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SUMEX UTILITY FUNCTIONS

by

Yutaka Nakamura

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Yutaka Nakamura
Institute of Socio-Economic Planning
University of Tsukuba 1-1-1 Tennoudai, Tsukuba
Ibaraki 305, Japan

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Abstract

This paper develops necessary and sufficient axioms for a decreasingly absolute risk averse von Neumann-Morgenstern utility function on the real line to be representable by a sumex functional form, i.e., sum of finite number of exponential or linear functions. Furthermore, we discuss constructive procedures that identify parameters in that form.

1 Introduction

In most decision analyses, risk properties such as (decreasing) risk aversion together with other relevant characteristics (e.g., continuity, monotonicity, boundedness, and so forth) are identified to restrict the von Neumann-Morgenstern utility function u on the real line, interpreted as wealth, to some class of admissible functional forms. Although such properties provide useful information on the shape of u, they are in most cases not sufficient to obtain a particular class of functional form except the constant risk properties including the constant absolute risk attitude and the constant proportional risk attitude. As is well known, the former characterizes linear and exponential utility functions, and the latter characterizes power and logarithmic utility functions.

To cope with such difficulties, Farquhar and Nakamura [2, 3] proposed a necessary and sufficient condition for u to belong to a class of *polynomial-exponential* (or *polynex*) functions which are representable as sums of products of polynomials and exponential functions, i.e.,

$$u(x) \;=\; \sum_{i=1}^{n} P_i(x)e^{\lambda_i x}, \tag{1}$$

where $P_i, i = 1, \ldots, n$, are polynomials and $\lambda_i, i = 1, \ldots, n$, are distinct parameters. Although their condition are restricted to the case of $n = 2$, they provide a constructive procedure to identify parameters of the polynex functions in that case. Bell [1] independently characterized the general case. However, his condition cannot be applied to distinguish among subclasses of the polynex functions.

Since the polynex functions include a wide class of utility functions exhibiting various attitudes toward risk at distinct wealth levels, it will be significant in both theoretical and practical point of view to identify conditions that distinguish subclasses of the polynex functions. One of the important subclasses consists of sumex utility functions that are polynex functions (1) with $P_i(x) = -a_i$ for nonzero λ_i, and $P_k(x) = a_k x + b$ for $\lambda_k = 0$. When $a_i > 0$ and $\lambda_i \leq 0$ for $i = 1, \ldots, n$, the sumex utility functions exhibit decreasingly absolute risk aversion. As is discussed in [1], there are polynex functions that are not sumex but decreasingly absolute risk averse when $n > 2$.

This paper develops necessary and sufficient axioms for a decreasingly absolute risk averse utility function to be representable by a sumex function. Furthermore, we discuss constructive procedures that identify parameters in that function. The paper is organized as follows. Section 2 introduces axioms and a representation theorem. Then Section 3 provides constructive procedures with the necessity proof of the axioms. Finally, Section 4 gives the sufficiency proof of the theorem.

2 Axioms and a Theorem

Throughout the paper we assume that the outcome set X is the real line, interpreted as wealth. The set of all simple probability measures, or gambles, on X is denoted by $\mathcal{P}$, so that each $p \in \mathcal{P}$ has $\sum_{x \in A} p(x) = 1$ for a finite $A \subset X$. Expected value of a gamble p will be denoted by $E(p)$. For $p, q \in \mathcal{P}$ and $0 \leq \lambda \leq 1$, let $\lambda p + (1-\lambda)q$ take on the value $\lambda p(x) + (1-\lambda)q(x)$ for each $x \in X$. $\mathcal{P}$ is convex in the sense that $\lambda p + (1-\lambda)q \in \mathcal{P}$ whenever $p, q \in \mathcal{P}$ and $0 < \lambda < 1$. Let $[x, \alpha, y]$ denote the binary gamble that yields the outcome x with probability α and the outcome y with probability $1-\alpha$. Furthermore, we shall use compound gambles of the form $[p, \alpha, q]$ for gambles p and q. Each outcome x will be identified with the gamble that yields x with probability 1. The augmented gamble $p + \Delta$ is defined as the gamble resulting from p by augmenting each of its outcomes by an amount Δ.

Let $\succ$ be a binary preference relation on $\mathcal{P}$ with $\succeq$ and $\sim$ defined in the usual way: for $p, q \in \mathcal{P}$, $p \succeq q$ if not $q \succ q$; $p \sim q$ if not $p \succ q$ and not $q \succ p$. We say that $\succ$ is an EU (expected utility) order if it satisfies the expected utility axioms, i.e., asymmetric weak order, independence, and continuity (see Fishburn [4]).

Definition 1 $\succ$ *is a DARA (decreasingly absolute risk averse) order iff it satisfies the following four axioms, which are understood as applying to all* $p \in \mathcal{P}$ *and all* $x, y \in X$.

Axiom 1 $\succ$ *on* $\mathcal{P}$ *is an EU order.*

Axiom 2 *If* $x > y$*, then* $x \succ y$*.*

Axiom 3 $p \sim a$ *for some* $a \in X$*.*

Axiom 4 *If $p \sim x$, then $E(p) + \Delta \succeq p + \Delta \succeq x + \Delta$ for all $\Delta > 0$.*

It easily follows that Axioms 1–4 hold if and only if there is a strictly increasing and concave function u on X such that for all $p, q \in \mathcal{P}$,

$$p \succ q \Longleftrightarrow \sum_{x \in X} p(x)u(x) > \sum_{x \in X} q(x)u(x).$$

Moreover, u is unique up to a positive linear transformation. We shall write $u \approx v$ when u is a positive linear transformation of v.

The *certainty equivalent* of a gamble p is a unique outcome, denoted $c(p)$, such that $p \sim c(p)$. We say that $c(p)$ or $\succ$ is *augmented invariant* if $c(p + \Delta) = c(p) + \Delta$ for all $\Delta > 0$. By $\phi(p)$, we shall denote the risk premium of a gamble p, i.e., $\phi(p) = E(p) - c(p)$. Thus Axiom 4 requires that $\phi(p) \geq \phi(p + \Delta) \geq 0$ for all $\Delta > 0$, so that $\succ$ is said to be decreasingly absolute risk averse.

Let $\succ_a$ be any binary relation on $\mathcal{P}$ with $\sim_a$ and $\succeq_a$ defined in the usual way. We also say that $\succ_a$ is a DARA order if it satisfies Axioms 1–4 in place of $\succ$. Let $c_a(p)$ and $\phi_a(p)$ denote respectively the certainty equivalent and risk premium of a gamble p with respect to (w.r.t.) $\succ_a$. The usual definition of comparative risk aversion says that for two DARA orders $\succ_a$ and $\succ_b$ on $\mathcal{P}$, $\succ_a$ is more risk averse than $\succ_b$ iff $\phi_a(p) \geq \phi_b(p)$ for all $p \in \mathcal{P}$. We shall use a weaker definition of comparative risk aversion as follows.

Definition 2 *For two DARA orders $\succ_a$ and $\succ_b$ on $\mathcal{P}$, $\succ_a$ is said to be asymptotically more risk averse than $\succ_b$ iff, for each pure gamble $p \in \mathcal{P}$, there exist an outcome x and a positive number $\theta > 0$ such that for all $\Delta > \theta$, $p + \Delta \succ_b x + \Delta \succ_a p + \Delta$.*

An outcome x is said to be an *augmented certainty equivalent* of a gamble p w.r.t. $\succ_a$ if $p + \theta \sim_a x + \theta$ for some $\theta > 0$. Let $C_a(p)$ denote the set of all augmented certainty equivalents of p w.r.t. $\succ_a$. Since p is simple, the least upper bound, denoted $c_a^+(p)$, of $C_a(p)$ exists. It is easy to see that $c_a^+(p)$ is augmented invariant. In terms of augmented certainty equivalents, $\succ_a$ is asymptotically more risk averse than $\succ_b$ if and only if $c_a^+(p) < c_b^+(p)$ for all $p \in \mathcal{P}$.

Given $0 \leq b \leq 1$ and $\delta > 0$, we say that the *exchange relation* of gambles p and q w.r.t. $\succ_a$ holds if

$$[p, b, q + \delta] \succ_a [q, b, p + \delta],$$

where b is said to be an *exchange probability.* Throughout the paper we shall fix δ in the exchange relations w.r.t. any binary relations $\succ_a$ on $\mathcal{P}$. The exchange relations will be used to generate new binary relations on $\mathcal{P}$.

Definition 3 *$\succ_b$ is an exchange order induced by $\succ_a$ iff for all $p, q \in \mathcal{P}$,*

$$p \succ_b q \Longleftrightarrow [p, b, q + \delta] \succ_a [q, b, p + \delta].$$

It is easy to see that if $\succ_a$ is an EU order, then the exchange order $\succ_b$ induced by $\succ_a$ is also an EU order. We shall sometimes write expected value $\sum_{x \in X} p(x)u(x)$ of a function u w.r.t. a gamble p by $u(p)$. Let u_a and u_b be utility functions representing $\succ_a$ and $\succ_b$, respectively, i.e., for all $p, q \in \mathcal{P}$, $p \succ_a q$ iff $u_a(p) > u_a(q)$, and $p \succ_b q$ iff $u_b(p) > u_b(q)$. Then it follows from Definition 3 that

$$u_b(x) = bu_a(x) - (1-b)u_a(x+\delta).$$

We say that u_b is an *exchange function* for $\succ_b$.

The following axioms will be used to identify the maximum number of distinct exponential functions in a sumex functional form.

Axiom 5(1) *The certainty equivalents w.r.t. $\succ$ are augmented invariant.*

Axiom 5(n) *$n > 1$ and there are exchange probabilities $\alpha_1, \ldots, \alpha_{n-1}$ such that for $k = 1, \ldots, n-1$, $\succ_{\alpha_k}$ is the exchange DARA order induced by $\succ_{\alpha_{k-1}}$ with $\succ_{\alpha_0} = \succ$, $\succ_{\alpha_k}$ is asymptotically more risk averse than $\succ_{\alpha_{k-1}}$, and $\succ_{\alpha_{n-1}}$ is augmented invariant.*

Discussions of Axiom 5(n) will be deferred to the next section.

We now state a representation theorem as follows.

Theorem 1 *Suppose that Axioms 1-4 hold. Let u be a utility function representing $\succ$. Then Axiom 5(n) holds if and only if there are real numbers $a_1, \ldots, a_n, b_1, \ldots, b_n$ such that $a_k > 0$ for all k, $0 \le b_1 < \ldots < b_n$, and*

$$u(x) \approx -\sum_{k=1}^{n} a_k \xi_k(x), \tag{2}$$

where $\xi_k(x) = \exp(-b_k x)$ for $k = 2, \ldots, n$, and $\xi_1(x) = \exp(-b_1 x)$ when $b_1 > 0$, or $-x$ otherwise.

The necessity proof of Axiom 5(n) will be given in the next section together with constructive procedures for assessing parameters in the representation. The sufficiency of the axiom will be proved in the last section.

3 Constructive Procedures

This section discusses two constructive procedures to assess b-parameters $b_1, \ldots, b_n$ in the sumex function (2), which are directly assessed from preference statements w.r.t. $\succ$ and $\sim$. Assessments of a-parameters $a_1, \ldots, a_n$ may be conducted indirectly by solving utility equality relations that come from a number of indifference statements in the assessments of b-parameters. First we show the necessity of Axiom 5(n), since it may be instructive first to know the relations between b-parameters and the exchange probabilities before assessment discussions.

Suppose that a DARA utility function u is represented by (2). Given $\delta > 0$, let us define the exchange probabilities as follows: for $k = 1, \ldots, n-1$,

$$\alpha_k = \frac{e^{-b_k\delta}}{1 + e^{-b_k\delta}}. \tag{3}$$

Thus $\frac{1}{2} \geq \alpha_1 > \ldots > \alpha_{n-1} > 0$. For $k = 1, \ldots, n-1$, let $\succ_{\alpha_k}$ be the exchange order induced by $\succ_{\alpha_{k-1}}$, where $\succ_{\alpha_0} = \succ$. Then it easily follows that the exchange function u_k for $\succ_{\alpha_k}$ is sumex, i.e.,

$$u_k(x) \approx -\sum_{i=k+1}^{n} c_{ki}\xi_i(x),$$

where $c_{ki} = a_i \Pi_{j=1}^{k}(e^{-b_j\delta} - e^{-b_i\delta}) > 0$. Therefore, each $\succ_{\alpha_k}$ is an DARA order and $\succ_{\alpha_{n-1}}$ is augmented invariant. Moreover, since $0 \leq b_1 < \ldots < b_n$, $\succ_{\alpha_k}$ is asymptotically more risk averse than $\succ_{\alpha_{k-1}}$. Hence Axiom 5(n) follows.

Since $\succ_{\alpha_k}$ is the exchange order induced by $\succ_{\alpha_{k-1}}$, each $\succ_{\alpha_k}$ statements can be translated into $\succ$ statements between $(k+1)$-stage gambles. Therefore, it will be a difficult task to consider multi-stage gambles for the assessment purpose to obtain a general sumex functional form (2). To help the task, a number of necessary conditions will be useful upon which our constructive procedures are based. We shall focus on the case of $n = 2$, since extension to the general case easily follows but becomes more complicated.

To assess b-parameters, we may have two approaches. The first is concerned with direct assessments of the exchange probabilities, so that b-parameters can be calculated by (3). The second uses direct assessments of the augmented certainty equivalents w.r.t. exchange orders. Suppose througout that $\succ$ is a DARA order, i.e., all risk premiums are nonnegative and decreasing when gambles are augmented.

For the first approach, we can proceed as follows. The first step is to identify the exchange probability α_1. Let $\succ_\alpha$ be the exchange order induced by $\succ$ with u_α an associated exchange function, i.e.,

$$\begin{aligned} u_\alpha(x) &= \alpha u(x) - (1-\alpha)u(x+\delta) \\ &= -\sum_{k=1}^{n} a_k a_{1k}\xi_k(x), \end{aligned}$$

where $a_{1k} = \alpha_1 - (1-\alpha_1)\xi_k(\delta)$. If (2) were to hold, then

$$a_{11} \geq 0 \text{ iff } \alpha_1 \geq \frac{e^{-b_1\delta}}{1 + e^{-b_1\delta}}.$$

Thus we must find α that satisfies $a_{11} = 0$. We note that $a_{11} < \ldots < a_{1n}$. If $a_{11} \geq 0$, then $\succ_\alpha$ is an DARA order. If $a_{11} < 0$, then u_α is no longer strictly increasing, so that there is an outcome x such that $x \succ_\alpha x + \delta$, i.e.,

$$[x, \alpha, x+2\delta] \succ [x+\delta, \alpha, x+\delta] = x + \delta.$$

Therefore, α_1 is given by the infimum of the set of exchange probabilities as follows.

$$\alpha_1 = \inf\{\alpha : x + \delta \succ [x, \alpha, x + 2\delta] \text{ for all } x \in X\}.$$

If $\succ_{\alpha_1}$ is not asymptotically more risk averse than $\succ$, then u cannot be sumex. Otherwise, we proceed to assess the exchange probability α_2 in the next step. Let $\succ_\alpha$ be the exchange order induced by $\succ_{\alpha_1}$. Similar arguments apply to identify α_2, i.e., it is the infimum of the set of exchange probabilities given as follows:

$$\inf\{\alpha : x + \delta \succ_{\alpha_1} [x, \alpha, x + 2\delta] \text{ for all } x \in X\} =$$
$$\inf\{\alpha : [x + \delta, \alpha_1 + (1 - \alpha_1)\alpha, x + 3\delta] \succ [x, \alpha_1\alpha, x + 2\delta] \text{ for all } x \in X\}.$$

For the second approach, we shall let all exchange probabilities be $\frac{1}{2}$ and focus on assessments of augmented certainty equivalents. The first step is to identify the set $\mathcal{C}_1(p)$ of all augmented certainty equivalents of a binary gamble $p = [x, \frac{1}{2}, y]$ w.r.t. $\succ$, where $x > y$. Then let $c_1^+(p) = \sup \mathcal{C}_1(p)$. Since $c^+(p)$ is augmented invariant, there is $b > 0$ such that

$$2e^{-bc_1^+(p)} = e^{-bx} + e^{-by}. \tag{4}$$

Therefore, define $b_1 = b$ and the exchange probability α_1 is obtained by (3).

The next step proceeds as follows, where similar arguments apply to identify b_2 and α_2. We note that the set $\mathcal{C}_2(p)$ of all augmented certainty equivalents of p w.r.t. $\succ_{\alpha_1}$ is characterized in terms of $\succ$ as the set of all outcomes x such that for some $\theta > 0$,

$$[p + \theta, \alpha_1, x + \theta + \delta] \sim [x + \theta, \alpha_1, p + \theta + \delta].$$

Then let $c_2^+(p) = \sup \mathcal{C}_2(p)$. If $c_2^+(p) = c_1^+(p)$ for some p, then $\succ_{\alpha_1}$ is not asymptotically more risk averse than $\succ$, so that u cannot be sumex. Otherwise, b_2 is obtained by (4), where $c_1^+(p)$ is replaced by $c_2^+(p)$.

4 Sufficiency Proof

Suppose that Axioms 1–4 hold. Then let u on X be a DARA utility function representing $\succ$. This section proves the sufficiency of Axiom 5(n). Suppose that Axiom 5(n) holds. When $n = 1$, the result is well known. Thus we assume $n > 1$. Then there are exchange probabilities $\alpha_1, \ldots, \alpha_{n-1}$ such that for $k = 1, \ldots, n-1$, $\succ_{\alpha_k}$ is the exchange DARA order induced by $\succ_{\alpha_{k-1}}$ with $\succ_{\alpha_0} = \succ$, $\succ_{\alpha_k}$ is asymptotically more risk averse than $\succ_{\alpha_{k-1}}$, and $\succ_{\alpha_{n-1}}$ is augmented invariant.

For $k = 1, \ldots, n-1$, let u_k be the exchange function for $\succ_{\alpha_k}$, i.e., for all $p, q \in \mathcal{P}$,

$$p \succ_{\alpha_k} q \iff u_k(p) > u_k(q).$$

Since $\succ_{\alpha_{n-1}}$ is augmented invariant and asymptotically more risk averse than $\succ$, there exists a $b_n > 0$ such that

$$u_{n-1}(x) \approx -e^{-b_n x}.$$

We prove the following claim, so that each u_k turns out to be sumex. Hence the conclusion of the theorem obtains.

Claim 1 *Suppose that $\succ_\beta$ is the exchange DARA order induced by a DARA $\succ_\alpha$ and is asymptotically more risk averse than $\succ_\alpha$. Let u_α and u_β be the exchange functions for $\succ_\alpha$ and $\succ_\beta$, respectively. If $d_1 > \ldots > d_m > 0$, $a_{\beta k} > 0$ for $1 \leq k \leq m$, and*

$$u_\beta(x) \approx -\sum_{k=1}^{m} a_{\beta k} e^{-d_k x},$$

then there are $a_{\alpha k} > 0, k = 1, \ldots, m+1$, and d_{m+1} such that $d_m > d_{m+1} \geq 0$ and

$$u_\alpha(x) \approx -\sum_{k=1}^{m+1} a_{\alpha k} e^{-d_k x}.$$

Proof. Suppose that the hypotheses of the claim hold. Let

$$u_\beta(x) = -\sum_{k=1}^{m} a_{\beta k} e^{-d_k x}.$$

Given u_α and some $a_k, k = 1, \ldots, m$, we define real valued function v and w on X by

$$\begin{aligned} v(x) &= -\sum_{k=1}^{m} a_k e^{-d_k x}, \\ u_\alpha(x) &= w(x) + v(x). \end{aligned}$$

Since $\succ_\beta$ is the exchange DARA order induced by $\succ_\alpha$,

$$u_\beta(x) = \beta u_\alpha(x) - (1-\beta) u_\alpha(x+\delta).$$

Thus substituting the expressions of u_α and u_β for this, we obtain

$$\beta w(x) - (1-\beta) w(x+\delta) = -\sum_{k=1}^{m} e^{-d_k x} \left\{ a_{\beta k} - a_k \left(\beta - (1-\beta) e^{-d_k \delta} \right) \right\}.$$

Let d and $a_k, k = 1, \ldots, m$ satisfy $\beta = e^{-d\delta}/(1+e^{-d\delta})$ and $a_{\beta k} = a_k \left(\beta - (1-\beta) e^{-d_k \delta} \right)$ when $\beta \neq (1-\beta) e^{-d_k \delta}$. We have two cases to examine: $d = d_i$ for some $1 \leq i \leq m$; $d \neq d_k$ for all $1 \leq k \leq m$. Suppose first that $d = d_i$ for some i. Then we obtain the following functional equation:

$$\beta w(x) - (1-\beta) w(x+\delta) = -a_{\beta i} e^{-d_i x}.$$

The solution $w(x)$ for $x > 0$ is given by

$$w(x) = -e^{-d_i x} \left(\frac{a_{\beta i} x}{\delta e^{-d_i \delta}} + b \right).$$

Since $u_\alpha(x) = w(x) + v(x)$, it readily follows that u_α is not a DARA order, i.e., it is decreasing for large x. This is a contradiction.

Suppose next that $d \neq d_k$ for all $1 \leq k \leq m$. For $0 \leq \theta < \delta$, let $I_\theta = \{\theta, 2\theta, 3\theta, \ldots\}$. Then we obtain the functional equation $e^{-d\delta}w(x) = w(x+\delta)$, whose solution for $x > 0$ is given as follows. For all $x \in I_\theta$ and all $0 \leq \theta < \delta$, there exist real numbers a_θ such that

$$w(x) = a_\theta e^{-dx}.$$

Since u_α is increasing and $\succ_\beta$ is asymptotically more risk averse than $\succ_\alpha$, we must have that $a_\theta < 0$ and $d_m > d$. We are to show that $w(x) \approx -e^{-dx}$, so that letting $d_{m+1} = d$, the desired result obtains.

For $p \in \mathcal{P}$, let ϕ_p^u, ϕ_p^v, and ϕ_p^w on X be respectively the risk premium functions for u_α, v, and w, i.e., for $\Delta > 0$,

$$\begin{aligned} u_\alpha(p+\Delta) &= u_\alpha(E(p)+\Delta-\phi_p^u(\Delta)), \\ v(p+\Delta) &= v(E(p)+\Delta-\phi_p^v(\Delta)), \\ w(p+\Delta) &= w(E(p)+\Delta-\phi_p^w(\Delta)). \end{aligned}$$

Since u_α and v are DARA, ϕ_p^u and ϕ_p^v are strictly decreasing for all pure gambles p. If $w(p) = w(x)$, then $w(p+\delta) = w(x+\delta)$. In general $w(p+N\delta) = w(x+N\delta)$ for all positive intergers N. Therefore, ϕ_p^w is periodic, i.e., for all $0 \leq \theta < \delta$ and all $x, y \in I_\theta$, $\phi_p^w(x) = \phi_p^w(y)$.

If ϕ_p^w is constant on $[0, \delta)$, then the desired result obtains. Thus we assume that $\phi_p^w(\theta) \neq \phi_p^w(0)$ for some $0 < \theta < \delta$. It follows from the definition that for all $x \in I_\theta$,

$$u_\alpha(E(p)+x-\phi_p^u(x)) = v(E(p)+x-\phi_p^v(x)) + w(E(p)+x-\phi_p^w(x)).$$

Noting that u_α, i.e., $v+w$, is sumex on I_θ and ϕ_p^w on I_θ is constant, it easily follows that for any $y \in I_\theta$,

$$\inf_{x \in I_\theta} \phi_p^u(x) = \phi_p^w(y).$$

Since ϕ_p^w is periodic, this implies that $\phi_p^u(x)$ can not be strictly decreasing for sufficiently large x. This is a contradiction. Hence ϕ_p^w on $[o, \delta)$ is constant.

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