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Abstract

In economies with one private good and one public good, the minimal provision mechanism is shown to be the unique mechanism satisfying strategy-proofness, voluntary participation, and the full-range property if the cost share rule has the convex property and shares no fixed cost. There is no longer such a satisfactory mechanism in case of one private good and several public goods. Furthermore, when public goods can be produced with fixed costs, the implied nonconvexity of cost share rules leads to the impossibility of mechanisms which are strategy-proof and induce voluntary participation.

1. Introduction

When a society provides public goods, it has to determine the level of public goods to produce and how to share the costs among individuals. A mechanism is a function that describes the decision-making based on preferences of individuals. Moulin (1991a) characterized the mechanisms which satisfy coalition-strategy-proofness, anonymity, and voluntary participation in one private good-one public good economies. His result relies on the assumption that public goods can be produced without fixed costs. It is more natural, however, to suppose that we need positive fixed costs to produce public goods. This paper incorporates the consideration of fixed costs, and presents contrasting impossibility results.

This paper explores mechanisms satisfying two basic conditions. One is strategy-proofness. A mechanism satisfies strategy-proofness if and only if it is a dominant strategy for each individual to reveal preferences truthfully. Moulin's result is quite interesting, because it is well-known that strategy-proofness is a severe requirement in general environments. The Gibbard (1973) -Satterthwaite (1975) theorem established that, under minor conditions, any strategy-proof mechanism must be dictatorial. Recently, Barberà and Peleg (1990) and Zhou (1991) proved similar powerful impossibility results. Another condition is voluntary participation. A mechanism satisfies voluntary participation if and only if allocations which individuals obtain after playing the mechanism are no worse than before. No individual lacks an incentive to participate in voluntary participation mechanisms. The main result of this paper is that strategy-proofness and voluntary participation are sufficient for impossibility results in more natural economic environments.

This paper concentrates on the analysis of mechanisms which determine the level of public goods, assuming that cost share rules are exogenously given. One interpretation of this setting is that the revision of tax rules is less frequently than public decisions. The class of cost share rules we deal with is restricted to a reasonable one. That is, we assume that cost share rules have the same properties (convexity, monotonicity, and so on) as the cost function. For example, the equal cost share rule (the costs are shared among individuals equally) and proportional cost share rules (the costs are shared among individuals according to a given proportion vector) belong to this class.

First, for the sake of a comparison, we consider the case that a cost function is convex and has no fixed costs. In economies with one private good and one public good, the minimal provision mechanism is shown to be the unique mechanism satisfying strategy-proofness, voluntary participation, and the full-range property for any cost share rule. The full-range property is a condition that any level of the public good is attainable by the mechanism. If we turn our attention to the case of one private good and several public goods, the result drastically changes. It is easily proven by the result of Zhou (1991) that there is no longer a satisfactory mechanism.

Next, we consider the case that a cost function is convex and has positive fixed costs. Since there exist fixed costs, the cost function includes a nonconvex portion. Thus, any cost share rule must be partially nonconvex. It is shown that this nonconvexity prevents us from constructing mechanisms satisfying strategy-proofness and voluntary participation. This result implies that we must restrict the range of mechanisms, if we seek strategy-proof and voluntary participation mechanisms. In other words, nonconvexity of cost share rules limits the variety of our choices, and therefore it is less desirable in terms of efficiency.

This paper is organized as follows. Section 2 contains notation and definitions. Section 3 is devoted to the case without fixed costs. Section 4 examines the case with fixed costs. Some remarks are stated in the last section.

2. Notation and Definitions

Let N be a society containing n ($n \geq 2$) individuals. There are two types of commodities x and y, where x is a (one-dimensional) private good and $y=(y_1, \cdots, y_m)$ is an m-dimensional vector of public goods. Public good i can be produced at any level y_i in $Y_i=[0, y_{i\,max}]$. The capacity $y_{i\,max}$ is finite for all i. Let $Y=\prod_{i=1}^{m} Y_i$ denote the space of levels of public goods which can possibly be provided. A cost function is given by $c(y_1,\cdots,y_m)$. Let $X=[0, \max\{c(y_1,\cdots,y_m)\}]$ denote the possible range of costs.

An allocation for individual i is $(y_1,\cdots,y_m;x_i)$, where x_i is individual i's cost share of producing $y=(y_1,\cdots,y_m)$ units of public goods. Each individual i has a preference on $Y\times X$, which is denoted by u_i. For each individual, let U denote the set of possible preferences, which consists of all continuous, strictly convex and monotonic (nondecreasing in $y_1, \cdots, y_m$ and nonincreasing in x_i) preferences on $Y\times X$. Since we deal with preferences that have utility function representations, we use the notation $u_i(a)\geq u_i(b)$ for any $a,b\in Y\times X$ that means "a is strictly preferred or indifferent to b at u_i." Let ">" and "=" be the strict relation and the indifference relation, respectively. Given a preference u_i in U and a set $B\subset Y\times X$, $\text{argmax}(u_i;B)$ denotes the unique maximal allocation of u_i on B, if the set of maximal allocations consists of one single allocation. A list of preferences $u=(u_1,\cdots,u_n)$ is called a preference profile. Let U^n, the Cartesian product of U, denote the set of preference profiles.

A *cost share function* for individual i specifies individual i's cost share for each production level (consumption level) of public goods.

$$f_i : Y\rightarrow X.$$

A *cost share rule* is a list of individual cost share functions.

$$f=(f_1,\cdots,f_n).$$

Definition 2.1 For a given cost function c, a cost share rule $f=(f_i)_{i\in N}$ is *feasible* if for all $y=(y_1,\cdots,y_m)\in Y$, $\sum_{i\in N} f_i(y_1,\cdots,y_m)\geq c(y_1,\cdots,y_m)$.

For a cost function c, let F_c denote the set of feasible cost share rules.

Example 2.2 (1) *The equal cost share rule*, $f_i=\frac{c}{n}$ for all i, is feasible.
(2) Any *proportional cost share rule*, $f_i=p_ic$ where $p_i>0$ is individual i's proportion of cost share and $\sum_{i\in N} p_i=1$, is feasible.

For a feasible cost share rule f in F_c, the set A_f of feasible outcomes is given by

$$A_f=\{(y_1,\cdots,y_m;x_1,\cdots,x_n) \mid y_i\in Y_i \text{ for all i and } x_i=f_i(y_1,\cdots,y_m) \text{ for all } i\in N\}.$$

The set A_f^i of feasible allocations for individual i is also given by

$$A_f^i=\{(y_1,\cdots,y_m;x_i) \mid y_i\in Y_i \text{ for all i and } x_i=f_i(y_1,\cdots,y_m)\}.$$

For a given feasible cost share rule $f\in F_c$, a *mechanism* G_f specifies an outcome in A_f for each preference profile $u=(u_1,\cdots,u_n)$ in U^n.

$$G_f : U^n\rightarrow A_f.$$

The range of G_f is denoted by $r(G_f)$. Let $\#r(G_f)$ and $\dim(r(G_f))$ denote the cardinality and the dimension of the projection of $r(G_f)$ on to Y, respectively.

Let G_f^i be a function that specifies an allocation for individual i corresponding to G_f. The range of G_f^i is denoted by $r^i(G_f^i)$.

We assume that each individual knows his/her preference, his/her cost share function and the structure of a mechanism he/she participates in.

Definition 2.3 A mechanism G_f satisfies *strategy-proofness* if for all $u=(u_1,\cdots,u_n)\in U^n$, $i\in N$ and $\overline{u_i}\in U$: $u_i(G_f^i(u))\geq u_i(G_f^i(\overline{u_i},u_{-i}))$, where $(\overline{u_i},u_{-i})=(u_1,\cdots,u_{i-1},\overline{u_i},u_{i+1},\cdots,u_n)$.

Strategy-proofness states that truthful revelation of preferences is always a dominant strategy. If a mechanism G_f is not strategy-proof, then there exist some $u=(u_1,\cdots,u_n)\in U^n$, $i\in N$ and $\overline{u_i}\in U$: $u_i(G_f^i(\overline{u_i},u_{-i}))>u_i(G_f^i(u))$, and therefore we say that individual i can manipulate G_f at u via $\overline{u_i}$.

Definition 2.4 A mechanism G_f satisfies *voluntary participation* if for all $u=(u_1,\cdots,u_n)\in U^n$ and $i\in N$: $u_i(G_f^i(u))\geq u_i(w_i)$, where $w_i=(0,\cdots,0;0)$ for all i.

Voluntary participation guarantees that the allocation after playing the mechanism is no worse than before for each individual.

The following lemma is very useful.

Lemma 2.5 (Barberà and Peleg (1990), Zhou (1991)) *Given any cost share rule* $f \in F_c$, *if a mechanism* G_f *is strategy-proof,* $(y;f_1(y),\cdots,f_n(y)) \in r(G_f)$ *and* $(y;f_i(y))=\mathrm{argmax}(u_i;r^i(G_f^i))$ *for all* $i \in N$, *then* $G_f(u)=(y;f_1(y),\cdots,f_n(y))$.

(Proof) Since $(y;f_1(y),\cdots,f_n(y)) \in r(G_f)$, we can choose $\overline{u}=(\overline{u_1},\cdots,\overline{u_n})$ such that $G_f(\overline{u})=(y;f_1(y),\cdots,f_n(y))$. Suppose by the way of contradiction that $G_f(u) \neq (y;f_1(y),\cdots,f_n(y))$. Let $z_i=G_f(u_1,\cdots,u_i,\overline{u_{i+1}},\cdots,\overline{u_n})$ for $i=0,\cdots,n$. Then $z_0=(y;f_1(y),\cdots,f_n(y))$ and $z_n \neq (y;f_1(y),\cdots,f_n(y))$. Hence there exists j $(1 \leq j \leq n)$ such that $z_{j-1}=(y;f_1(y),\cdots,f_n(y))$ and $z_j \neq (y;f_1(y),\cdots,f_n(y))$. Therefore individual j can manipulate G_f at $(u_1,\cdots,u_j,\overline{u_{j+1}},\cdots,\overline{u_n})$ via $\overline{u_j}$. **Q.E.D.**

3. The Minimal Provision Mechanism: The Case without Fixed Costs

In this section we consider the case that the cost function is strictly increasing, convex and $c(0,\cdots,0)=0$. That is, the cost function has no fixed costs, and the average cost is increasing. Our assumption on cost share rules is that they should have the same properties as the cost function.

Assumption 3.1 Each cost share rule $f=(f_i)_{i \in N}$ satisfies the following property: each f_i is strictly increasing, convex, and $f_i(0,\cdots,0)=0$.

Assumption 3.1 states that each individual cost share function should be strictly increasing, convex and share no fixed costs. Let $\overline{F_c}$ be the class of feasible cost share rules satisfying Assumption 3.1. For example, the equal cost share rule and all other proportional cost share rules belong to $\overline{F_c}$. There are many feasible cost share rules which do not satisfy Assumption 3.1. We restrict the class of cost share rules for two reasons. One is that the similarity of shapes between the cost function and the cost share rules is plausible and socially acceptable. Another is that nonconvexity of cost share rules often induces negative results as shown in section 4.

Notice that once any cost share rule is given, the useful information about each preference reduces to its restriction on the allocation set A_f^i. Moreover, since there is a one to one and onto projection from A_f^i to Y, we can regard the restricted preferences on A_f^i as preferences on Y.

Consider economies with one private good x and one public good y. The maximal allocation on A_f^i is uniquely determined for any f_i and $u_i \in U$, since the individual cost share function f_i is convex and preference u_i is strictly convex.

For any given $f=(f_i)_{i \in N} \in \overline{F_c}$ and $u_i \in U$, let $(y_i^*(u_i);f_i(y_i^*(u_i)))=\text{argmax}(u_i, A_f^i)$. For any $u=(u_i)_{i \in N} \in U^n$, let $y^*(u)=\min y_i^*(u_i)$.

Definition 3.2 *The minimal provision mechanism* G_f^* specifies an outcome $(y^*(u);f_1(y^*(u)),\cdots,f_n(y^*(u)))$ for each preference profile $u \in U^n$.

Although we define the minimal provision mechanism in terms of preference revelation mechanisms, it works in a simple manner as follows. Each individual has only to reveal his maximal allocation $(y_i^*(u_i);f_i(y_i^*(u_i)))$ of $u_i \in U$ on A_f^i. Then, the society selects the minimum $y_i^*(u_i)$ as the level of the public good to produce and shares the costs among individuals according to f.

Lemma 3.3 *For any cost share rule $f \in \overline{F_c}$, the minimal provision mechanism G_f^* satisfies strategy-proofness (coalition-strategy-proofness[1]) and voluntary participation.*

(Proof) Consider the restriction of preference u_i in U to the individual i's allocation set A_f^i. Let V denote the set of all preferences on A_f^i obtained by such restriction. The set V consists of all single peaked preferences on A_f^i, since each individual cost share function f_i is convex and preference u_i is strictly convex.[2] Then, the usual argument on single peakedness proves that G_f^* is strategy-proof (coalition-strategy-proof). Further, since $0 \leq y^*(u) \leq y_i^*(u_i)$ for all $u=(u_i)_{i \in N} \in U^n$ and $i \in N$, single peakedness guarantees that $u_i(y^*(u);f_i(y^*(u))) \geq u_i(0;0)$.
Q.E.D.

Definition 3.4 A mechanism G_f satisfies *the full-range property* if for all y, $0 \leq y \leq y_{max}$, there exists a preference profile u such that $G_f(u)=(y;f_1(y),\cdots,f_n(y))$.

Theorem 3.5 *For any cost share rule $f \in \overline{F_c}$, the minimal provision mechanism G_f^* is the unique mechanism satisfying strategy-proofness, voluntary participation and the full-range property.*

(Proof) It follows from Lemma 3.3 that G_f^* satisfies strategy-proofness and voluntary participation. It is clear that G_f^* satisfies the full-range property. We prove uniqueness.

Suppose by way of contradiction that there exists a mechanism G_f other than G_f^* that satisfies the premises of the theorem. We consider two possible cases (see Figure 1 and 2).

First, suppose that there exists $u \in U^n$ such that $G_f(u)=(\overline{y^1};f_1(\overline{y^1}),\cdots,f_n(\overline{y^1}))$, where $\overline{y^1}<y^*(u)$. Without loss of generality, we can assume $y^*(u)=y_1^*(u_1) \leq y_2^*(u_2) \leq \cdots \leq y_n^*(u_n)$ by

1 A mechanism G_f satisfies *coalition-strategy-proofness* if for all $u=(u_1,\cdots,u_n) \in U^n$, coalition $T \subset N$ and coalition profile $\overline{u_T} \in U^{\#T}$, $u_i(G_f^i(\overline{u_T},u_{-T}))>u_i(G_f^i(u))$ for some $i \in T$ implies $u_j(G_f^j(\overline{u_T},u_{-T}))<u_j(G_f^j(u))$ for some $j \in T$, where $(\overline{u_T},u_{-T})$ is the profile with i-th element $\overline{u_i}$ if $i \in T$ and u_i if $i \notin T$.

2 A preference u_i is *single peaked* on A_f^i if $y'<y''<y_i^*(u_i)$ implies $u_i(y';f_i(y'))<u_i(y'';f_i(y''))<u_i(y_i^*;f_i(y_i^*(u_i)))$ and $y_i^*(u_i)<y'<y''$ implies $u_i(y_i^*;f_i(y_i^*(u_i)))>u_i(y';f_i(y'))>u_i(y'';f_i(y''))$.

permuting indexes of individuals. Then $y^*(u)=y_1^*(u_1)\neq y_n^*(u_n)$, otherwise $G_f(u)=(y^*(u);f_1(y^*(u)),\cdots,f_n(y^*(u)))$ by Lemma 2.5. Let j be the smallest index such that $y^*(u)\neq y_j^*(u_j)$. For $k=j,\cdots,n$, let $\overline{u_k}$ be a preference such that $\operatorname{argmax}(\overline{u_k};A_f^k)=(y^*(u);f_k(y^*(u)))$ and $\overline{u_k}(y_k^*(u_k);f_k(y_k^*(u_k)))=\overline{u_k}(0;0)$. Then $G_f(u_1,\cdots,u_{j-1},\overline{u_j},u_{j+1},\cdots,u_n)=(\overline{y^2};f_1(\overline{y^2}),\cdots,f_n(\overline{y^2}))$, where $\overline{y^2}\in[0,y_j^*(u_j)]$, otherwise voluntary participation is not satisfied for individual j. Hence $\overline{y^2}\leq\overline{y^1}$, otherwise individual j can manipulate G_f at u via $\overline{u_j}$. Similarly, $G_f(u_1,\cdots,u_{j-1},\overline{u_j},\overline{u_{j+1}},u_{j+2},\cdots,u_n)=(\overline{y^3};f_1(\overline{y^3}),\cdots,f_n(\overline{y^3}))$, where $\overline{y^3}\in[0,y_{j+1}^*(u_{j+1})]$, otherwise voluntary participation is not satisfied for individual j+1. Hence $\overline{y^3}\leq\overline{y^2}$, otherwise individual j+1 can manipulate G_f at $(u_1,\cdots,u_{j-1},\overline{u_j},u_{j+1},\cdots,u_n)$ via $\overline{u_{j+1}}$. Applying this argument successively, we obtain $G_f(u_1,\cdots,u_{j-1},\overline{u_j},\cdots,\overline{u_n})=(\overline{y};f_1(\overline{y}),\cdots,f_n(\overline{y}))$, where $\overline{y}\leq\overline{y^1}<y^*(u)$. However, since G_f satisfies the full-range property, $G_f(u_1,\cdots,u_{j-1},\overline{u_j},\cdots,\overline{u_n})=(y^*(u);f_1(y^*(u)),\cdots,f_n(y^*(u)))$ by Lemma 2.5. It is a contradiction.

Next, suppose that there exists $u\in U^n$ such that $G_f(u)=(\widetilde{y^1};f_1(\widetilde{y^1}),\cdots,f_n(\widetilde{y^1}))$, where $y^*(u)<\widetilde{y^1}$. Without loss of generality, we assume $y^*(u)=y_1^*(u_1)\leq y_2^*(u_2)\leq\cdots\leq y_n^*(u_n)$. Let $\widetilde{u_1}$ be a preference such that $\operatorname{argmax}(\widetilde{u_1};A_f^1)=(y^*(u);f_1(y^*(u)))$, and for some $\widetilde{y^2}$ such that $y^*(u)<\widetilde{y^2}<\widetilde{y^1}$, $\widetilde{u_1}(\widetilde{y^2};f_1(\widetilde{y^2}))=\widetilde{u_1}(0;0)$. Voluntary participation requires $G_f(\widetilde{u_1},u_2,\cdots,u_n)=(\widetilde{y^3};f_1(\widetilde{y^3}),\cdots,f_n(\widetilde{y^3}))$, where $\widetilde{y^3}\in[0,\widetilde{y^2}]$. Pick $\widetilde{y^4}$ such that $u_1(\widetilde{y^4};f_1(\widetilde{y^4}))=u_1(\widetilde{y^1};f_1(\widetilde{y^1}))$, if any. If there is no such $\widetilde{y^4}$, then individual 1 can manipulate G_f at u via $\widetilde{u_1}$. If $\widetilde{y^3}\in(\widetilde{y^4},\widetilde{y^2}]$, then individual 1 can manipulate G_f at u via $\widetilde{u_1}$. Then $\widetilde{y^3}$ must be in $[0,\widetilde{y^4}]$. Hence $\widetilde{y^3}<y^*(\widetilde{u_1},u_2,\cdots,u_n)$ holds. Applying the first argument to preference profile $(\widetilde{u_1},u_2,\cdots,u_n)$ leads to a contradiction. **Q.E.D.**

Moulin (1991a) studied mechanisms which determine both the level of public goods to produce and the cost share. He characterized the mechanisms satisfying coalition-strategy-proofness, voluntary participation, anonymity, and the full-range property. The proof of the theorem consisted of two steps. First, he showed, from the result in Moulin (1991b), that any mechanism satisfying coalition-strategy-proofness and anonymity shares the costs equally.[3] Next, he showed, from the result in Moulin (1980) and Barberà and Jackson (1991), that any mechanism satisfying all four conditions provides public goods minimally.

On the other hand, this paper studies mechanisms which determine only the level of public goods, assuming that the cost share rule is exogenously given. However we use only strategy-proofness, voluntary participation and the full-range property. All the results in this paper can be applied to Moulin's second step, because our class of cost share rules includes the equal cost share rule.

Theorem 3.5 can be interpreted as a characterization of mechanisms on single peaked preferences. Moulin (1980) and Barberà and Jackson (1991) present alternative characterizations of strategy-proof mechanisms on single peaked preferences.

If we drop the full-range property, we obtain three types of results according to the property of the possible space of public goods. That is, there are cases of "no mechanism", "the unique mechanism" and "several mechanisms". These facts are immediate from consideration in section 4.

Next, consider economies with one private good x and two or more public goods $y_1, \cdots, y_m$ ($m \geq 2$). In contrast to the case of one private good and one public good, we can derive the following negative result from the general result by Zhou (1991).

Theorem 3.6 *For any cost share rule $f \in \overline{F_c}$, there exists no mechanism G_f satisfying strategy-proofness, voluntary participation and $dim(r(G_f)) \geq 2$.*[4]

(Proof) Consider the restriction of preference u_i in U to the individual i's allocation set A^i_f. Let V denote the set of all preferences on A^i_f obtained by such restriction. The set V consists of all continuous and strictly convex preferences on A^i_f, since each cost share

[3] A mechanism G_f satisfies *anonymity* if for all $u \in U^n$, $u_i \equiv u_j$ for some i, $j \in N$ implies $u_i(G^i_f(u)) = u_j(G^j_f(u))$.

[4] If the range of mechanisms is only one-dimensional (that is, the range is a straight line through the origin with respect to Y), the minimal provision mechanism is the unique mechanism satisfying strategy-proofness, voluntary participation and the full-range property along a one-dimensional line.

function f_i is convex and each preference u_i is strictly convex. In such an environment, the general result by Zhou proves that strategy-proofness and $\dim(r(G_f))\geq 2$ imply dictatorship.[5] Dictatorship and voluntary participation are inconsistent in our model. To see that, suppose that $i\in N$ is a dictator. Let $u=(u_i)_{i\in N}\in U^n$ be some preference profile in which $\mathrm{argmax}(u_i;r^i(G_f^i))=(y_1,\cdots,y_m;x_i)\neq(0,\cdots,0;0)$ and $\mathrm{argmax}(u_j;A_f^j)=(0,\cdots,0;0)$ for some j ($\neq$i). A dictatorial mechanism G_f specifies $(y;f_1(y),\cdots,f_n(y))$, where $y=(y_1,\cdots,y_m)$, for preference profile u. Voluntary participation is not satisfied for individual j, because $u_j(0,\cdots,0;0)>u_j(G_f^j(u))=u_j(y;f_j(y))$. **Q.E.D.**

4. Impossibility Results: The Case with Fixed Costs

In this section we consider the case that the cost function is strictly increasing, convex and has positive fixed costs. No cost share rule considered in section 3 is feasible in such a situation, since it is not feasible sufficiently near the origin. We deal with cost share rules in which each individual cost share function f_i is of the form $f_i^f+f_i^v$, where f_i^f is an individual fixed cost share function and f_i^v is an individual variable cost share function. We impose the following assumption on cost share rules.

Assumption 4.1 Each cost share rule $f=(f_i)_{i\in N}=(f_i^f+f_i^v)_{i\in N}$ satisfies the following properties: for each $i\in N$, $f_i^f(y_1,\cdots,y_m)=0$ if $(y_1,\cdots,y_m)=(0,\cdots,0)$, $f_i^f(y_1,\cdots,y_m)=F_i$ if $(y_1,\cdots,y_m)\neq(0,\cdots,0)$, where each F_i is a positive constant value, f_i^v is strictly increasing, convex and $f_i^v(0,\cdots,0)=0$.

Let $\widehat{F_c}$ be the class of feasible cost share rules satisfying Assumption 4.1. The equal cost share rule and all other proportional cost share rules belong to $\widehat{F_c}$. Notice that any cost share rule f in $\widehat{F_c}$ divides fixed costs among every individual, namely, every individual has to pay

[5] A mechanism G_f is *dictatorial* if there exists individual i, who is called a dictator, such that for all $u=(u_1,\cdots,u_n)\in U^n$ and $a_i\in r^i(G_f^i)$: $u_i(G_f^i(u))\geq u_i(a_i)$.

for fixed costs. Thus, each f_i is not convex on the whole domain Y. In particular, f_i is not convex near the origin. Notice also that f_i is not continuous at the origin.

Consider economies with one private good x and one public good y. Since each individual has nonzero fixed cost share, the individual cost share function f_i is neither continuous nor convex on the whole domain. The restriction of any preference u_i in U to the individual allocation set A_f^i is no longer single peaked. Hence the minimal provision mechanism does not work as in the case of no fixed costs. However, if we restrict the range of the mechanism properly, the minimal provision mechanism may satisfy strategy-proofness and voluntary participation. The following two lemmas indicate how to restrict the range of the mechanisms.

Lemma 4.2 *Given any cost share rule $f \in \widehat{F}_c$, if a mechanism G_f is strategy-proof, then $r(G_f)$ is closed.*[6]

(Proof) Choose any $(y;f_1(y),\cdots,f_n(y)) \in$ Closure$(r(G_f))$. If y=0, $(y;f_1(y),\cdots,f_n(y))$ is an isolated point of A_f, and thus $(y;f_1(y),\cdots,f_n(y)) \in r(G_f)$. We consider the case of $y \in (0,y_{max})$. Pick $u=(u_i)_{i\in N}$ such that $\text{argmax}(u_i;A_f^i)=(y;f_i(y))$ for all $i \in N$ and $u_i \equiv u_j$ for all i, $j \in N$. Suppose toward a contradiction that $G_f(u)=(\hat{y};f_1(\hat{y}),\cdots,f_n(\hat{y}))$, where $\hat{y} \neq y$. Since each individual variable cost share function is convex, it is continuous. We can choose y' and y'' such that $y \in (y',y'')$, $u_i(y';f_i(y'))=u_i(y'';f_i(y''))>u_i(\hat{y};f_i(\hat{y}))$ and $u_i(y';f_i(y'))>u_i(0;0)$. Since $(y;f_1(y),\cdots,f_n(y)) \in$ Closure$(r(G_f))$, there is some $(\bar{y};f_1(\bar{y}),\cdots,f_n(\bar{y})) \in r(G_f)$ such that $\bar{y}$ is between y' and y'', and thus $u_i(\bar{y};f_i(\bar{y}))>u_i(y';f_i(y'))=u_i(y'';f_i(y''))>u_i(\hat{y};f_i(\hat{y}))$. For any $i \in N$, pick $\overline{u_i}$ such that $\text{argmax}(\overline{u_i};A_f^i)=(\bar{y};f_i(\bar{y}))$. Then $G_f(\overline{u_1},u_2,\cdots,u_n)=(\overline{y^1};f_1(\overline{y^1}),\cdots,f_n(\overline{y^1}))$, where $\overline{y^1} \in Y\backslash(y',y'')$, otherwise individual 1 can manipulate G_f at u via $\overline{u_1}$. Similarly, $G_f(\overline{u_1},\overline{u_2},u_3,\cdots,u_n)=(\overline{y^2};f_1(\overline{y^2}),\cdots,f_n(\overline{y^2}))$, where $\overline{y^2} \in Y\backslash(y',y'')$, otherwise individual 2 can

[6] The closedness of the range of mechanisms is a necessary condition of strategy-proofness in several environments. See Barberà and Peleg (1990), Zhou (1991), and Barberà and Jackson (1991).

manipulate G_f at $(\overline{u_1},u_2,\cdots,u_n)$ via $\overline{u_2}$. Applying this argument successively, we obtain $G_f(\overline{u})=((\overline{y^n};f_1(\overline{y^n}),\cdots,f_n(\overline{y^n}))$, where $\overline{y^n}\in Y\backslash(y',y")$. It contradicts the fact that $G_f(\overline{u})=((\overline{y};f_1(\overline{y}),\cdots,f_n(\overline{y}))$ by Lemma 2.5. Therefore $G_f(u)=(y;f_1(y),\cdots,f_n(y))$, and thus $(y;f_1(y),\cdots,f_n(y))\in r(G_f)$. The remaining case of $y=y_{max}$ is similar. **Q.E.D.**

Lemma 4.3 *Given any cost share rule $f\in\widehat{F_c}$, if a mechanism G_f satisfies voluntary participation, then $0\in r(G_f)$.*

(Proof) It is straightforward from the definition of voluntary participation and the existence of preference u_i such that $\mathrm{argmax}(u_i;A_f^i)=(0;0)$. **Q.E.D.**

These range conditions are necessary and sufficient for the existence of strategy-proof and voluntary participation mechanisms in the case of no fixed costs. However, they are not sufficient in the case with fixed costs.

Definition 4.4 A cost share rule $f=(f_i)_{i\in N}\in\widehat{F_c}$ is *essentially convex* on $W\subset Y$ if each f_i is convex on W.[7]

In the above definition it is not required that W be a convex set in Y. For any cost share rule $f=(f_i)_{i\in N}\in\widehat{F_c}$ and any closed set $W\subset Y$, let $A_f(W)=\{(y;f_1(y),\cdots,f_n(y))\mid y\in W\}$ and $A_f^i(W)=\{(y;f_i(y))\mid y\in W\}$. The maximal allocation on $A_f^i(W)$ consists of one single allocation or two allocations for any $u_i\in U$, when f is essentially convex on W. Denote such allocations by $(y_i^\dagger(u_i)|_w;f_i(y_i^\dagger(u_i)|_w))$ and $(y_i^*(u_i)|_w;f_i(y_i^*(u_i)|_w))$, where $y_i^\dagger(u_i)|_w\le y_i^*(u_i)|_w$. Notice that if $y_i^\dagger(u_i)|_w\neq y_i^*(u_i)|_w$ for some $u_i\in U$, $(y;f_1(y),\cdots,f_n(y))\notin A_f(W)$ for any y such that $y_i^\dagger(u_i)|_w<y<y_i^*(u_i)|_w$. For any $u=(u_i)_{i\in N}\in U^n$, let $y^+(u)|_w=\min y_i^\dagger(u_i)|_w$ and $y^*(u)|_w=\min$

[7] For the equal cost share rule and any proportional cost share rule, f is essentially convex on $W\subset Y$ if and only if the cost function c is convex on W.

$y_i^*(u_i)|_w$. We define the class of the minimal provision type mechanisms. Each minimal provision type mechanism G_f into $A_f(W)$, where $W \subset Y$, specifies an outcome $(y^+(u)|_w; f_1(y^+(u)|_w), \cdots, f_n(y^+(u)|_w))$ or an outcome $(y^*(u)|_w; f_1(y^*(u)|_w), \cdots, f_n(y^*(u)|_w))$ for each preference profile u in U^n. As a special case of the minimal provision type mechanisms, we define the following mechanism.

Definition 4.5 *The minimal provision mechanism* G_f^* *into* $A_f(W)$, where W is a closed set in Y, specifies an outcome $(y^*(u)|_w; f_1(y^*(u)|_w), \cdots, f_n(y^*(u)|_w))$ for each preference profile $u \in U^n$.

Lemma 4.6 *For any cost share rule $f \in \widehat{F}_c$ and closed set $W \subset Y$ such that f is essentially convex on W and $0 \in W$, the minimal provision mechanism G_f^* into $A_f(W)$ satisfies strategy-proofness (coalition-strategy-proofness) and voluntary participation.*

(Proof) It is immediate from Lemma 3.3. **Q.E.D.**

Notice that there are some other mechanisms satisfying strategy-proofness (coalition-strategy-proofness) and voluntary participation in the class of the minimal provision type mechanisms. However, the allocations they induce are Pareto inferior to those of the minimal provision mechanism. We say that a mechanism G_f Pareto dominates $\overline{G_f}$ if for all $u \in U^n$ and $i \in N$: $u_i(G_f^i(u)) \geq u_i(\overline{G_f^i}(u))$.

Theorem 4.7 *For any cost share rule $f \in \widehat{F}_c$ and closed set $W \subset Y$ such that f is essentially convex on W and $0 \in W$, any mechanism G_f satisfying strategy-proofness, voluntary participation and $r(G_f)=A_f(W)$ is Pareto dominated by the minimal provision mechanism G_f^* into $A_f(W)$.*

(Proof) It is immediate from Theorem 3.5 that any mechanism G_f satisfying strategy-proofness, voluntary participation and $r(G_f)=A_f(W)$ belongs to the class of the minimal provision type mechanisms. Clearly, $y^+(u)|_W \leq y^*(u)|_W \leq y_i^*(u_i)|_W$ for all $u=(u_i)_{i\in N}\in U^n$ and $i\in N$. Single peakedness implies that G_f is Pareto dominated by G_f^* into $A_f(W)$. **Q.E.D.**

We characterize the counterpart of Theorem 4.7. It is impossible to construct strategy-proof and voluntary participation mechanisms if the range of the mechanism includes three outcomes whose allocations are nonconvex for every individual. We define nonconvexity of cost share rules on a triple T, three distinct points in Y.

Definition 4.8 A cost share rule $f=(f_i)_{i\in N}\in \widehat{F}_c$ is *essentially nonconvex* on a triple $T\subset Y$ if each individual cost share function f_i is nonconvex on T, that is,
$f_i(y^2)>\frac{y^3-y^2}{y^3-y^1}f_i(y^1)+\frac{y^2-y^1}{y^3-y^1}f_i(y^3)$ for $T=\{y^1,y^2,y^3\in Y \mid y^1<y^2<y^3\}$.[8]

Theorem 4.9 *For any cost share rule* $f\in\widehat{F}_c$, *there exists no mechanism* G_f *satisfying strategy-proofness, voluntary participation and such that* $r(G_f)$ *contains* $A_f(T)$ *for some triple T on which f is essentially nonconvex.*

(Proof) Suppose by way of contradiction that there exist a mechanism G_f and some triple T which satisfy the premises of the theorem. Since each individual variable cost share function f_i^V is convex, $0\in Y$ is included in this triple T. Let 0, y' and y" (0<y'<y") be the elements of T. Let S(y'), S(y") and S([y',y"]) be some closed segments in Y including y', y" and [y',y"], respectively. Denote the boundary points of S by Bd(S). If $y''\neq y_{max}$, pick S(y'), S(y") and S([y',y"]) such that $y'\notin Bd(S(y'))$, $y''\notin Bd(S(y''))$, $S(y')\cap S(y'')=\emptyset$,

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[8] For the equal cost share rule and any proportional cost share rule, f is essentially nonconvex on some triple $T\subset Y$ if and only if the cost function c is nonconvex on T. For any subset $W\subset Y$, the equal cost share rule and any proportional cost share rule are either essentially convex on W or essentially nonconvex on some triple $T\subset W$.

$S(y')\subset S([y',y''])$, $Bd(S(y'))\cap Bd(S([y',y'']))=\emptyset$, $S(y'')\subset S([y',y''])$, $Bd(S(y''))\cap Bd(S([y',y'']))=\emptyset$, $0\notin Bd(S([y',y'']))$ and that f is essentially nonconvex on $\{0\}\cap Bd(S(y''))$. If $y''=y_{max}$, pick $S(y')$, $S(y'')$ and $S([y',y''])$ such that $y'\notin Bd(S(y'))$, $\{y''\}\neq S(y'')$, $S(y')\cap S(y'')=\emptyset$, $S(y')\subset S([y',y''])$, $Bd(S(y'))\cap Bd(S([y',y'']))=\emptyset$, $0\notin Bd(S([y',y'']))$ and that f is essentially nonconvex on $\{0\}\cap Bd(S(y''))$. We can find the following four types of preferences for each individual (see Figure 3 - Figure 6).

u_i: $\operatorname{argmax}(u_i;A_f^i)=(y';f_i(y'))$ and $u_i(y;f_i(y))=u_i(0;0)$ for all $y\in Bd(S(y'))$.

$\tilde{u}_i$: $\operatorname{argmax}(\tilde{u}_i;A_f^i)=(y'';f_i(y''))$, $\tilde{u}_i(y;f_i(y))=\tilde{u}_i(0;0)$ for all $y\in Bd(S([y',y'']))\setminus\{y''\}$ and $\tilde{u}_i(\tilde{y}^1;f_i(\tilde{y}^1))=\tilde{u}_i(\tilde{y}^2;f_i(\tilde{y}^2))$ for $\tilde{y}^1,\tilde{y}^2\in Bd(S(y''))\setminus\{y''\}$.

$\widehat{u}_i$: $\operatorname{argmax}(\widehat{u}_i;A_f^i)=(y';f_i(y'))$, $\widehat{u}_i(y;f_i(y))=\widehat{u}_i(0;0)$ for all $y\in Bd(S([y',y'']))\setminus\{y''\}$ and $\widehat{u}_i(\widehat{y}^1;f_i(\widehat{y}^1))=\widehat{u}_i(\widehat{y}^2;f_i(\widehat{y}^2))$ for $\widehat{y}^1,\widehat{y}^2\in Bd(S(y'))$.

$\overline{u}_i$: $\operatorname{argmax}(\overline{u}_i;A_f^i)=(y'';f_i(y''))$ and $\overline{u}_i(y;f_i(y))=\overline{u}_i(0;0)$ for all $y\in Bd(S(y''))\setminus\{y''\}$.

Notice that $\tilde{u}_i(y;f_i(y))>\tilde{u}_i(0;0)$ for all $y\in S(y')$ and $\widehat{u}_i(y;f_i(y))>\widehat{u}_i(0;0)$ for all $y\in S(y'')$. By Lemma 2.5, $G_f(u_1,\cdots,u_n)=(y';f_1(y'),\cdots,f_n(y'))$. Voluntary participation requires $G_f(u_1,\cdots,u_{n-1},\tilde{u}_n)=(y;f_1(y),\cdots,f_n(y))$, where $y\in\{0\}\cup S(y')$. If $y=0$, individual n can manipulate G_f at $(u_1,\cdots,u_{n-1},\tilde{u}_n)$ via u_n since $\tilde{u}_n(y';f_n(y'))>\tilde{u}_n(0;0)$. Hence $y\in S(y')$. Again, voluntary participation requires $G_f(u_1,\cdots,u_{n-2},\tilde{u}_{n-1},\tilde{u}_n)=(\bar{y};f_1(\bar{y}),\cdots,f_n(\bar{y}))$, where $\bar{y}\in\{0\}\cup S(y')$. If $\bar{y}=0$, individual n-1 can manipulate G_f at $(u_1,\cdots,u_{n-2},\tilde{u}_{n-1},\tilde{u}_n)$ via u_{n-1} since $\tilde{u}_{n-1}(y;f_{n-1}(y))>\tilde{u}_{n-1}(0;0)$ for all $y\in S(y')$. Hence $\bar{y}\in S(y')$. Applying this argument successively, we obtain $G_f(u_1,\tilde{u}_2,\cdots,\tilde{u}_n)=(y;f_1(y),\cdots,f_n(y))$, where $y\in S(y')$. (1)

Similarly, by Lemma 2.5, $G_f(\overline{u}_1,\overline{u}_2,\tilde{u}_3,\cdots,\tilde{u}_n)=(y'';f_1(y''),\cdots,f_n(y''))$. Voluntary participation requires $G_f(\widehat{u}_1,\overline{u}_2,\tilde{u}_3,\cdots,\tilde{u}_n)=(y;f_1(y),\cdots,f_n(y))$, where $y\in\{0\}\cup S(y'')$. If $y=0$, individual 1 can manipulate G_f at $(\widehat{u}_1,\overline{u}_2,\tilde{u}_3,\cdots,\tilde{u}_n)$ via $\overline{u}_1$ since $\widehat{u}_1(y'';f_1(y''))>\widehat{u}_1(0;0)$. Hence $y\in S(y'')$. (2)

Voluntary participation requires $G_f(\widehat{u}_1,\tilde{u}_2,\tilde{u}_3,\cdots,\tilde{u}_n)=(y;f_1(y),\cdots,f_n(y))$, where $y\in\{0\}\cup S([y',y''])$. If $y=0$, individual 1 can manipulate G_f at $(\widehat{u}_1,\tilde{u}_2,\tilde{u}_3,\cdots,\tilde{u}_n)$ via $\tilde{u}_1$ since $G_f(\tilde{u}_1,\tilde{u}_2,\tilde{u}_3,\cdots,\tilde{u}_n)=(y'';f_1(y''),\cdots,f_n(y''))$ by Lemma 2.5, and $\widehat{u}_1(y'';f_1(y''))>\widehat{u}_1(0;0)$. Hence

$y \in S([y',y''])$. If $y \notin S(y')$, it follows from (1) that individual 1 can manipulate G_f at $(\widehat{u_1},\widetilde{u_2},\widetilde{u_3},\cdots,\widetilde{u_n})$ via u_1 since $\widehat{u_1}(y;f_1(y))>\widehat{u_1}(\bar{y};f_1(\bar{y}))$ for all $y \in S(y')$ and $\bar{y} \in S([y',y''])\backslash S(y')$. If $y \notin S(y'')$, it follows from (2) that individual 2 can manipulate G_f at $(\widehat{u_1},\widetilde{u_2},\widetilde{u_3},\cdots,\widetilde{u_n})$ via $\overline{u_2}$ since $\widetilde{u_2}(y;f_2(y))>\widetilde{u_2}(\bar{y};f_2(\bar{y}))$ for all $y \in S(y'')$ and $\bar{y} \in S([y',y''])\backslash S(y'')$. Therefore y must be in S(y') and S(y''). It contradicts the assumption that $S(y')\cap S(y'')=\varnothing$. **Q.E.D.**

Consider economies with one private good and several public goods. We can prove the same negative result as in the case of no fixed costs.

Theorem 4.10 *For any cost share rule $f \in \widehat{F_c}$, there exists no mechanism G_f satisfying strategy-proofness, voluntary participation and $dim(r(G_f))\geq 2$.*

(Proof) Suppose toward a contradiction that there exists a mechanism G_f which satisfies the premises of the theorem. Then, $(0,\cdots,0;0,\cdots,0)\in r(G_f)$ by Lemma 4.3. Since $\dim(r(G_f))\geq 2$, we can pick y', y''$\in$ Y such that $(y';f_1(y'),\cdots,f_n(y'))\in r(G_f)$, $(y'';f_1(y''),\cdots,f_n(y''))\in r(G_f)$ and 0, y', y'' are not on a straight line. Let S(y') and S(y'') be some closed sets in Y including y' and y'', respectively. Let S({y',y''}) be some closed set in Y including S(y') and S(y''). If $y',y'' \notin Bd(Y)$, pick sufficiently large strictly convex sets S(y'), S(y'') and S({y',y''}) such that $y' \notin Bd(S(y'))$, $y'' \notin Bd(S(y''))$, $S(y')\cap S(y'')=\varnothing$, $Bd(S(y'))\cap Bd(S(\{y',y''\}))=\varnothing$, $Bd(S(y''))\cap Bd(S(\{y',y''\}))=\varnothing$ and $0 \notin Bd(S(\{y',y''\}))$.[9] Here, a sufficiently large set implies a set including points near the origin in order to construct preferences used in the previous theorem. If $y' \in Bd(Y)$, $y'' \notin Bd(Y)$, pick sufficiently large strictly convex sets S(y'), S(y'') and S({y',y''}) such that $y' \notin Closure(Bd(S(y'))\backslash Bd(Y))$, $y'' \notin Bd(S(y''))$, $S(y')\cap S(y'')=\varnothing$, $Closure(Bd(S(y'))\backslash Bd(Y))\cap Closure(Bd(S(\{y',y''\}))\backslash Bd(Y))=\varnothing$,

[9] A set $S\subset Y$ is *strictly convex* if it is convex and it is strictly convex except boundary points of Y.

$Bd(S(y''))\cap Bd(S(\{y',y''\}))=\emptyset$ and $0\notin Bd(S(\{y',y''\}))$. The case of $y'\notin Bd(Y)$, $y''\in Bd(Y)$ or $y', y''\in Bd(Y)$ is similar. The rest of the proof is the same as Theorem 4.9. **Q.E.D.**

As a corollary of Theorem 4.9 and Theorem 4.10, we have a simple negative result. Let $\widetilde{F_c}$ be the subclass of $\widehat{F_c}$ satisfying the following Assumption 4.11.

Assumption 4.11 Each individual variable cost share function f_i^y is linear.

Corollary 4.12 *For any cost share rule $f\in\widetilde{F_c}$, there exists no mechanism G_f satisfying strategy-proofness, voluntary participation and $\#r(G_f)\geq 3$.*

(Proof) It is immediate from Theorem 4.9 when the range of the mechanism is one-dimensional, and from Theorem 4.10 when the range is two or more dimensional.
Q.E.D.

5. Concluding Remarks

We have three remarks. First, our analysis concentrated on convex cost share rules (with or without fixed costs). We may be able to construct mechanisms satisfying strategy-proofness, voluntary participation, and some further criteria on other types of cost share rules. It seems likely, however, that such satisfactory cost share rules do not exist, because partial nonconvexity prevents us from constructing strategy-proof and voluntary participation mechanisms. The proofs of Theorem 4.9 and Theorem 4.10 can be applied to a much broader class of cost share rules to derive impossibility results. Second, we dealt with pure public goods and showed that the minimal provision mechanism had a good performance in economies with one private good and one public good. This positive result did not hold when we perturbed the model slightly: (i) a case of one private good and several public goods, or (ii) a case with fixed costs. Moulin (1991a) considered partial excludability of public goods and characterized that *the serial mechanism*, which was defined in excludable public good economies, had a better performance than *the conservative equal cost mechanism*, which was defined in pure public good economies. The partial exclusion may improve the performance of mechanisms under the above-mentioned situations. Third, in the literature of the private provision of public goods, Serizawa (1992) characterized strategy-proof and individually rational mechanisms in the two-individual case.[10] On the other hand, Saijo (1991) proved a negative result that there is no mechanism satisfying strategy-proofness and autarkic individual rationality.[11] The existence of strategy-proof and individual rational mechanisms whose range is more than one-dimensional with respect to public goods is still an open problem.[12] Our results have a close relationship to these problems.

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[10] Individual rationality is essentially the same notion as voluntary participation.

[11] Autarkic individual rationality is a stronger property than individual rationality. It states that the allocation he receives is no worse than the best allocation that can be achieved by himself.

[12] Moreno and Walker (1991) have shown a negative result that strategy-proofness, conditional unanimity, and the condition that the range of mechanisms is two or more dimensional with respect to Y imply dictatorship, though they have dealt with the preference domain without requiring monotonicity.

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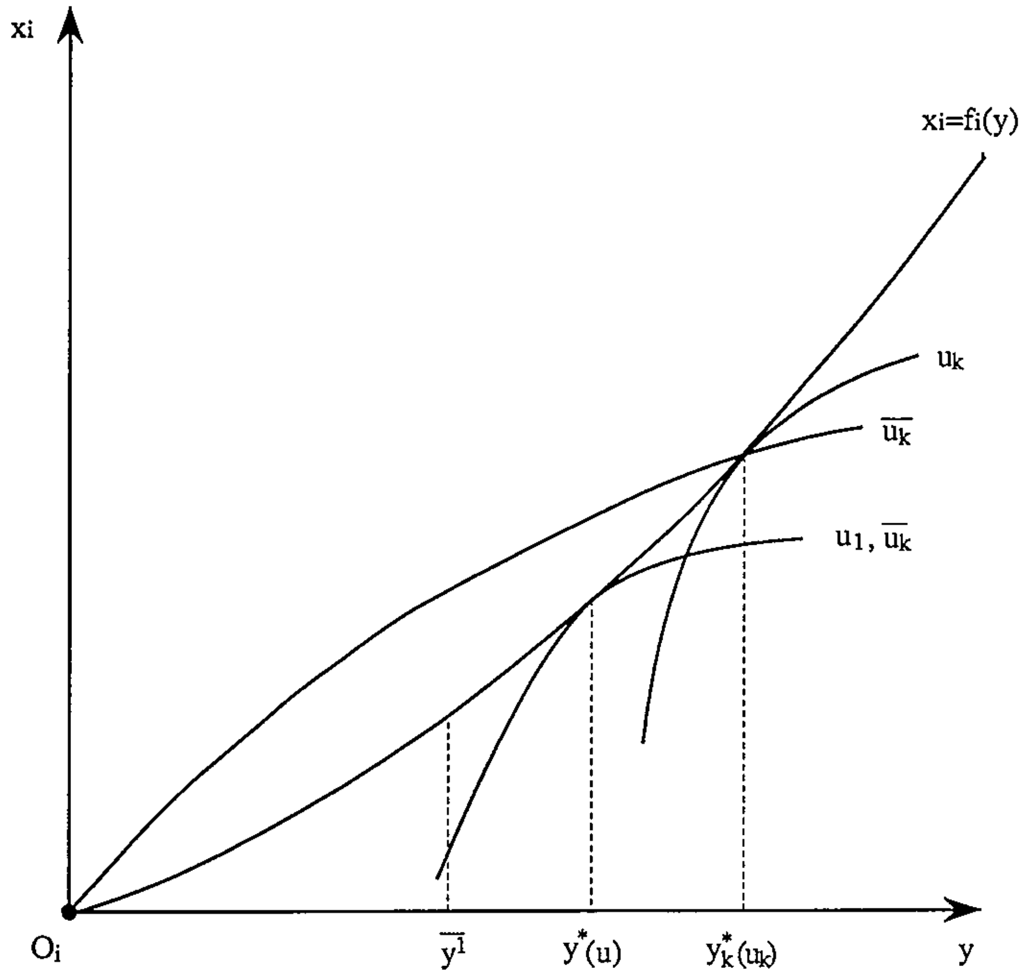

Figure 1.

An illustration of the proof of Theorem 3.5.

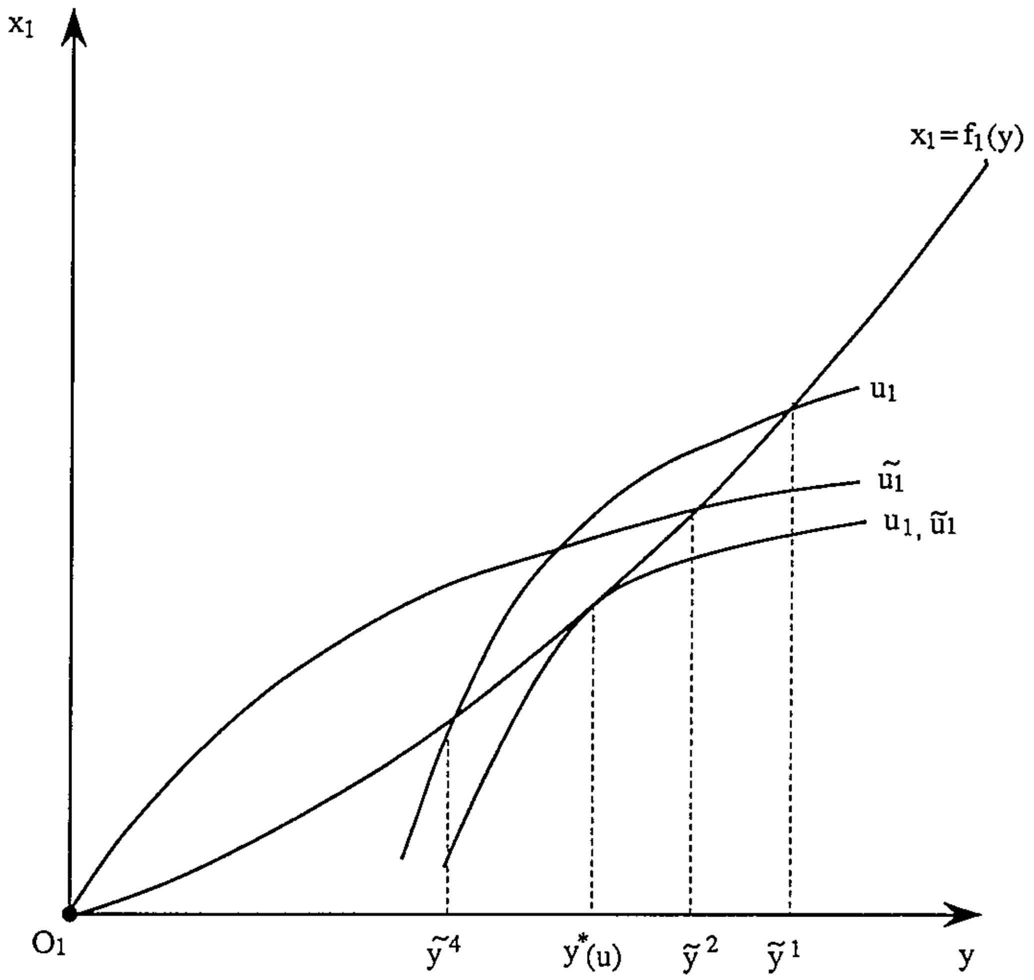

Figure 2.

An illustration of the proof of Theorem 3.5.

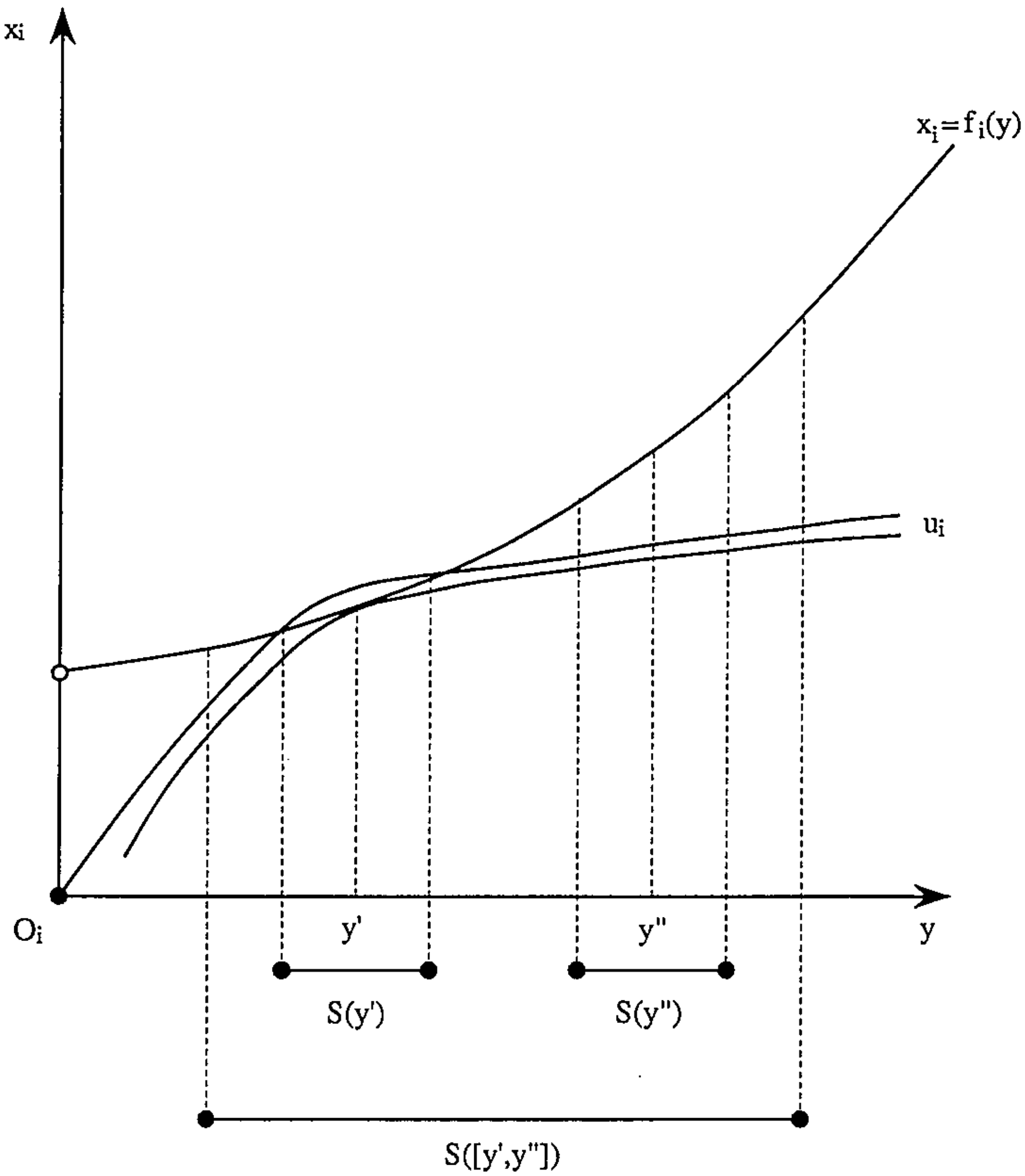

Figure 3.

An example of preference u_i.

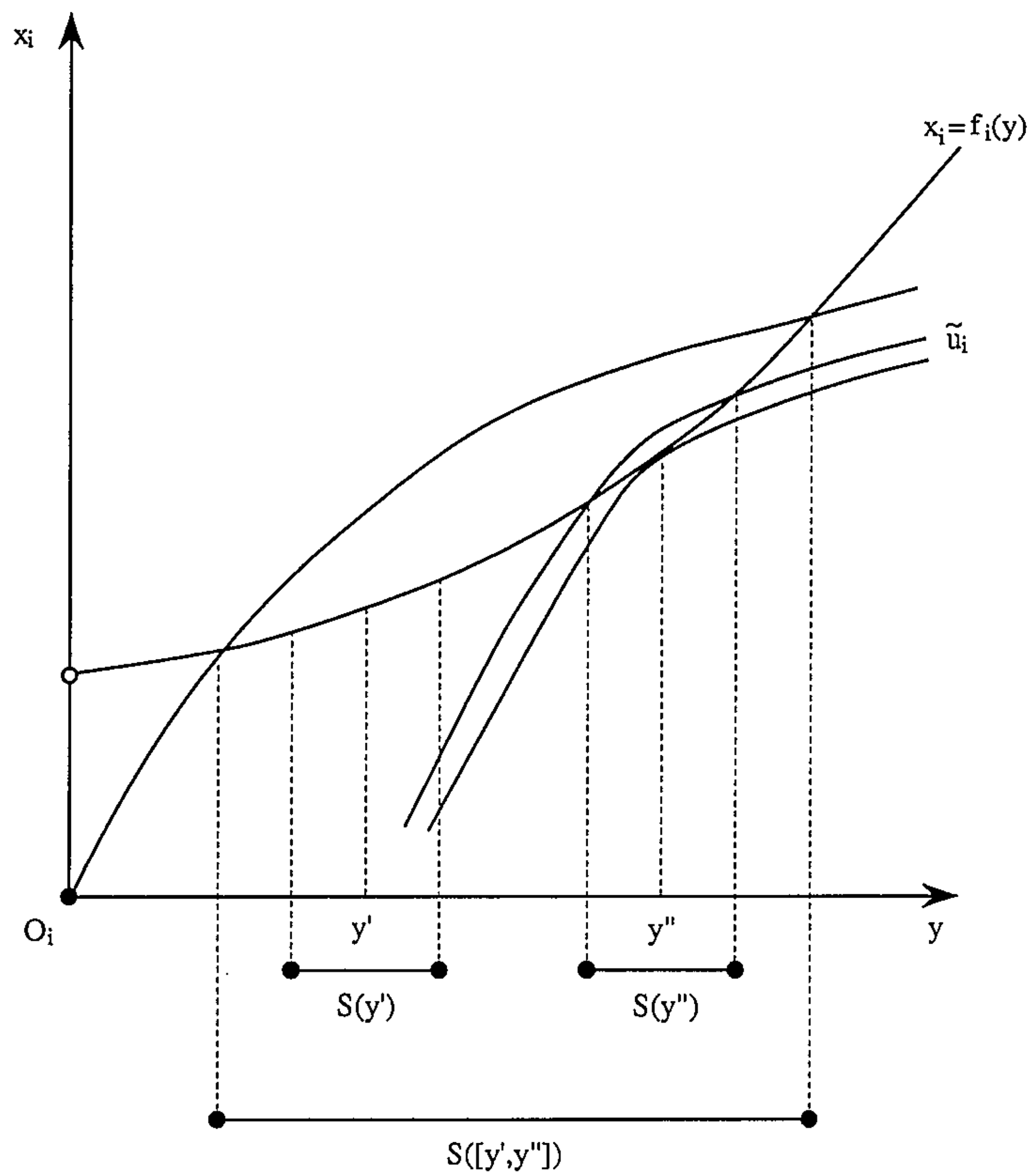

Figure 4.

An example of preference $\tilde{u}_i$.

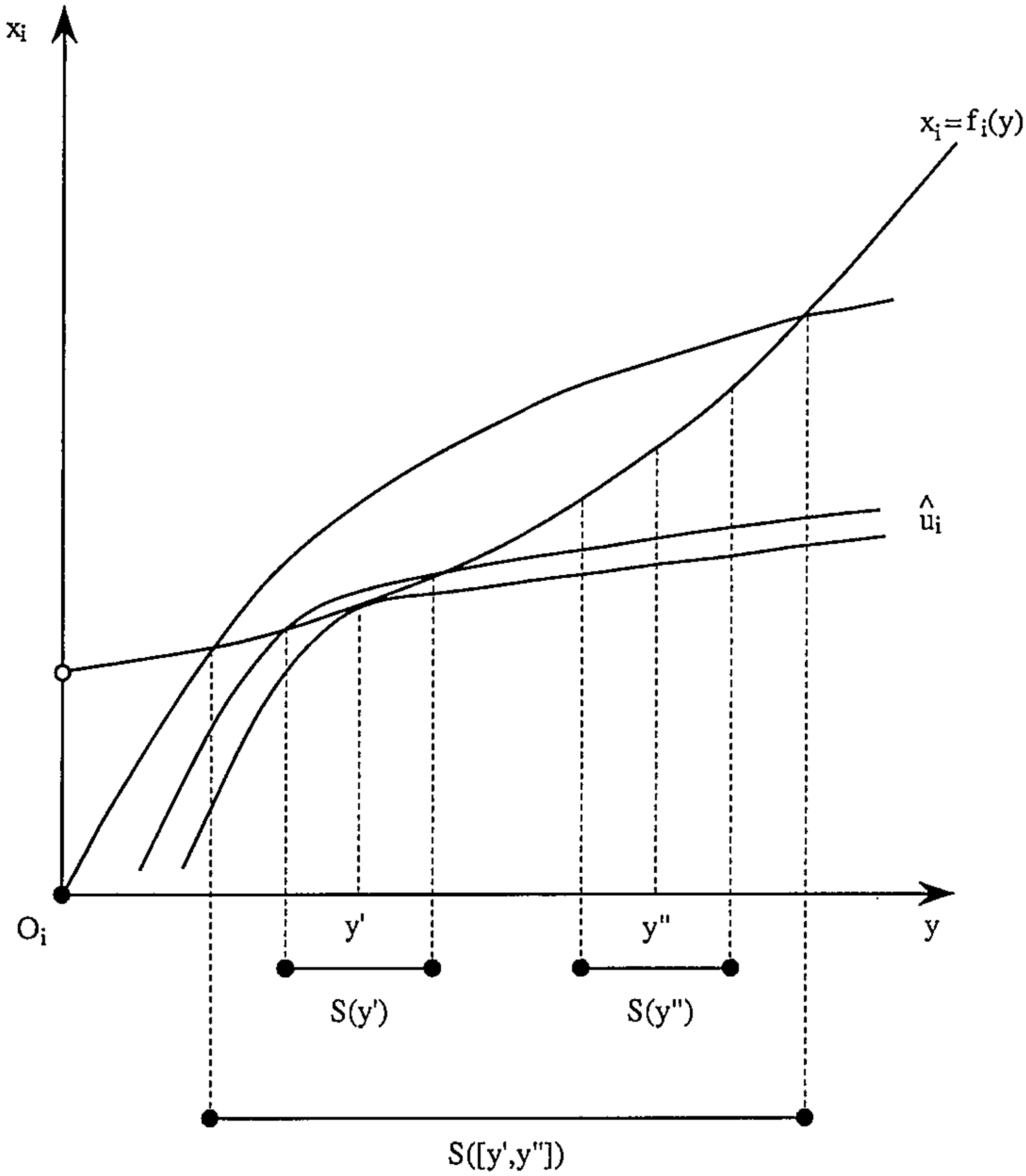

Figure 5.

An example of preference $\hat{u}_i$.

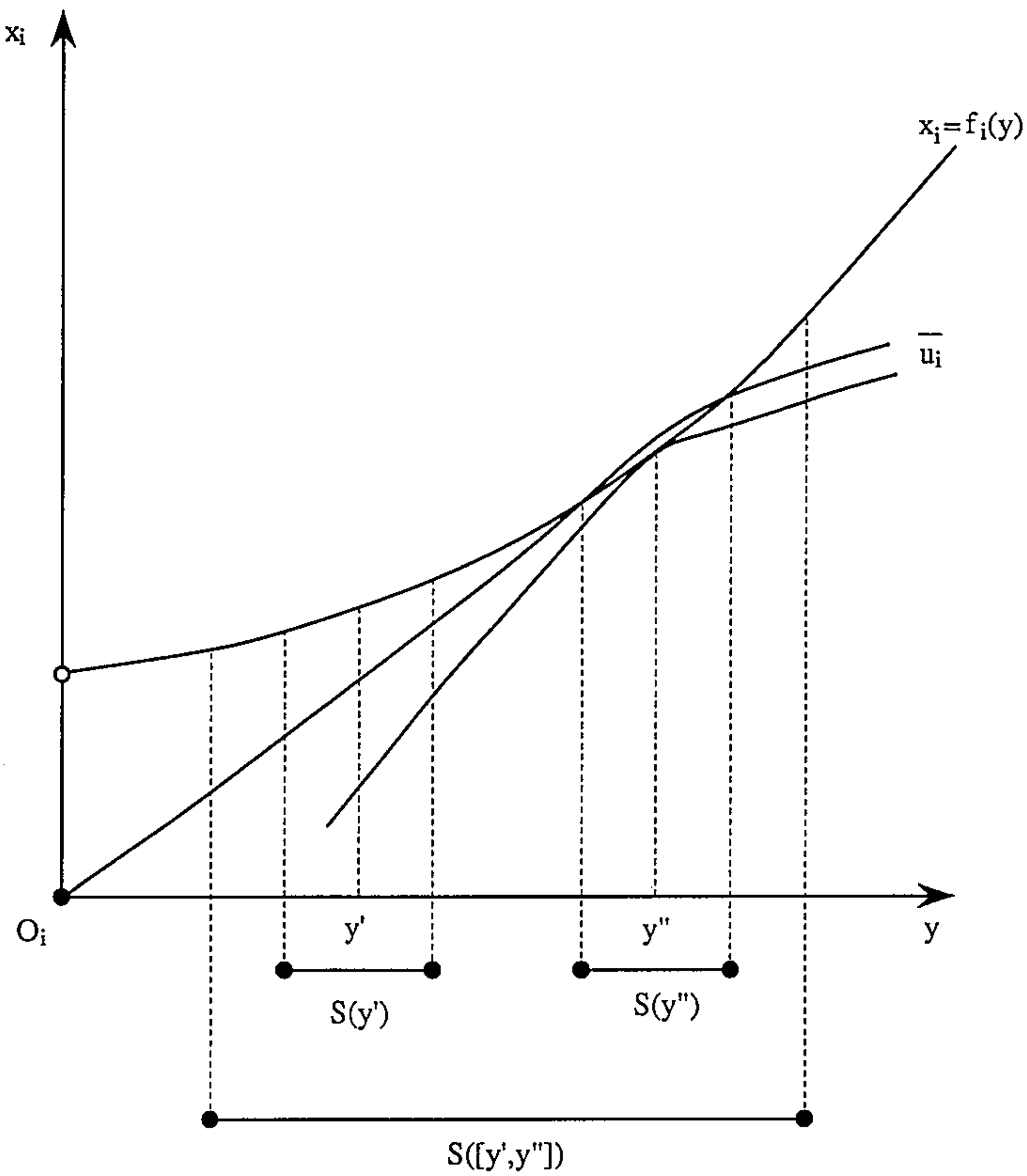

Figure 6.

An example of preference $\overline{u}_i$.