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**Abstract**

The concept of signed poset has recently been introduced by V. Reiner as a generalization of ordinary poset (partially ordered set). We consider the problem of finding a minimum-weight ideal of a signed poset and show that the problem can be reduced to a problem of finding a minimum-weight ideal of an appropriately defined ordinary poset and hence to a minimum-cut problem. We also consider the case when the weight of an ideal is defined in terms of two weight functions. The problem is also reduced to a minimum-cut problem. We reveal the relationship between the minimum-weight ideal problem and a certain bisubmodular function minimization problem.

**Key words**: Signed posets, Ideals, Minimum-weight ideals, Posets, Minimum cuts
**AMS subject classifications**: 90C27; 05C85, 90B10, 06A07
**Abbreviated title**: The minimum-weight ideal problem for signed posets

## 1. Introduction

The concept of signed poset has recently been introduced by V. Reiner [10]. A signed poset is a kind of bidirected graph ([5]). A *bidirected graph* $G = (V, A; \partial)$ has a vertex set $V$, an arc set $A$ and a boundary operator $\partial$ such that for each arc $a \in A$

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$\partial a$ is given by one of the following three: for some vertices $v, w \in V$
(1) $\partial a = v - w$ ($a$ has a tail at $v$ and a head at $w$),
(2) $\partial a = v + w$ ($a$ has two tails, one at $v$ and the other at $w$),
(3) $\partial a = -v - w$ ($a$ has two heads, one at $v$ and the other at $w$).
Here, if $v = w$, the arc $a$ is called a *selfloop*. We do not allow selfloops of type (1). We sometimes regard the boundaries $\partial a$ as vectors in $\mathbf{R}^V$. Figure 1.1 shows an example of a bidirected graph on $V = \{1, 2, 3, 4\}$.

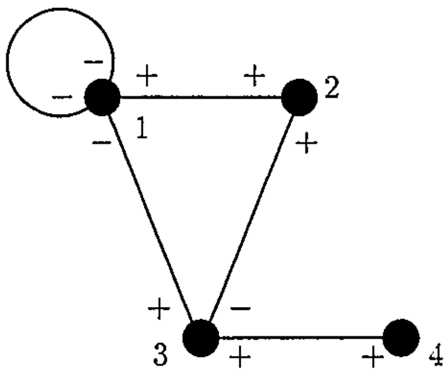


Figure 1.1. An example of a bidirected graph.

We say an arc $a$ is *incident to* a vertex $v$ (and a vertex $w$) if $\partial a = \pm v \pm w$, and $a$ is *positively* (or *negatively*) incident to $v$ if the coefficient of $v$ in $\partial a$ is positive (or negative). Two arcs $a_1, a_2$ are said to be *oppositely incident to* a vertex $v$ if one of the two is positively incident to $v$ and the other is negatively incident to $v$.

A *signed poset* $\mathcal{P} = (V, A; \partial)$ is a bidirected graph with a vertex set $V$, an arc set $A$ and a boundary operator $\partial$ such that
(i) for any two arcs $a_1, a_2 \in A$ we have $\partial a_1 \neq -\partial a_2$,
(ii) for any $a_1, a_2 \in A$ oppositely incident to a common vertex there exists an arc $a_3 \in A$ satisfying $\partial a_3 = \partial a_1 + \partial a_2$,
(iii) for any two selfloops $a_1, a_2 \in A$ incident to distinct vertices there exits an arc $a_3 \in A$ satisfying $2\partial a_3 = \partial a_1 + \partial a_2$.

For a bidirected graph $G = (V, A; \partial)$ we say an arc $a \in A$ is *redundant* if $\partial a$ can be expressed as a nonnegative linear combination of the other $\partial a'$ ($a' \in A - \{a\}$). For a signed poset $\mathcal{P} = (V, A; \partial)$ there exists a unique maximal bidirected subgraph that has no redundant arcs, which is called the *Hasse diagram* of $\mathcal{P}$ (see [10]). Figure 1.1 is in fact an example of the Hasse diagram of a signed poset.

Denote by $3^V$ the collection of all the ordered pairs $(X, Y)$ of disjoint subsets $X$ and $Y$ of $V$. Each element $(X, Y) \in 3^V$ can be made correspond to its characteristic vector $\chi_{(X,Y)} \in \mathbf{R}^V$ defined by $\chi_{(X,Y)}(v) = 1$ if $v \in X$, $= -1$ if $v \in Y$ and $= 0$ otherwise. We call each element of $3^V$ a *signed set.* An *ideal* of a signed poset $\mathcal{P} = (V, A; \partial)$ is a signed set $(X, Y) \in 3^V$ such that for any arc $a \in A$

$$\langle \partial a, (X, Y) \rangle \leq 0, \tag{1.1}$$

where $\langle \cdot, \cdot \rangle$ denotes the (canonical) inner product and $\partial a$ and $(X, Y)$ should be regarded as vectors in $\mathbf{R}^V$ under natural correspondences. In Figure 1.1 $(\{1,3\},\{2,4\})$ is an ideal but $(\{1,3\},\{4\})$ is not.

We can easily see that the concept of signed poset is a generalization of that of (ordinary) poset. Posets play a fundamental rôle in a lot of practical problems arising in scheduling problems, network optimization etc. Though we have not yet found concrete practical applications of signed posets, we expect that there should be many possible applications of signed posets. For theoretical applications see, e.g., [3] and [6].

In Section 2 we show that the minimum-weight ideal problem for signed posets can be reduced to the minimum-weight ideal problem for ordinary posets, so that it can be solved by any minimum-cut algorithm for two-terminal networks (see [8], [9] and [1]). In Section 3 we also consider the relationship between the minimum-weight ideal problem and the problem of minimizing bisubmodular functions and show that the problem of minimizing so-called box-bisubmodular functions can also be reduced to a minimum-cut problem by the same reduction.

# 2. The Minimum-Weight Ideal Problem

Given a signed poset $\mathcal{P} = (V, A; \partial)$ and a weight function $w : V \to \mathbf{R}$, consider the following problem:

$$(P) \qquad \text{Minimize} \quad w(X,Y) = \sum_{v \in X} w(v) - \sum_{v \in Y} w(v)$$
$$\text{subject to} \quad (X,Y) \in \mathcal{I}(\mathcal{P}), \tag{2.1}$$

where $\mathcal{I}(\mathcal{P})$ is the set of all the ideals of the signed poset $\mathcal{P}$.

To solve this problem construct a directed graph $\hat{G}(\mathcal{P}) = (\hat{V}, \hat{A}; \hat{\partial})$ as follows.

$$\hat{V} = \{(v,+) \mid v \in V, \forall a \in A : \partial a \neq 2v\} \cup \{(v,-) \mid v \in V, \forall a \in A : \partial a \neq -2v\}. \tag{2.2}$$

The arc set $\hat{A}$ consists of the following arcs $\hat{a}$.
(i) $\hat{a} \in \hat{A}$ with $\hat{\partial}\hat{a} = (v,+) - (w,+)$ if and only if $\exists a \in A : \partial a = v - w$ and $(v,+),(w,+) \in \hat{V}$.
(ii) $\hat{a} \in \hat{A}$ with $\hat{\partial}\hat{a} = (v,+) - (w,-)$ if and only if $\exists a \in A : \partial a = v + w, v \neq w$ and $(v,+),(w,-) \in \hat{V}$.

(iii) $\hat{a} \in \hat{A}$ with $\hat{\partial}\hat{a} = (v,-) - (w,+)$ if and only if $\exists a \in A : \partial a = -v - w, v \neq w$ and $(v,-),(w,+) \in \hat{V}$.
(iv) $\hat{a} \in \hat{A}$ with $\hat{\partial}\hat{a} = (v,-) - (w,-)$ if and only if $\exists a \in A : \partial a = -v + w$ and $(v,-),(w,-) \in \hat{V}$.
This construction is almost the same as in our previous paper [4]. Here, note that we have no arc $\hat{a} \in \hat{A}$ such that $\partial\hat{a} = (v,\pm) - (w,\pm)$ with $v = w$. Also, it should be noted that each arc of $\mathcal{P}$ corresponds to at most two arcs of $\hat{G}(\mathcal{P})$. For example, (ii) means that for an arc $a \in A$ with $\partial a = v + w$ and $v \neq w$, if $(v,+),(w,-) \in \hat{V}$, then there exists an arc $\hat{a} \in \hat{A}$ with $\hat{\partial}\hat{a} = (v,+) - (w,-)$, and if $(w,+),(v,-) \in \hat{V}$, then there exists an arc $\hat{a}' \in \hat{A}$ with $\hat{\partial}\hat{a}' = (w,+) - (v,-)$. See Figure 2.1, which is the Hasse diagram of the poset $\hat{G}(\mathcal{P})$ corresponding to the signed poset represented by the Hasse diagram shown in Figure 1.1, where the fact that $\hat{G}(\mathcal{P})$ is an ordinary poset will be shown in Theorem 2.1.

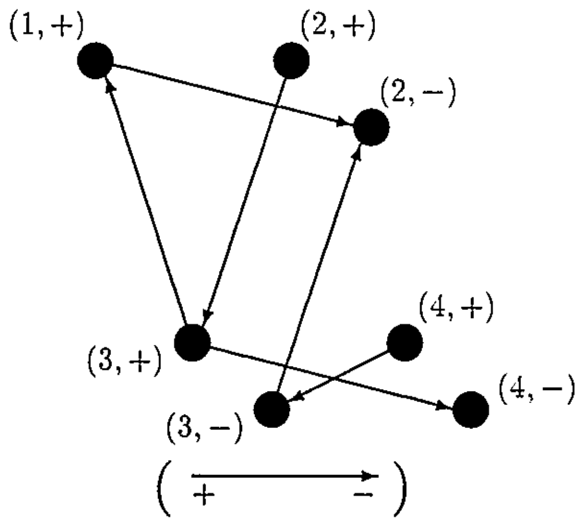


Figure 2.1. A directed graph representing the Hasse diagram of the poset $\hat{G}(\mathcal{P})$.

**Theorem 2.1:** *The (ordinary) directed graph $\hat{G}(\mathcal{P})$ defined above is transitive and acyclic (i.e., $\hat{G}(\mathcal{P})$ represents an ordinary poset).*
(Proof) Suppose that there exists two arcs $\hat{a}_1, \hat{a}_2 \in \hat{A}$ such that

$$\hat{\partial}\hat{a}_1 = (v_0,\sigma_0) - (v_1,\sigma_1), \qquad \hat{\partial}\hat{a}_2 = (v_1,\sigma_1) - (v_2,\sigma_2), \tag{2.3}$$

where $(v_i,\sigma_i) \in \hat{V}$ $(i = 0,1,2)$. Then, by the definition of $\hat{G}(\mathcal{P})$ there exist arcs $a_1, a_2 \in A$ such that

$$\partial a_1 = \sigma_0 v_0 - \sigma_1 v_1, \qquad \partial a_2 = \sigma_1 v_1 - \sigma_2 v_2. \tag{2.4}$$

Since $\mathcal{P}$ is a signed poset, there exists an arc $a_3 \in A$ with $\partial a_3 = \sigma_0 v_0 - \sigma_2 v_2$. If $v_0 = v_2$, then $\sigma_0 = -\sigma_2$ and hence there is a selfloop at $v_0$ with sign $\sigma_0$, which contradicts that $(v_0, \sigma_0) \in \hat{V}$. Hence, $v_0 \neq v_2$ and by the definition of $\hat{G}(\mathcal{P})$ there exists an arc $\hat{a}_3$ such that $\hat{\partial}\hat{a}_3 = (v_0, \sigma_0) - (v_2, \sigma_2)$. Therefore, $\hat{G}(\mathcal{P})$ is transitive.

Next, we show $\hat{G}(\mathcal{P})$ is acyclic. Suppose, on the contrary, that there is a directed cycle

$$\hat{C} = ((v_0, \sigma_0), \hat{a}_1, (v_1, \sigma_1), \cdots, \hat{a}_k, (v_k, \sigma_k)) \tag{2.5}$$

in $\hat{G}(\mathcal{P})$, where $(v_i, \sigma_i) \in \hat{V}$ $(i = 0, 1, \cdots, k)$, $\hat{a}_i \in \hat{A}$ $(i = 1, \cdots, k)$ with $\hat{\partial}\hat{a}_i = (v_{i-1}, \sigma_{i-1}) - (v_i, \sigma_i)$ $(i = 1, \cdots, k)$ and $(v_0, \sigma_0) = (v_k, \sigma_k)$. Here, note that $k \geq 2$ by the definition of $\hat{G}(\mathcal{P})$. It follows that there exist arcs $a_i \in A$ $(i = 1, \cdots, k)$ in $\mathcal{P}$ such that

$$\partial a_i = \sigma_{i-1} v_{i-1} - \sigma_i v_i \quad (i = 1, \cdots, k). \tag{2.6}$$

Then, from the definition of signed graph there exists an arc $a_0 \in A$ such that $\partial a_0 = \sigma_0 v_0 - \sigma_{k-1} v_{k-1}$, which contradicts the existence of arc $a_k$, for which $\partial a_0 = -\partial a_k$. Therefore, $\hat{G}(\mathcal{P})$ is acyclic. □

**Theorem 2.2**: *Let $\hat{J}$ be an (ordinary order) ideal of the poset $\hat{G}(\mathcal{P})$ and define*

$$\hat{J}_0 = \{(v, +), (v, -) \mid v \in V, \{(v, +), (v, -)\} \subseteq \hat{J}\}. \tag{2.7}$$

*Then, $\hat{J} - \hat{J}_0$ is also an ideal of $\hat{G}(\mathcal{P})$.*

(Proof) Let $\hat{J}$ be an ideal of the poset $\hat{G}(\mathcal{P})$ and $\hat{J}_0$ be the set defined by (2.7). Suppose, on the contrary, that $\hat{J} - \hat{J}_0$ is not an ideal of the poset $\hat{G}(\mathcal{P})$. Then, there exist $(v, \sigma) \in \hat{J}_0$ and $(w, \tau) \in \hat{J} - \hat{J}_0$ such that

$$\hat{\partial}\hat{a} = (w, \tau) - (v, \sigma) \tag{2.8}$$

for some $\hat{a} \in \hat{A}$. By the definition of $\hat{G}(\mathcal{P})$ there exists an arc $a \in A$ such that $\partial a = \tau w - \sigma v$. If there exists an arc $a' \in A$ with $\partial a' = -2\tau w$, then there exists a selfloop $a'' \in A$ with $\partial a'' = -2\sigma v$, which contradicts $(v, -\sigma) \in \hat{J}_0 \subseteq \hat{V}$. Therefore, there is no selfloop $a' \in A$ with $\partial a' = -2\tau w$, so that we have $(w, -\tau) \in \hat{V}$. It follows from the definition of $\hat{G}(\mathcal{P})$ that there is an arc $\hat{a}' \in \hat{A}$ such that

$$\hat{\partial}\hat{a}' = (v, -\sigma) - (w, -\tau). \tag{2.9}$$

Since $\hat{J}$ is an ideal of $\hat{G}(\mathcal{P})$ and $(v, -\sigma) \in \hat{J}$, we have $(w, -\tau) \in \hat{J}$ and hence, $(w, \pm) \in \hat{J}_0$, a contradiction. □

We call the ideal $\hat{J} - \hat{J}_0$ in Theorem 2.2 a *reduced ideal* of the poset $\hat{G}(\mathcal{P})$. For a reduced ideal $\hat{J}$ of $\hat{G}(\mathcal{P})$ we can make it correspond to the ideal $(X, Y)$ of the signed poset $\mathcal{P}$ defined by

$$X = \{v \mid (v, +) \in \hat{J}\}, \qquad Y = \{v \mid (v, -) \in \hat{J}\}. \tag{2.10}$$

Indeed, suppose that $a \in A$ is an arc such that $\partial a = v + \sigma w$ and $v \in X$ for some $\sigma \in \{+, -\}$. The arc $a$ cannot be a selfloop since $(v, +) \in \hat{V}$. We have $(w, -\sigma) \in \hat{V}$ since otherwise there would exist a positive selfloop at $v$, which would contradict $(v, +) \in \hat{V}$. Hence, there is an arc $\hat{a} \in \hat{A}$ such that $\hat{\partial}\hat{a} = (v, +) - (w, -\sigma)$. Since $\hat{J}$ is an ideal of $\hat{G}(\mathcal{P})$, we have $(w, -\sigma) \in \hat{J}$ and hence $\langle \partial a, (X, Y) \rangle = 0$. The case when $\partial a = -v + \sigma w$ and $v \in Y$ for some $\sigma \in \{+, -\}$ can be treated similarly. It follows that $(X, Y)$ defined by (2.10) is an ideal of $\mathcal{P}$.

**Theorem 2.3**: *The above correspondence* (2.10) *gives a one-to-one and onto mapping from the set of all the reduced ideals of the poset* $\hat{G}(\mathcal{P})$ *to the set of all the ideals of the signed poset* $\mathcal{P}$.

(Proof) First, we show that the correspondence is onto. Suppose that $(X, Y) \in 3^V$ is an ideal of $\mathcal{P}$. Then, we must have

$$(v, +) \in \hat{V} \quad (v \in X), \qquad (v, -) \in \hat{V} \quad (v \in Y). \tag{2.11}$$

We will show that $\hat{J} \subseteq \hat{V}$ defined by

$$\hat{J} = \{(v, +) \mid v \in X\} \cup \{(v, -) \mid v \in Y\} \tag{2.12}$$

is an ideal of the poset $\hat{G}(\mathcal{P})$, from which follows the fact that the correspondence is onto. Here, note that an ideal $\hat{J}$ of the form (2.12) is a reduced ideal and $(X, Y)$ clearly corresponds to $\hat{J}$.

Now, let $\hat{a}$ be an arc in $\hat{A}$ such that $\hat{\partial}\hat{a} = (v, \sigma) - (w, \tau)$ and $(v, \sigma) \in \hat{J}$. Then there is an arc $a \in A$ such that $\partial a = \sigma v - \tau w$. Suppose that $\sigma = +$. Then, by the definition of $\hat{J}$ we have $v \in X$. Since $(X, Y)$ is an ideal of $\mathcal{P}$ we have

$$w \in X, \qquad \tau = +, \tag{2.13}$$

or

$$w \in Y, \qquad \tau = -. \tag{2.14}$$

Hence, $(w, \tau) \in \hat{J}$. It follows that $\hat{J}$ is a (reduced) ideal of $\hat{G}(\mathcal{P})$. The case when $\sigma = -$ can be treated similarly.

Next, we show that the correspondence is one to one. For each $i = 1, 2$ let $(X_i, Y_i)$ be the ideal of the signed poset $\mathcal{P}$ that corresponds to a reduced ideal $\hat{J}_i$ of the poset $\hat{G}(\mathcal{P})$. Suppose $\hat{J}_1 - \hat{J}_2 \neq \emptyset$, without loss of generality. Let $(v, \sigma) \in \hat{J}_1 - \hat{J}_2$. If $\sigma = +$, then $v \in X_1$ but $v \notin X_2$, otherwise $v \in Y_1$ but $v \notin Y_2$. Hence, $(X_1, Y_1) \neq (X_2, Y_2)$. □

Based on the above theorems, defining a weight $\hat{w} : \hat{V} \to \mathbf{R}$ by

$$\hat{w}((v, +)) = w(v) \qquad ((v, +) \in \hat{V}), \tag{2.15}$$

$$\hat{w}((v, -)) = -w(v) \qquad ((v, -) \in \hat{V}). \tag{2.16}$$

Problem $(P)$ is reduced to the problem of finding a minimum-weight (ordinary) ideal of the poset $\hat{G}(\mathcal{P})$ with respect to the weight $\hat{w}$. Note that the weight of an ideal $\hat{J}$ of $\hat{G}(\mathcal{P})$ is equal to that of its corresponding reduced ideal $\hat{J} - \hat{J}_0$.

# 3. The Problem of Minimizing Bisubmodular Functions

Define two binary operations, $\sqcup$ (*reduced union*) and $\sqcap$ (*intersection*), on $3^V$ as follows. For each $(X_i, Y_i) \in 3^V$

$$(X_1, Y_1) \sqcup (X_2, Y_2) = ((X_1 \cup X_2) - (Y_1 \cup Y_2), (Y_1 \cup Y_2) - (X_1 \cup X_2)), \quad (3.1)$$

$$(X_1, Y_1) \sqcap (X_2, Y_2) = (X_1 \cap X_2, Y_1 \cap Y_2). \quad (3.2)$$

Note that $3^V$ is closed with respect to $\sqcup$ and $\sqcap$. Also, it is known (see [2]) that the set $\mathcal{I}(\mathcal{P})$ of all the ideals of a signed poset is closed with respect to $\sqcup$ and $\sqcap$. For a $\sqcup, \sqcap$-closed family $\mathcal{F} \subseteq 3^V$ a function $f : \mathcal{F} \to \mathbf{R}$ is called a *bisubmodular* (or *delta-submodular*) *function* if for each $(X_i, Y_i) \in \mathcal{F}$

$$f(X_1, Y_1) + f(X_2, Y_2) \geq f((X_1, Y_1) \sqcup (X_2, Y_2)) + f((X_1, Y_1) \sqcap (X_2, Y_2)). \quad (3.3)$$

For a weight function $w : V \to \mathbf{R}$ define for each ideal $(X, Y)$ of a poset $\mathcal{P} = (V, A; \partial)$

$$f(X, Y) = w(X) - w(Y), \quad (3.4)$$

where $w(Z) = \sum_{v \in Z} w(v)$ for any $Z \subseteq V$. Then $f$ satisfies (3.3) with equality for each $(X, Y) \in \mathcal{I}(\mathcal{P})$. Such an $f$ is called a *bimodular function.* Hence, the minimum-weight ideal problem for signed posets is equivalent to the problem of minimizing bimodular functions.

Now, consider two weight functions $w^+, w^- : V \to \mathbf{R}$ satisfying $w^+ \geq w^-$ and define

$$f(X, Y) = w^+(X) - w^-(Y) \quad (3.5)$$

for each $(X, Y) \in \mathcal{I}(\mathcal{P})$. We can easily show that $f : \mathcal{I}(\mathcal{P}) \to \mathbf{R}$ is a bisubmodular function but not necessarily a bimodular function. It is called a *box-bisubmodular function* in [7]. The problem of minimizing the box-bisubmodular function, or the problem of finding a minimum-weight ideal of the signed poset with the weight defined by (3.5), is reduced to a minimum-cut problem as follows. Consider the directed graph (poset) $\hat{G}(\mathcal{P}) = (\hat{V}, \hat{A}; \hat{\partial})$ defined in Section 2. Define the weight function $\hat{w} : \hat{V} \to \mathbf{R}$ by

$$\hat{w}((v, +)) = w^+(v) \qquad ((v, +) \in \hat{V}), \quad (3.6)$$

$$\hat{w}((v, -)) = -w^-(v) \qquad ((v, -) \in \hat{V}). \quad (3.7)$$

We can easily see that for an ideal $I$ of $\hat{G}(\mathcal{P})$ and its corresponding reduced ideal $I'$ we have

$$\hat{w}(I) \geq \hat{w}(I'). \quad (3.8)$$

Therefore, the problem is reduced to finding a minimum-weight (reduced) ideal, which is further reduced to a minimum-cut problem.

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