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On Structures of Bisubmodular Polyhedra

by

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Abstract

A bisubmodular polyhedron is defined in terms of a so-called bisubmodular function on a family of ordered pairs of disjoint subsets of a finite set. We examine the structures of bisubmodular polyhedra in terms of signed poset and exchangeability graph. We give characterizations of boundedness and pointedness of bisubmodular polyhedra and also give a characterization of extreme points together with an $O(n^2)$ algorithm for discerning whether a given point is an extreme point, where n is the cardinality of the underlying set. The algorithm also determines the signed poset structure associated with the given point if it is an extreme point. We examine the greedy algorithm over possibly unbounded bisubmodular polyhedra and show an optimality condition in terms of exchangeability graph. We also give characterizations of faces and their dimensions and provide the adjacency relation of extreme points in terms of the Hasse diagrams of the associated signed posets. Moreover, we investigate the connectivity and the decomposition into connected components.

Keywords: bisubmodular polyhedra, bisubmodular functions, greedy algorithm, signed posets.

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1. Introduction

F. D. J. Dunstan and D. J. A. Welsh [10] first considered a general class of polyhedra for which a greedy algorithm works. The class of polyhedra is exactly that of bisubmodular polyhedra determined by bisubmodular functions, which was pointed out by M. Nakamura [15]. R. Chandrasekaran and S. N. Kabadi ([8], [14]) first considered such polyhedra in terms of bisubmodular function, without mentioning the result of [10]. They called such a polyhedron a (poly-)pseudomatroid. The same or similar concepts as pseudomatroid were independently considered by A. Bouchet ([5], [6]), A. Dress and T. Havel ([9]), L. Qi ([16]) and M. Nakamura ([15]).

For a finite set V a bisubmodular function f is a function on a $\sqcup, \sqcap$-closed family $\mathcal{F} \subseteq 3^V$ that satisfies a certain kind of submodularity inequality (precise definitions of the terms not yet defined here will be given in Section 2). A bisubmodular polyhedron is defined by a system of linear inequalities with $\{0, \pm 1\}$-coefficients and the right-hand sides given by a bisubmodular function. The authors have recently found ([1]) that $\sqcup, \sqcap$-closed families can be represented by signed posets introduced by V. Reiner [17]. We shall examine the structures of bisubmodular polyhedra in terms of signed poset and, more generally, the so-called exchangeability graph.

Section 2 gives basic notions and facts on bisubmodular functions, $\sqcup, \sqcap$-closed families, signed posets and exchangeability graphs. In Section 3.1 we show characterizations of pointedness and boundedness of bisubmodular polyhedra. A characterization of extreme points is given in Section 3.2, where an algorithm for discerning whether a given point is an extreme point is also provided. The algorithm determines the signed poset associated with the given point if it is an extreme point. In Section 3.3 we consider linear optimization problems over a bisubmodular polyhedron, characterize optimal solutions in terms of exchangeability graph, and examine the greedy algorithm for general bisubmodular polyhedra. Characterizations of faces and their dimensions are given in Section 3.4. Section 3.5 reveals the adjacency relation of extreme points in terms of the Hasse diagram of a signed poset. Finally, in Section 3.6 we investigate the connectivity and the decomposition of a bisubmodular system into connected components.

It should be noted here that the term, bisubmodular, was first used by A. Schrijver [18] in a way slightly different from the one defined in this paper. Also, bisubmodular functions defined here are called generalized submodular in [8], directed submodular in [16] and, very recently, delta-submodular in [7].

2. Definitions and Preliminaries

In this section we give basic notions and results that are useful for examining the structures of bisubmodular polyhedra.

2.1. Bisubmodular systems and bisubmodular polyhedra

Throughout this paper V is a nonempty finite set and $\mathbf{R}$ is the set of reals. The set of all the mappings from a set A to a set B is denoted by B^A. For any finite set X we denote by $|X|$ the cardinality of X.

Denote by 3^V the set of all the ordered pairs of disjoint subsets of V, i.e., $3^V = \{(X,Y) \mid X,Y \subseteq V, X \cap Y = \emptyset\}$. Each element (X,Y) of 3^V can be identified with the $\{0,\pm 1\}$-vector $\chi_{(X,Y)}$ defined by

$$\chi_{(X,Y)}(v) = \begin{cases} 1 & \text{if } v \in X \\ -1 & \text{if } v \in Y \\ 0 & \text{otherwise} \end{cases} \qquad (v \in V). \tag{2.1}$$

Because of this we call each element of 3^V a *signed set.* We have two binary operations, *reduced union* $\sqcup$ and *intersection* $\sqcap$, on 3^V defined as

$$(X_1,Y_1) \sqcup (X_2,Y_2) = ((X_1 \cup X_2) - (Y_1 \cup Y_2), (Y_1 \cup Y_2) - (X_1 \cup X_2)), \tag{2.2}$$

$$(X_1,Y_1) \sqcap (X_2,Y_2) = (X_1 \cap X_2, Y_1 \cap Y_2) \tag{2.3}$$

for each $(X_1,Y_1),(X_2,Y_2) \in 3^V$. We call $\mathcal{F} \subseteq 3^V$ a $\sqcup,\sqcap$-*closed family* if it is closed under operations $\sqcup$ and $\sqcap$. For any $\mathcal{F} \subseteq 3^V$ we also call $\mathcal{F}$ a *family on* V.

A function $f\colon \mathcal{F} \to \mathbf{R}$ on a $\sqcup,\sqcap$-closed family $\mathcal{F} \subseteq 3^V$ is said to be *bisubmodular* if f satisfies

$$f(X_1,Y_1) + f(X_2,Y_2) \geq f((X_1,Y_1) \sqcup (X_2,Y_2)) + f((X_1,Y_1) \sqcap (X_2,Y_2)) \tag{2.4}$$

for any $(X_1,Y_1),(X_2,Y_2) \in \mathcal{F}$.

For any $(X_i,Y_i) \in 3^V$ $(i = 1,2)$ we write $(X_1,Y_1) \sqsubseteq (X_2,Y_2)$ if $X_1 \subseteq X_2$ and $Y_1 \subseteq Y_2$. The binary relation $\sqsubseteq$ on 3^V is a partial order. We also write $(X_1,Y_1) \sqsubset (X_2,Y_2)$ if $(X_1,Y_1) \sqsubseteq (X_2,Y_2)$ and $(X_1,Y_1) \neq (X_2,Y_2)$. It is shown in [1, Lemma 4.1] that all the maximal elements (X,Y) in a $\sqcup,\sqcap$-closed family $\mathcal{F} \subseteq 3^V$ (maximal with respect to $\sqsubseteq$) give the same set $X \cup Y$. We call such a set $X \cup Y$ the *support* of $\mathcal{F}$ and denote it by $\mathrm{Supp}(\mathcal{F})$. If we have $\mathrm{Supp}(\mathcal{F}) = V$, we say $\mathcal{F}$ is a *spanning* family on V and $\mathcal{F}$ *spans* V. If $|\mathrm{Supp}(\mathcal{F})| = |V| - 1$, we call $\mathcal{F}$ *pre-spanning.*

A $\sqcup,\sqcap$-closed family $\mathcal{F} \subseteq 3^V$ with $(\emptyset,\emptyset) \in \mathcal{F}$ is called *simple* if for each distinct $v,w \in \mathrm{Supp}(\mathcal{F})$ there exists an element $(X,Y) \in \mathcal{F}$ that separates v and w, i.e., $v \in X \cup Y$ and $w \notin X \cup Y$, or $v \notin X \cup Y$ and $w \in X \cup Y$. We call $\mathcal{F}$ with $(\emptyset,\emptyset) \in \mathcal{F}$ *pre-simple* if each distinct two vertices in $\mathrm{Supp}(\mathcal{F})$ except for one fixed pair of vertices are separated by an element of $\mathcal{F}$.

A pair $(\mathcal{F},f)$ of a spanning $\sqcup,\sqcap$-closed family $\mathcal{F} \subseteq 3^V$ on V with $(\emptyset,\emptyset) \in \mathcal{F}$ and a bisubmodular function $f\colon \mathcal{F} \to \mathbf{R}$ with $f(\emptyset,\emptyset) = 0$ is called a *bisubmodular system* on V. The *bisubmodular polyhedron* $\mathrm{P}_*(f)$ associated with the bisubmodular system $(\mathcal{F},f)$ on V is defined by

$$\mathrm{P}_*(f) = \{x \mid x \in \mathbf{R}^V, \forall (X,Y) \in \mathcal{F}\colon x(X,Y) \leq f(X,Y)\}. \tag{2.5}$$

where for any $X \subseteq V$ $x(X) = \sum_{e \in X} x(e)$, $x(\emptyset) = 0$, and for any $(X, Y) \in 3^V$

$$x(X, Y) = x(X) - x(Y). \tag{2.6}$$

It should be noted that we always have $\mathrm{P}_*(f) \neq \emptyset$ (cf. [12, Section 3.5.b]).

For a bisubmodular system $(\mathcal{F}, f)$ on V let (S, T) be any maximal element of $\mathcal{F}$. Note that $S \cup T = V$ since $\mathcal{F}$ spans V. We call such a pair (S, T) an *orthant.* The nonempty face of $\mathrm{P}_*(f)$ given by

$$\mathrm{B}_{(S,T)}(f) = \{x \mid x \in \mathrm{P}_*(f), x(S, T) = f(S, T)\} \tag{2.7}$$

is called the *base polyhedron* of $(\mathcal{F}, f)$ *in the orthant* (S, T). The polyhedron $\mathrm{B}_{(S,T)}(f)$ has the same combinatorial structure as the base polyhedron of a submodular system (see [12, Section 3.5.b]).

Let $\mathcal{F}$ be a $\sqcup, \sqcap$-closed family on V. A sequence of signed sets $(U_i, W_i) \in \mathcal{F}$ $(i = 0, 1, \cdots, k)$ is called a *chain* of $\mathcal{F}$ if it satisfies

$$(U_0, W_0) \sqsubset (U_1, W_1) \sqsubset \cdots \sqsubset (U_k, W_k). \tag{2.8}$$

Here, k is the *length* of the chain. A chain $\mathcal{C}$ of $\mathcal{F}$ is called a *maximal chain* of $\mathcal{F}$ if there is no chain $\mathcal{C}'$ of $\mathcal{F}$ such that $\mathcal{C}'$ contains $\mathcal{C}$ as a proper subchain. Every maximal chain of $\mathcal{F}$ has the same length (also see Theorem 2.5 shown below). If $\mathcal{F}$ is simple and spanning, then the length of any maximal chain of $\mathcal{F}$ is equal to $|V|$.

A function $f : \mathcal{F} \to \mathbf{R}$ on a $\sqcup, \sqcap$-closed family is called *bimodular* if (2.4) holds with equality for each $(X_i, Y_i) \in \mathcal{F}$ $(i = 1, 2)$. Bimodular functions are characterized as follows.

Lemma 2.1: *Suppose that f is a bimodular function on a simple and spanning $\sqcup, \sqcap$-closed family $\mathcal{F}$ on V with $f(\emptyset, \emptyset) = 0$. Then there exists a unique vector $\nu \in \mathbf{R}^V$ such that*

$$f(X, Y) = \sum_{v \in X} \nu(v) - \sum_{v \in Y} \nu(v). \tag{2.9}$$

(Proof) Let $\mathcal{C}$: $(\emptyset, \emptyset) = (S \cap U_0, T \cap U_0) \sqsubset (S \cap U_1, T \cap U_1) \sqsubset \cdots \sqsubset (S \cap U_n, T \cap U_n) = (S, T)$ be any maximal chain of $\mathcal{F}$, where note that $\emptyset = U_0 \subset U_1 \subset \cdots \subset U_n = V$ and $|U_i - U_{i-1}| = 1$ $(i = 1, 2, \cdots, n)$ with $n = |V|$ due to the assumption. Define a vector $\nu \in \mathbf{R}^V$ by

$$\nu(e_i) = \begin{cases} f(S \cap U_i, T \cap U_i) - f(S \cap U_{i-1}, T \cap U_{i-1}) & (\text{if } \{e_i\} = S \cap (U_i - U_{i-1})) \\ f(S \cap U_{i-1}, T \cap U_{i-1}) - f(S \cap U_i, T \cap U_i) & (\text{if } \{e_i\} = T \cap (U_i - U_{i-1})) \end{cases} \tag{2.10}$$

for $i = 1, \cdots, n$. We can easily show that for any $(X,Y) \in \mathcal{F}$ with $(X,Y) \sqsubseteq (S,T)$ equation (2.9) holds (see [12, Lemma 7.5]). Then we also have for any $(X,Y) \in \mathcal{F}$

$$\begin{aligned} f(X,Y) &= -f(S,T) + f(S-Y, T-X) + f(S \cap X, T \cap Y) && (2.11) \\ &= -\sum_{e \in S} \nu(e) + \sum_{e \in T} \nu(e) + \sum_{e \in S-Y} \nu(e) - \sum_{e \in T-X} \nu(e) \\ &\quad + \sum_{e \in S \cap X} \nu(e) - \sum_{e \in T \cap Y} \nu(e) && (2.12) \\ &= \sum_{e \in X} \nu(e) - \sum_{e \in Y} \nu(e) && (2.13) \end{aligned}$$

where (2.11) follows from the bimodularity of f on $\mathcal{F}$ and note that (2.9) holds for $(X,Y) = (S,T), (S-Y, T-X), (S \cap X, T \cap Y) \sqsubseteq (S,T)$. Moreover, since $\mathcal{F}$ is simple and spanning, such a representation of f is unique. □

For a bisubmodular system $(\mathcal{F}, f)$ on V define

$$\mathcal{F}^\diamond = \{(Y,X) \mid (X,Y) \in \mathcal{F}\}, \tag{2.14}$$

$$f^\diamond(Y,X) = f(X,Y) \quad ((X,Y) \in \mathcal{F}). \tag{2.15}$$

We call $(\mathcal{F}^\diamond, f^\diamond)$ the *dual bisubmodular system* of $(\mathcal{F}, f)$ on V. It follows that

$$\mathrm{P}_*(f) = -\mathrm{P}_*(f^\diamond). \tag{2.16}$$

A function $g : \mathcal{F} \to \mathbf{R}$ is called *bisupermodular* if $-g$ is bisubmodular. Also, $(\mathcal{F}, g)$ is called a *bisupermodular system* on V if $(\mathcal{F}, -g)$ is a bisubmodular system on V. The *bisupermodular polyhedron* $\mathrm{P}_*(g)$ associated with a bisupermodular system $(\mathcal{F}, g)$ on V is defined by

$$\mathrm{P}_*(g) = \{x \mid x \in \mathbf{R}^V, \forall (X,Y) \in \mathcal{F} : x(X,Y) \geq g(X,Y)\}. \tag{2.17}$$

For a bisubmodular system $(\mathcal{F}, f)$ we can easily see that

$$\mathrm{P}_*(-f) = -\mathrm{P}_*(f), \quad \mathrm{P}_*(f) = \mathrm{P}_*(-f^\diamond), \quad \mathrm{P}_*(-f) = \mathrm{P}_*(f^\diamond). \tag{2.18}$$

2.2. $\sqcup, \sqcap$-closed families and signed posets

Let $G = (V, A; \partial)$ be a bidirected graph ([11]) with a vertex set V and an arc set A, where ∂ is a mapping from A to a free module with a base V and for each arc a we have one of the following three for some vertices v and w:

(1) $\partial a = v - w$ (arc a has a tail at v and a head at w),

(2) $\partial a = v + w$ (arc a has two tails, one at v and the other at w),

(3) $\partial a = -v - w$ (arc a has two heads, one at v and the other at w).

If $v = w$, we call arc a a *selfloop*. Throughout this paper we do not allow any selfloop a of type (1). Also, we do not allow any parallel arcs (i.e., arcs a and a' with $\partial a = \partial a'$) in bidirected graphs. We call a selfloop of type (2) (or (3)) a *positive* (or *negative*) selfloop. We say an arc a is *incident to* a vertex v (and w) if $\partial a = \pm v \pm w$, and a is *positively* (or *negatively*) *incident to* v if the coefficient of v in ∂a is positive (or negative). Furthermore, two arcs a and a' are said to be *oppositely incident to* a vertex v if one of the two is positively incident to v and the other is negatively incident to v.

A bidirected graph $G = (V, A; \partial)$ is called a *signed poset* if it satisfies the following three (see [17] and [1]).

(i) There are no two arcs $a_1, a_2 \in A$ such that $\partial a_1 = -\partial a_2$.

(ii) For any two arcs $a_1, a_2 \in A$ that are oppositely incident to a common vertex there exists an arc $a_3 \in A$ such that $\partial a_3 = \partial a_1 + \partial a_2$.

(iii) For any two selfloops $a_1, a_2 \in A$ incident to distinct vertices there exists an arc $a_3 \in A$ such that $2\partial a_3 = \partial a_1 + \partial a_2$.

For a signed poset $\mathcal{P} = (V, A; \partial)$ a signed set $(X, Y) \in 3^V$ is called an *ideal* of $\mathcal{P}$ if we have

$$\langle \partial a, (X, Y) \rangle \leq 0 \tag{2.19}$$

for any arc $a \in A$, where $\langle \cdot, \cdot \rangle$ denotes the ordinary (canonical) inner product and ∂a ($a \in A$) and (X, Y) should be regarded as $\{0, \pm 1\}$-vectors in $\{0, \pm 1\}^V$ under natural correspondences. Denote by $\mathcal{I}(\mathcal{P})$ the collection of all the ideals of the signed poset $\mathcal{P}$.

We have the following

Lemma 2.2 ([1]): *For any signed poset $\mathcal{P} = (V, A; \partial)$ the collection $\mathcal{I}(\mathcal{P})$ of all the ideals of $\mathcal{P}$ is a simple and spanning $\sqcup, \sqcap$-closed family on V.* □

Let us regard $\mathcal{I}(\cdot)$ as a mapping from the set of all the signed posets on V to the set of all the simple and spanning $\sqcup, \sqcap$-closed families on V. Then we also have

Theorem 2.3 ([1]): *The mapping $\mathcal{I}(\cdot)$ is a bijection.* □

For a bidirected graph $G = (V, A; \partial)$ an arc a is said to be *redundant* if ∂a can be expressed as a nonnegative linear combination of the other $\partial a'$ ($a' \in A - \{a\}$). For a signed poset $\mathcal{P} = (V, A; \partial)$ there exists a unique maximal subgraph of $\mathcal{P}$ that has no redundant arc. This is called *the Hasse diagram* of $\mathcal{P}$ (see [17]).

Consider any bidirected graph $G = (V, A; \partial)$. The *transitive closure* of G, denoted by $\tilde{G}$, is the bidirected graph constructed from G by repeating the following operations (1) and (2) until no new arc can be generated:

(1) For any two arcs a_1, a_2 in G that are not both selfloops and are oppositely incident to a common vertex, if there is no arc a_3 in G such that $\partial a_3 = \partial a_1 + \partial a_2$, then add such an arc a_3 to G.

(2) For any two selfloops a_1, a_2 in G incident to distinct vertices, if there is no arc a_3 in G such that $2\partial a_3 = \partial a_1 + \partial a_2$, then add such an arc a_3 to G.
The Hasse diagram of a signed poset is a unique minimal bidirected graph whose transitive closure is the given signed poset.

For any bidirected graph $G = (V, A; \partial)$ we call a signed set $(X, Y) \in 3^V$ an *ideal* of G if (2.19) holds for each arc $a \in A$. Denote by $\mathcal{I}(G)$ the set of all the ideals of G.

Lemma 2.4 ([3]): *For any bidirected graph $G = (V, A; \partial)$ $\mathcal{I}(G)$ is a $\sqcup, \sqcap$-closed family of signed sets in 3^V.* □

For any signed set $(X, Y) \in 3^V$ and any subset U of V the *reflection of* (X, Y) by U is the signed set, denoted by $(X, Y) : U$, defined as follows (see [7]).

$$(X, Y) : U = ((X - U) \cup (Y \cap U), (Y - U) \cup (X \cap U)). \qquad (2.20)$$

Moreover, for any family $\mathcal{F} \subseteq 3^V$ and any subset $U \subseteq V$ the *reflection* $\mathcal{F} : U$ of $\mathcal{F}$ by U is the family of the reflections by U of signed sets in $\mathcal{F}$.

The following theorem is also fundamental.

Theorem 2.5 ([1]): *For any $\sqcup, \sqcap$-closed family $\mathcal{F} \subseteq 3^V$ there uniquely exists a partition $\Pi(\mathcal{F})$ of* Supp$(\mathcal{F})$ *such that two elements $v, w \in$* Supp$(\mathcal{F})$ *belong to the same component of $\Pi(\mathcal{F})$ if and only if there is no $(X, Y) \in \mathcal{F}$ that separates v and w. Furthermore, each component $W \in \Pi(\mathcal{F})$ is uniquely decomposed into two parts W_1 and W_2 such that for any $(X, Y) \in \mathcal{F}$ with $(W_1 \cup W_2) \cap (X \cup Y) \neq \emptyset$ we have either $W_1 \subseteq X$ and $W_2 \subseteq Y$, or $W_1 \subseteq Y$ and $W_2 \subseteq X$. Here, either W_1 or W_2 (but not both) may be empty.* □

If either W_1 or W_2 is empty for each $W \in \Pi(\mathcal{F})$ in Theorem 2.5, we call $\mathcal{F}$ *homogeneous.* If $\mathcal{F}$ is not homogeneous, defining $U = \cup\{W_2 \mid W \in \Pi(\mathcal{F})\}$, the reflection $\mathcal{F} : U$ is homogeneous. Then, $\mathcal{F} : U$ naturally defines a simple and spanning $\sqcup, \sqcap$-closed family on $\Pi(\mathcal{F})$ and, hence, is represented by a signed poset due to Theorem 2.3. This is the essential part of the signed Birkhoff Theorem discussed in [1]. For the homogenization of $\mathcal{F}$ we can choose either W_1 or W_2 for each component $W \in \Pi(\mathcal{F})$. The signed poset on $\Pi(\mathcal{F})$ is unique up to homogenization.

For a bisubmodular system $(\mathcal{F}, f)$ on V and a subset U of V we define the *reflection* of $(\mathcal{F}, f)$ by U as the bisubmodular system $(\mathcal{F} : U, f : U)$ on V, where

$$(f : U)((X, Y) : U) = f(X, Y) \qquad ((X, Y) \in \mathcal{F}). \qquad (2.21)$$

We denote the reflection $(\mathcal{F} : U, f : U)$ by $(\mathcal{F}, f) : U$.

For any vector $x \in \mathbf{R}^V$ and any subset U of V define the *reflection* $x : U$ of x by U by

$$(x:U)(v) = \begin{cases} x(v) & \text{if } v \in V - U \\ -x(v) & \text{if } v \in U. \end{cases} \qquad (v \in V). \tag{2.22}$$

Moreover, for any set $Q \subseteq \mathbf{R}^V$ and any subset U of V we define the *reflection* $Q : U$ of Q by U by

$$Q : U = \{x : U \mid x \in Q\}. \tag{2.23}$$

We can easily see that for any $U \subseteq V$

$$\mathrm{P}_*(f) : U = \mathrm{P}_*(f : U) \tag{2.24}$$

(also see [7]).

2.3. Exchangeability graphs and $\sqcup, \sqcap$-closed families

For any $x \in \mathrm{P}_*(f)$ and any $v \in V$, if we have

$$\forall \alpha > 0 \colon x + \alpha\chi_v \notin \mathrm{P}_*(f), \tag{2.25}$$

we say x is *positively saturated at* v, where χ_v is a unit vector in $\{0,1\}^V$ defined by $\chi_v(v) = 1$ and $\chi_v(w) = 0$ for $w \in V - \{v\}$. Similarly, we say x is *negatively saturated at* v if

$$\forall \alpha > 0 \colon x - \alpha\chi_v \notin \mathrm{P}_*(f). \tag{2.26}$$

Denote by $\mathrm{sat}^{(+)}(x)$ (or $\mathrm{sat}^{(-)}(x)$) the set of elements of V at which x is positively (or negatively) saturated. Note that we may have $\mathrm{sat}^{(+)}(x) \cap \mathrm{sat}^{(-)}(x) \neq \emptyset$. We call $\mathrm{sat}^{(+)}$ and $\mathrm{sat}^{(-)}$ the *signed saturation functions* ([13]), which generalize the saturation function for polymatroids and submodular systems (see [12]).

For $x \in \mathrm{P}_*(f)$, define $\mathcal{F}(x) \subseteq \mathcal{F}$ by

$$\mathcal{F}(x) = \{(X,Y) \mid (X,Y) \in \mathcal{F}, x(X,Y) = f(X,Y)\}. \tag{2.27}$$

We can easily show the following (see [14], [7] for the case when $\mathcal{F} = 3^E$).

Lemma 2.6: *$\mathcal{F}(x)$ is closed with respect to $\sqcup$ and $\sqcap$.* □

Note that we have $v \in \mathrm{sat}^{(+)}(x)$ (or $v \in \mathrm{sat}^{(-)}(x)$) if and only if there exists some $(X,Y) \in \mathcal{F}(x)$ such that $v \in X$ (or $v \in Y$). Therefore, for any $v \in \mathrm{sat}^{(+)}(x)$ define

$$\mathrm{dep}(x, +v) = \sqcap\{(X,Y) \mid v \in X, (X,Y) \in \mathcal{F}(x)\}, \tag{2.28}$$

and for any $v \in \mathrm{sat}^{(-)}(x)$ define

$$\mathrm{dep}(x, -v) = \sqcap\{(X,Y) \mid v \in Y, (X,Y) \in \mathcal{F}(x)\}. \tag{2.29}$$

We call dep the *signed dependence function* ([13]), which generalizes the dependence function for polymatroids and submodular systems (see [12]).

For convenience, we also define $\mathrm{dep}(x, +v) = (\emptyset, \emptyset)$ for $v \in V - \mathrm{sat}^{(+)}(x)$ and $\mathrm{dep}(x, -v) = (\emptyset, \emptyset)$ for $v \in V - \mathrm{sat}^{(-)}(x)$.

For any signed set $Z = (X, Y) \in 3^V$ we define

$$Z^+ = X, \quad Z^- = Y. \tag{2.30}$$

We call $Z^+(= X)$ $(Z^-(= Y))$ the *positive* (*negative*) *part* of the signed set $Z = (X, Y)$. Note that $\mathrm{dep}(x, +v)$ $(\mathrm{dep}(x, -v))$ is a unique minimal element of $\mathcal{F}(x)$ whose positive (negative) part contains v.

We can easily see that for any $v \in \mathrm{sat}^{(+)}(x)$,

$$\mathrm{dep}(x, +v)^+ = \{w \mid w \in V, \exists \alpha > 0 : x + \alpha(\chi_v - \chi_w) \in \mathrm{P}_*(f)\}, \tag{2.31}$$

$$\mathrm{dep}(x, +v)^- = \{w \mid w \in V, \exists \alpha > 0 : x + \alpha(\chi_v + \chi_w) \in \mathrm{P}_*(f)\}. \tag{2.32}$$

Similarly, for any $e \in \mathrm{sat}^{(-)}(x)$,

$$\mathrm{dep}(x, -v)^+ = \{w \mid w \in V, \exists \alpha > 0 : x + \alpha(-\chi_v - \chi_w) \in \mathrm{P}_*(f)\}, \tag{2.33}$$

$$\mathrm{dep}(x, -v)^- = \{w \mid w \in V, \exists \alpha > 0 : x + \alpha(-\chi_v + \chi_w) \in \mathrm{P}_*(f)\}. \tag{2.34}$$

Define a bidirected graph $G(\mathcal{F}(x)) = (V, A(x); \partial)$ associated with $\mathcal{F}(x)$ as follows.

(1) For each $v \in V$,

(1a) there is a selfloop a at v with $\partial a = 2v$ if and only if $v \in V - \mathrm{sat}^{(+)}(x)$,

(1b) there is a selfloop a at v with $\partial a = -2v$ if and only if $v \in V - \mathrm{sat}^{(-)}(x)$.

(2) For each distinct $v, w \in V$,

(2a) there is an arc a with $\partial a = v - w$ if and only if $w \in \mathrm{dep}(x, +v)^+$ or $v \in \mathrm{dep}(x, -w)^-$,

(2b) there is an arc a with $\partial a = v + w$ if and only if $w \in \mathrm{dep}(x, +v)^-$ or $v \in \mathrm{dep}(x, +w)^-$,

(2c) there is an arc a with $\partial a = -v - w$ if and only if $w \in \mathrm{dep}(x, -v)^+$ or $v \in \mathrm{dep}(x, -w)^+$.

We call $G(\mathcal{F}(x))$ *the exchangeability (bidirected) graph* associated with $x \in \mathrm{P}_*(f)$. The collection of all the ideals of the exchangeability graph $G(\mathcal{F}(x))$ is $\mathcal{F}(x)$ as shown below.

Lemma 2.7: *Let $(X, Y) \in 3^V$ be an ideal of the exchangeability graph $G(\mathcal{F}(x)) = (V, A(x); \partial)$. Then we have for each $v \in X$*

$$v \in \mathrm{sat}^{(+)}(x), \quad \mathrm{dep}(x, +v) \sqsubseteq (X, Y) \tag{2.35}$$

and for each $v \in Y$

$$v \in \mathrm{sat}^{(-)}(x), \quad \mathrm{dep}(x, -v) \sqsubseteq (X, Y). \tag{2.36}$$

(Proof) This lemma easily follows from the definitions of ideal and exchangeability graph $G(\mathcal{F}(x))$ □

Theorem 2.8: *The set of all the ideals of $G(\mathcal{F}(x)) = (V, A(x); \partial)$ coincides with $\mathcal{F}(x)$.*

(Proof) Suppose that $(X, Y) \in 3^V$ is an ideal of $G(\mathcal{F}(x)) = (V, A(x); \partial)$. Then we have from Lemma 2.7

$$(X, Y) = (\sqcup_{v \in X} \mathrm{dep}(x, +v)) \sqcup (\sqcup_{v \in Y} \mathrm{dep}(x, -v)). \tag{2.37}$$

It follows from Lemma 2.6 that $(X, Y) \in \mathcal{F}(x)$.

Conversely, suppose $(X, Y) \in \mathcal{F}(x)$. Then by the definition of signed saturation function we have

$$X \subseteq \mathrm{sat}^{(+)}(x), \tag{2.38}$$

$$Y \subseteq \mathrm{sat}^{(-)}(x). \tag{2.39}$$

Also, by the definition of signed dependence function we have

$$\mathrm{dep}(x, +v) \sqsubseteq (X, Y) \quad (v \in X), \tag{2.40}$$

$$\mathrm{dep}(x, -v) \sqsubseteq (X, Y) \quad (v \in Y). \tag{2.41}$$

It follows from (2.38)~(2.41) that (X, Y) is an ideal of $G(\mathcal{F}(x))$. □

3. Structures of Bisubmodular Polyhedra

We examine the structures of bisubmodular polyhedra by means of exchangeability graphs, signed posets etc. prepared in the previous section.

3.1. Pointedness and boundedness

The *linearity space* of $\mathrm{P}_*(f)$ is the solution set of the following system of linear equations:

$$x(X, Y) = 0 \quad ((X, Y) \in \mathcal{F}). \tag{3.1}$$

The bisubmodular polyhedron $\mathrm{P}_*(f)$ is *pointed* if and only if (3.1) has a unique solution $x = \mathbf{0}$. Recall that $\mathcal{F}$ is spanning due to the definition of bisubmodular system $(\mathcal{F}, f)$.

Theorem 3.1: *A bisubmodular polyhedron* $\mathrm{P}_*(f)$ *is pointed if and only if* $\mathcal{F}$ *is simple (and spanning).*
(Proof) Note that $\mathcal{F}$ is spanning by definition. We can easily see that (3.1) is equivalent to

$$x(W_1) - x(W_2) = 0 \quad (W \in \Pi(\mathcal{F})), \tag{3.2}$$

where $\Pi(\mathcal{F})$ is the partition of $\mathrm{Supp}(\mathcal{F})$ and $\{W_1, W_2\}$ is the bipartition of $W \in \Pi(\mathcal{F})$ that appeared in Theorem 2.5. $\mathrm{P}_*(f)$ is pointed if and only if (3.2) has a unique solution $x = \mathbf{0}$. This is the case if and only if $\mathrm{Supp}(\mathcal{F}) = V$ and $|W| = 1$ for each $W \in \Pi(\mathcal{F})$, i.e., $\mathcal{F}$ is spanning and simple. □

It should be noted that (3.2) gives an efficient representation of the linearity space of $\mathrm{P}_*(f)$ and that the dimension of the linearity space is equal to $|V| - |\Pi(\mathcal{F})|$.

Now, suppose that the underlying family $\mathcal{F}$ is simple and spanning. Therefore, $\mathcal{F}$ can be represented by a signed poset $\mathcal{P}(\mathcal{F}) = (V, A; \partial)$ uniquely defined from $\mathcal{F}$ and $\mathcal{F}$ is the collection of all the ideals of $\mathcal{P}(\mathcal{F})$ (see Theorem 2.3). The *characteristic cone* $\mathrm{C}(\mathcal{F})$ of the bisubmodular polyhedron $\mathrm{P}_*(f)$ is given by

$$\mathrm{C}(\mathcal{F}) = \{x \mid x \in \mathbf{R}^V, \forall (X, Y) \in \mathcal{F}\colon x(X, Y) \leq 0\}. \tag{3.3}$$

$\mathrm{P}_*(f)$ is *bounded* if and only if the system of inequalities appearing in the right-hand side of (3.3) has a unique solution $x = \mathbf{0}$. If the arc set A of the signed poset $\mathcal{P}(\mathcal{F}) = (V, A; \partial)$ is nonempty, then for any arc $a \in A$ the vector $x = \partial a$ (considered as a vector in $\mathbf{R}^V$) satisfies the inequalities in (3.3). Hence, $\mathrm{P}_*(f)$ is not bounded. On the other hand, if $A = \emptyset$, then we have $\mathcal{F} = 3^V$, so that $\mathrm{P}_*(f)$ is bounded.

Consequently, we have

Theorem 3.2: *A bisubmodular polyhedron* $\mathrm{P}_*(f)$ *is bounded if and only if we have* $\mathcal{F} = 3^V$. □

Consider a capacity function c on the arc set A of the poset $\mathcal{P}(\mathcal{F})$ such that $c(a) = +\infty$ for all $a \in A$. A feasible flow φ in the (bidirected) network $\mathcal{N} = (\mathcal{P}(\mathcal{F}), c)$ is a function $\varphi : A \to \mathbf{R}$ such that $0 \leq \varphi(a) \leq c(a)$ $(a \in A)$. The *boundary* $\partial\varphi$ of a feasible flow φ in $\mathcal{N}$ is the vector in $\mathbf{R}^V$ defined by

$$\partial\varphi = \sum\{\varphi(a)\partial a \mid a \in A\}, \tag{3.4}$$

where ∂a $(a \in A)$ are regarded as vectors in $\mathbf{R}^V$. From a result in [2] we can show that the set of the boundaries of all the feasible flows in $\mathcal{N}$ is exactly the characteristic cone $\mathrm{C}(\mathcal{F})$ in (3.3).

From this we have

Theorem 3.3: *Suppose that $\mathcal{F}$ is simple and spanning and let $\mathcal{P}(\mathcal{F}) = (V, A; \partial)$ be the signed poset on V representing $\mathcal{F}$. Then, the characteristic cone $\mathrm{C}(\mathcal{F})$ of $\mathrm{P}_*(f)$ is generated by $\{\partial a \mid a \in A\}$. Moreover, extreme rays of $\mathrm{C}(\mathcal{F})$ are exactly given by ∂a for arcs a of the Hasse diagram of $\mathcal{P}(\mathcal{F})$.* □

The latter part of Theorem 3.3 directly follows from the definition of Hasse diagram.

3.2. Extreme points and signed posets

Associated with a point $x \in \mathrm{P}_*(f)$ we have a $\sqcup, \sqcap$-closed family $\mathcal{F}(x)$. Note that $x \in \mathrm{P}_*(f)$ is an extreme point of $\mathrm{P}_*(f)$ only if $\mathcal{F}(x)$ spans V. Also, if $\mathcal{F}(x)$ spans V, f restricted to $\mathcal{F}(x)$ defines a bisubmodular system on V and x is the only one possible extreme point, if any, of the associated bisubmodular polyhedron. Therefore, we have from Theorem 3.1

Theorem 3.4: *A point $x \in \mathrm{P}_*(f)$ is an extreme point of $\mathrm{P}_*(f)$ if and only if $\mathcal{F}(x)$ is simple and spanning.* □

It follows from Theorem 3.4 that for each extreme point $x \in \mathrm{P}_*(f)$ we have a signed poset $\mathcal{P}(\mathcal{F}(x))$ representing the simple and spanning $\mathcal{F}(x)$. It follows from Theorem 2.8 that for each extreme point the signed poset $\mathcal{P}(\mathcal{F}(x))$ is essentially the same as the exchangeability graph $G(\mathcal{F}(x))$; they have the same set of ideals. In terms of these signed posets (or exchangeability graphs) we can characterize the adjacency of extreme points (see Section 3.5).

The signed poset $\mathcal{P}(\mathcal{F}(x))$ can be constructed if we are given $\mathrm{dep}(x, \pm v)$ ($v \in V$). We can obtain these $\mathrm{dep}(x, \pm v)$ ($v \in V$) by adapting an algorithm of Bixby, Cunningham and Topkis [4] for polymatroids to bisubmodular systems (also see [12, p. 62]). The following algorithm consists of two parts, Algorithm I and Algorithm II. Algorithm I discerns whether a given $x \in \mathbf{R}^V$ is an extreme point of $\mathrm{P}_*(f)$ and, if x is an extreme point, Algorithm II finds all $\mathrm{dep}(x, \pm v)$ ($v \in V$), using the output of Algorithm I. It should be noted that we do not need Algorithm II for polymatroids or submodular systems.

Suppose we are given a vector $x \in \mathbf{R}^V$.

Algorithm I
Step 1: Put $S \leftarrow (\emptyset, \emptyset)$.
Step 2: For each $i = 1, 2, \cdots, |V|$ do the following (2-1) and (2-2).
(2-1) If there exists no element $v \in V - (S^+ \cup S^-)$ such that
(i) $(S^+ \cup \{v\}, S^-) \in \mathcal{F}(x)$ or
(ii) $(S^+, S^- \cup \{v\}) \in \mathcal{F}(x)$,
then stop (x is not an extreme point of $\mathrm{P}_*(f)$).

Otherwise let v be an element of $V - (S^+ \cup S^-)$ that satisfies (i) or (ii) and put $v_i \leftarrow v$.
If (i) is satisfied by $v = v_i$, then put $S \leftarrow (S^+ \cup \{v_i\}, S^-)$ and $\sigma(v_i) \leftarrow +1$.
Otherwise put $S \leftarrow (S^+, S^- \cup \{v_i\})$ and $\sigma(v_i) \leftarrow -1$.
(2-2) Put $T \leftarrow S$ and for each $j = 1, 2, \cdots, i-1$ do the following $(*)$:
$(*)$ If $(T^+ - \{v_{i-j}\}, T^-) \in \mathcal{F}(x)$ when $\sigma(v_{i-j}) = +1$,
or if $(T^+, T^- - \{v_{i-j}\}) \in \mathcal{F}(x)$ when $\sigma(v_{i-j}) = -1$,
then put $T \leftarrow (T^+ - \{v_{i-j}\}, T^-)$ or $T \leftarrow (T^+, T^- - \{v_{i-j}\})$ according as $\sigma(v_{i-j}) = +1$ or $\sigma(v_{i-j}) = -1$.
If $\sigma(v_i) = +1$, then put $\mathrm{dep}(x, +v_i) \leftarrow T$.
Otherwise put $\mathrm{dep}(x, -v_i) \leftarrow T$.
(End)

When Algorithm I terminates with an orthant $S = (S^+, S^-)$, we move to Algorithm II given as follows. Here, we assume $S^+ = \emptyset$ to simplify the description of the algorithm. If necessary, consider the reflections $(\mathcal{F}, f) : S^+$ and $x : S^+$ of the inputs of Algorithm I and the reflections $\mathrm{dep}(x, \pm v_i) : S^+$ $(i = 1, 2, \cdots, |V|)$ of the outputs.

Now, we have the orthant $(\emptyset, V) \in \mathcal{F}(x)$ and $\mathrm{dep}(x, -v_i)$ $(i = 1, 2, \cdots, |V|)$. Define a binary relation $\preceq$ on V by $u \preceq v$ if and only if $v \in \mathrm{dep}(x, -u)^-$. We can easily see that the binary relation $\preceq$ is a partial order and it gives an ordinary poset $\mathcal{P}_0 = (V, \preceq)$. For each $v \in V$ denote by $D(v)$ the principal lower (order) ideal of v in $\mathcal{P}_0$, i.e.,

$$D(v) = \{u \mid u \in V, u \preceq v\}. \tag{3.5}$$

We can obtain all $D(v)$ $(v \in V)$ in $\mathrm{O}(|V|^2)$ time, which are used in Algorithm II.

Algorithm II
Step 1: Put $W \leftarrow V$.
Step 2: For each $v \in V$, if $(D(v), \emptyset) \in \mathcal{F}(x)$, then put $\mathrm{dep}(x, +v) \leftarrow (D(v), \emptyset)$ and $W \leftarrow W - \{v\}$.
Step 3: For each $v \in W$, if $(D(v), W - D(v)) \notin \mathcal{F}(x)$, then put $\mathrm{dep}(x, +v) \leftarrow (\emptyset, \emptyset)$, otherwise do the following $(**)$.
$(**)$ Let $u_1, u_2, \cdots, u_k$ be the elements of $W - D(v)$ arranged in the topological order in $\mathcal{P}_0$ restricted on $W - D(v)$ (i.e., $u_i \prec u_j$ implies $i < j$).
Put $T \leftarrow (D(v), W - D(v))$.
For each $i = 1, 2, \cdots, k$, if $(T^+, T^- - \{u_i\}) \in \mathcal{F}(x)$, then put $T \leftarrow (T^+, T^- - \{u_i\})$.
Put $\mathrm{dep}(x, +v) \leftarrow T$.
(End)

The validity of Algorithm I can be shown as follows. If x is an extreme point of $\mathrm{P}_*(f)$, then $\mathcal{F}(x)$ is simple and spanning due to Theorem 3.4 and hence any maximal chain

$$\mathcal{C} : (\emptyset, \emptyset) = (S_0^+, S_0^-) \sqsubset (S_1^+, S_1^-) \sqsubset \cdots \sqsubset (S_n^+, S_n^-) \tag{3.6}$$

of signed sets $(S_i^+, S_i^-) \in \mathcal{F}(x)$ $(i = 0, 1, \cdots, n)$ satisfies $|S_i^+ \cup S_i^-| = i$ $(i = 0, 1, \cdots, n)$ with $n = |V|$. Therefore, if x is an extreme point of $\mathrm{P}_*(f)$, a maximal chain of $\mathcal{F}(x)$ of length n is found by Step (2-1). Moreover, if Step (2-1) finds a maximal chain of $\mathcal{F}(x)$ of length n, then $\mathcal{F}(x)$ is simple and spanning, so that x is an extreme point of $\mathrm{P}_*(f)$. This proves the validity of Step (2-1).

Now, let us consider Step (2-2). At an execution of Step (2-2) when (i) of Step (2-1) is satisfied by $v = v_i$, we have $\mathrm{dep}(x, +v_i) \neq (\emptyset, \emptyset)$ and from Step (2-1) we have obtained a chain

$$(\emptyset, \emptyset) = (S_0^+, S_0^-) \sqsubset (S_1^+, S_1^-) \sqsubset \cdots \sqsubset (S_i^+, S_i^-). \tag{3.7}$$

Because of the definition of $\mathrm{dep}(x, +v_i)$ we have

$$\mathrm{dep}(x, +v_i) \sqsubseteq (S_i^+, S_i^-). \tag{3.8}$$

It follows from (3.7) and (3.8) that the distinct members of $\mathrm{dep}(x, +v_i) \sqcup (S_j^+, S_j^-)$ $(j = 0, 1, \cdots, i)$ form a maximal chain from $\mathrm{dep}(x, +v_i)$ to (S_i^+, S_i^-). Therefore, removing possible elements from (S_i^+, S_i^-) one by one, we reach $\mathrm{dep}(x, +v_i)$. This validates Step (2-2). The validity of Step (2-2) when (ii) is satisfied can be shown similarly.

We show the validity of Algorithm II. We assume that the orthant $(\emptyset, V)$ is obtained by Algorithm I. Under this assumption the signed poset $\mathcal{P}(\mathcal{F}(x))$ does not contain any arcs a of type $\partial a = -v - w$. Hence, in particular, $\mathcal{P}(\mathcal{F}(x))$ does not contain any negative selfloops. Therefore, if $w \in \mathrm{dep}(x, +v)^+$, we have $v \in \mathrm{dep}(x, -w)^-$ (see [1, Lemma 3.3]). This means that Algorithm I has already obtained all the arcs a of type $\partial a = v - w$ for $v, w \in V$ in $\mathcal{P}(\mathcal{F}(x))$ by means of $\mathrm{dep}(x, -v)$ $(v \in V)$. The subgraph of $\mathcal{P}(\mathcal{F}(x))$ induced by the set of arcs a of type $\partial a = v - w$ $(v, w \in V)$ is isomorphicly expressed by the poset $\mathcal{P}_0 = (V, \preceq)$ defined above. For each $v \in V$, if there is not a positive selfloop incident to v in $\mathcal{P}(\mathcal{F}(x))$, we have $(D(v), V - D(v)) \in \mathcal{F}(x)$ since there is no arc a such that $\partial a = u + w$ for $u, w \in D(v)$ (by the assumption) or $\partial a = u - w$ for $u \in D(v)$ and $w \in V - D(v)$ (by the definition of $D(v)$). Also, we can easily see that there is no $(X, Y) \in \mathcal{F}(x)$ such that $v \in X \subset D(v)$ (strict inclusion). Therefore, we have only to delete from $(D(v), V - D(v))$ as many elements in $V - D(v)$ as possible to obtain $\mathrm{dep}(x, +v)$. If there is no arc a of type $\partial a = u + w$ in $\mathcal{P}(\mathcal{F}(x))$ that is incident to any vertex of $D(v)$, we have $(D(v), \emptyset) \in \mathcal{F}(x)$ and hence $\mathrm{dep}(x, +v) = (D(v), \emptyset)$. This case is treated by Step 2 of Algorithm II. Note also that if $(D(v), \emptyset) \notin \mathcal{F}(x)$, then there is an arc a in $\mathcal{P}(\mathcal{F}(x))$ such that $\partial a = u + w$ and u or w belongs to $D(v)$. The set $W \subseteq V$ obtained after Step 2 is the set of the vertices to which some arc a in $\mathcal{P}(\mathcal{F}(x))$ of type $\partial a = u + w$ is incident. It follows that for each $v \in W$, if there is no positive selfloop incident to v, then we have $(D(v), W - D(v)) \in \mathcal{F}(x)$. We see that any maximal chain

$$\mathrm{dep}(x, +v)^- = T_0^- \subset T_1^- \subset \cdots \subset T_l^- = W - D(v) \tag{3.9}$$

of upper (order) ideals of $\mathcal{P}_0 = (V, \preceq)$ from $\mathrm{dep}(x, +v)^-$ to $W - D(v)$ gives a maximal chain

$$\mathrm{dep}(x, +v) = (T_0^+, T_0^-) \sqsubset (T_0^+, T_1^-) \cdots \sqsubset (T_0^+, T_l^-) = (D(v), W - D(v)) \qquad (3.10)$$

of ideals of $\mathcal{P}(\mathcal{F}(x))$ from $\mathrm{dep}(x, +v)$ to $(D(v), W - D(v))$. Also, note that a maximal chain in (3.9) is formed by different T^-'s appearing in (**) of Step 3 since we have $(T^+, T^- - \{u_i\}) \in \mathcal{F}(x)$ (at the iteration for i) if and only if $u_i \notin \mathrm{dep}(x, +v)^-$, due to the ordering of u_i $(i = 1, 2, \cdots, k)$. Therefore, $\mathrm{dep}(x, +v)$ for each $v \in W$ is obtained by Step 3. It should be noted that a topological ordering of elements of $W - D(v)$ required in (**) of Step 3 can be obtained in $\mathrm{O}(|V|)$ time for each $v \in W$ if we have once obtained a topological ordering of W, which requires $\mathrm{O}(|V|^2)$ time.

The total running time of Algorithms I and II is $\mathrm{O}(|V|^2)$ if we assume an oracle for function evaluations for f, while the algorithm of Bixby, Cunningham and Topkis [4] also requires $\mathrm{O}(|V|^2)$ time for polymatroids.

For any $x \in \mathrm{P}_*(f)$ the *tangent cone* $\mathrm{TC}(f, x)$ of $\mathrm{P}_*(f)$ at x is given by

$$\mathrm{TC}(f, x) = \{x \mid x \in \mathbf{R}^V, \forall (X, Y) \in \mathcal{F}(x) : x(X, Y) \leq 0\}. \qquad (3.11)$$

Therefore, from Theorems 3.3 and 3.4 we have the following

Theorem 3.5: *For any extreme point $x \in \mathrm{P}_*(f)$ the extreme rays of the tangent cone $\mathrm{TC}(f, x)$ are exactly given by the boundaries of arcs of the Hasse diagram of the signed poset $\mathcal{P}(\mathcal{F}(x))$ representing $\mathcal{F}(x)$.* □

By an argument similar to the one around (3.4) we can show

Corollary 3.6: *For any point $x \in \mathrm{P}_*(f)$ the tangent cone $\mathrm{TC}(f, x)$ is generated by the boundaries of arcs of the exchangeability graph $G(\mathcal{F}(x))$.* □

3.3. Linear optimization and the greedy algorithm

For a bisubmodular system $(\mathcal{F}, f)$ on V let us consider the following linear optimization problem:

$$\begin{aligned} (P_w) \qquad & \text{Maximize } \sum_{v \in V} w(v) x(v) \\ & \text{subject to } x \in \mathrm{P}_*(f), \end{aligned} \qquad (3.12)$$

where $w \in \mathbf{R}^V$ is a weight vector. A greedy algorithm is given in [8], [14], [15] and [10] for the case $\mathcal{F} = 3^V$. We give a characterization of optimal solutions of this problem and examine the greedy algorithm in terms of exchangeability graph, which is applicable to the case when $\mathcal{F} \neq 3^V$ as well. For simplicity we assume that $\mathcal{F}$ is simple. Recall that $\mathcal{F}$ is spanning by the definition of $(\mathcal{F}, f)$.

For the signed poset $\mathcal{P}(\mathcal{F}) = (V, A; \partial)$ corresponding to $\mathcal{F}$ define

$$\mathrm{C}^*(\mathcal{F}) = \{z \mid z \in \mathbf{R}^V, \forall a \in A : \langle \partial a, z \rangle \leq 0\}. \tag{3.13}$$

We see from (3.3) and Theorem 3.3 that $\mathrm{C}^*(\mathcal{F})$ is the dual cone of the characteristic cone $\mathrm{C}(\mathcal{F})$. Therefore, we have the following

Theorem 3.7: *Problem* (P_w) *has a finite optimal solution if and only if* $w \in \mathrm{C}^*(\mathcal{F})$. □

Note that we can easily check whether $w \in \mathrm{C}^*(\mathcal{F})$ holds, using the signed poset $\mathcal{P}(\mathcal{F})$. It should also be noted that the cone $\mathrm{C}^*(\mathcal{F})$ is generated by $\chi_{(X,Y)}$ $((X,Y) \in \mathcal{F})$ (cf. [17, Proposition 3.1]).

For any $x \in \mathrm{P}_*(f)$ we have the exchangeability graph $G(\mathcal{F}(x)) = (V, A(x); \partial)$ associated with x. Define a cone

$$\mathrm{C}^*(\mathcal{F}(x)) = \{z \mid z \in \mathbf{R}^V, \forall a \in A(x) : \langle \partial a, z \rangle \leq 0\}. \tag{3.14}$$

It follows from Corollary 3.6 that $\mathrm{C}^*(\mathcal{F}(x))$ is the dual cone of the tangent cone $\mathrm{TC}(f, x)$ at x of $\mathrm{P}_*(f)$. The cone $\mathrm{C}^*(\mathcal{F}(x))$ is generated by $\chi_{(X,Y)}$ $((X,Y) \in \mathcal{F}(x))$.

Optimal solutions of Problem (P_w) are characterized by the following theorem.

Theorem 3.8: *A vector* $x \in \mathrm{P}_*(f)$ *is an optimal solution of Problem* (P_w) *if and only if* $w \in \mathrm{C}^*(\mathcal{F}(x))$.
(Proof) The "only if" part follows from the definitions of exchangeability graph and of the cone $\mathrm{C}^*(\mathcal{F}(x))$. The "if" part also easily follows from the fact that $\mathrm{C}^*(\mathcal{F}(x))$ is the dual cone of the tangent cone $\mathrm{TC}(f, x)$ at x of $\mathrm{P}_*(f)$. □

Suppose that $w \in \mathrm{C}^*(\mathcal{F})$. Let the distinct positive values of $|w(v)|$ $(v \in V)$ be given by

$$w_1 > w_2 > \cdots > w_p (> 0). \tag{3.15}$$

Define

$$U_i = \{v \mid v \in V, w(v) \geq w_i\} \quad (i = 1, 2, \cdots, p), \tag{3.16}$$

$$W_i = \{v \mid v \in V, -w(v) \geq w_i\} \quad (i = 1, 2, \cdots, p). \tag{3.17}$$

Since $w \in \mathrm{C}^*(\mathcal{F})$, for each $i = 1, 2, \cdots, p$ we have $(U_i, W_i) \in \mathcal{F}$. Moreover, since

$$(U_1, W_1) \sqsubset (U_2, W_2) \sqsubset \cdots \sqsubset (U_p, W_p) \tag{3.18}$$

is a chain of $\mathcal{F}$, it can be extended to a maximal chain of $\mathcal{F}$ as

$$(\emptyset, \emptyset) = (\hat{U}_1, \hat{W}_1) \sqsubset (\hat{U}_2, \hat{W}_2) \sqsubset \cdots \sqsubset (\hat{U}_n, \hat{W}_n), \tag{3.19}$$

where the length n of the chain is equal to $|V|$ since $\mathcal{F}$ is simple and spanning. Suppose that

$$\{v_i\} = (\hat{U}_i \cup \hat{W}_i) - (\hat{U}_{i-1} \cup \hat{W}_{i-1}) \quad (i = 1, 2, \cdots, n). \tag{3.20}$$

Then define

$$x(v_i) = \begin{cases} f(\hat{U}_i, \hat{W}_i) - f(\hat{U}_{i-1}, \hat{W}_{i-1}) & \text{if } \{v_i\} = \hat{U}_i - \hat{U}_{i-1} \\ f(\hat{U}_{i-1}, \hat{W}_{i-1}) - f(\hat{U}_i, \hat{W}_i) & \text{if } \{v_i\} = \hat{W}_i - \hat{W}_{i-1} \end{cases} \tag{3.21}$$

for each $i = 1, 2, \cdots, n$. We have

$$x(\hat{U}_i, \hat{W}_i) = f(\hat{U}_i, \hat{W}_i) \quad (i = 1, 2, \cdots, n). \tag{3.22}$$

We can easily see that $x \in \mathrm{P}_*(f)$ and $w \in \mathrm{C}^*(\mathcal{F}(x))$. Therefore, the vector x given by (3.21) is an optimal solution of Problem (P_w) due to Theorem 3.8. This gives the *greedy algorithm* for Problem (P_w), which generalizes the one when $\mathcal{F} = 3^V$.

3.4. Faces

We adapt the general technique developed in [12, Section 3.3.d] to bisubmodular polyhedra.

For any $\mathcal{G} \subseteq \mathcal{F}$ define

$$\begin{aligned} \mathrm{F}(\mathcal{G}) \;=\; \{x \mid x \in \mathbf{R}^E, \; & \forall (X,Y) \in \mathcal{G} \colon x(X,Y) = f(X,Y), \\ & \forall (X,Y) \in \mathcal{F} - \mathcal{G} \colon x(X,Y) \le f(X,Y)\}, \end{aligned} \tag{3.23}$$

$$\begin{aligned} \mathrm{F}^\circ(\mathcal{G}) \;=\; \{x \mid x \in \mathbf{R}^E, \; & \forall (X,Y) \in \mathcal{G} \colon x(X,Y) = f(X,Y), \\ & \forall (X,Y) \in \mathcal{F} - \mathcal{G} \colon x(X,Y) < f(X,Y)\}. \end{aligned} \tag{3.24}$$

Also, define

$$\mathbf{G} = \{\mathcal{G} \mid \mathcal{G} \text{ is a } \sqcup, \sqcap\text{-closed subfamily of } \mathcal{F} \text{ with } (\emptyset, \emptyset) \in \mathcal{G}, \mathrm{F}^\circ(\mathcal{G}) \neq \emptyset\}. \tag{3.25}$$

Lemma 3.9: *The collection* $\mathbf{G}$ *of subfamilies of* $\mathcal{F}$ *defined by* (3.25) *is given by*

$$\mathbf{G} = \{\mathcal{F}(x) \mid x \in \mathrm{P}_*(f)\}, \tag{3.26}$$

where for each $x \in \mathrm{P}_*(f)$ $\mathcal{F}(x)$ *is defined by* (2.27).

(Proof) If $\mathcal{G} \in \mathbf{G}$, then for any $x \in \mathrm{F}^\circ(\mathcal{G})$ we have $\mathcal{G} = \mathcal{F}(x)$ by the definition (3.24). Conversely, for any $x \in \mathrm{P}^*(f)$ $\mathcal{F}(x)$ is a $\sqcup, \sqcap$-closed subfamily of $\mathcal{F}$ with $(\emptyset, \emptyset) \in \mathcal{G}$ and $x \in \mathrm{F}^\circ(\mathcal{F}(x))$. Hence, $\mathcal{F}(x) \in \mathbf{G}$. □

From Lemma 3.9 we have the following

Theorem 3.10: *The collection* $\mathbf{F}$ *of all the nonempty faces of* $\mathrm{P}_*(f)$ *is given by* $\{\mathrm{F}(\mathcal{G}) \mid \mathcal{G} \in \mathbf{G}\}$. *Also,*
(i) *If* $\mathcal{G}_1, \mathcal{G}_2 \in \mathbf{G}$ *and* $\mathcal{G}_1 \neq \mathcal{G}_2$, *then* $\mathrm{F}(\mathcal{G}_1) \neq \mathrm{F}(\mathcal{G}_2)$.
(ii) *For any* $\mathcal{G}_1, \mathcal{G}_2 \in \mathbf{G}$, $\mathcal{G}_1 \subseteq \mathcal{G}_2$ *if and only if* $\mathrm{F}(\mathcal{G}_1) \supseteq \mathrm{F}(\mathcal{G}_2)$.
In other words, $\mathrm{F}(\cdot)$ *defined by* (3.23) *determines an anti-order isomorphism from* $\mathbf{G}$ *to* $\mathbf{F}$, *where* $\mathbf{G}$ *and* $\mathbf{F}$ *are considered as posets relative to set inclusion.* □

Also, the dimensions of faces are given as follows.

Theorem 3.11: *For any* $\mathcal{G} \in \mathbf{G}$ *we have*

$$\dim\mathrm{F}(\mathcal{G}) = |V| - |\Pi(\mathcal{G})|, \tag{3.27}$$

where $\dim\mathrm{F}(\mathcal{G})$ *is the dimension of* $\mathrm{F}(\mathcal{G})$ *and* $\Pi(\mathcal{G})$ *is the partition of* $\mathrm{Supp}(\mathcal{G})$ *defined as in Theorem* 2.5.
(Proof) For any $\mathcal{G} \in \mathbf{G}$ the dimension of the face $\mathrm{F}(\mathcal{G})$ is equal to the dimension of the linearity space formed by the solution vectors of the following system of linear equations:

$$x(X,Y) = 0 \quad ((X,Y) \in \mathcal{G}), \tag{3.28}$$

which is equivalent to

$$x(W_1) - x(W_2) = 0 \quad (W \in \Pi(\mathcal{G})) \tag{3.29}$$

as in the proof of Theorem 3.1. The present theorem easily follows from this. □

It should be noted that $|\Pi(\mathcal{G})|$ is equal to the length of any maximal chain of $\mathcal{G}$. Theorem 3.4 also follows from Theorem 3.11.

We also have

Corollary 3.12: *For any* $\mathcal{G} \in \mathbf{G}$ *let*

$$\mathcal{C}\colon (\emptyset,\emptyset) = (U_0^+, U_0^-) \sqsubset (U_1^+, U_1^-) \sqsubset \cdots \sqsubset (U_k^+, U_k^-) \tag{3.30}$$

be a maximal chain of $\mathcal{G}$. *Then we have*

$$\mathrm{F}(\mathcal{G}) = \mathrm{F}(\mathcal{C}). \tag{3.31}$$

(Proof) It follows from Theorem 3.11 and (3.30) that the coefficient matrix of the system of linear equations

$$x(X,Y) = 0 \quad ((X,Y) \in \mathcal{G}) \tag{3.32}$$

and that of its subsystem

$$x(X,Y) = 0 \quad ((X,Y) \in \mathcal{C}) \tag{3.33}$$

have the same rank $k = |\Pi(\mathcal{G})|$, the length of the maximal chain $\mathcal{C}$. Therefore, (3.32) and (3.33) determine the same solution set. Relation (3.31) follows from this fact. □

Since f is bimodular on $\mathcal{G} \in \mathbf{G}$, Corollary 3.12 also follows from Lemma 2.1.

3.5. Adjacency of extreme points

From Theorem 3.5 the edges, one-dimensional faces, of $\mathrm{P}_*(f)$ incident to an extreme point $x \in \mathrm{P}_*(f)$ are characterized as follows. Denote by $\mathcal{H}(x)$ the Hasse diagram of the signed poset $\mathcal{P}(\mathcal{F}(x)))$ (or equivalently, of the exchangeability graph $G(\mathcal{F}(x)))$.

Theorem 3.13: *Let x be any extreme point of $\mathrm{P}_*(f)$. For any arc a of the Hasse diagram $\mathcal{H}(x)$ associated with x let $G'(x,a)$ be the bidirected graph obtained by adding to $G(\mathcal{F}(x))$ the arc $\bar{a}$ with its boundary $\partial\bar{a} = -\partial a$. Then, we have $\mathcal{I}(G'(x,a)) \in \mathbf{G}$ and $\mathrm{F}(\mathcal{I}(G'(x,a)))$ is an edge of $\mathrm{P}_*(f)$ incident to x. Conversely, every edge incident to x is given for some arc a of $\mathcal{H}(x)$ in this way.*

(Proof) Note that edges incident to an extreme point x correspond to extreme rays of the tangent cone $\mathrm{TC}(f,x)$ given in Theorem 3.5. Therefore, for any arc a of the Hasse diagram $\mathcal{H}(x)$ a point $y = x + \epsilon\partial a$ with sufficiently small positive real ϵ lies on (the relative interior of) the edge corresponding to the arc a. We can easily see that

$$\mathcal{F}(y) \subset \mathcal{F}(x), \tag{3.34}$$

$$\mathcal{F}(x) - \mathcal{F}(y) = \{(X,Y) \mid (X,Y) \in \mathcal{F}(x), \langle\partial a, (X,Y)\rangle < 0\}. \tag{3.35}$$

Adding to $G(\mathcal{F}(x))$ the arc $\bar{a}$ with $\partial\bar{a} = -\partial a$ removes from $\mathcal{F}(x)$ $(= \mathcal{I}(G(\mathcal{F}(x))))$ exactly the signed sets in (3.35) and gives $\mathcal{F}(y) = \mathcal{I}(G'(x,a))$. This proves the present theorem. □

Now, we can give a characterization of the adjacency of extreme points.

Theorem 3.14: *Two distinct extreme points x_1 and x_2 of $\mathrm{P}_*(f)$ are adjacent if and only if there exist arcs a_1 of $\mathcal{H}(x_1)$ and a_2 of $\mathcal{H}(x_2)$ such that $\tilde{G}'(x_1,a_1) = \tilde{G}'(x_2,a_2)$, where $\tilde{G}'(x_1,a_1)$ and $\tilde{G}'(x_2,a_2)$ are the transitive closures of $G'(x_1,a_1)$ and $G'(x_2,a_2)$, respectively.*

(Proof) Immediate from Theorem 3.13. Note that $\mathcal{I}(G'(x_1,a_1)) = \mathcal{I}(G'(x_2,a_2))$ if and only if $\tilde{G}'(x_1,a_1) = \tilde{G}'(x_2,a_2)$. □

Suppose that x_1 and x_2 are adjacent vertices. Then the arcs a_1 and a_2 in the above theorem satisfy $\partial a_1 = -\partial a_2$ and may be (both) non-selfloops or selfloops. If they are both non-selfloops, then x_1 and x_2 are extreme points of the base polyhedron in some common orthant (S,T), i.e., $\mathcal{F}(x_1)$ and $\mathcal{F}(x_2)$ have a common maximal element (S,T) with $S \cup T = V$. Also, $\mathcal{I}(G'(x_1,a_1))(= \mathcal{I}(G'(x_2,a_2)))$ is spanning and pre-simple. On the other hand, if a_1 and a_2 are both selfloops, then x_1 and x_2 are not extreme points of the base polyhedron in any common orthant but $\mathcal{F}(x_1)$ and $\mathcal{F}(x_2)$ have a common chain of length $n-1$ that is a maximal chain of $\mathcal{I}(G'(x_1,a_1)) = \mathcal{I}(G'(x_2,a_2))$. In this case $\mathcal{I}(G'(x_1,a_1))(= \mathcal{I}(G'(x_2,a_2)))$ is pre-spanning and simple.

3.6. Connectivity and connected components

We say that a bisubmodular system $(\mathcal{F}, f)$ on V is *connected* if there is no $(X,Y) \in 3^V$ such that $(X,Y),(Y,X) \in \mathcal{F}$, $f(X,Y)+f(Y,X)=0$ and $X \cup Y \neq \emptyset, V$. Also, we call a bisubmodular system $(\mathcal{F}, f)$ on V *fully connected* if there is no $(X,Y) \in 3^V$ such that $(X,Y),(Y,X) \in \mathcal{F}$, $f(X,Y)+f(Y,X)=0$ and $(X,Y) \neq (\emptyset,\emptyset)$. It should be noted that $(X,Y),(Y,X) \in \mathcal{F}$ and $f(X,Y)+f(Y,X)=0$ imply $x(X,Y)=f(X,Y)$ and $x(Y,X)=f(Y,X)$, i.e., $(X,Y),(Y,X) \in \mathcal{F}(x)$, for any $x \in \mathrm{P}_*(f)$ and, conversely, that $(X,Y),(Y,X) \in \mathcal{F}(x)$ for some $x \in \mathrm{P}_*(f)$ implies $(X,Y),(Y,X) \in \mathcal{F}$ and $f(X,Y)+f(Y,X)=0$, and hence $(X,Y),(Y,X) \in \mathcal{F}(y)$ for any $y \in \mathrm{P}_*(f)$.

For any signed set (X,Y) we call (Y,X) the *reversal* of (X,Y).

Lemma 3.15: *Suppose that* $\mathrm{P}_*(f) = \mathrm{F}(\mathcal{G})$ *for some* $\mathcal{G} \in \mathbf{G}$, *where* $\mathbf{G}$ *is defined by* (3.25). *Then* $\mathcal{G}$ *is closed with respect to reversal.*

(Proof) Let x be a vector in $\mathrm{P}_*(f)$ such that $\mathcal{F}(x)=\mathcal{G}$. For any $(X,Y) \in \mathcal{G}$ we have $x(X,Y)=f(X,Y)$. Since $y(X,Y)=f(X,Y)$ holds for any $y \in \mathrm{P}_*(f)$, it follows from the definition of signed saturation function that

$$X \cup Y \subseteq \mathrm{sat}^{(+)}(x) \cap \mathrm{sat}^{(-)}(x). \tag{3.36}$$

Therefore, similarly from (3.36) and (2.31)~(2.34) we have

$$\forall e \in Y: \ \mathrm{dep}(x,+e) \sqsubseteq (Y,X), \tag{3.37}$$

$$\forall e \in X: \ \mathrm{dep}(x,-e) \sqsubseteq (Y,X). \tag{3.38}$$

This implies $(Y,X) \in \mathcal{F}(x)$, i.e., $(Y,X) \in \mathcal{G}$. □

From this lemma we can show

Theorem 3.16: *Suppose that* $\mathrm{P}_*(f) = \mathrm{F}(\mathcal{G})$ *for some* $\mathcal{G} \in \mathbf{G}$ *and that* $\mathrm{Supp}(\mathcal{G}) \neq \emptyset$. *Then* $\mathcal{B}$ *defined by*

$$\mathcal{B} = \{X \cup Y \mid (X,Y) \in \mathcal{G}\} \tag{3.39}$$

is a Boolean lattice with respect to the set union and intersection as the lattice operations and has the set $\Pi(\mathcal{G})$ *of its atoms.*

(Proof) For any $(X_i,Y_i) \in \mathcal{G}$ $(i=1,2)$, since $\mathcal{G}$ is $\sqcup,\sqcap$-closed, we have $((X_1,Y_1) \sqcup (X_2,Y_2)) \sqcup (X_1,Y_1) \in \mathcal{G}$. Denote this signed set by (U_1,W_1). Then, $(X_1 \cup Y_1) \cup (X_2 \cup Y_2) = U_1 \cup W_1$. Hence, $\mathcal{B}$ is closed with respect to set union $\cup$. Also, since (Y_2,X_2) belongs to $\mathcal{G}$ due to Lemma 3.15, we have $((X_1,Y_1) \sqcap (X_2,Y_2)) \sqcup ((X_1,Y_1) \sqcap (Y_2,X_2)) \in \mathcal{G}$. Denote this signed set (U_2,W_2). Then, $(X_1 \cup Y_1) \cap (X_2 \cup Y_2) = U_2 \cup W_2$. Hence, $\mathcal{B}$ is closed with respect to set intersection $\cap$. Moreover, for any $(X,Y) \in \mathcal{G}$ let (S,T) be a maximal element of $\mathcal{G}$ such that $(X,Y) \sqsubseteq (S,T)$. Note that $S \cup T = \mathrm{Supp}(\mathcal{G})$.

Since $(Y,X) \in \mathcal{G}$ due to Lemma 3.15, we have $(S,T) \sqcup (Y,X) \in \mathcal{G}$. Denote this signed set by (U_3, W_3). Then we have $(S \cup T) - (X \cup Y) = U_3 \cup W_3$, so that $\mathcal{B}$ is complemented with $\mathrm{Supp}(\mathcal{G})$ as the whole set. Therefore, $\mathcal{B}$ is a Boolean lattice with $\Pi(\mathcal{G})$ being the set of its atoms. □

Theorem 3.17: *A bisubmodular system* $(\mathcal{F}, f)$ *on* V *is fully connected if and only if we have* $\{(\emptyset,\emptyset)\} \in \mathbf{G}$, *i.e., there is a vector* x *in* $\mathrm{P}_*(f)$ *such that*

$$x(X,Y) < f(X,Y) \tag{3.40}$$

for any $(X,Y) \in \mathcal{F}$ *with* $(X,Y) \neq (\emptyset,\emptyset)$.
(Proof) *The "if" part*: If there is a vector $x \in \mathrm{P}_*(f)$ satisfying (3.40) for any $(X,Y) \in \mathcal{F}$ with $(X,Y) \neq (\emptyset,\emptyset)$, then we have

$$f(X,Y) + f(Y,X) > x(X,Y) + x(Y,X) = 0 \tag{3.41}$$

for any $(X,Y) \in \mathcal{F}$ with $(Y,X) \in \mathcal{F}$ and $(X,Y) \neq (\emptyset,\emptyset)$.

The "only if" part: Suppose that $(\mathcal{F}, f)$ is fully connected. We have $\mathrm{F}(\mathcal{G}) = \mathrm{P}_*(f)$ for some $\mathcal{G} \in \mathbf{G}$. We show $\mathcal{G} = \{(\emptyset,\emptyset)\}$. Note that $(\emptyset,\emptyset) \in \mathcal{G}$. Suppose that there exists another $(X,Y) \in \mathcal{G}$ with $(X,Y) \neq (\emptyset,\emptyset)$. Then from Lemma 3.15 we have $(Y,X) \in \mathcal{G}$. Let x be a vector in $\mathrm{P}_*(f)$ such that $\mathcal{F}(x) = \mathcal{G}$. It follows that

$$f(X,Y) + f(Y,X) = x(X,Y) + x(Y,X) = 0. \tag{3.42}$$

This contradicts the assumption that $(\mathcal{F}, f)$ is fully connected. □

It follows from Theorems 3.11 and 3.17 that a bisubmodular system $(\mathcal{F}, f)$ on V is fully connected if and only if $\mathrm{P}_*(f)$ is full-dimensional, i.e., $\dim \mathrm{P}_*(f) = |V|$.

Suppose that $(\mathcal{F}, f)$ is connected but not fully connected. Then define

$$\mathcal{F}_0 = \{(X,Y) \mid (X,Y), (Y,X) \in \mathcal{F}, f(X,Y) + f(Y,X) = 0\}. \tag{3.43}$$

If there exist two distinct $(S_i, T_i) \in \mathcal{F}_0$ $(i = 1,2)$ such that $(S_1,T_1) \neq (T_2,S_2)$, then

$$\begin{aligned} 0 &= f(S_1,T_1) + f(T_1,S_1) + f(S_2,T_2) + f(T_2,S_2) \\ &\geq f((S_1,T_1) \sqcup (S_2,T_2)) + f((S_1,T_1) \sqcap (S_2,T_2)) \\ &\quad + f((T_1,S_1) \sqcup (T_2,S_2)) + f((T_1,S_1) \sqcap (T_2,S_2)) \\ &\geq 0. \end{aligned} \tag{3.44}$$

This implies $(S_1,T_1) \sqcup (S_2,T_2)$ $(= (S_1,T_1) \sqcap (S_2,T_2)) \in \mathcal{F}_0$, which contradicts the assumption that $(\mathcal{F}, f)$ is connected. Consequently, there exists a unique pair of signed sets $(S,T), (T,S) \in \mathcal{F}_0$ such that $S \cup T = V$. From this we have the following.

Theorem 3.18: *Let $(\mathcal{F}, f)$ be a connected but not fully connected bisubmodular system on V. Then there exists a unique orthant (S,T) (unique up to reversal) such that*

$$\mathrm{P}_*(f) = \mathrm{B}_{(S,T)}(f), \tag{3.45}$$

where $\mathrm{B}_{(S,T)}(f)$ is the base polyhedron of $(\mathcal{F}, f)$ in the orthant (S,T). □

From Theorem 3.18 and a connectivity result on base polyhedra (see [12]) we also have

Corollary 3.19: *A bisubmodular system $(\mathcal{F}, f)$ on V is connected but not fully connected if and only if $\{(\emptyset,\emptyset),(S,T),(T,S)\} \in \mathbf{G}$ for an orthant (S,T).* □

For a bisubmodular system $(\mathcal{F}, f)$ on V and a subset U of V such that there exist $(X,Y) \in \mathcal{F}$ with $U = X \cup Y$, the *restriction* $(\mathcal{F}^U, f^U)$ of $(\mathcal{F}, f)$ to U is the bisubmodular system on U defined by

$$\mathcal{F}^U = \{(X,Y) \mid (X,Y) \in \mathcal{F}, X \cup Y \subseteq U\}, \tag{3.46}$$

$$f^U(X,Y) = f(X,Y) \quad ((X,Y) \in \mathcal{F}^U). \tag{3.47}$$

Theorem 3.20: *For a bisubmodular system $(\mathcal{F}, f)$ on V there uniquely exists a partition P^* of V such that the following three hold:*
(i) *For each $U \in P^*$ and $(X,Y) \in \mathcal{F}$ we have $(X \cap U, Y \cap U) \in \mathcal{F}$.*
(ii) *Restrictions $(\mathcal{F}^U, f^U)$ $(U \in P^*)$ are connected and only one of them may be fully connected.*
(iii) *For any $(X,Y) \in \mathcal{F}$,*

$$f(X,Y) = \sum_{U \in P^*} f^U(X \cap U, Y \cap U). \tag{3.48}$$

(Proof) Suppose that $\mathcal{G} \in \mathbf{G}$ satisfies $\mathrm{F}(\mathcal{G}) = \mathrm{P}_*(f)$. If $\mathcal{G} = \{(\emptyset,\emptyset)\}$, the present theorem holds with $P^* = \{V\}$. So, suppose $\mathcal{G} \neq \{(\emptyset,\emptyset)\}$. Let W be the support of $\mathcal{G}$ and $\Pi(\mathcal{G}) = \{U_1, \cdots, U_k\}$. Let (S,T) be a maximal element of $\mathcal{G}$. From Theorem 3.16 we have $(S \cap U, T \cap U), (T \cap U, S \cap U) \in \mathcal{G}$ for each $U \in \Pi(\mathcal{G})$. Then we have for any $(X,Y) \in \mathcal{F}$ and $U \in \Pi(\mathcal{G})$

$$\begin{aligned}&(X \cap U, Y \cap U)\\ &= (S \cap X \cap U, T \cap Y \cap U) \sqcup (T \cap X \cap U, S \cap Y \cap U)\\ &= ((S \cap U, T \cap U) \sqcap (X,Y)) \sqcup ((T \cap U, S \cap U) \sqcap (X,Y)) \in \mathcal{F}.\end{aligned} \tag{3.49}$$

Also, since $(S,T),(T,S) \in \mathcal{G}$, we have for any $(X,Y) \in \mathcal{F}$

$$\begin{aligned}&(X - W, Y - W)\\ &= ((X,Y) \sqcup (S,T)) \sqcap ((X,Y) \sqcup (T,S)) \in \mathcal{F}.\end{aligned} \tag{3.50}$$

Moreover,

$$\begin{aligned} f(S,T) &= f(S,T) + f(T\cap U_1, S\cap U_1) + \cdots + f(T\cap U_k, S\cap U_k) \\ &\quad + f(S\cap U_1, T\cap U_1) + \cdots + f(S\cap U_k, T\cap U_k) \\ &\geq f(S\cap U_1, T\cap U_1) + \cdots + f(S\cap U_k, T\cap U_k) \\ &\geq f(S,T). \end{aligned} \tag{3.51}$$

Hence, we have $f(S,T) = f(S\cap U_1, T\cap U_1) + \cdots + f(S\cap U_k, T\cap U_k)$. Similarly, we have $f(T,S) = f(T\cap U_1, S\cap U_1) + \cdots + f(T\cap U_k, S\cap U_k)$. Therefore,

$$\begin{aligned} 2f(X,Y) &= f(X,Y) + f(S,T) + f(X,Y) + f(T,S) \\ &= f(X,Y) + f(S\cap U_1, T\cap U_1) + \cdots + f(S\cap U_k, T\cap U_k) \\ &\quad + f(X,Y) + f(T\cap U_1, S\cap U_1) + \cdots + f(T\cap U_k, S\cap U_k) \\ &\geq f(S\cap X\cap U_1, T\cap Y\cap U_1) + \cdots + f(S\cap X\cap U_k, T\cap Y\cap U_k) \\ &\quad + f((S,T)\sqcup(X,Y)) \\ &\quad + f(T\cap X\cap U_1, S\cap Y\cap U_1) + \cdots + f(T\cap X\cap U_k, S\cap Y\cap U_k) \\ &\quad + f((T,S)\sqcup(X,Y)) \\ &\geq f(X\cap U_1, Y\cap U_1) + \cdots + f(X\cap U_k, Y\cap U_k) \\ &\quad + f(X,Y) + f(X-W, Y-W) \\ &\geq f(X\cap W, Y\cap W) + f(X,Y) + f(X-W, Y-W) \\ &\geq 2f(X,Y). \end{aligned} \tag{3.52}$$

It follows from (3.52) that

$$f(X,Y) = f(X\cap U_1, Y\cap U_1) + \cdots + f(X\cap U_k, Y\cap U_k) + f(X-W, Y-W). \tag{3.53}$$

Also, for each U_i ($i = 1, 2, \cdots, k$) and $(X,Y) \in \mathcal{F}$ such that $X\cup Y \subseteq U_i$ and $(Y,X) \in \mathcal{F}$, we have from (3.53)

$$0 \leq f(X,Y) + f(Y,X) = f(X\cap U_i, Y\cap U_i) + f(Y\cap U_i, X\cap U_i) \tag{3.54}$$

If (3.54) holds with equality, then (X,Y) must be a member of $\mathcal{G}$. Therefore, we must have $X\cup Y = U_i$ or $X\cup Y = \emptyset$ by the definition of U_i. Hence, $(\mathcal{F}^{U_i}, f^{U_i})$ is connected, but not fully connected since $U_i \in \Pi(\mathcal{G})$. Moreover, if $W \neq V$, then put $U_{k+1} = V - W$. For any $(X,Y) \in \mathcal{F}$ with $X\cup Y \subseteq U_{k+1}$ (3.54) holds for $i = k+1$. If it holds with equality, we have $(X,Y) = (\emptyset,\emptyset)$ since $\mathrm{Supp}(\mathcal{G}) = W$. It follows that $(\mathcal{F}^{U_{k+1}}, f^{U_{k+1}})$ is fully connected. Therefore, putting $P^* = \Pi(\mathcal{G})$ when $W = V$ or putting $P^* = \Pi(\mathcal{G}) \cup \{V - W\}$ when $W \neq V$, (i), (ii) and (iii) hold.

Conversely, if we are given a partition P^* satisfying (i), (ii) and (iii), we can easily see that $\Pi(\mathcal{G}) \subseteq P^*$ and possibly only one component (i.e., $V - \mathrm{Supp}(\mathcal{G})$ if nonempty) of P^* does not belong to $\Pi(\mathcal{G})$. Therefore, the partition P^* is uniquely determined by $\mathcal{G}$. □

Each $(\mathcal{F}^U, f^U)$ for $U \in P^*$ in Theorem 3.20 is called a *connected component* of $(\mathcal{F}, f)$.

Theorem 3.20 together with Theorems 3.17 and 3.18 shows that after an appropriate reflection any bisubmodular system is a direct sum of a fully connected bisubmodular system and connected submodular systems.

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