No.587

LEXICOGRAPHIC ADDITIVITY FOR MULTI-ATTRIBUTE PREFERENCES ON FINITE SETS

by

Yutaka Nakamura

June 1994

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June 3, 1994

Abstract

This paper explores lexicographically additive representations of multi-attribute preferences on finite sets. Lexicographic additivity combines a lexicographic feature with local value tradeoffs. Tradeoff structures are governed by either transitive or nontransitive additive conjoint measurement. Alternatives are locally traded off when they are close enough within threshold associated with a dominant subset of attributes.

1 Introduction

Many decision problems involve a number of conflicting objectives. For example, an individual, faced with a decision on moving to a new house, will have to balance objectives such as price, closeness to work, facilities available in the neighborhood, lot size and so on. If the house with the lowest price would also lead to the long distance to the work place, then some judgment would have to be made about the tradeoffs among the factors and the relative importance of the objectives.

Price may be the most important objective in the sense that price differences beyond threshold control the final decision. Then closeness to work may be the second important objective so that within threshold of price differences larger differences of distances to the work place control the decision. When both price and closeness lie within their associated thresholds, the third factor, say lot size, may come into play to make decision. This consideration leads to a lexicographic order. Although the basic idea is intuitively appealing, lexicographic orders are refutable as violating actual preferences, since value tradeoffs are entirely ignored.

Luce (1978) is the first who proposed a model that combines a lexicographic feature with local value tradeoffs. His model deals with two-attribute preferences, say price and

*Institute of Socio-Economic Planning, University of Tsukuba, 1-1-1, Tennoudai, Tsukuba, Ibaraki 305, Japan

closeness to work, where price is the dominant attribute. Then the model says that when utility differences of price lie within threshold, the final decision is made by value tradeoffs between price and closeness to work applying additive conjoint measurement. Fishburn (1980) subsequently provided an axiomatic characterization for a lexicographic additive difference model for two-attribute preferences in which value tradeoffs are made by the additive difference model.

This paper axiomatically examines lexicographic additive representations of n-attribute preferences on finite sets, where value tradeoffs are governed by nontransitive additive conjoint measurement. Let decision alternatives be described by n-tuples $x = (x_1, \ldots, x_n)$ in the product set $X = X_1 \times \ldots \times X_n$. By $\succ$ we denote the binary preference relation on X, read as *is preferred to*. Then a lexicographic additive model yields skew-symmetric functions ϕ_i on $X_i \times X_i$ for $i = 1, \ldots, n$ and nonnegative threshold values $\mu_1, \ldots, \mu_n$ such that for all $x, y \in X$,

$$
\begin{aligned}
x \succ y \quad \text{iff} \quad & \phi_1(x_1, y_1) > \mu_1, \\
\text{or} \quad & \phi_1(x_1, y_1) + \phi_2(x_2, y_2) > \mu_2 \\
& \qquad \text{when } |\phi_1(x_1, y_1)| \leq \mu_1, \\
& \vdots \\
\text{or} \quad & \phi_1(x_1, y_1) + \cdots + \phi_n(x_n, y_n) > \mu_n \\
& \qquad \text{when } |\sum_{i=1}^{k} \phi_i(x_i, y_i)| \leq \mu_k \text{ for } k = 1, \ldots, n-1.
\end{aligned}
$$

Skew-symmetry of ϕ_i means $\phi_i(x_i, y_i) = -\phi_i(y_i, x_i)$ for all $x_i, y_i \in X_i$.

The paper is organized as follows. Section 2 states more formally the lexicographic additive representations and discusses relationships to lexicographic or additive value tradeoff models for finite sets. Then Section 3 presents necessary and sufficient axioms and the representation theorems. The proofs of the theorems will appear in the Section 4. Finally Sections 5 concludes the paper.

2 Lexicographic Additivity

Throughout our discussion we shall let X denote a finite set of decision alternatives. In the multi-attribute context we suppose that there are finitely many attributes, say n, that can be used to differentiate among the objects in X. Let $\mathcal{N} = \{1, \ldots, n\}$. For each $i \in \mathcal{N}$, there is a finite set X_i whose elements are specific levels of attribute i. Elements in X_i might be numbers, vectors of numbers, qualitative descriptions of various kinds, and so forth.

We shall take X to be the set of all n-tuples $x = (x_1, \ldots, x_n)$ in the product set $X_1 \times \ldots \times X_n$, i.e., $X = X_1 \times \ldots \times X_n$. The i th element of x will be always denoted by x_i. We shall sometimes write $(x_i)_{i \in \mathcal{N}}$ in place of $(x_1, \ldots, x_n)$. Although some elements in X might be unrealizable or infeasible at the time of decision, we assume at least that a decision maker can make meaningful comparisons between all pairs of n-tuple in the product set.

Let $\succ$ be a binary preference relation on X with symmetric complement $\sim$, i.e., for all $x, y \in X$, $x \sim y$ if neither $x \succ y$ nor $y \succ x$. For the time being we assume that $\succ$ is asymmetric, i.e., $x \succ y$ and $y \succ x$ cannot both hold for any $x, y \in X$, since asymmetry will be a consequence of our axioms introduced in the next section.

Let $\Re$ be the set of all real numbers with $\Re^n = \Re \times \ldots \times \Re$ (n times), and also let $\Re_+$ be the set of all nonnegative real numbers with $\Re_+^n = \Re_+ \times \ldots \times \Re_+$ (n times). A zero vector in $\Re^n$ or $\Re_+^n$ will be denoted by 0. We define two binary *lexicographic relations* $>_L \subset \Re^n \times \Re^n$ and $>_\ell \subset \Re^n \times \Re_+^n$ as follows. For real vectors $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$ in $\Re^n$,

$$a >_L b \iff \{i \in \mathcal{N} : a_i \neq b_i\} \neq \emptyset \text{ and } a_i > b_i \text{ for the first } i \text{ in this set.}$$

For real vectors $a = (a_1, \ldots, a_n)$ in $\Re^n$ and $\mu = (\mu_1, \ldots, \mu_n)$ in $\Re_+^n$,

$$a >_\ell \mu \iff \{i \in \mathcal{N} : |a_i| > \mu_i\} \neq \emptyset \text{ and } a_i > \mu_i \text{ for the first } i \text{ in this set.}$$

We note that for $a, b \in \Re^n$, $a - b >_\ell 0$ if and only if $a >_L b$.

When the preference $\succ$ obeys a lexicographic order, the importance ordering on n attributes is reflected in the lexicographic ordering on the decision alternatives. On the other hand, when the preference is not fully lexicographic, the importance ordering may not be explicitly incorporated in the prefernece ordering because of value tradeoffs. Since different people may have different feeling on the attribute importance, we shall propose axioms which avoid direct reference to the importance ordering. Thus an importance ordering on n attributes will be implicitly reflected by a permutation σ that is one-to-one mapping from $\mathcal{N}$ to $\mathcal{N}$.

We say that $(X, \succ)$ has a *lexicographically additive* (LA) model if there exist a permutation σ on $\mathcal{N}$, a skew symmetric function ϕ_i on $X_i \times X_i$ for each $i \in \mathcal{N}$ and a threshold vector $\mu \in \Re_+^n$ such that, for all $x, y \in X$,

$$x \succ y \iff \left(\sum_{i=1}^{k} \phi_{\sigma(i)}(x_{\sigma(i)}, y_{\sigma(i)})\right)_{k \in \mathcal{N}} >_\ell \left(\mu_{\sigma(k)}\right)_{k \in \mathcal{N}}, \tag{1}$$

and that it has a *locally semiordered* LA model if it has an LA model (1), where each $\phi_i(x_i, y_i)$ is decomposed as $u_i(x_i) - u_i(y_i)$ for some real valued function u_i on X_i, i.e., for all $x, y \in X$,

$$x \succ y \iff \left(\sum_{i=1}^{k} u_{\sigma(i)}(x_{\sigma(i)}) - \sum_{i=1}^{k} u_{\sigma(i)}(y_{\sigma(i)})\right)_{k \in \mathcal{N}} >_\ell \left(\mu_{\sigma(k)}\right)_{k \in \mathcal{N}}. \tag{2}$$

Throughout the rest of the section, we shall assume that $(X, \succ)$ has LA model (1).

For notational convenience, for $x \in X$ and $I = \{i_1, \ldots, i_k\} \subseteq \mathcal{N}$, let x_I denote the k-tuple $(x_{i_1}, \ldots, x_{i_k})$, and $X_I = X_{i_1} \times \ldots \times X_{i_k}$. The complement $\mathcal{N} \setminus I$ of I will be denoted by I^c. We also let (x_I, y_{I^c}) denote the n-tuple $(a_1, \ldots, a_n)$, where $a_i = x_i$ if $i \in I$, and $a_i = y_i$ if $i \notin I$. Then for each $I \subseteq \mathcal{N}$, we define a binary *dominant relation* $\succ_I$ on X_I as follows. For all $x_I, y_I \in X_I$,

$$x_I \succ_I y_I \iff (x_I, a_{I^c}) \succ (y_I, b_{I^c}) \text{ for all } a_{I^c}, b_{I^c} \in X_{I^c}.$$

Let $\sim_I$ be the symmetric complement of $\succ_I$. The dominant relation $\succ_I$ applies when preference of an alternative x to an alternative y is unaffected whatever changes are made in the attribute levels in I^c that each alternative can achieve. Nonempty $\succ_I$ implies that there is no tradeoff between I and I^c. Thus we might say that some of the attributes in I definitely have more influence on the preference $\succ$ than the attributes in I^c, i.e., the attribute set I dominates its complement. However, those domination might come out not from the importance ordering of the attributes but from negligible small difference of the attribute levels.

Given a permutation σ in LA model (1), let $I_k = \{\sigma(1), \ldots, \sigma(k)\}$ for $k = 1, \ldots, n$. Then the set of dominant relations $\{\succ_{I_1}, \ldots, \succ_{I_n}\}$, some of which may be empty, constitutes a hierarchy of domination in the sense that for all $x, y \in X$ and $k = 1, \ldots, n$,

$$x_{I_k} \succ_{I_k} y_{I_k} \iff \left(\sum_{i=1}^{j} \phi_{\sigma(i)}(x_{\sigma(i)}, y_{\sigma(i)}) \right)_{j \in \mathcal{N}_k} >_\ell \left(\mu_{\sigma(j)} \right)_{j \in \mathcal{N}_k},$$

where $\mathcal{N}_k = \{1, \ldots, k\}$. If $\succ_{I_k}$ is not empty, then $(X_{I_k}, \succ_{I_k})$ also has an LA model. For an attribute set $I \subseteq \mathcal{N}$, if there exists the largest dominant set I_k which is included in I, then for all $x, y \in X$, $x_I \succ_I y_I$ iff $x_{I_k} \succ_{I_k} y_{I_k}$, since the attributes in $I \setminus I_k$ have no influence on $\succ$.

For $I \subseteq \mathcal{N}$, let $\succ_I^*$ denote a binary *tradeoff relation* on X_I defined as follows. For all $x_I, y_I \in X_I$,

$$x_I \succ_I^* y_I \iff x_I \succ_I y_I \text{ and } x_J \sim_J y_J \text{ for all } J \subset I.$$

Asymmetry of $\succ_I^*$ follows from asymmetry of $\succ_I$. The tradeoff relation $\succ_I^*$ is said to hold when I is a dominant set of attributes, and the domination of I comes not from the domination of any proper subset of I but from the holistic considerations of the attribute levels in I, namely value tradeoffs. Throughout let $|I|$ denote the cardinality of I, which should not be confused with absolute values. If $|I| = k$ and the tradeoff relation $\succ_I^*$ is not empty, then $I = \{\sigma(1), \ldots, \sigma(k)\}$ and for all $x_I, y_I \in X_I$,

$$x_I \succ_I^* y_I \iff \sum_{i=1}^{k} \phi_{\sigma(i)}(x_{\sigma(i)}, y_{\sigma(i)}) > \mu_{\sigma(k)},$$

which is a nontransitive additive conjoint model with constant threshold. When semiordered LA model (2) applies, we have

$$x_I \succ_I^* y_I \iff \sum_{i=1}^{k} u_{\sigma(i)}(x_{\sigma(i)}) > \sum_{i=1}^{k} u_{\sigma(i)}(y_{\sigma(i)}) + \mu_{\sigma(k)},$$

so that $\succ_I^*$ is a semiorder.

In what follows we shall see in detail several specializations of LA model (1), namely lexicographic or additive models. Suppose that the threshold vector vanishes, i.e., $\mu = 0$. Then LA model (1) reduces to the following *nontransitive lexicographic* model: for all $x, y \in X$,

$$x \succ y \Longleftrightarrow \left(\phi_{\sigma(k)}(x_{\sigma(k)}, y_{\sigma(k)})\right)_{k\in\mathcal{N}} >_\ell 0.$$

An axiomatization of this model appears in Fishburn (1976). On the other hand, semiordered LA model (2) reduces to

$$\begin{aligned} x \succ y &\Longleftrightarrow \left(u_{\sigma(k)}(x_{\sigma(k)}) - u_{\sigma(k)}(y_{\sigma(k)})\right)_{k\in\mathcal{N}} >_\ell 0 \\ &\Longleftrightarrow \left(u_{\sigma(k)}(x_{\sigma(k)})\right)_{k\in\mathcal{N}} >_L \left(u_{\sigma(k)}(y_{\sigma(k)})\right)_{k\in\mathcal{N}}, \end{aligned}$$

which is a *weakly ordered lexicographic* model. A necessary and sufficient axiom system for this model is found in Fishburn (1975). If this model represents the preference $\succ$, then finiteness of X implies that there also exists a *locally semiordered* lexicographic model given by

$$x \succ y \Longleftrightarrow \left(u_{\sigma(k)}(x_{\sigma(k)}) - u_{\sigma(k)}(y_{\sigma(k)})\right)_{k\in\mathcal{N}} >_\ell (\mu_{\sigma(k)})_{k\in\mathcal{N}}.$$

However, the converse is not necessarily true, since $\succ$ may not be transitive.

The locally semiordered lexicographic model is a special case of the following nontransitive lexicographic model with threshold vector μ:

$$x \succ y \Longleftrightarrow \left(\phi_{\sigma(k)}(x_{\sigma(k)}, y_{\sigma(k)})\right)_{k\in\mathcal{N}} >_\ell (\mu_{\sigma(k)})_{k\in\mathcal{N}},$$

which is easily shown to be tantamount to a nontransitive lexicographic model without threshold, i.e.,

$$x \succ y \Longleftrightarrow \left(\phi^*_{\sigma(k)}(x_{\sigma(k)}, y_{\sigma(k)})\right)_{k\in\mathcal{N}} >_\ell 0,$$

where for $k = 1, \ldots, n$,

$$\begin{aligned} \phi^*_k(x_k, y_k) &= \phi_k(x_k, y_k) \text{ if } |\phi_k(x_k, y_k)| > \mu_k \\ &= 0 \qquad\qquad \text{otherwise.} \end{aligned}$$

Since X is assumed to be finite, we can also say that all the lexicographic models discussed in the preceding paragraphs can be translated into additive models. Fishburn (1970) argued that if a weakly ordered lexicographic model holds, then additive utilities exist in the following sense. For all $x, y \in X$,

$$x \succ y \Longleftrightarrow \sum_{i=1}^{n} v_i(x_i) > \sum_{i=1}^{n} v_i(y_i).$$

A similar argument applies to nontransitive lexicographic models as follows. Given skew-symmetric functions ϕ_i on $X_i \times X_i$ for $i = 1, \ldots, n$, let

$$\begin{aligned} M_i &= \max\{\phi_i(x_i, y_i) : x_i, y_i \in X_i \text{ and } \phi_i(x_i, y_i) > 0\}, \\ m_i &= \min\{\phi_i(x_i, y_i) : x_i, y_i \in X_i \text{ and } \phi_i(x_i, y_i) > 0\}. \end{aligned}$$

Under appropriate multiplicative transformations of every ϕ_i, we obtain that $m_{\sigma(k)} = \sum_{i=k+1}^{n} M_{\sigma(i)}$ for $k = 1, \ldots, n$, so that for all $x, y \in X$,

$$x \succ y \iff \sum_{i=1}^{n} \phi_i(x_i, y_i) > 0,$$

which is a nontransitive additive conjoint measurement model (for axiomatizations of this model, see Bouyssou (1986), Fishburn (1991), and Vind (1991)).

3 Axioms and Theorems

Under finiteness assumption this section presents necessary and sufficient axioms for both LA and locally semiordered LA models. First we assume the following nonnecessary structural axiom.

Axiom 1 *$n \geq 2$, $X = X_1 \times \ldots \times X_n$, and each X_i is finite.*

A necessary order axiom is stated as follows.

Axiom 2 *$\succ$ on X is irreflexive.*

To state the remaining axioms, we shall use sequences of binary relations $E^*_{\Omega_m}$ and E_{Ω_m} on X^m for all integers $m > 1$, where X^m is the m-fold Cartesian product of X with itself, and $\Omega_m = (I_1, \ldots, I_m)$ is a sequence of nonempty subsets of $\mathcal{N}$ such that $I_i \subseteq I_j$ or $I_j \subseteq I_i$ for all distinct i and j. When $\Omega_m = (\mathcal{N}, \ldots, \mathcal{N})$, the relation $E^*_{\Omega_m}$ was recently introduced by Fishburn (1991) to axiomatize nontransitive additive conjoint measurement.

Definition 1 *For all $x^1, \ldots, x^m, y^1, \ldots, y^m \in X$, all $\Omega_m = (I_1, \ldots, I_m)$, and all $m > 1$, $(x^1, \ldots, x^m) E^*_{\Omega_m} (y^1, \ldots, y^m)$ if and only if it is true for each i that for all $a, b \in X$,*

$$|\{j : (x_i^j, y_i^j) = (a_i, b_i) \text{ and } i \in I_j\}| = |\{j : (x_i^j, y_i^j) = (b_i, a_i) \text{ and } i \in I_j\}|.$$

This definition utilizes skew-symmetry of real valued functions ϕ_i on $X_i \times X_i$ for $i = 1, \ldots, n$ to obtain that for all $x^1, \ldots, x^m, y^1, \ldots, y^m \in X$,

$$(x^1, \ldots, x^m) E^*_{\Omega_m} (y^1, \ldots, y^m) \Longrightarrow \sum_{k \in \{j : i \in I_j\}} \phi_i(x_i^k, y_i^k) = 0 \text{ for } i = 1, \ldots, n.$$

Thus this leads to the following important implication, namely, the summation property,

$$\sum_{k=1}^{m} \phi_{I_k}(x^k, y^k) = 0,$$

where $\phi_{I_k}(x, y) = \sum_{i \in I_k} \phi_i(x_i, y_i)$ for all $x, y \in X$.

A necessary and sufficient axiom for LA model (1) is given by

Axiom 3 *For all* $x^1, \ldots, x^{2m}, y^1, \ldots, y^{2m} \in X$, *all* $\Omega_{2m} = (I_1, \ldots, I_m, I_1, \ldots, I_m)$, *and all* $m \geq 1$, *if* $(x^1, \ldots, x^{2m}) E^*_{\Omega_{2m}} (y^1, \ldots, y^{2m})$, *then it is false that* $x^j \sim_{I_j} y^j$ *and* $x^{m+j} \succ^*_{I_j} y^{m+j}$ *for every* $1 \leq j \leq m$.

By the summation property,

$$(x^1, \ldots, x^{2m}) E^*_{\Omega_{2m}} (y^1, \ldots, y^{2m}) \Longrightarrow \sum_{j=1}^{m} \left(\phi_{I_j}(x^j, y^j) + \phi_{I_j}(x^{m+j}, y^{m+j}) \right) = 0.$$

Suppose on the contrary that $x^j \sim_{I_j} y^j$ and $x^{m+j} \succ^*_{I_j} y^{m+j}$ for every $1 \leq j \leq m$. Let $|I_j| = n_j$ for $1 \leq j \leq m$, so $I_j = \{\sigma(1), \ldots, \sigma(n_j)\}$. Then it follows from LA model (1) and the definition of tradeoff relations that for $j = 1, \ldots, m$,

$$\phi_{I_j}(x^{m+j}, y^{m+j}) > \mu_{n_j} \text{ and } |\phi_{I_j}(x^j, y^j)| \leq \mu_{n_j}.$$

Therefore, we have

$$\sum_{j=1}^{m} \left(\phi_{I_j}(x^j, y^j) + \phi_{I_j}(x^{m+j}, y^{m+j}) \right) > 0,$$

which contradicts the summation property. Hence the necessity of the axiom obtains.

Asymmetry of $\succ$ follows from Axioms 2 and 3 as follows. Let $\Omega_4 = (\mathcal{N}, \mathcal{N}, \mathcal{N}, \mathcal{N})$. Then $(x, y, x, y) E^*_{\Omega_4} (x, y, y, x)$. By Axiom 2, $x \sim x$ and $y \sim y$. Suppose $x \succ y$. Then by Axiom 3, not$(y \succ x)$.

Under the finiteness assumption, the LA representation theorem is given by

Theorem 1 *Suppose that Axioms 1 holds. Then Axioms 2 and 3 hold if and only if* $(X, \succ)$ *has LA model (1).*

The sufficiency of the axiom will be proved in the next section.

Next we present a necessary and sufficient axiom for locally semiordered LA model (2). To state the axiom, we shall use the following definition of binary relations E_{Ω_m} on X^m.

Definition 2 *For all* $x^1, \ldots, x^m, y^1, \ldots, y^m \in X$, *all* $\Omega_m = (I_1, \ldots, I_m)$, *and all* $m > 1$, $(x^1, \ldots, x^m) E_{\Omega_m} (y^1, \ldots, y^m)$ *if and only if it is true for each* i *that*

$$\{x_i^j : i \in I_j \text{ and } 1 \leq j \leq m\} = \{y_i^j : i \in I_j \text{ and } 1 \leq j \leq m\}.$$

Enumerate $\{j : i \in I_j\}$ as $k_{i1}, \ldots, k_{in_j}$. Then the definition requires that $x_i^{k_{i1}}, \ldots, x_i^{k_{in_j}}$ be a permutation of $y_i^{k_{i1}}, \ldots, y_i^{k_{in_j}}$. For $i = 1, \ldots, n$, let u_i be a real valued function on X_i. Then for all $x^1, \ldots, x^m, y^1, \ldots, y^m \in X$,

$$(x^1, \ldots, x^m) E_{\Omega_m} (y^1, \ldots, y^m) \Longrightarrow \sum_{k \in \{j : i \in I_j\}} u_i(x_i^k) = \sum_{k \in \{j : i \in I_j\}} u_i(y_i^k) \text{ for } i = 1, \ldots, n.$$

Thus the summation property is given by

$$\sum_{k=1}^{m} u_{I_k}(x^k) = \sum_{k=1}^{m} u_{I_k}(y^k),$$

where $u_{I_k}(x) = \sum_{i \in I_k} u_i(x_i)$ for all $x \in X$.

Axiom 3 is strengthened to the following.

Axiom 4 *For all $x^1, \ldots, x^{2m}, y^1, \ldots, y^{2m} \in X$, all $\Omega_{2m} = (I_1, \ldots, I_m, I_1, \ldots, I_m)$, and all $m \geq 1$, if $(x^1, \ldots, x^{2m}) E_{\Omega_{2m}} (y^1, \ldots, y^{2m})$, then it is false that $x^j \sim_{I_j} y^j$ and $x^{m+j} \succ^*_{I_j} y^{m+j}$ for every $1 \leq j \leq m$.*

By the summation property,

$$(x^1, \ldots, x^{2m}) E_{\Omega_{2m}} (y^1, \ldots, y^{2m}) \Longrightarrow \sum_{j=1}^{m} \left(u_{I_j}(x^j) + u_{I_j}(x^{m+j}) \right) = \sum_{j=1}^{m} \left(u_{I_j}(y^j) + u_{I_j}(y^{m+j}) \right).$$

Suppose on the contrary that $x^j \sim_{I_j} y^j$ and $x^{m+j} \succ^*_{I_j} y^{m+j}$ for every $1 \leq j \leq m$. Let $|I_j| = n_j$ for $1 \leq j \leq m$, so $I_j = \{\sigma(1), \ldots, \sigma(n_j)\}$. Then it follows from locally semiordered LA model (2) and the definition of tradeoff relations that for $j = 1, \ldots, m$,

$$u_{I_j}(x^{m+j}) - u_{I_j}(y^{m+j}) > \mu_{n_j} \text{ and } |u_{Ij}(x^j) - u_{I_j}(y^j)| \leq \mu_{n_j}.$$

Therefore we have

$$\sum_{j=1}^{m} \left(u_{I_j}(x^j) + u_{I_j}(x^{m+j}) \right) > \sum_{j=1}^{m} \left(u_{I_j}(y^j) + u_{I_j}(y^{m+j}) \right),$$

which contradicts the summation property. Hence the necessity of the axiom obtains.

When $\Omega_{2m} = \{\mathcal{N}, \ldots, \mathcal{N}\}$, this axiom together with Axioms 1 and 2 is necessary and sufficient for the existence of the semiordered additive conjoint measurement (see Fishburn (1970)), i.e.,

$$x \succ y \Longleftrightarrow \sum_{i}^{n} u_i(x_i) > \sum_{i}^{n} u_i(y_i) + 1.$$

The locally semiordered LA representation is stated in the following theorem.

Theorem 2 *Suppose that Axiom 1 holds. Then Axioms 2 and 4 hold if and only if $(X, \succ)$ has locally semiordered LA model (2).*

The sufficiency proof of the axiom will appear in the next section.

4 Sufficiency Proofs

This section provides sufficiency proofs of Theorems 1 and 2 in the preceding section. The necessities of the axioms were noted in that section. Our sufficiency proofs use the following restricted version of the familiar lemma for the existence of a solution to a finite system of linear inequalities (see Fishburn (1970)). When $a = (a_1, \ldots, a_N)$ and $b = (b_1, \ldots, b_N)$ are N dimensional vectors of real numbers, denote the inner product by $a \cdot b = \sum_{i=1}^{N} a_i b_i$. A real vector is called *rational* if each component is a rational number, and is called *integral* if each of its components is an integer.

Lemma 1 *Let $a^1, \ldots, a^M$ be N dimensional rational real vectors. Then either there is an N dimensional integral real vector ρ such that for $k = 1, \ldots, M$,*

$$\rho \cdot a^k > 0$$

or else there are nonnegative integers $r_1, \ldots, r_M$, at least one of which is positive, such that for $j = 1, \ldots, N$,

$$\sum_{k=1}^{M} r_k a_j^k = 0.$$

Throughout the section, we shall assume that Axiom 1 holds. Then $X = X_1 \times \ldots \times X_n$ is finite. It easily follows from the definition of the dominant relations that for all $I, J \subseteq \mathcal{N}$ and all $x, y \in X$,

$$\begin{aligned} [I \subseteq J, x_I \succ_I y_I] &\implies x_J \succ_J y_J, \\ [I^c \supseteq J, x_I \succ_I y_I] &\implies \succ_J = \emptyset. \end{aligned}$$

Those properties will be used without explicit mention to specify a permutation σ from $\succ$.

Sufficiency Proof of Theorem 1. Throughout the proof, Axioms 2 and 3 are assumed to hold. Thus $\succ$ is asymmetric. We also assume that X_i for some i has two or more elements and $\succ$ is not empty. Otherwise, the representations of Theorems 1 and 2 trivially obtain. With no loss of generality we treat the X_i as mutually disjoint for notational convenience.

We specify a hierarchy of dominant relations, $\succ_{I_1}, \ldots, \succ_{I_n}$, such that there exists a permutation σ on $\{1, \ldots, n\}$ for which $I_k = \{\sigma(1), \ldots, \sigma(k)\}$ for $k = 1, \ldots, n$. Suppose first that $\succ_{\{i_1\}}$ is not empty. Then it must be unique. At this stage we take $\sigma(1) = i_1$. At the second stage, we identify $i_2 \neq i_1$ such that $\succ^*_{\{i_1, i_2\}}$ is not empty. Note also that i_2 is uniquely determined if it exists, and let $\sigma(2) = i_2$. If $\succ^*_{\{i_1, i\}} = \emptyset$ for all $i \neq i_1$, then $\sigma(2)$ is unspecified for the time being, and we proceed to identify the set $\{i_1, i, j\}$ such that $\succ^*_{\{i_1, i, j\}} \neq \emptyset$. If that set exists, we can take $\sigma(2) = i$ and $\sigma(3) = j$. If again $\succ^*_{\{i_1, i, j\}} = \emptyset$ for all i, j, then $\sigma(2)$ and $\sigma(3)$ are both unspecified in this stage. This process goes up to the n-th stage and can identify all $\sigma(1), \ldots, \sigma(n)$, since $\succ^*_{\mathcal{N}}$ is not empty.

Suppose next that $\succ_{\{i\}} = \emptyset$ for all $i = 1, \ldots, n$. Then $\sigma(1)$ is unspecified for the time being, and we proceed to identify the set I_2 as $\{i, j\}$ for $i \neq j$ such that $\succ_{\{i,j\}} \neq \emptyset$, which is unique if it exists. Then at the second stage we can take $\sigma(1) = i$ and $\sigma(2) = j$. If $\succ_{\{i,j\}} = \emptyset$ for all distinct i and j, then $\sigma(1)$ and $\sigma(2)$ are unspecified in this stage. This process goes up to the n-th stage to identify $\sigma(1), \ldots, \sigma(n)$ as in the proceding paragraph.

With no loss of generality we shall assume that an identified permutation is an identity map, i.e., $\sigma(i) = i$ for all $i = 1, \ldots, n$. For notational convenience, let $\succ_k = \succ_{I_k}$ and $\succ_k = \succ^*_{I_k}$ for $k = 1, \ldots, n$.

For $i = 1, \ldots, n$, let

$$\begin{aligned} X_i &= \{x_{i1}, x_{i2}, \ldots, x_{iN_i}\}, \\ X_i^* &= \{(x_{ij}, x_{ik}) : 1 \leq j < k \leq N_i\}. \end{aligned}$$

Enumerate X_i^* as $t^{i1}, t^{i2}, \ldots, t^{iK_i}$, where $K_i = |X_i^*|$.

Suppose that LA model (1) holds with skew-symmetric functions ϕ_i on $X_i \times X_i$ and threshold values $\mu_i \geq 0$ for $i = 1, \ldots, n$. We define K_i dimensional row vectors ρ^i, $i = 1, \ldots, n$ and an n dimensional row vector μ by

$$\begin{aligned}\rho^i &= (\phi_i(t^{i1}), \ldots, \phi_i(t^{iK_i})),\\ \mu &= (\mu_1, \ldots, \mu_n).\end{aligned}$$

Then let $\rho = (\rho^1, \ldots, \rho^n, \mu)$ be a $(K+n)$ dimensional row vector, where $K = \sum_{i=1}^n K_i$. We note that for all $x, y \in X$,

$$\begin{aligned}x \succ y &\iff x \succ_k^* y \text{ for some } 0 < k \leq n \text{ and}\\ &\qquad x \sim_j y \text{ for all } 0 < j < k;\\ x \sim y &\iff x \sim_j y \text{ for all } 0 < j \leq n.\end{aligned}$$

Then LA model (1) requires that there exist a vector ρ such that

$$\begin{aligned}x \succ y &\iff \sum_{i=1}^{k} \rho^i \cdot a^i > \mu_k \text{ for some } 0 < k \leq n \text{ and}\\ &\qquad |\sum_{i=1}^{j} \rho^i \cdot a^i| \leq \mu_j \text{ for all } 0 < j < k;\\ x \sim y &\iff |\sum_{i=1}^{j} \rho^i \cdot a^i| \leq \mu_j \text{ for all } 0 < j \leq n,\end{aligned}$$

where $a^i \in \{-1, 0, 1\}^n$ for $i = 1, \ldots, n$.

Since X is finite, all the equalities that the model generates from $\sim_j$ statements can be slightly modified to make them strict inequalities by increasing slightly the threshold values μ_j without disturbing any other strict inequalities that the model generates from $\succ_k^*$ statements. If $\mu_j = 0$ then similar slight increase can make it positive. Hence, if the vector ρ exists, then there is such a vector with strict inequalities holding and $\mu_i > 0$ for $i = 1, \ldots, n$. The modified model yields that for all $x, y \in X$, $x \succ_k^* y$ if and only if for some $(K+n)$ dimensional vector $a(x,y)$,

$$\rho \cdot a(x,y) \;\; > 0 \quad (\text{i.e., } \sum_{i=1}^{k} \phi_i(x_i, y_i) - \mu_k > 0);$$

and $x \sim_k y$ if and only if for $1 \leq j \leq k$ and some $(K+n)$ dimensional vectors $b^j(x,y)$ and $c^j(x,y)$,

$$\begin{aligned}\rho \cdot b^j(x,y) &> 0 \quad (\text{i.e., } \sum_{i=1}^{j} \phi_i(x_i, y_i) + \mu_j > 0)\\ \rho \cdot c^j(x,y) &> 0 \quad (\text{i.e., } \sum_{i=1}^{j} \phi_i(y_i, x_i) + \mu_j > 0)\end{aligned}$$

where $a(x,y), b^j(x,y), c^j(x,y) \in \{-1,0,1\}^{K+n}$. Clearly, $a(x,y)$ has at most $k+1$ nonzero elements when $x \succ_k^* y$, and $b^j(x,y)$ and $c^j(x,y)$ have at most $j+1$ nonzero elements when $x \sim_k y$ and $1 \leq j \leq k$.

For $k = 1, \ldots, n$, we define

$$\begin{aligned} P_k &= \{(x,y) \in X \times X : x \succ_k^* y\}, \\ J_k &= \{(x,y) \in X \times X : x \sim_k y\}, \end{aligned}$$

and enumerate P_k as $t_1^k, \ldots, t_{L_k}^k$, and half of J_k as $s_1^k, \ldots, s_{M_k}^k$ by using only one of (x,y) and (y,x) when $x \sim_k y$. Then our system of linear inequalities is stated as follows. For $k = 1, \ldots, n$,

$$\begin{aligned} \rho \cdot a(t_i^k) &> 0 \quad (i = 1, \ldots, L_k) \\ \rho \cdot b^k(s_i^k) &> 0 \quad (i = 1, \ldots, M_k) \\ \rho \cdot c^k(s_i^k) &> 0 \quad (i = 1, \ldots, M_k) \end{aligned}$$

whenever P_k and J_k are not empty.

The proof is completed by establishing that the system of linear inequalities has a ρ solution. Suppose on the contrary that there is no ρ solution. Then it follows from Lemma 1 that there are nonnegative integers α_{ki}, β_{kj}, and γ_{kj} for $i = 1, \ldots, L_k, j = 1, \ldots M_k$, and $k = 1, \ldots, n$, at least one of which is positive, such that

$$\sum_{k=1}^{n} \left(\sum_{i=1}^{L_k} \alpha_{ki} a(t_i^k) + \sum_{i=1}^{M_k} \beta_{ki} b^k(s_i^k) + \sum_{i=1}^{M_k} \gamma_{ki} c^k(s_i^k) \right) = 0.$$

For $k = 1, \ldots, n$, the $(K+k)$-th elements of $a(t_i^k), b^k(s_i^k)$, and $c^k(s_i^k)$ refer to μ_k, so that $a(t_i^k)_{K+k} = -1$ for $i = 1, \ldots, L_k$, and $b^k(s_i^k)_{K+k} = c^k(s_i^k)_{K+k} = 1$ for $i = 1, \ldots, M_k$. Therefore, we have that for $k = 1, \ldots, n$,

$$\sum_{i=1}^{L_k} \alpha_{ki} = \sum_{i=1}^{M_k} (\beta_{ki} + \gamma_{ki}),$$

and let $m_k = \sum_{i=1}^{L_k} \alpha_{ki}$ and $m = \sum_{k=1}^{n} m_k$.

For $k = 1, \ldots, n$, we define

$$\begin{aligned} P_k^* &= \{(x,y) \in P_k : t_i^k = (x,y) \text{ and } \alpha_{ki} \neq 0\}, \\ J_k^* &= \{(x,y) \in J_k : s_i^k = (x,y) \text{ and } \beta_{ki} + \gamma_{ki} \neq 0\}. \end{aligned}$$

List the elements of P_k^* and J_k^* with α_{ki} repeates for t_i^k and $\beta_{ki} + \gamma_{ki}$ repeats for s_i^k, and enumerate them as $(x^{k1}, y^{k1}), \ldots, (x^{km_k}, y^{km_k})$ and $(z^{k1}, w^{k1}), \ldots, (z^{km_k}, w^{km_k})$, respectively. Then we have m_k preference statements and m_k indifference statements as follows.

$$\begin{aligned} &x^{k1} \succ_k^* y^{k1}, \ldots, x^{km_k} \succ_k^* y^{km_k}, \\ &z^{k1} \sim_k w^{k1}, \ldots, z^{km_k} \sim_k w^{km_k}. \end{aligned}$$

Consider $\Omega_{2m} = (J_1, \ldots, J_m, J_1, \ldots, J_m)$, where for $k = 1, \ldots, m$,

$$J_k = \{1, \ldots, j\} \text{ if } m_1 + \cdots + m_{j-1} < k \leq m_1 + \cdots + m_j.$$

Then we have an obvious violation of Axiom 3.

Sufficiency Proof of Theorem 2. Throughout the proof Axioms 2 and 4 are assumed to hold. The proof proceeds as in the sufficiency proof of Theorem 1 with minor changes. Thus we show only such changes without going into detail. For $i = 1, \ldots, n$, let $X_i = \{x_{i1}, x_{i2}, \ldots, x_{iN_i}\}$. We define N_i dimensional row vectors $\rho^1, \ldots, \rho^n$ and an n dimensional row vector μ by

$$\begin{aligned} \rho^i &= (u_i(x_{i1}), \ldots, u_i(x_{iN_i})), \\ \mu &= (\mu_1, \ldots, \mu_n). \end{aligned}$$

Then let $\rho = (\rho^1, \ldots, \rho^n, \mu)$ be an $(N + n)$ dimensional row vector, where $N = \sum_{i=1}^{k} N_i$.

The modified locally semiordered LA model yields that for all $x, y \in X$, $x \succ_k^* y$ if and only if for some $(K + n)$ dimensional vector $a(x, y)$,

$$\rho \cdot a(x,y) \quad > 0 \quad (\text{i.e., } \sum_{i=1}^{k} u_i(x_i) - \sum_{i=1}^{k} u_i(y_i) - \mu_k > 0);$$

and $x \sim_k y$ if and only if for $1 \leq j \leq k$ and some $(K + n)$ dimensional vectors $b^j(x, y)$ and $c^j(x, y)$,

$$\begin{aligned} \rho \cdot b^j(x,y) \quad &> 0 \quad (\text{i.e., } \sum_{i=1}^{j} u_i(x_i) - \sum_{i=1}^{j} u_i(y_i) + \mu_j > 0) \\ \rho \cdot c^j(x,y) \quad &> 0 \quad (\text{i.e., } \sum_{i=1}^{j} u_i(y_i) - \sum_{i=1}^{j} u_i(x_i) + \mu_j > 0) \end{aligned}$$

where $a(x, y), b^j(x, y), c^j(x, y) \in \{-1, 0, 1\}^{N+n}$.

With those minor changes, we can exactly follows the sufficiency proof of Theorem 1, so that the rest of the proof will be omitted.

5 Conclusions

The aim of this paper has been to introduce lexicographic additivity in multi-attribute preference modelling and to explore axiomatic characterizations. We focused on finite sets and applied a solution theory to a finite system of linear inequalities to show the existence of lexicographic additive conjoint measurement models. As in any other finite set axiomatizations, our axioms are necessary and sufficient for the representations, but not testable. Topological or algebraic extensions of lexicographic additivity to infinite sets might give more testable and interpretable axiom systems.

References

Bouyssou, D. (1986) Some remarks on the notion of compensation in MCDM. *European Journal of Operational Research*, **26**, 150–160.

Fishburn, P.C. (1970) *Utility Theory for Decision Making.* Wiley, New York.

Fishburn, P.C. (1975) Axioms for lexicographic preferences. *Review of Economic Studies*, **42**, 415–419.

Fishburn, P.C. (1976) Noncompensatory preferences. *Synthese*, **33**, 393–403.

Fishburn, P.C. (1980) Lexicographic additive differences. *Journal of Mathematical Psychology*, **21**, 191–218.

Fishburn, P.C. (1991) Nontransitive additive conjoint measurement. *Journal of Mathematical Psychology*, **35**, 1–40.

Luce, R.D. (1978) Lexicographic tradeoff structures. *Theory and Decision*, **9**, 187–193.

Vind, K. (1991) Independent preferences. *Journal of Mathematical Economics*, **20**, 119-135.