b(2)^2 b(3)b(4)}{(1-\rho)^{12}} + \frac{2100\lambda^2 b(3)^2 b(4)}{(1-\rho)^{11}} + \frac{1575\lambda^2 b(2)b(4)^2}{(1-\rho)^{11}} + \frac{6930\lambda^3 b(2)^3 b(5)}{(1-\rho)^{12}} + \frac{2520\lambda^2 b(2)b(3)b(5)}{(1-\rho)^{11}} \\
&\quad + \frac{126\lambda b(4)b(5)}{(1-\rho)^{10}} + \frac{630\lambda^2 b(2)^2 b(6)}{(1-\rho)^{11}} + \frac{84\lambda b(3)b(6)}{(1-\rho)^{10}} + \frac{36\lambda b(2)b(7)}{(1-\rho)^{10}} + \frac{b(8)}{(1-\rho)^9} \\
E[\Theta^9] &= \frac{2027025\lambda^7 b(2)^8}{(1-\rho)^{17}} + \frac{4729725\lambda^6 b(2)^6 b(3)}{(1-\rho)^{16}} + \frac{3153150\lambda^5 b(2)^4 b(3)^2}{(1-\rho)^{15}} + \frac{600600\lambda^4 b(2)^2 b(3)^3}{(1-\rho)^{14}} + \frac{15400\lambda^3 b(3)^4}{(1-\rho)^{13}} \\
&\quad + \frac{945945\lambda^5 b(2)^5 b(4)}{(1-\rho)^{15}} + \frac{900900\lambda^4 b(2)^3 b(3)b(4)}{(1-\rho)^{14}} + \frac{138600\lambda^3 b(2)b(3)^2 b(4)}{(1-\rho)^{13}} + \frac{51975\lambda^3 b(2)^2 b(4)^2}{(1-\rho)^{13}} + \frac{5775\lambda^2 b(3)b(4)^2}{(1-\rho)^{12}} \\
&\quad + \frac{135135\lambda^4 b(2)^4 b(5)}{(1-\rho)^{14}} + \frac{83160\lambda^3 b(2)^2 b(3)b(5)}{(1-\rho)^{13}} + \frac{4620\lambda^2 b(3)^2 b(5)}{(1-\rho)^{12}} + \frac{6930\lambda^2 b(2)b(4)b(5)}{(1-\rho)^{12}} + \frac{126\lambda b(5)^2}{(1-\rho)^{11}} \\
&\quad + \frac{13860\lambda^3 b(2)^3 b(6)}{(1-\rho)^{13}} + \frac{4620\lambda^2 b(2)b(3)b(6)}{(1-\rho)^{12}} + \frac{210\lambda b(4)b(6)}{(1-\rho)^{11}} + \frac{990\lambda^2 b(2)^2 b(7)}{(1-\rho)^{12}} + \frac{120\lambda b(3)b(7)}{(1-\rho)^{11}} \\
&\quad + \frac{45\lambda b(2)b(8)}{(1-\rho)^{11}} + \frac{b(9)}{(1-\rho)^{10}} \\
E[\Theta^{10}] &= \frac{34459425\lambda^8 b(2)^9}{(1-\rho)^{19}} + \frac{91891800\lambda^7 b(2)^7 b(3)}{(1-\rho)^{18}} + \frac{75675600\lambda^6 b(2)^5 b(3)^2}{(1-\rho)^{17}} + \frac{21021000\lambda^5 b(2)^3 b(3)^3}{(1-\rho)^{16}} + \frac{1401400\lambda^4 b(2)b(3)^4}{(1-\rho)^{15}} \\
&\quad + \frac{18918900\lambda^6 b(2)^6 b(4)}{(1-\rho)^{17}} + \frac{23648625\lambda^5 b(2)^4 b(3)b(4)}{(1-\rho)^{16}} + \frac{6306300\lambda^4 b(2)^2 b(3)^2 b(4)}{(1-\rho)^{15}} + \frac{200200\lambda^3 b(3)^3 b(4)}{(1-\rho)^{14}} \\
&\quad + \frac{1576575\lambda^4 b(2)^3 b(4)^2}{(1-\rho)^{15}} + \frac{450450\lambda^3 b(2)b(3)b(4)^2}{(1-\rho)^{14}} + \frac{5775\lambda^2 b(4)^3}{(1-\rho)^{13}} + \frac{2837835\lambda^5 b(2)^5 b(5)}{(1-\rho)^{16}} + \frac{2522520\lambda^4 b(2)^3 b(3)b(5)}{(1-\rho)^{15}} \\
&\quad + \frac{360360\lambda^3 b(2)b(3)^2 b(5)}{(1-\rho)^{14}} + \frac{270270\lambda^3 b(2)^2 b(4)b(5)}{(1-\rho)^{14}} + \frac{27720\lambda^2 b(3)b(4)b(5)}{(1-\rho)^{13}} + \frac{8316\lambda^2 b(2)b(5)^2}{(1-\rho)^{13}} + \frac{315315\lambda^4 b(2)^4 b(6)}{(1-\rho)^{15}} \\
&\quad + \frac{180180\lambda^3 b(2)^2 b(3)b(6)}{(1-\rho)^{14}} + \frac{9240\lambda^2 b(3)^2 b(6)}{(1-\rho)^{13}} + \frac{13860\lambda^2 b(2)b(4)b(6)}{(1-\rho)^{13}} + \frac{462\lambda b(5)b(6)}{(1-\rho)^{12}} + \frac{25740\lambda^3 b(2)^3 b(7)}{(1-\rho)^{14}} \\
&\quad + \frac{7920\lambda^2 b(2)b(3)b(7)}{(1-\rho)^{13}} + \frac{330\lambda b(4)b(7)}{(1-\rho)^{12}} + \frac{1485\lambda^2 b(2)^2 b(8)}{(1-\rho)^{13}} + \frac{165\lambda b(3)b(8)}{(1-\rho)^{12}} + \frac{55\lambda b(2)b(9)}{(1-\rho)^{12}} + \frac{b(10)}{(1-\rho)^{11}}
\end{aligned}$$

Appendix F. Moments of the Waiting Time in an LCFS M/G/1 System

$$\begin{aligned}
E[W] &= \frac{\lambda b^{(2)}}{2(1-\rho)} \\
E[W^2] &= \frac{\lambda^2 b^{(2)^2}}{2(1-\rho)^3} + \frac{\lambda b^{(3)}}{3(1-\rho)^2} \\
E[W^3] &= \frac{3\lambda^3 b^{(2)^3}}{2(1-\rho)^5} + \frac{3\lambda^2 b^{(2)} b^{(3)}}{2(1-\rho)^4} + \frac{\lambda b^{(4)}}{4(1-\rho)^3} \\
E[W^4] &= \frac{15\lambda^4 b^{(2)^4}}{2(1-\rho)^7} + \frac{10\lambda^3 b^{(2)^2} b^{(3)}}{(1-\rho)^6} + \frac{4\lambda^2 b^{(3)^2}}{3(1-\rho)^5} + \frac{2\lambda^2 b^{(2)} b^{(4)}}{(1-\rho)^5} + \frac{\lambda b^{(5)}}{5(1-\rho)^4} \\
E[W^5] &= \frac{105\lambda^5 b^{(2)^5}}{2(1-\rho)^9} + \frac{175\lambda^4 b^{(2)^3} b^{(3)}}{2(1-\rho)^8} + \frac{25\lambda^3 b^{(2)} b^{(3)^2}}{(1-\rho)^7} + \frac{75\lambda^3 b^{(2)^2} b^{(4)}}{4(1-\rho)^7} + \frac{25\lambda^2 b^{(3)} b^{(4)}}{6(1-\rho)^6} \\
&\quad + \frac{5\lambda^2 b^{(2)} b^{(5)}}{2(1-\rho)^6} + \frac{\lambda b^{(6)}}{6(1-\rho)^5} \\
E[W^6] &= \frac{945\lambda^6 b^{(2)^6}}{2(1-\rho)^{11}} + \frac{945\lambda^5 b^{(2)^4} b^{(3)}}{(1-\rho)^{10}} + \frac{420\lambda^4 b^{(2)^2} b^{(3)^2}}{(1-\rho)^9} + \frac{70\lambda^3 b^{(3)^3}}{3(1-\rho)^8} + \frac{210\lambda^4 b^{(2)^3} b^{(4)}}{(1-\rho)^9} \\
&\quad + \frac{105\lambda^3 b^{(2)} b^{(3)} b^{(4)}}{(1-\rho)^8} + \frac{15\lambda^2 b^{(4)^2}}{4(1-\rho)^7} + \frac{63\lambda^3 b^{(2)^2} b^{(5)}}{2(1-\rho)^8} + \frac{6\lambda^2 b^{(3)} b^{(5)}}{(1-\rho)^7} + \frac{3\lambda^2 b^{(2)} b^{(6)}}{(1-\rho)^7} + \frac{\lambda b^{(7)}}{7(1-\rho)^6} \\
E[W^7] &= \frac{10395\lambda^7 b^{(2)^7}}{2(1-\rho)^{13}} + \frac{24255\lambda^6 b^{(2)^5} b^{(3)}}{2(1-\rho)^{12}} + \frac{7350\lambda^5 b^{(2)^3} b^{(3)^2}}{(1-\rho)^{11}} + \frac{980\lambda^4 b^{(2)} b^{(3)^3}}{(1-\rho)^{10}} + \frac{11025\lambda^5 b^{(2)^4} b^{(4)}}{4(1-\rho)^{11}} \\
&\quad + \frac{2205\lambda^4 b^{(2)^2} b^{(3)} b^{(4)}}{(1-\rho)^{10}} + \frac{490\lambda^3 b^{(3)^2} b^{(4)}}{3(1-\rho)^9} + \frac{245\lambda^3 b^{(2)} b^{(4)^2}}{2(1-\rho)^9} + \frac{441\lambda^4 b^{(2)^3} b^{(5)}}{(1-\rho)^{10}} + \frac{196\lambda^3 b^{(2)} b^{(3)} b^{(5)}}{(1-\rho)^9} \\
&\quad + \frac{49\lambda^2 b^{(4)} b^{(5)}}{4(1-\rho)^8} + \frac{49\lambda^3 b^{(2)^2} b^{(6)}}{(1-\rho)^9} + \frac{49\lambda^2 b^{(3)} b^{(6)}}{6(1-\rho)^8} + \frac{7\lambda^2 b^{(2)} b^{(7)}}{2(1-\rho)^8} + \frac{\lambda b^{(8)}}{8(1-\rho)^7} \\
E[W^8] &= \frac{135135\lambda^8 b^{(2)^8}}{2(1-\rho)^{15}} + \frac{180180\lambda^7 b^{(2)^6} b^{(3)}}{(1-\rho)^{14}} + \frac{138600\lambda^6 b^{(2)^4} b^{(3)^2}}{(1-\rho)^{13}} + \frac{30800\lambda^5 b^{(2)^2} b^{(3)^3}}{(1-\rho)^{12}} + \frac{2800\lambda^4 b^{(3)^4}}{3(1-\rho)^{11}} \\
&\quad + \frac{41580\lambda^6 b^{(2)^5} b^{(4)}}{(1-\rho)^{13}} + \frac{46200\lambda^5 b^{(2)^3} b^{(3)} b^{(4)}}{(1-\rho)^{12}} + \frac{8400\lambda^4 b^{(2)} b^{(3)^2} b^{(4)}}{(1-\rho)^{11}} + \frac{3150\lambda^4 b^{(2)^2} b^{(4)^2}}{(1-\rho)^{11}} + \frac{420\lambda^3 b^{(3)} b^{(4)^2}}{(1-\rho)^{10}} \\
&\quad + \frac{6930\lambda^5 b^{(2)^4} b^{(5)}}{(1-\rho)^{12}} + \frac{5040\lambda^4 b^{(2)^2} b^{(3)} b^{(5)}}{(1-\rho)^{11}} + \frac{336\lambda^3 b^{(3)^2} b^{(5)}}{(1-\rho)^{10}} + \frac{504\lambda^3 b^{(2)} b^{(4)} b^{(5)}}{(1-\rho)^{10}} + \frac{56\lambda^2 b^{(5)^2}}{5(1-\rho)^9} \\
&\quad + \frac{840\lambda^4 b^{(2)^3} b^{(6)}}{(1-\rho)^{11}} + \frac{336\lambda^3 b^{(2)} b^{(3)} b^{(6)}}{(1-\rho)^{10}} + \frac{56\lambda^2 b^{(4)} b^{(6)}}{3(1-\rho)^9} + \frac{72\lambda^3 b^{(2)^2} b^{(7)}}{(1-\rho)^{10}} + \frac{32\lambda^2 b^{(3)} b^{(7)}}{3(1-\rho)^9} \\
&\quad + \frac{4\lambda^2 b^{(2)} b^{(8)}}{(1-\rho)^9} + \frac{\lambda b^{(9)}}{9(1-\rho)^8} \\
E[W^9] &= \frac{2027025\lambda^9 b^{(2)^9}}{2(1-\rho)^{17}} + \frac{6081075\lambda^8 b^{(2)^7} b^{(3)}}{2(1-\rho)^{16}} + \frac{2837835\lambda^7 b^{(2)^5} b^{(3)^2}}{(1-\rho)^{15}} + \frac{900900\lambda^6 b^{(2)^3} b^{(3)^3}}{(1-\rho)^{14}} + \frac{69300\lambda^5 b^{(2)} b^{(3)^4}}{(1-\rho)^{13}} \\
&\quad + \frac{2837835\lambda^7 b^{(2)^6} b^{(4)}}{4(1-\rho)^{15}} + \frac{2027025\lambda^6 b^{(2)^4} b^{(3)} b^{(4)}}{2(1-\rho)^{14}} + \frac{311850\lambda^5 b^{(2)^2} b^{(3)^2} b^{(4)}}{(1-\rho)^{13}} + \frac{11550\lambda^4 b^{(3)^3} b^{(4)}}{(1-\rho)^{12}} + \frac{155925\lambda^5 b^{(2)^3} b^{(4)^2}}{2(1-\rho)^{13}} \\
&\quad + \frac{51975\lambda^4 b^{(2)} b^{(3)} b^{(4)^2}}{2(1-\rho)^{12}} + \frac{1575\lambda^3 b^{(4)^3}}{4(1-\rho)^{11}} + \frac{243243\lambda^6 b^{(2)^5} b^{(5)}}{2(1-\rho)^{14}} + \frac{124740\lambda^5 b^{(2)^3} b^{(3)} b^{(5)}}{(1-\rho)^{13}} + \frac{20790\lambda^4 b^{(2)} b^{(3)^2} b^{(5)}}{(1-\rho)^{12}} \\
&\quad + \frac{31185\lambda^4 b^{(2)^2} b^{(4)} b^{(5)}}{2(1-\rho)^{12}} + \frac{1890\lambda^3 b^{(3)} b^{(4)} b^{(5)}}{(1-\rho)^{11}} + \frac{567\lambda^3 b^{(2)} b^{(5)^2}}{(1-\rho)^{11}} + \frac{31185\lambda^5 b^{(2)^4} b^{(6)}}{2(1-\rho)^{13}} + \frac{10395\lambda^4 b^{(2)^2} b^{(3)} b^{(6)}}{(1-\rho)^{12}} \\
&\quad + \frac{630\lambda^3 b^{(3)^2} b^{(6)}}{(1-\rho)^{11}} + \frac{945\lambda^3 b^{(2)} b^{(4)} b^{(6)}}{(1-\rho)^{11}} + \frac{189\lambda^2 b^{(5)} b^{(6)}}{5(1-\rho)^{10}} + \frac{1485\lambda^4 b^{(2)^3} b^{(7)}}{(1-\rho)^{12}} + \frac{540\lambda^3 b^{(2)} b^{(3)} b^{(7)}}{(1-\rho)^{11}} \\
&\quad + \frac{27\lambda^2 b^{(4)} b^{(7)}}{(1-\rho)^{10}} + \frac{405\lambda^3 b^{(2)^2} b^{(8)}}{4(1-\rho)^{11}} + \frac{27\lambda^2 b^{(3)} b^{(8)}}{2(1-\rho)^{10}} + \frac{9\lambda^2 b^{(2)} b^{(9)}}{2(1-\rho)^{10}} + \frac{\lambda b^{(10)}}{10(1-\rho)^9}
\end{aligned}$$

$$\begin{aligned}
E[W^{10}] = & \frac{34459425 \lambda^{10} b(2)^{10}}{2(1-\rho)^{19}} + \frac{57432375 \lambda^9 b(2)^8 b(3)}{(1-\rho)^{18}} + \frac{63063000 \lambda^8 b(2)^6 b(3)^2}{(1-\rho)^{17}} + \frac{26276250 \lambda^7 b(2)^4 b(3)^3}{(1-\rho)^{16}} \\
& + \frac{3503500 \lambda^6 b(2)^2 b(3)^4}{(1-\rho)^{15}} + \frac{200200 \lambda^5 b(3)^5}{3(1-\rho)^{14}} + \frac{13513500 \lambda^8 b(2)^7 b(4)}{(1-\rho)^{17}} + \frac{23648625 \lambda^7 b(2)^5 b(3) b(4)}{(1-\rho)^{16}} \\
& + \frac{10510500 \lambda^6 b(2)^3 b(3)^2 b(4)}{(1-\rho)^{15}} + \frac{1001000 \lambda^5 b(2) b(3)^3 b(4)}{(1-\rho)^{14}} + \frac{7882875 \lambda^6 b(2)^4 b(4)^2}{4(1-\rho)^{15}} + \frac{1126125 \lambda^5 b(2)^2 b(3) b(4)^2}{(1-\rho)^{14}} \\
& + \frac{57750 \lambda^4 b(3)^2 b(4)^2}{(1-\rho)^{13}} + \frac{28875 \lambda^4 b(2) b(4)^3}{(1-\rho)^{13}} + \frac{4729725 \lambda^7 b(2)^6 b(5)}{2(1-\rho)^{16}} + \frac{3153150 \lambda^6 b(2)^4 b(3) b(5)}{(1-\rho)^{15}} \\
& + \frac{900900 \lambda^5 b(2)^2 b(3)^2 b(5)}{(1-\rho)^{14}} + \frac{30800 \lambda^4 b(3)^3 b(5)}{(1-\rho)^{13}} + \frac{450450 \lambda^5 b(2)^3 b(4) b(5)}{(1-\rho)^{14}} + \frac{138600 \lambda^4 b(2) b(3) b(4) b(5)}{(1-\rho)^{13}} \\
& + \frac{5775 \lambda^3 b(4)^2 b(5)}{2(1-\rho)^{12}} + \frac{20790 \lambda^4 b(2)^2 b(5)^2}{(1-\rho)^{13}} + \frac{2310 \lambda^3 b(3) b(5)^2}{(1-\rho)^{12}} + \frac{315315 \lambda^6 b(2)^5 b(6)}{(1-\rho)^{15}} + \frac{300300 \lambda^5 b(2)^3 b(3) b(6)}{(1-\rho)^{14}} \\
& + \frac{46200 \lambda^4 b(2) b(3)^2 b(6)}{(1-\rho)^{13}} + \frac{34650 \lambda^4 b(2)^2 b(4) b(6)}{(1-\rho)^{13}} + \frac{3850 \lambda^3 b(3) b(4) b(6)}{(1-\rho)^{12}} + \frac{2310 \lambda^3 b(2) b(5) b(6)}{(1-\rho)^{12}} + \frac{35 \lambda^2 b(6)^2}{(1-\rho)^{11}} \\
& + \frac{32175 \lambda^5 b(2)^4 b(7)}{(1-\rho)^{14}} + \frac{19800 \lambda^4 b(2)^2 b(3) b(7)}{(1-\rho)^{13}} + \frac{1100 \lambda^3 b(3)^2 b(7)}{(1-\rho)^{12}} + \frac{1650 \lambda^3 b(2) b(4) b(7)}{(1-\rho)^{12}} + \frac{60 \lambda^2 b(5) b(7)}{(1-\rho)^{11}} \\
& + \frac{2475 \lambda^4 b(2)^3 b(8)}{(1-\rho)^{13}} + \frac{825 \lambda^3 b(2) b(3) b(8)}{(1-\rho)^{12}} + \frac{75 \lambda^2 b(4) b(8)}{2(1-\rho)^{11}} + \frac{275 \lambda^3 b(2)^2 b(9)}{2(1-\rho)^{12}} + \frac{50 \lambda^2 b(3) b(9)}{3(1-\rho)^{11}} \\
& + \frac{5 \lambda^2 b(2) b(10)}{(1-\rho)^{11}} + \frac{\lambda b(11)}{11(1-\rho)^{10}}
\end{aligned}$$

