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Optimal Distribution of City Sizes in a Region

by

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ABSTRACT

This paper first proposes an optimal spatial distribution model of population sizes in a country. The objective function to be examined consists of the amount of inter-action benefit which is formulated by accessibility, and the amount of intra-action congestion cost which is measured by population density. Second, by this optimization model, the optimal population distribution is obtained and the necessary and sufficient conditions for the optimal solution is given. Third, based upon the data analysis of population distribution in Japanese prefectures in 1975, it is shown that Japanese population is suburbanizing and that this suburbanization would lead to the optimal population distribution. Finally, by use of this model, the optimal grid system population distribution of the Tokyo Metropolitan Area is obtained and analyzed.

1. INTRODUCTION

Since the work of Auerbach (1913), a considerable amount of studies of city size distribution has appeared, for example, the rank-size rule (Zipf, 1949), and the Parato or log-normal city size distribution. (A good review is provided by Richardson (1973).) To understand these empirical "laws", a variety of models has been proposed, such as the entropy model (Curry, 1964), the order statistics model (Okabe, 1979), the central place theory (Beckmann, 1958), the stochastic theory (Simon, 1955) and so forth. In these models, however, a spatial aspect of city size distribution is not always taken into account explicitly. In this paper we first formulate a spatial distribution model of city sizes as an optimization model where "inter-action benefit" and "intra-action congestion cost" are optimized under given geographical conditions (i.e.; distances between cities and inhabitable areas of cities are given). With this model, second, the optimal distribution of regional population in Japan is obtained and its empirical implications are discussed by comparing with the actual regional populations, third, the optimal distribution of Tokyo Metropolitan population is obtained and examined in comparison to the actual regional populations.

2. AN OPTIMIZATION MODEL

Consider a closed region whose total population is given by a fixed amount P and assume that all people P have to reside in one of n cities (including towns and villages) of the region. To allocate P population in n cities, we optimize an objective function which is given by a linear combination of "inter-action benefit" and "intra-action congestion cost. By "inter-action benefit" we imply the benefit derived from the accessibility which shortens commuting time, facilitates commodity and information transfer, and so forth. Mathematically we consider that the inter-action benefit perceived by an inhabitant in city i is proportional to the accessibility $\sum_{j=1}^{n} P_j/d_{ij}^{\lambda}$ of city i [where P_i is a population size of city i; d_{ij} is a distance between cities i and j (note that d_{ii} is given by the average intra-urban distance shown by Koshizuka, 1978); G and λ are positive constants]. The inter-action benefit of city i is, therefore, given by

$$A_i = \sum_{j=1}^{n} \frac{P_j}{d_{ij}^{\lambda}} \cdot P_i \,, \qquad i = 1, 2, \ldots, n. \tag{1}$$

In a sense, since this inter-action benefit can be considered the gravity model's sum total, we may say that its maximization means a maximization of migration or commodity flows. We may also say that it means a maximization of interaction between cities.

By "intra-action congestion cost" we imply the congestion cost, such as traffic congestion, unhealthy housing, air pollution, and so forth. Mathematically we assume that the intra-action congestion cost perceived by an inhabitant in city i is proportional to the population density P_i/S_i of city i, (where S_i is an inhabitable area of city i,) and hence the intra-action congestion cost of city i is given by

$$C_i = \frac{P_i}{S_i} \cdot P_i , \qquad i = 1, 2, \ldots, n. \tag{2}$$

It is noted that the smaller intra-action congestion cost is desirable.

With the inter-action benefit A_i and intra-action congestion cost C_i defined above, we now fix an objective function to be a linear combination of the total inter-action benefit $\sum_{i=1}^{n} A_i$ and intra-action congestion cost $\sum_{i=1}^{n} C_i$.
To sum up, our model is formulated as:

$$\begin{aligned} \text{Max. } \Phi &= \kappa' \sum_{i=1}^{n} \sum_{j=1}^{n} G \frac{P_i P_j}{d_{ij}^{\lambda}} - \sum_{i=1}^{n} \frac{P_i^2}{S_i} \\ &= \kappa \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{P_i P_j}{d_{ij}^{\lambda}} - \sum_{i=1}^{n} \frac{P_i^2}{S_i} , \end{aligned} \tag{3}$$

subject to

$$\sum_{i=1}^{n} P_i = P , \tag{4}$$

$$P_i \geqq 0 , \qquad i = 1, 2, \ldots, n. \tag{5}$$

It is noted that the above optimization problem takes a form of the quadratic programming. The parameter κ (= κ'G) indicates the degree of relative importance of the inter-action benefit to the intra-action congestion cost. The parameter κ assumes an important role between inter-action benefit, which might be maximized in view of economic activities, and intra-action congestion cost, which might be minimized in view of human activities. Because trade-off occurs between these two factors, we shall consider changes of the parameter κ in the next section.

Alternatively equations (3), (4) are written as the Lagrange function:

$$\text{Max. } L = \kappa\, {}^{t}\mathbf{p}D\mathbf{p} - {}^{t}\mathbf{p}S\mathbf{p} + ({}^{t}\mathbf{p} - {}^{t}\mathbf{t})\boldsymbol{\mu} , \tag{6}$$

where

$$D = \begin{bmatrix} 1/d_{11}^{\lambda} & \cdots\cdots & 1/d_{1n}^{\lambda} \\ \vdots & 1/d_{ij}^{\lambda} & \\ 1/d_{n1}^{\lambda} & \cdots\cdots & 1/d_{nn}^{\lambda} \end{bmatrix} , \qquad S = \begin{bmatrix} 1/S_1 & & & & 0 \\ & \ddots & & & \\ & & 1/S_i & & \\ & & & \ddots & \\ 0 & & & & 1/S_n \end{bmatrix} ,$$

$$^t\mathbf{p} = [P_1, P_2, \ldots\ldots, P_n] ,$$

$$^t\mathbf{t} = [P/n, \ldots\ldots\ldots, P/n] ,$$

$$\boldsymbol{\mu} = [\mu, \ldots\ldots\ldots\ldots, \mu] ,$$

$^t\mathbf{p}$ is a transposed vector of $\mathbf{p}$, and μ is the Lagrange multiplier. By taking the first derivative with respect to $\mathbf{p}$, the first-order condition is given by

$$2\kappa D\mathbf{p} - 2S\mathbf{p} + \boldsymbol{\mu} = \mathbf{0} . \tag{7}$$

Upon solving equation (7), the optimal population distribution $\mathbf{p}^*$ is obtained as

$$\mathbf{p}^* = \frac{1}{2} (S - \kappa D)^{-1}\boldsymbol{\mu} . \tag{8}$$

It is noted that equation (7) is a necessary condition. The necessary and sufficient conditions are that the determinant of $(S - \kappa D)$ is non-zero and that matrix $(S - \kappa D)$ is non-negative definite, (see Konno and Yamashita, 1978). In addition, the non-negative condition (5) should be satisfied. The non-negative condition of $\mathbf{p}^*$ is the same as the Hawkins-Simon's condition, which is given by

$$|H_k| > 0, \qquad k = 1, 2, \ldots, n, \tag{9}$$

where $H = S - \kappa D$, and

$$H_k = \begin{bmatrix} \frac{1}{S_1} - \frac{\kappa}{d_{11}^{\lambda}}, & -\frac{\kappa}{d_{12}^{\lambda}}, & \ldots, & -\frac{\kappa}{d_{1k}^{\lambda}} \\ -\frac{\kappa}{d_{21}^{\lambda}}, & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ -\frac{\kappa}{d_{k1}^{\lambda}}, & \ldots\ldots\ldots, & & \frac{1}{S_k} - \frac{\kappa}{d_{kk}^{\lambda}} \end{bmatrix}, \quad k=1, 2, \ldots, n.$$

It can be shown, however, that this condition is equivalent to the non-negative definite condition (see Appendix).

As is seen in equation (9), since S_i and d_{ij} are given, the non-negative condition depends on parameter κ. Hence, we shall examine the range of κ that guarantees the non-negative population $\mathbf{p}^*$. By multiplying each row of equation (7), we obtain

$$2\kappa M\mathbf{p} - 2I\mathbf{p} + \boldsymbol{\nu} = 0\,, \tag{10}$$

where

$$M = \begin{bmatrix} \frac{S_1}{d_{11}^{\lambda}}, & \frac{S_1}{d_{12}^{\lambda}}, & \ldots\ldots, & \frac{S_1}{d_{1n}^{\lambda}} \\ \vdots & & \frac{S_i}{d_{ij}^{\lambda}} & \vdots \\ \frac{S_n}{d_{n1}^{\lambda}}, & \frac{S_n}{d_{n2}^{\lambda}}, & \ldots\ldots, & \frac{S_n}{d_{nn}^{\lambda}} \end{bmatrix}, \qquad \boldsymbol{\nu} = \begin{bmatrix} S_1\mu \\ \vdots \\ S_i\mu \\ \vdots \\ S_n\mu \end{bmatrix},$$

and I is a unit matrix. It then follows from this equation that

$$\left(\frac{I}{\kappa} I - M\right)\mathbf{p} = \frac{1}{2}\,\boldsymbol{\nu}\,. \tag{11}$$

It is well known that this problem is equivalent to a non-negative eigen value problem which can be solved by the Frobenius' theorem. His theorem shows that the range of κ is determined by the maximum eigen value of M, (see Nikaido, 1960), and that the range of κ is given by

$$\frac{1}{\sigma(M)} > \kappa > 0\;, \tag{12}$$

where $\sigma(M)$ is the maximum eigen value of M. Within this range it is guaranteed that there exists the positive optimal solution $\mathbf{p}^*$ of equation (8).

3. OPTIMAL POPULATIONS OF PREFECTURES IN JAPAN

Having obtained the optimal populations in a theoretical context, let us now examine its empirical implications by use of Japanese data. For convenience, we use the data of 46 prefectural population of D. I. D. (Densely Inhabited District) from 1966 to 1975 (Japanese Bureau of Statistics, 1977) because the amount of city-based data is too large to analyze. (There are more than 600 cities in Japan.)

There are two parameters λ and κ that cannot be obtained by the data. We may estimate the values of the two parameters by use of multiple regression analysis or canonical correlation analysis, but these are not suitable for the purpose of our study. Hence, we fix $\lambda = 2$ and change the value of parameter κ within $0 < \kappa < 1/\sigma(M)$ (this is later subjected to sensitivity analysis). As a matter of fact, it is not possible to definitively determine the value of λ. For example, according to an analysis of inter-prefectural migration (1966 - 1975), the value of λ by use of the gravity model was estimated at 1.3; for inter-prefectural automobile flow it was estimated at 3.1 (see Moriguchi, 1974); for the inter-prefectural commodity flow of cement it was estimated at 5.0. While recognizing the possible advantages of setting the value of λ within the above values, we set $\lambda = 2$ (the original value in Isaac Newton's Gravity Model) so as to standardize the dimension between A_i and C_i (population^2/km^2). We would like to analyze, on a later occasion, the impact of changes in the value of λ.

First, to see the level of the inter-action benefit, the absolute value of $\sum_i A_i$ [where $\lambda = 2$] is calculated and is shown in Figure 2a. This figure shows that the value is increasing over time. Since the model assumes the constant total population, the comparison of the inter-action benefit between years may require a certain normalization. To do so, we use the relative population size P_i defined by P_i/P instead of the absolute population size P_i. The change of the relative inter-action benefit is depicted in Figure 3a. Like Figure 2a, the value is increasing with a decreasing rate over time. This fact may imply that people migrate from rural areas to the Metropolises (see heavy line in Figure 1). Second, concerning the intra-action congestion cost, the absolute and relative values of $\sum_i C_i$ are respectively shown in Figure 2b and 3b. Although the value in Figure 2b is increasing, Figure 3b indicates that the value is decreasing with a decreasing rate in the period of 1966-1975. These phenomena observed in Figure 3a and 3b may show that the population distribution is approaching the stationary state.

To obtain the optimal regional population of Japan, the parameter value κ must be specified. As it is, however, almost impossible to determine the value of κ using the range $0 < \kappa < 1/\sigma(M)$ (Equation (12)), we divide the range into five equal segments as follows:

$$\kappa_m = \frac{m-1}{5}\frac{1}{\sigma(M)}$$

$$= \frac{m-1}{5}\frac{1}{1.82}, \qquad m = 1, 2, \ldots, 6. \tag{13}$$

These segments are then individually subjected to sensitivity analysis. After completion of the calculations stated above, we can obtain six optimal population distributions P^*_m (m = 1, 2, ..., 6) for each value κ_1 (Table 1). The significant point to be considered is the change in the value of κ_m rather than the determination of the true value of κ_m.

From Table 1, we can observe that the populations of Tokyo, Kanagawa, Saitama and Chiba prefectures (these are within the Tokyo Metropolitan Area) become larger in accordance with the increase of the value κ_m (i.e. in accordance with the relative increase in accessibility sum-total $\sum_{i=1}^{n} A_i$ rather than with an increase in density sum-total $\sum_{i=1}^{n} C_i$. This increase of κ_m means that the benefits of accessibility are relatively more important than the costs of density congestion. This is likely due to shortened time distances through the improvement of transportation services rather than by the relief of housing and air pollntion problems. At the same time, we can observe that the populations of prefectures such as Hokkaido, Kagoshima, Nagasaki and Miyazaki (situated at the extremities of the Japanese Archipelago) become smaller as the value of κ_m increases. Hence, we can consider that the increase in κ_m (the relative increase in accessibility rather than the decrease in density) leads to population agglomeration in the Tokyo Metropolitan Area, and that the decrease in κ_m (the ralative decrease in density rather than the increase in accessibility) leads to population decentralization.

Interestingly, we can also observe that the populations of Osaka, Hyogo and Nara prefectures (within the Osaka Metropolitan Area) first increase and later decrease in accordance with the increase of the value κ_m.

Next if we define a norm N_m of a vector as

$$N_m = | \mathbf{p} - \mathbf{p}_m^* |$$

$$= \sqrt{\sum_{i=1}^{46} (P_i - P_{im}^*)^2}, \qquad m = 1, 2, \ldots, 6, \qquad (14)$$

we can compare the similarity between the actual population distribution of 1975 $\mathbf{p}$ and six optimal population distributions $\mathbf{p}_m^*$ (m = 1, ..., 6). Figure 4 shows that the value of the norm N_m is lowest when m = 4 (i.e. κ_4 = 0.330) are limitations in setting the optimal population distribution as $\mathbf{p}_4^*$, however we first thought that the optimal distribution would be obtained by maximizing accessibility sum-total and minimizing density sum-total. Hence, it is thought that $\mathbf{p}_4^*$ is closer to the actual population distribution $\mathbf{p}$ than any other optimal population distribution. More specifically, although the value κ_m may be within κ_3 to κ_5, there is no available method to determine the precise value of κ. Hence, for simplicity's sake, we call $\mathbf{p}_4^*$ an optimal population distribution.

To compare the optimal population with the actual population, Figure 5 is depicted. (The correlation coefficient is 0.953.) Provided that the parameter value is determined by the lowest value of N_m, we may say that prefectures whose ratio P_i/P^*_{i4} is smaller than 1.0 may potentially accommodate more population, while prefectures whose ratio is greater than 1.0 have excessive populations. The former prefectures, such as Shiga, Ibaraki, Nara and Tochigi are situated around the Tokyo or Osaka metropolitan areas and they have in-migration flows. The phenomena of migrating into those prefectures may be reasonable because they have great accessibility but less population density. The latter prefectures, such as Yamagata, Kagoshima, Iwate and Nagano, are situated far from the metropolitan areas. In those prefectures, we observe out-migration flows which may be due to either less accessibility or more population density or both.

Finally, corresponding to the optimal population distribution P^*_4, we calculated the optimal distribution of population density for all prefectures and classified them into four groups as shown in Figure 6. Note that this population density distribution has a positive linear relationship with the optimal accessibility distribution because the first-order condition given by equation (7) is alternatively rewritten as

$$\frac{P^*_i}{S_i} = \kappa \sum_{j=1}^{n} \frac{P^*_j}{d^{\lambda}_{ij}} + \mu \ . \tag{15}$$

Hence, we can regard Figure 6 as the optimal accessibility distribution. Figure 6 clearly shows that there exist two population density cores; the Tokyo Metropolis and the Osaka Metropolis. In the following section, we examine the larger core (the Tokyo Metropolitan Area; see Figure 1) and illustrate its suburbanization by use of the same optimization model.

4. OPTIMAL POPULATIONS OF TOKYO METROPOLITAN AREA

By use of grid system data for 1975 (National Land Agency, 1977) for the Tokyo Metropolitan Area (120km x 120km, the Tokyo Station is about the center of this area), we examine the empirical implications of the optimal population distribution. In this case, we consider this area a closed region and divide it into 144 equal square sectors (each sector is 10km x 10kmp 100km^2 area; see Figure 7). As there is no inhabitable area (kajūchi menseki) data for this grid system, we use built-up area (tatemono yōchi menseki) data instead, and because of a lack of development area in five regions (water and mountain areas) we eliminate these regions and consider 139 regions. With $n = 139$ and $\lambda = 2$, we calculate the maximum eigen value $\sigma(M')$, then

$$\kappa'_m = \frac{m-1}{5} \cdot \frac{1}{3.74}, \qquad m = 1, 2, \ldots, 6. \tag{16}$$

Using these six values of κ'_m, we can obtain six optimal population distributions $\mathbf{p}'^*_m (m = 1, 2, \ldots, 6)$ as shown in Table 2. From this table, we can see that with an increase in the value κ'_m, population concentrates in the central sectors, while with a decrease in κ'_m, population disperses to all sectors.

Calculating the norms N'_m $(m = 1, 2, \ldots, 6)$ by use of equation (14), we can determine an optimal population distribution $\mathbf{p}'^*_5$ ($m = 5$, $\kappa'_5 = 0.214$ and the correlation coefficient is 0.943) by the same method as stated in the previous section. By depicting the actual population distribution $\mathbf{P}'$ in Figure 8 and the optimal population distribution $\mathbf{p}'^*_5$ in Figure 9, we can observe slight differences between them. The main difference, as indicated in Figure 10, is that $\mathbf{p}'$ is skewed southward in comparison with $\mathbf{p}'^*_5$. Assuming that the actual population distribution $\mathbf{p}'$ will approach the optimal population distribution $\mathbf{p}'^*_5$, we may predict that sectors whose actual population exceeds the optimal population ($P'_i - P'^*_{i5}$) by more than 100,000 may experience future population decrease. Conversely, sectors whose actual population is less than the optimal population by more than 100,000 may experience future population increase. These predictions may not be unreasonable if we remember that the southern parts of the Tokyo Metropolitan Area developed first and that now the nothern parts of the Tokyo Metropolitan Area are being developed. At a later date we would like to undertake a more detailed analysis of these trends by introducing population dynamics.

5. CONCLUDING RAMARKS

In this paper, we first showed the optimal spatial distribution of city sizes in a region by optimizing the objective function consists of "inter-action benefit" represented by accessibility and "intra-action congestion cost" measured by population density. By examining this optimization model, it is shown that the necessary and sufficient conditions for optimality are given by equation (12) and the optimal solution is given by (8).

Second, in this model, an increase in the parameter value κ, which indicates a relative increase in accessibility rather than population density, leads, at the national level, to population agglomeration in the Tokyo Metropolis and the Osaka Metropolis. When applied to the Tokyo Metropolitan Area through the use of a grid system, it leads to population agglomeration in the center of the Area.

Third, with this model, the optimal population distribution of Japanese prefectures is obtained using certain reasonable parameter values. The result is tabulated in Table 1 and compared with the actual population. From this examination we may draw the following two conclusions:

(1) in 1975, prefectures around the metropolitan areas have less populations than the optimal populations, and the most of these prefectures have population increase due to in-migration;

(2) in 1975, prefectures situated far from the metropolitan areas have more populations than the optimal populations, and the most of these prefectures have population decrease due to out-migration.

Finally, with this model, the optimal population distribution of the Tokyo Metropolitan Area is obtained using certain reasonable parameter values. The result is tabulated and compared with the actual population distribution in Table 2 and illustrated in Figures 8, 9 and 10. From these Figures we may say that:

(1) in 1975, as the northern parts of the Tokyo Metropolitan Area are less populated than the optimal populations, we may expect these areas may experience relatively greater population increase than the southern parts of the Tokyo Metropolitan Area;

(2) in 1975, as the southern parts of the Tokyo

Metropolitan Area are more populated than the optimal populations, we may expect these areas may experience relatively greater population decrease than the northern parts of the Tokyo Metropolitan Area.

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[APPENDIX]

We shall prove that the Hawkins - Simon's condition [equation (9)] is equivalent to the non-negative definite condition as follows (see Kan, 1979).

It is obvious when n = 1, because

$$\Phi_1 = \frac{1}{S_1} - \frac{\kappa}{d_{11}^{\lambda}} P_1^{\,2} \tag{a}$$

Assume that this is true for an integer n-1. H is given by quadratic form

$$\begin{aligned}\Phi &= {}^t\mathbf{p}_n H \mathbf{p}_n \\ &= {}^t\mathbf{p}_{n-1} H_{n-1} \mathbf{p}_{n-1} + 2P_n\, {}^t\mathbf{h}_{n-1} \mathbf{p}_{n-1} + h_{nn} P_n^{\,2} \\ &= {}^t\mathbf{q}_{n-1} H_{n-1} \mathbf{q}_{n-1} + (h_{nn} - {}^t\mathbf{h}_{n-1} H_{n-1}^{-1} \mathbf{h}_{n-1}) P_n^{\,2}\,,\end{aligned} \tag{b}$$

where $\mathbf{q}_{n-1} = \mathbf{p}_{n-1} + P_n H_{n-1}^{-1} \mathbf{h}_{n-1}$, (c)

$$H = \begin{bmatrix} H_{n-1} & \mathbf{h}_{n-1} \\ {}^t\mathbf{h}_{n-1} & h_{nn} \end{bmatrix}, \quad \mathbf{p} = \begin{bmatrix} \mathbf{p}_{n-1} \\ P_n \end{bmatrix}, \quad \mathbf{p}_{n-1} = \begin{bmatrix} P_1 \\ \vdots \\ P_{n-1} \end{bmatrix},$$

$$\mathbf{h}_{n-1} = \begin{bmatrix} -\dfrac{\kappa}{d_{1n}^{\lambda}} \\ \vdots \\ -\dfrac{\kappa}{d_{n-1n}^{\lambda}} \end{bmatrix}, \qquad h_{nn} = \frac{1}{S_n} - \frac{\kappa}{d_{nn}^{\lambda}}\,.$$

As H_{n-1} is non-negative definite,

$$h_{nn} - {}^t\mathbf{h}_{n-1}H_{n-1}^{-1}\mathbf{h}_{n-1} \geqq 0 \tag{d}$$

is necessary condition for H being non-negative definite.

Conversely if $|H_k| > 0$ $(k = 1, 2, \ldots, n-1)$ and $(h_{nn} - {}^t\mathbf{h}_{n-1}H_{n-1}^{-1}\mathbf{h}_{n-1}) \geqq 0$, equation (b) is non-negative definite. If we express

$$G = \begin{bmatrix} I & -H_{n-1}^{-1}\mathbf{h}_{n-1} \\ {}^t0 & 1 \end{bmatrix},$$

then

$${}^tGHG = \begin{bmatrix} H_{n-1} & 0 \\ {}^t0 & h_{nn} - {}^t\mathbf{h}_{n-1}H_{n-1}^{-1}\mathbf{h}_{n-1} \end{bmatrix}, \tag{e}$$

which is the Jordan's normal form. Therefore, as $|G| = 1$

$$\begin{aligned} |H| &= |{}^tGHG| \\ &= |H_{n-1}|\ (h_{nn} - {}^t\mathbf{h}_{nn-1}H_{n-1}^{-1}\mathbf{h}_{n-1}) . \end{aligned} \tag{f}$$

Compared with both sides, it can be shown that

$$|H_{n-1}| > 0,\ (h_{nn} - {}^t\mathbf{h}_{n-1}H_{n-1}^{-1}\mathbf{h}_{n-1}) \geqq 0, \text{ if and only if}$$

$$|H_{n-1}| > 0,\ |H_n| > 0. \tag{g}$$

Combined with equation (b), the proof is completed.

REFERENCES

Auerbach F, 1913 "Das Gesetz der Bevölkerungskonzentration" Petermann's Geographische Mitteilungen 59-I 74-77

Beckmann M J, 1958 "City Hierarchies and the Distribution of City Size" Economic Development and Cultural Change 6 243-248

Curry L, 1964 "The Random Spatial Economy: An Explanation in Settlement Theory" Annals of the Association of American Geographers 54 138-146

Japanese Bureau of Statistics, 1977 Statistical Yearbook (Office of Prime Minister, Tokyo)

Kan T, 1979 Bekutoru to Gyōretsu(Vectors and Matrices) (Shinyōsha, Tokyo) 143-146

Konno H, Yamashita H, 1978 Hisenkei keikakuhō(Non-Linear Programming) (Nikkagiren, Tokyo)

Koshizuka T, 1978 "On the Random Distance within an Area" Journal of the Operations Research Society of Japan 21(2) 302-319

Moriguchi S, Okudaira K, 1974 Kōtsumō to Jinkōbunpu ni kansuru Kenkyū (A report on the Transportation Networks and the Population Distribution) (National Land Agency, Tokyo)

National Land Agency, 1977 A Report on the Grid System Data (Tokyo)

Nikaido F, 1960 Introduction to Sets and Mappings in Modern Economics (Baifūkan, Tokyo)

Okabe A, 1979 "An Expected Rank-Size Rule: A Theoretical Relationship between the Rank-Size Rule and City Size Distributions" Regional Science and Urban Economics 9 31-40

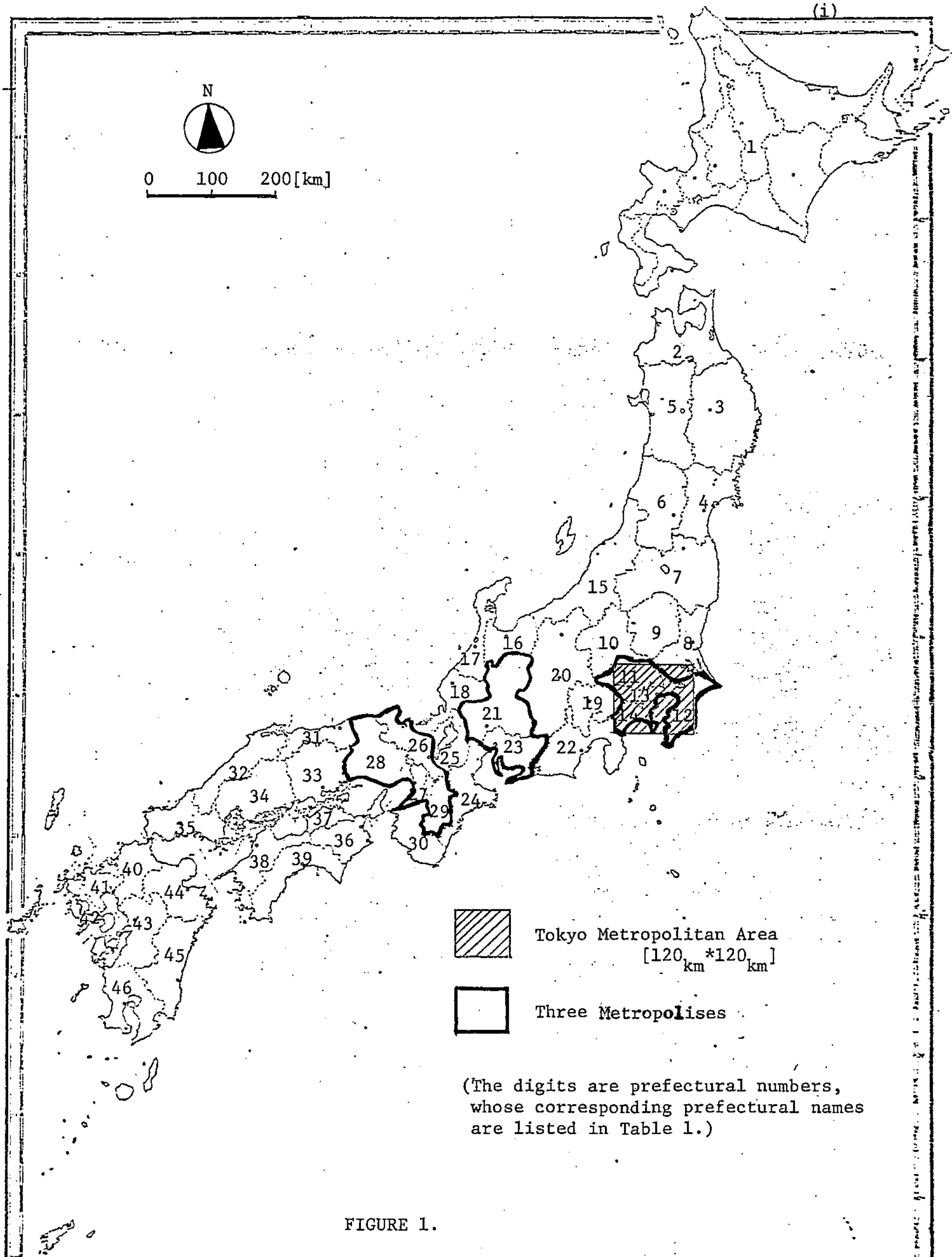
Richardson H W, 1973 The Economics of Urban Size (Saxon House, Westmead)

Simon H A, 1955 "On a Class of Skew Distribution Function" Biometrika 42 425-440

Tabuchi T, 1980 "Optimization of the Prefectural Population Distribution through a Consideration of Accessibility and Density" City Planning Review 110 8-13

Zipf G K, 1949 Human Behavior and the Principle of Least Effort (Addison-Wesley, Cambridge)

[FIGURES AND TABLES]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Tokyo Metropolitan Area
[120km*120km]
Three Metropolises
(The digits are prefectural numbers, whose corresponding prefectural names are listed in Table 1.)
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FIGURE 1.

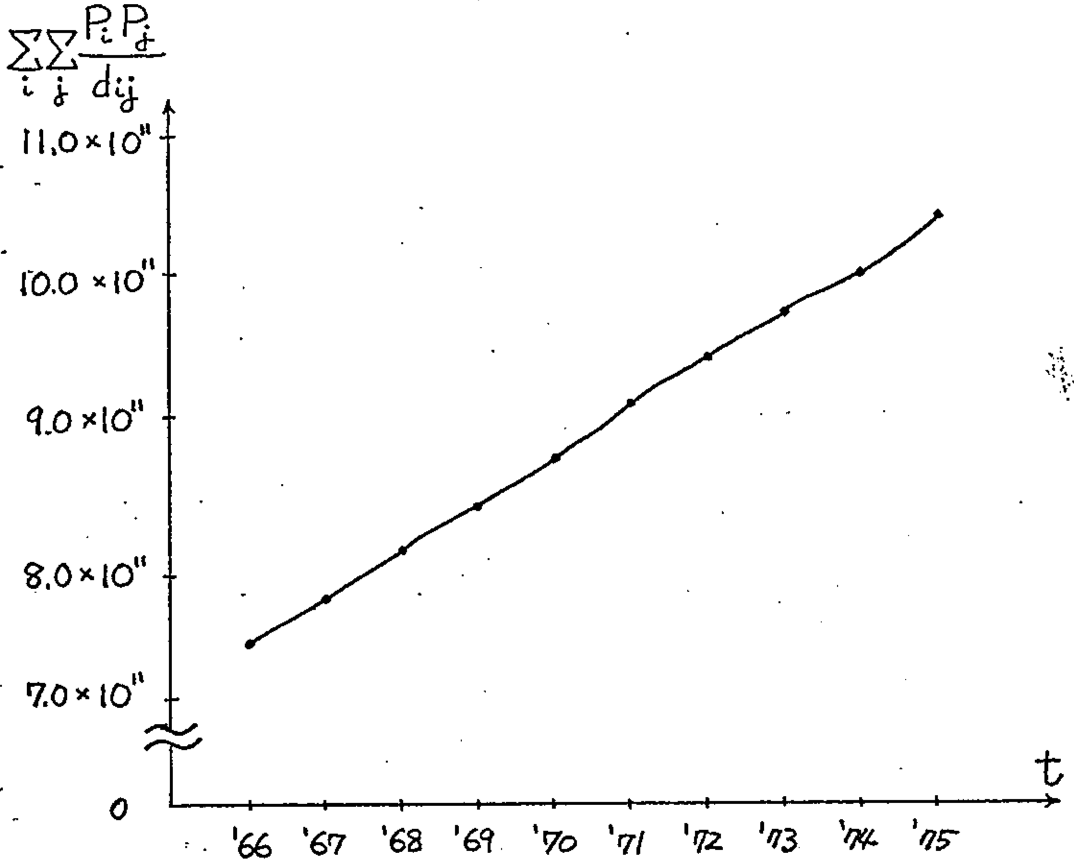


FIGURE 2a.

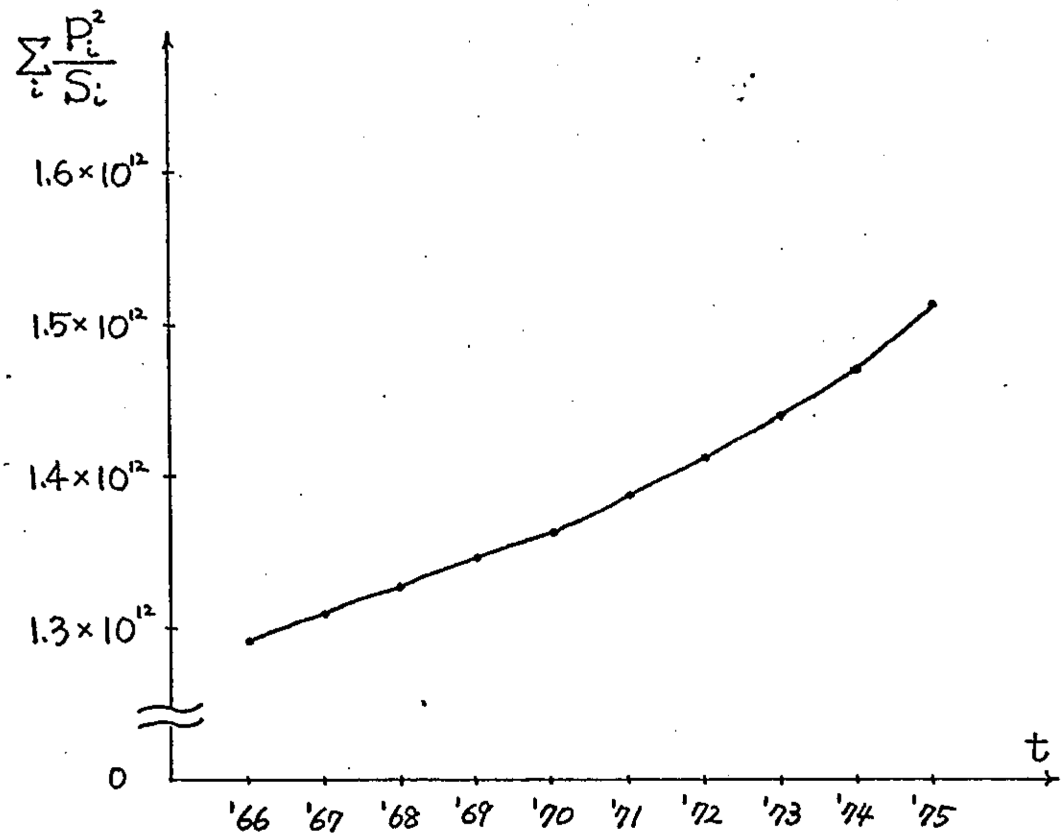


FIGURE 2b.

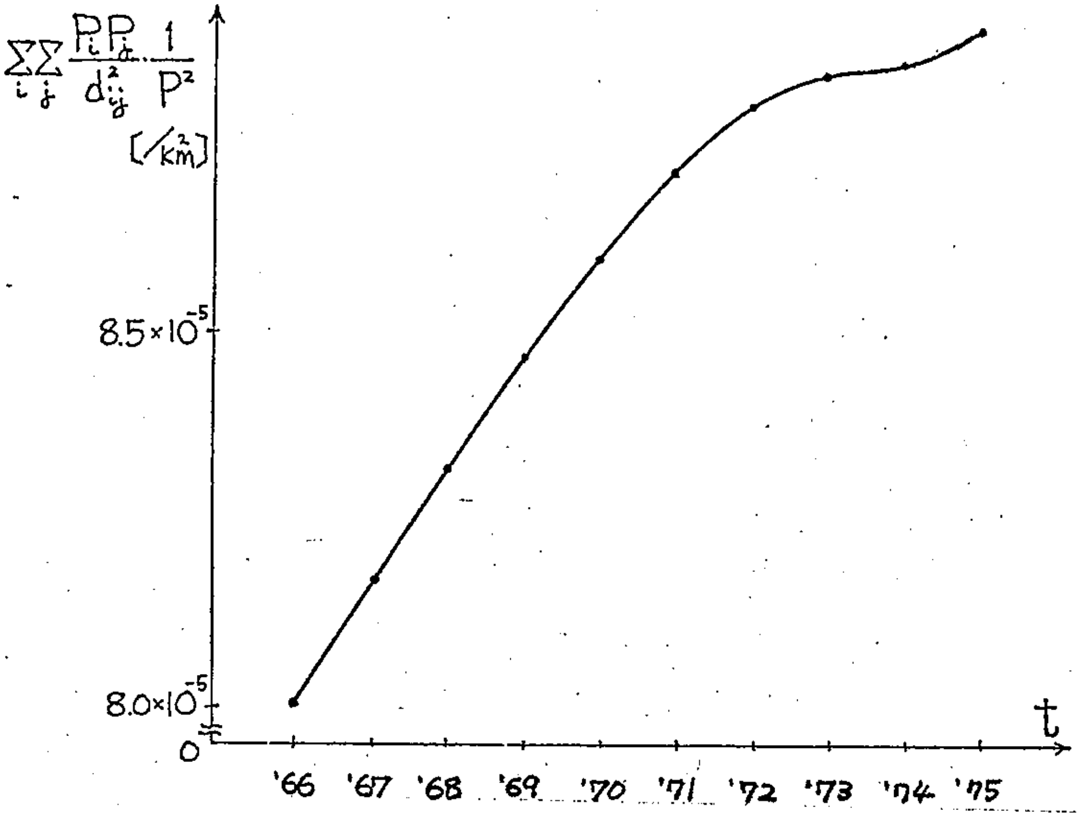


FIGURE 3a.

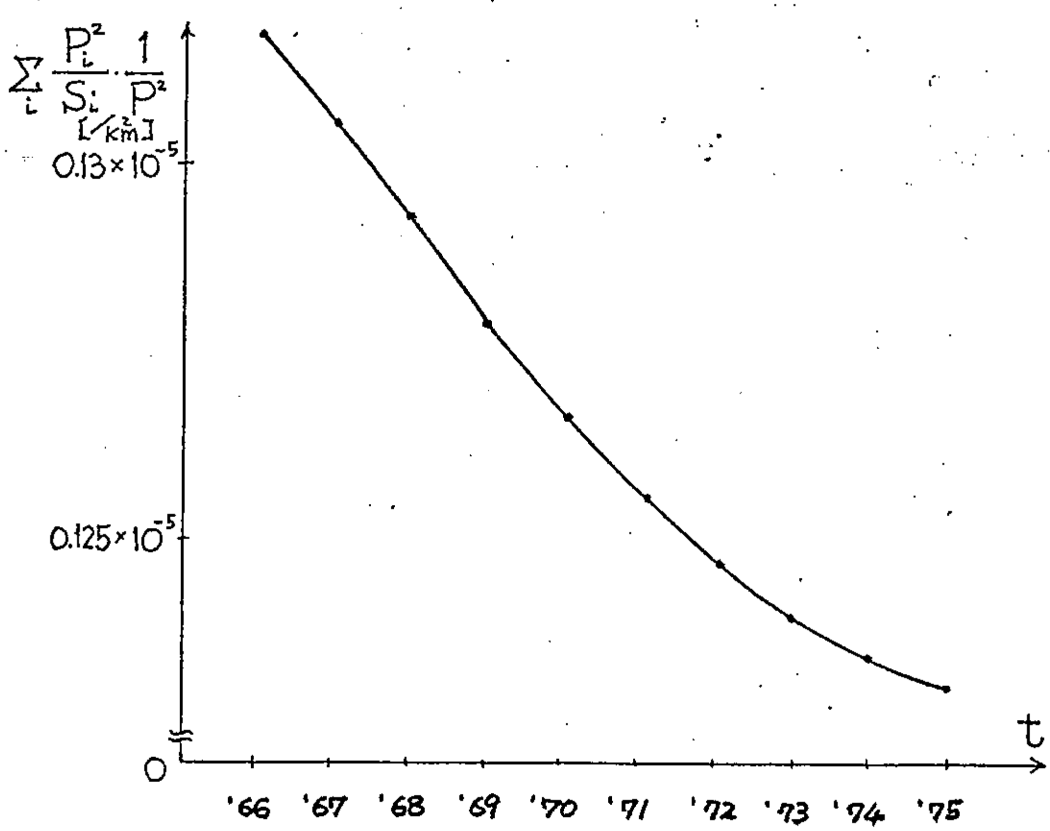


FIGURE 3b.

	Prefecture	P^*_{i1}	P^*_{i2}	P^*_{i3}	P^*_{i4}	P^*_{i5}	P^*_{i6}	P_i
1	Hokkaido	3437519.	3095107.	2688324.	2175065.	1445511.	34277.	3240194.
2	Aomori	553511.	499450.	434931.	353153.	236396.	9638.	562118.
3	Iwate	297363.	268869.	234724.	191270.	128999.	7693.	356878.
4	Miyagi	1111862.	1013266.	892592.	735853.	506932.	53847.	911768.
5	Akita	306284.	276918.	241734.	196966.	132815.	7827.	340938.
6	Yamagata	258706.	235414.	207090.	170527.	117429.	12846.	435679.
7	Fukushima	1128943.	1034660.	919500.	770390.	553869.	130016.	557846.
8	Ibaraki	1802436.	1734712.	1648670.	1536568.	1383563.	1142942.	548449.
9	Tchigi	1425200.	1375626.	1307412.	1211218.	1066801.	801100.	535058.
10	Gumma	954052.	916062.	868436.	806868.	722695.	586882.	601294.
11	Saitama	3989785.	4303517.	4745618.	5438514.	6742650.	10324184.	3112732.
12	Chiba	2554212.	2637151.	2778900.	3038612.	3590771.	5242291.	2393833.
13	Tokyo	6195321.	7406142.	9109951.	11744029.	16561838.	29354125.	11278685.
14	Kanagawa	3932664.	4316650.	4869563.	5749102.	7412972.	12011204.	5400872.
15	Niigata	861385.	786333.	695551.	578999.	410797.	82494.	952533.
16	Toyama	519418.	481124.	432182.	365732.	264071.	52846.	402461.
17	Ishikawa	634974.	584991.	522325.	438646.	312297.	51774.	406597.
18	Fukui	256216.	237126.	213083.	180521.	129908.	19959.	287630.
19	Yamanashi	273713.	262680.	249515.	233404.	212891.	184776.	241164.
20	Nagano	435049.	404579.	367240.	318752.	248290.	111466.	580811.
21	Gifu	781996.	747988.	697610.	617588.	472393.	108735.	677859.
22	Shizuoka	1691582.	1585372.	1452207.	1274720.	1008646.	471477.	1614295.
23	Aichi	5179446.	5078411.	4858956.	4416713.	3469667.	793764.	3634569.
24	Mie	1011727.	971854.	912046.	813850.	624730.	105128.	588050.
25	Shiga	825494.	806191.	769408.	698368.	544864.	87162.	245590.
26	Kyoto	1465932.	1440555.	1388194.	1277815.	1014459.	139287.	1829521.
27	Osaka	4422621.	4707203.	4941052.	4988485.	4376412.	617505.	7682085.
28	Hyogo	3439110.	3440699.	3390061.	3206595.	2627388.	324807.	3455442.
29	Nara	993954.	1055719.	1109155.	1123940.	992679.	145875.	470144.
30	Wakayama	402823.	377790.	345283.	298833.	219786.	21899.	448070.
31	Tottori	277309.	254501.	226179.	188479.	131067.	9064.	151707.
32	Shimane	176689.	161124.	142044.	117080.	79992.	4084.	167995.
33	Okayama	1227004.	1143582.	1033358.	877079.	622857.	43650.	560615.
34	Hiroshima	1858935.	1719878.	1538824.	1287812.	893755.	45615.	1478187.
35	Yamaguchi	868093.	796673.	706295.	584568.	399363.	15033.	645924.
36	Tokushima	429586.	399131.	359751.	304882.	216641.	16073.	200201.
37	Kagawa	424330.	398143.	362201.	309510.	221280.	15489.	297291.
38	Ehime	677642.	621604.	551177.	456811.	313506.	14792.	560621.
39	Kochi	338303.	309148.	273121.	225612.	154481.	7796.	288369.
40	Fukuoka	2667763.	2479245.	2225670.	1864651.	1287672.	41748.	2517806.
41	Saga	239896.	222370.	199069.	166257.	114363.	3417.	204864.
42	Nagasaki	689191.	631472.	558589.	460767.	312778.	8591.	604950.
43	Kumamoto	717060.	655052.	577725.	475182.	321788.	9497.	547993.
44	Oita	661322.	606466.	537176.	444051.	302742.	10608.	412375.
45	Miyazaki	528753.	479693.	420063.	342982.	230530.	6844.	334046.
46	Kagoshima	369076.	334009.	291701.	237434.	158922.	4126.	528144.

TABLE 1.

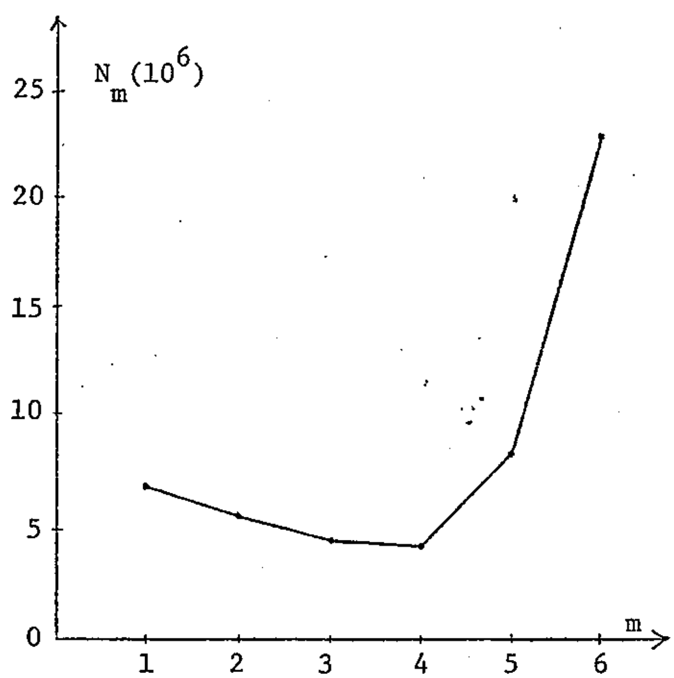
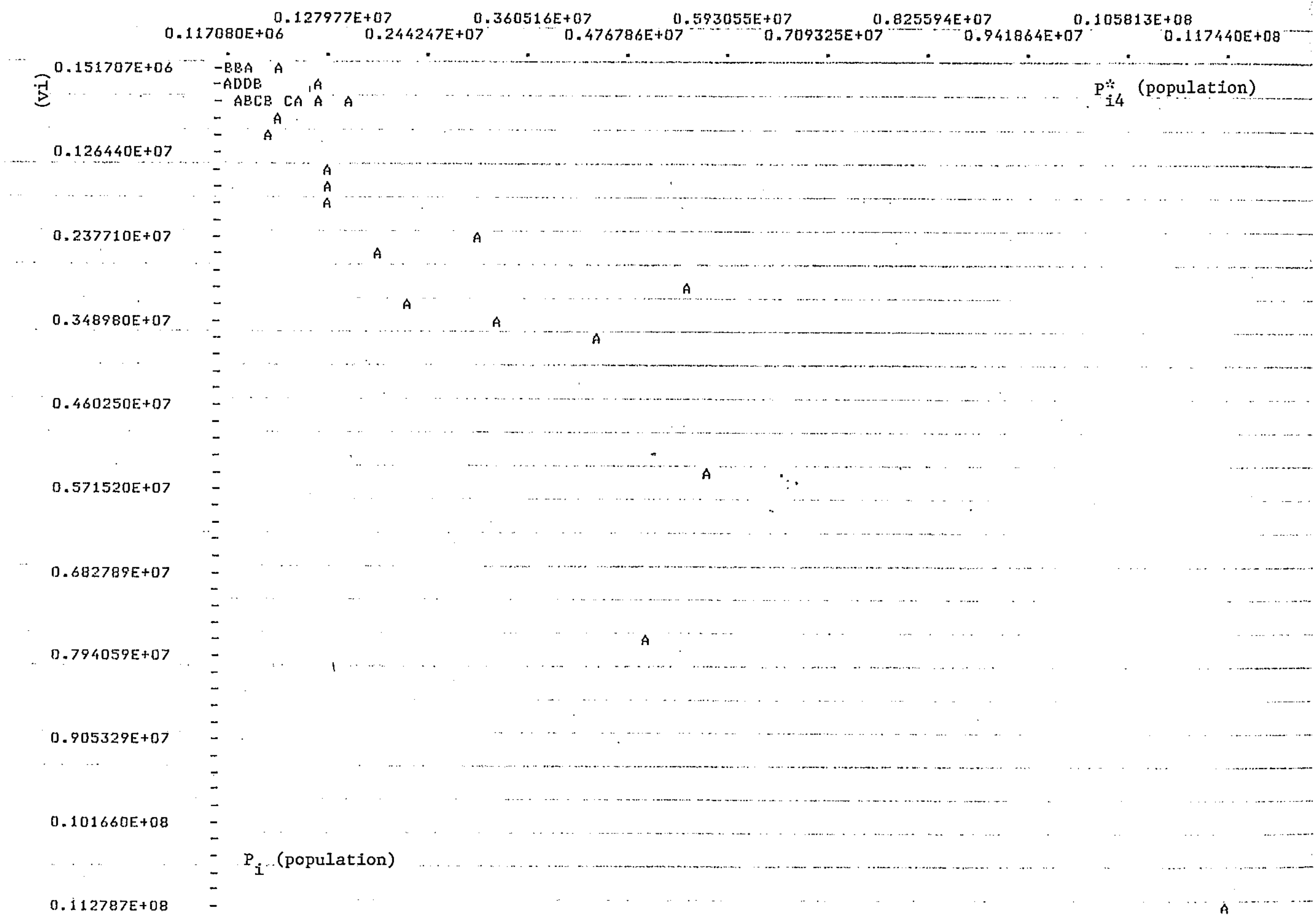


FIGURE 4.

(vi)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0.127977E+07
0.244247E+07
0.360516E+07
0.476786E+07
0.593055E+07
0.709325E+07
0.825594E+07
0.941864E+07
0.105813E+08
0.117440E+08
P^{*}_{i4} (population)
0.151707E+06
0.126440E+07
0.237710E+07
0.348980E+07
0.460250E+07
0.571520E+07
0.682789E+07
0.794059E+07
0.905329E+07
0.101660E+08
0.112787E+08
P_i (population)

FIGURE 5.

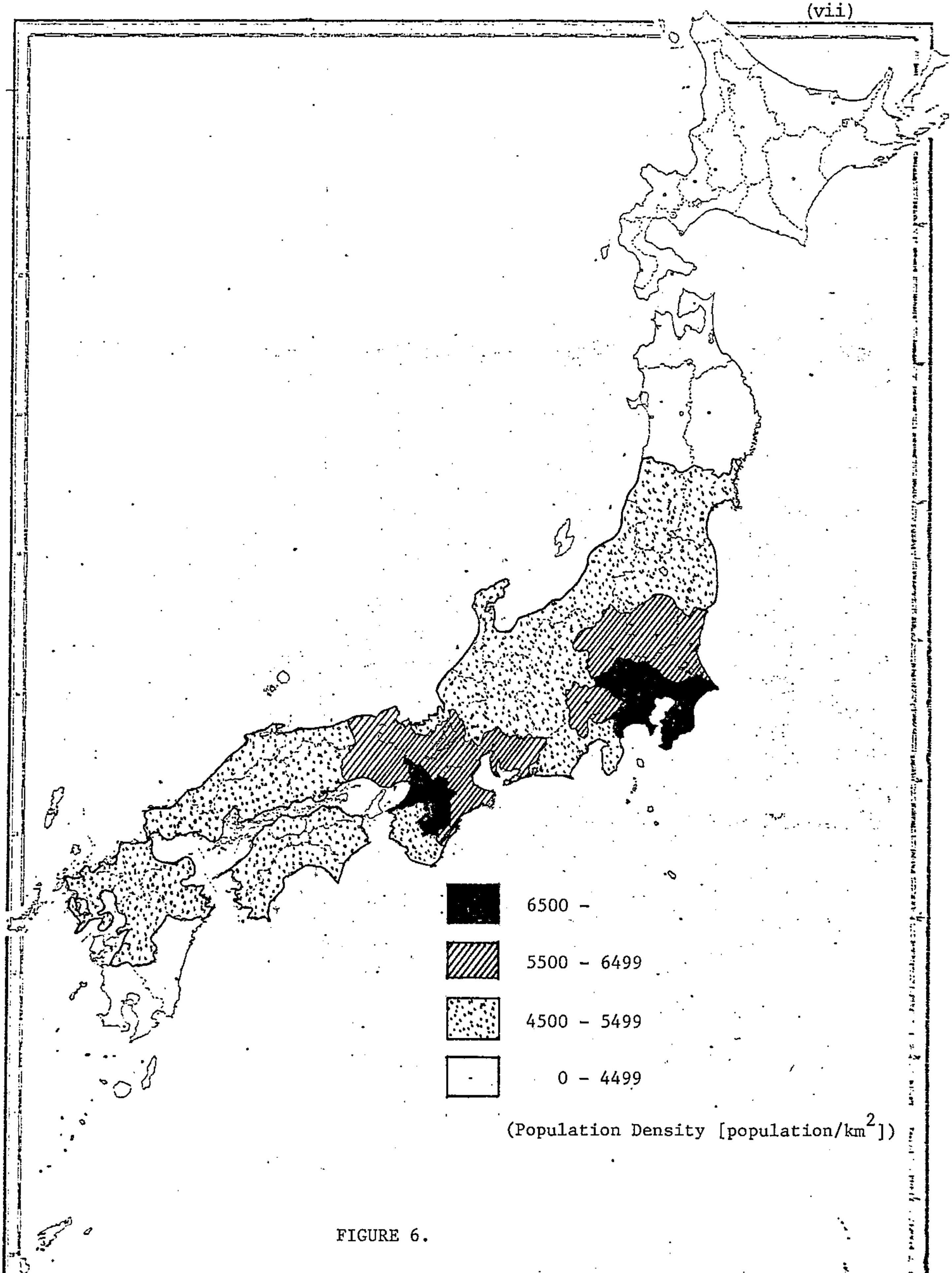
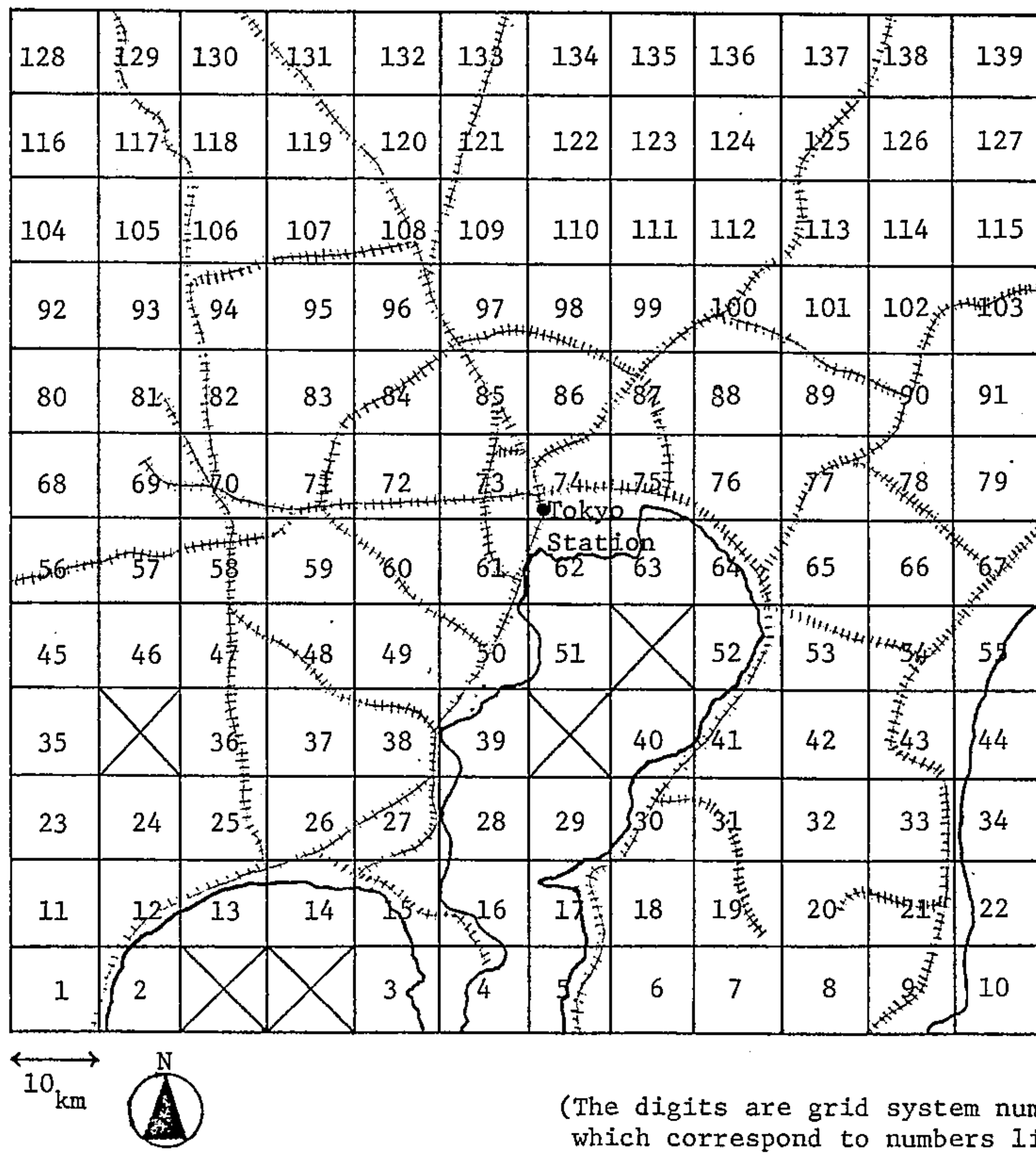

6500 -
5500 - 6499
4500 - 5499
0 - 4499
(Population Density [population/km^2])

FIGURE 6.



(The digits are grid system numbers, which correspond to numbers listed in Table 2.)

Japan National Railways

FIGURE 7.

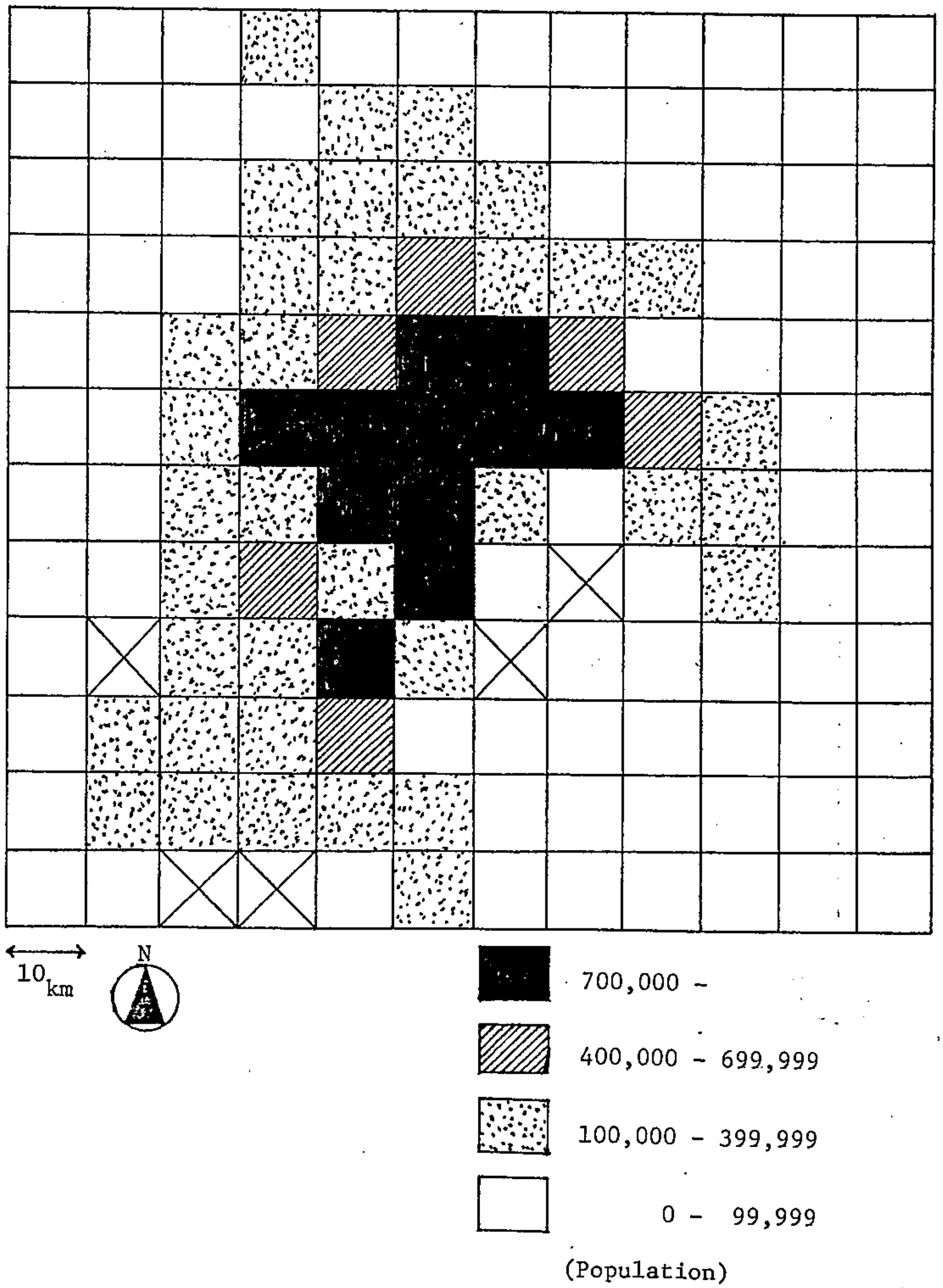

10 km
N
700,000 -
400,000 - 699,999
100,000 - 399,999
0 - 99,999
(Population)

FIGURE 8.

TABLE 2.

No.	P'^{*}_{i1}	P'^{*}_{i2}	P'^{*}_{i3}	P'^{*}_{i4}	P'^{*}_{i5}	P'^{*}_{i6}	P'_{i}
1	43704.	37824.	31267.	23719.	14596.	2673.	15836.
2	29771.	25922.	21587.	16540.	10363.	2177.	27477.
3	28580.	25968.	22807.	18798.	13424.	5552.	16163.
4	152546.	137474.	115731.	98194.	70239.	30789.	162469.
5	12823.	11331.	9834.	8040.	5763.	2820.	5787.
6	17148.	15219.	13053.	10541.	7472.	3408.	11719.
7	9050.	7977.	6787.	5425.	3792.	1674.	8697.
8	11551.	10120.	8542.	6752.	4625.	1901.	5955.
9	49777.	43352.	38097.	27633.	17259.	3503.	16496.
10	23102.	20223.	18947.	13093.	8323.	1928.	11851.
11	59066.	52599.	44962.	35605.	23506.	6525.	32159.
12	270439.	243291.	210672.	169648.	115733.	37619.	184216.
13	177101.	117612.	105246.	88016.	63160.	24418.	119871.
14	198748.	176530.	159914.	136456.	101577.	45544.	182215.
15	251981.	236852.	218103.	186639.	142628.	71661.	225522.
16	200775.	186762.	168918.	145204.	111784.	60496.	240950.
17	58127.	51831.	46479.	39667.	30687.	17676.	31752.
18	60018.	54551.	48269.	40785.	31372.	18495.	43640.
19	29414.	26435.	23086.	19200.	14465.	8224.	14279.
20	15600.	13055.	11921.	9712.	7073.	3676.	10680.
21	44299.	38056.	32619.	25228.	16020.	3600.	19642.
22	56208.	49489.	41766.	32573.	21046.	5375.	12413.
23	44061.	39422.	33901.	27075.	18160.	5511.	19034.
24	140400.	128555.	113458.	93503.	65701.	23761.	100958.
25	360109.	334322.	300296.	253562.	185828.	79533.	205425.
26	408890.	368513.	358459.	312936.	241534.	122074.	301078.
27	496247.	473672.	438751.	387490.	306564.	177875.	636360.
28	71588.	69684.	66283.	60663.	51257.	34700.	77330.
29	15719.	15170.	14399.	13277.	11560.	8725.	8687.
30	130159.	122806.	114072.	103251.	89007.	68487.	91804.
31	73117.	67535.	61227.	53871.	44866.	32941.	24964.
32	28818.	26117.	23101.	19657.	15544.	10268.	22688.
33	84311.	74413.	62949.	49197.	31827.	8045.	39816.
34	24412.	22010.	19088.	15401.	10495.	3424.	6435.
35	2387.	2170.	1908.	1570.	1111.	431.	459.
36	156833.	150804.	141360.	126501.	102443.	61092.	101423.
37	449541.	440139.	424805.	399749.	357678.	282821.	394936.
38	570173.	567715.	561216.	547729.	521504.	469945.	779618.
39	284610.	284773.	284164.	282216.	277811.	268440.	383754.
40	31676.	29648.	27425.	24948.	22108.	18689.	15323.
41	104556.	92927.	79355.	62904.	41848.	12559.	65244.
42	68116.	61440.	53310.	43027.	29301.	9428.	21939.
43	140995.	127760.	111539.	90884.	63012.	22278.	54996.
44	49182.	44752.	39303.	32332.	22907.	9071.	17301.
45	10122.	9278.	8250.	6941.	5174.	2574.	4096.
46	24769.	23295.	21404.	18868.	15256.	9646.	8251.
47	244717.	237714.	227965.	213727.	191658.	154532.	121749.
48	566983.	588852.	609507.	630599.	651243.	669128.	516643.
49	424772.	453277.	486920.	523372.	582559.	659794.	360781.
50	753920.	771572.	794619.	843860.	915315.	1037589.	1405790.
51	39536.	39032.	38730.	38801.	39580.	41797.	19815.
52	97411.	89759.	79912.	66755.	48225.	19995.	38590.
53	207206.	191299.	170585.	142632.	102933.	42041.	140009.
54	98959.	91118.	81091.	67789.	49194.	21074.	45107.
55	128134.	116810.	103149.	85977.	63117.	29988.	30885.
56	47634.	43823.	39437.	33977.	26844.	16657.	26908.
57	46443.	44490.	42028.	38782.	34235.	27274.	20088.
58	305569.	304014.	301985.	299046.	294299.	285698.	289426.
59	274726.	293577.	316467.	345277.	383437.	438054.	276646.
60	556121.	616578.	693920.	797184.	943327.	1168318.	758317.
61	946240.	1030845.	1143830.	1301287.	1533470.	1905536.	1749443.
62	117417.	123325.	131776.	144321.	163920.	196948.	175880.
63	25127.	23630.	21701.	19113.	15447.	9828.	47482.
64	251386.	236365.	215763.	188489.	142934.	73312.	307200.
65	277703.	259478.	234759.	200117.	149195.	68659.	209837.
66	81453.	75432.	67587.	56993.	41930.	18706.	29382.
67	88360.	80780.	71515.	59737.	43910.	20806.	33981.
68	9765.	8922.	8043.	6911.	5452.	3406.	5483.
69	40131.	38179.	35840.	32949.	29133.	23651.	21240.
70	321764.	321925.	322312.	323040.	324269.	326223.	224505.
71	536473.	569873.	612980.	670824.	752778.	878235.	721111.
72	737605.	827497.	945052.	1104998.	1334964.	1693499.	1068228.
73	957553.	1087457.	1256277.	1485807.	1817047.	2337208.	2081460.
74	798814.	857773.	937211.	1048435.	1212889.	1476503.	1671732.
75	429416.	420634.	407592.	388022.	357721.	308114.	798920.
76	421438.	403852.	377388.	337959.	276665.	175305.	501697.
77	186723.	176219.	161494.	140256.	108249.	56549.	107923.
78	36559.	33831.	30326.	25648.	19066.	9041.	32272.
79	94910.	86306.	76173.	63444.	46666.	22617.	24621.
80	5359.	4889.	4347.	3695.	2470.	1732.	5610.
81	39774.	37379.	34593.	31226.	26928.	20957.	21535.
82	172433.	170919.	169103.	166857.	163928.	159733.	144658.
83	340818.	358625.	381198.	410763.	452816.	514215.	329313.
84	508726.	563205.	633936.	729380.	865964.	1074683.	490823.
85	805721.	898438.	1019517.	1180920.	1419650.	1785964.	1181796.
86	635074.	682359.	744139.	828173.	949015.	1137564.	893587.
87	402503.	391627.	375293.	350413.	311240.	245719.	494291.
88	169933.	163353.	152967.	136632.	110354.	65738.	96940.
89	78833.	73787.	66964.	57429.	43415.	21218.	23374.
90	59423.	54433.	48248.	40243.	29328.	13160.	45331.
91	53707.	48774.	42863.	35508.	25845.	12067.	16281.
92	357.	325.	287.	241.	184.	104.	980.
93	10479.	9757.	8901.	7860.	6539.	4689.	8662.
94	142549.	157692.	157778.	144277.	134183.	119345.	80712.
95	342961.	349129.	356352.	364822.	375147.	388437.	220536.
96	483337.	429197.	460889.	500398.	553011.	628149.	366013.
97	521111.	559098.	606212.	666875.	749133.	869722.	508648.
98	385474.	401466.	420814.	445147.	477433.	523820.	304635.
99	333435.	315723.	291954.	257644.	208995.	125580.	250510.
100	199704.	187435.	170696.	146657.	110709.	52941.	130832.
101	145044.	133702.	119124.	99634.	72127.	30083.	56357.
102	63591.	57930.	50946.	41995.	29886.	12122.	25409.
103	70379.	63492.	55209.	45146.	32135.	13874.	33063.
104	61685.	55570.	48536.	40151.	29419.	15261.	47768.
105	1766.	1646.	1480.	1276.	1014.	647.	6341.
106	131469.	125065.	117164.	107025.	93232.	72814.	57149.
107	266034.	263516.	259148.	251828.	239457.	217478.	140063.
108	416317.	423267.	429444.	433996.	435187.	429116.	234012.
109	412119.	421914.	431350.	440294.	447447.	451045.	244392.
110	387618.	383420.	377897.	369096.	357941.	336648.	228557.
111	203276.	189775.	171676.	146582.	110020.	52561.	48847.
112	122061.	113128.	101280.	84776.	60997.	23929.	45652.
113	109795.	100484.	88980.	73413.	51642.	18765.	49628.
114	70259.	63521.	55329.	44593.	31237.	11397.	22050.
115	44299.	39687.	34281.	27711.	19295.	7486.	19552.
116	73951.	66212.	57316.	46728.	33450.	15409.	46028.
117	20721.	18858.	16742.	14100.	10689.	5918.	15763.
118	113487.	106043.	96842.	85045.	69094.	45711.	48949.
119	165050.	159295.	151095.	139211.	121328.	92540.	71757.
120	276036.	270244.	261136.	246873.	223994.	185118.	142546.
121	291941.	283280.	273756.	258609.	235216.	196486.	159587.
122	135394.	130308.	123403.	113785.	99780.	77834.	41019.
123	181248.	164169.	143195.	116369.	80081.	26768.	59197.
124	88003.	80289.	70444.	57448.	39421.	12436.	30071.
125	275188.	202857.	175690.	141386.	95217.	29834.	98637.
126	43823.	39161.	33988.	27286.	18485.	5977.	15271.
127	17982.	15968.	13642.	10859.	7353.	2569.	11836.
128	32629.	28958.	24778.	19854.	13749.	5550.	18758.
129	68354.	61354.	53274.	43613.	31450.	14848.	37207.
130	151713.	138705.	123187.	103989.	78947.	43521.	58966.
131	304021.	281053.	253300.	218723.	172769.	108514.	154278.
132	188129.	174498.	159964.	141081.	115158.	76822.	78159.
133	142901.	133301.	121629.	106948.	87031.	58103.	74764.
134	143815.	131710.	117982.	100985.	79199.	49579.	51014.
135	143615.	132123.	118141.	100298.	78385.	41671.	47116.
136	87408.	79735.	70384.	58574.	42867.	20759.	32362.
137	144687.	130172.	112964.	91829.	64465.	26094.	58795.
138	50134.	44548.	38624.	30314.	20524.	7149.	22237.

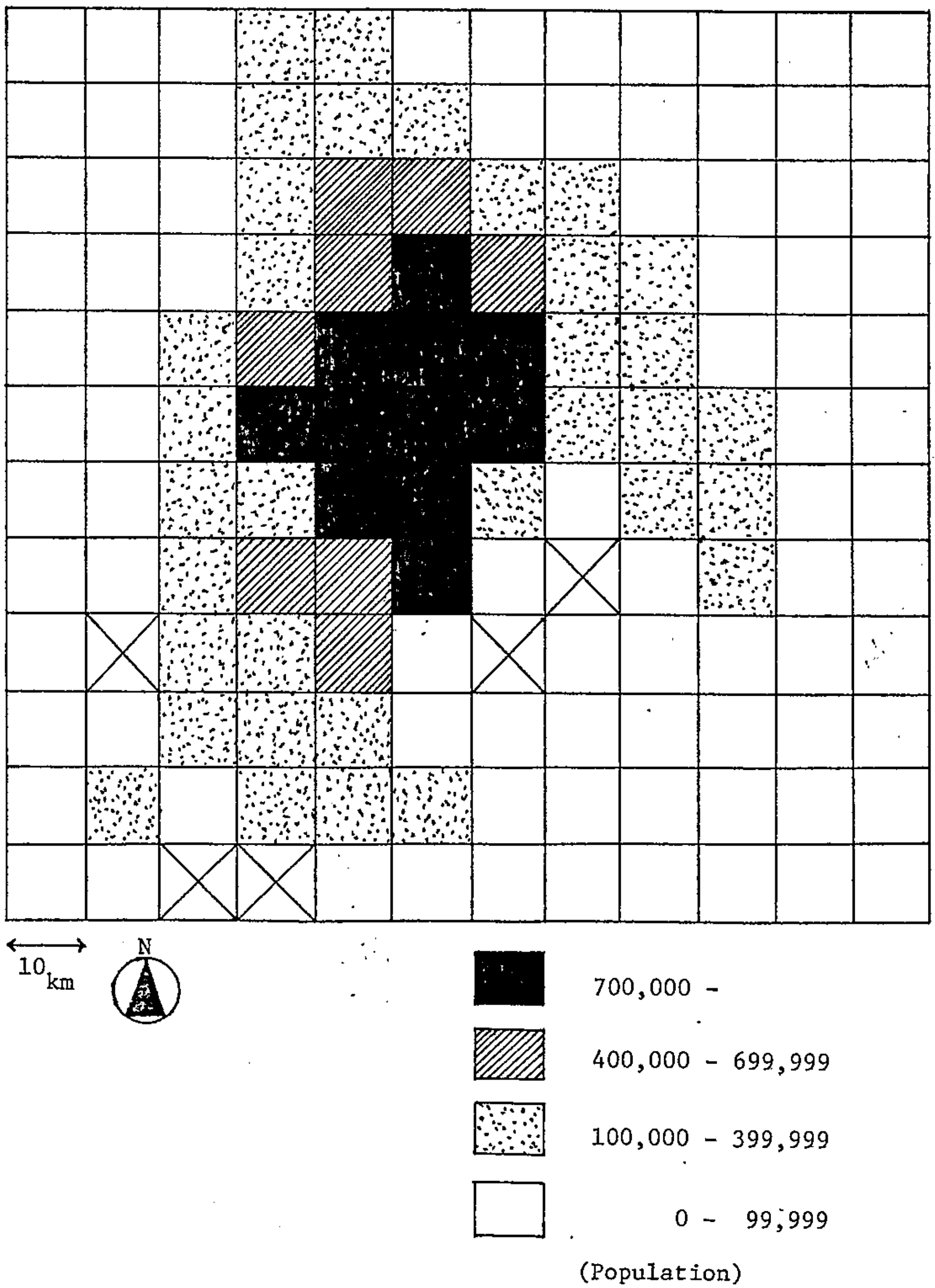

10 km
N
700,000 -
400,000 - 699,999
100,000 - 399,999
0 - 99,999
(Population)

FIGURE 9.

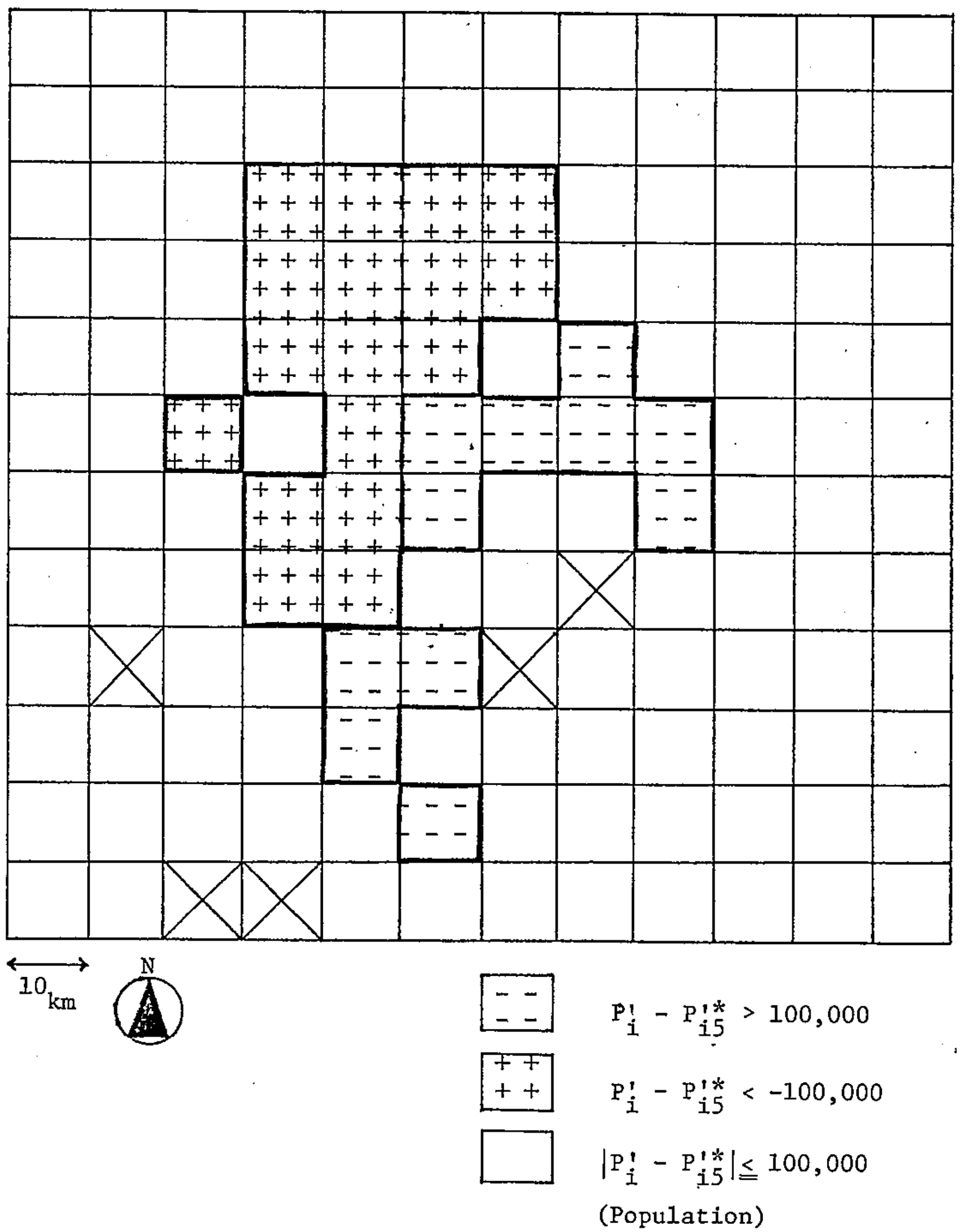

10km
N
$P_i^! - P_{i5}^{!*} > 100{,}000$
$P_i^! - P_{i5}^{!*} < -100{,}000$
$|P_i^! - P_{i5}^{!*}| \leq 100{,}000$
(Population)

FIGURE 10.