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$\sqcup,\sqcap$-closed Families and Signed Posets

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Abstract

Recently, V. Reiner has introduced the concept of signed poset and showed the so-called signed Birkhoff theorem that is a signed analogue of the well-known Birkhoff theorem on the relationship between the set of ideals of a poset and a distributive lattice. For a finite nonempty set V we consider a family $\mathcal{F} \subseteq 3^V \equiv \{(X,Y) \mid X,Y \subseteq V, X \cap Y = \emptyset\}$ that is closed with respect to the reduced union $\sqcup$ and the intersection $\sqcap$, where for each $(X_i, Y_i) \in \mathcal{F}$ $(i = 1,2)$ we define

$$\begin{aligned}(X_1,Y_1) \sqcup (X_2,Y_2) &= ((X_1 \cup X_2) - (Y_1 \cup Y_2), (Y_1 \cup Y_2) - (X_1 \cup X_2)),\\ (X_1,Y_1) \sqcap (X_2,Y_2) &= (X_1 \cap X_2, Y_1 \cap Y_2).\end{aligned}$$

We show that there exists a one-to-one correspondence between the set of all the simple and spanning $\sqcup,\sqcap$-closed families $\mathcal{F} \subseteq 3^V$ on V with $(\emptyset,\emptyset) \in \mathcal{F}$ and the set of all the signed posets $\mathcal{P}$ on V such that each such $\mathcal{F}$ is the set of all the ideals of the corresponding signed poset $\mathcal{P}$. This is a strengthening of the signed Birkhoff theorem of V. Reiner.

1. Introduction

Recently, V. Reiner [6] has introduced the concept of signed poset and showed the so-called signed Birkhoff theorem that is a signed analogue of the well-known Birkhoff theorem on the relationship between the set of ideals of a poset and a distributive lattice. The signed Birkhoff theorem [6, Theorem 4.8] asserts that for a finite lattice

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$\mathcal{L}$ with the maximum element $\hat{1}$, $\mathcal{L} - \{\hat{1}\}$ is isomorphic to the set of ideals of some signed poset $\mathcal{P}$ if and only if $\mathcal{L}$ is B_n-distributive, where $\mathcal{P}$ is determined by $\mathcal{L}$ up to isomorphism as a signed poset (see [6] for the terminology).

On the other hand, the authors [1] have been interested in a family $\mathcal{F} \subseteq 3^V \equiv \{(X, Y) \mid X, Y \subseteq V, X \cap Y = \emptyset\}$ for a finite nonempty set V that is closed with respect to the reduced union $\sqcup$ and the intersection $\sqcap$. For each $(X_i, Y_i) \in \mathcal{F}$ $(i = 1, 2)$ the reduced union $(X_1, Y_1) \sqcup (X_2, Y_2)$ and the intersection $(X_1, Y_1) \sqcap (X_2, Y_2)$ are, respectively, defined by

$$(X_1, Y_1) \sqcup (X_2, Y_2) = ((X_1 \cup X_2) - (Y_1 \cup Y_2), (Y_1 \cup Y_2) - (X_1 \cup X_2)), \quad (1.1)$$

$$(X_1, Y_1) \sqcap (X_2, Y_2) = (X_1 \cap X_2, Y_1 \cap Y_2). \quad (1.2)$$

We call such a family $\mathcal{F}$ a $\sqcup, \sqcap$-*closed family.* $\sqcup, \sqcap$-closed families have been introduced as domains of bisubmodular functions (see [1], [4]). (A bisubmodular function $f : \mathcal{F} \to \mathbf{R}$ is a function satisfying

$$f(X_1, Y_1) + f(X_2, Y_2) \geq f((X_1, Y_1) \sqcup (X_2, Y_2)) + f((X_1, Y_1) \sqcap (X_2, Y_2)) \quad (1.3)$$

for each $(X_i, Y_i) \in \mathcal{F}$ $(i = 1, 2)$. Bisubmodular functions are generalizations of rank functions of polymatroids and other related polyhedra (see, e.g., [2], [3] and [5]).

Each $(X, Y) \in \mathcal{F}$ can be identified with its characteristic vector $\chi_{(X,Y)} \in \{0, \pm 1\}^V$ defined by

$$\chi_{(X,Y)}(v) = \begin{cases} 1 & \text{if } v \in X \\ -1 & \text{if } v \in Y \\ 0 & \text{otherwise} \end{cases} \quad (1.4)$$

for each $v \in V$.

We say $\mathcal{F}$ is *simple* if for each distinct $v, w \in V$

(1) there is no $(X, Y) \in \mathcal{F}$ such that $v, w \in X \cup Y$ or

(2) there exists some $(X, Y) \in \mathcal{F}$ such that either $v \in X \cup Y$ and $w \notin X \cup Y$, or $v \notin X \cup Y$ and $w \in X \cup Y$.

Also, $\mathcal{F}$ is *spanning* if there exists some $(X, Y) \in \mathcal{F}$ such that $X \cup Y = V$. We can show that a $\sqcup, \sqcap$-closed family $\mathcal{F}$ is spanning if and only if for each $v \in V$ there exists some $(X, Y) \in \mathcal{F}$ such that $v \in X \cup Y$ (see Lemma 4.1 in Section 4).

We shall show that there exists a one-to-one correspondence between the set of all the simple and spanning $\sqcup, \sqcap$-closed families $\mathcal{F} \subseteq 3^V$ on V with $(\emptyset, \emptyset) \in \mathcal{F}$ and the set of all the signed posets $\mathcal{P}$ on V such that each such $\mathcal{F}$ is the set of all the ideals of the corresponding signed poset $\mathcal{P}$. This will be a strengthening of the signed Birkhoff theorem of V. Reiner [6]. We also discuss the representations of general non-simple or non-spanning $\sqcup, \sqcap$-closed families.

2. Signed Posets and Ideals

Following Reiner [6], we define a signed poset and its ideal in an equivalent but slightly different way in terms of bidirected graph.

For a finite vertex set V and a finite arc set A we are given a boundary operator ∂ on A as follows. For each arc $a \in A$ there exist some vertices $v, w \in V$ such that one of the following three holds:

(1) $\partial a = v + w$ (arc a has two tails, one at v and one at w),

(2) $\partial a = -v - w$ (arc a has two heads, one at v and one at w),

(3) $\partial a = v - w$ (arc a has a tail at v and a head at w),

where the right-hand sides should be regarded as elements of the free module over the set $\mathbf{Z}$ of integers with a base V, and if $v = w$, we do not allow (3). Also, we assume that there do not exist any two distinct arcs $a, a' \in A$ such that $\partial a = \partial a'$. When $\partial a = \pm v \pm w$, we say arc a is incident to v (and w). If arc a is incident to only one vertex v, arc a is called a *selfloop* at v. We call $G = (V, A; \partial)$ a *bidirected graph.*

For the bidirected graph $G = (V, A; \partial)$ we further assume the following:

(i) There are no two arcs $a_1, a_2 \in A$ such that $\partial a_1 = -\partial a_2$.

(ii) For any two non-selfloop arcs $a_1, a_2 \in A$, if the sum of the absolute values of the coefficients of the vertices in $\partial a_1 + \partial a_2$ is two, then there exists an arc $a_3 \in A$ such that $\partial a_3 = \partial a_1 + \partial a_2$.

(iii) For any selfloop $a_1 \in A$ and non-selfloop $a_2 \in A$, if the sum of the absolute values of the coefficients of the vertices in $\partial a_1 + 2\partial a_2$ is two, then there exists a selfloop $a_3 \in A$ such that $\partial a_3 = \partial a_1 + 2\partial a_2$.

(iv) For any two selfloops $a_1, a_2 \in A$ incident to distinct vertices there exists an arc $a_3 \in A$ such that $2\partial a_3 = \partial a_1 + \partial a_2$.

Then, we call the bidirected graph $G = (V, A; \partial)$ a *signed poset* on V and denote it $\mathcal{P} = (V, A; \partial)$ as well. This definition is equivalent to the one given by Reiner [6]. A signed poset $\mathcal{P} = (V, A; \partial)$ on V is uniquely determined by $\partial A \equiv \{\partial a \mid a \in A\}$, i.e., a poset is uniquely determined up to the relabeling of the arc set.

For each $a \in A$ we denote by $\partial^+ a$ ($\partial^- a$) the set of the vertices that have positive (negative) coefficients in ∂a.

Now, we define an ideal of the signed poset $\mathcal{P} = (V, A; \partial)$ as follows. An element $(X, Y) \in 3^V$ is an *ideal* of $\mathcal{P}$ if it satisfies

$$< \partial a, (X, Y) > \leq 0 \quad (a \in A), \tag{2.1}$$

where ∂a and (X, Y) should be regarded as integer vectors in $\mathbf{Z}^V$ in a natural way, and $< \cdot, \cdot >$ as the ordinary inner product. Inequality (2.1) means that $(\partial^+ a \cap X) \cup (\partial^- a \cap Y) \neq \emptyset$ implies $(\partial^- a \cap X) \cup (\partial^+ a \cap Y) \neq \emptyset$. In Reiner's definition of ideal the inequality sign in (2.1) is reversed but we adopt the above definition due only to the consistency with our system of notations for ordinary posets and ideals (cf. [4]).

3. $\sqcup,\sqcap$-closed Families and Their Representations

Now, let us consider a simple and spanning $\sqcup,\sqcap$-closed family $\mathcal{F} \subseteq 3^V$ with $(\emptyset,\emptyset) \in \mathcal{F}$. (Recall the definition of a simple and spanning $\sqcup,\sqcap$-closed family given in Section 1).

For each $v \in V$ define

$$F(+v) = \sqcap\{(X,Y) \mid v \in X, (X,Y) \in \mathcal{F}\} \tag{3.1}$$

if there exists some $(X,Y) \in \mathcal{F}$ such that $v \in X$, and define

$$F(-v) = \sqcap\{(X,Y) \mid v \in Y, (X,Y) \in \mathcal{F}\} \tag{3.2}$$

if there exists some $(X,Y) \in \mathcal{F}$ such that $v \in Y$. If there is no $(X,Y) \in \mathcal{F}$ such that $v \in X$ (or $v \in Y$), then we define $F(+v) = \emptyset$ (or $F(-v) = \emptyset$). Note that since $\mathcal{F}$ is a spanning family on V, $F(+v)$ or $F(-v)$ is nonempty for any $v \in V$. Also, for any $W = (X,Y) \in \mathcal{F}$ we define

$$W^+ = X, \quad W^- = Y. \tag{3.3}$$

Given a simple and spanning $\sqcup,\sqcap$-closed family $\mathcal{F} \subseteq 3^V$ on V, we construct a bidirected graph $G(\mathcal{F})$ as follows. $G(\mathcal{F})$ has the vertex set V (since $\mathcal{F}$ is spanning). The arc set A is constructed by the following procedures (1)~(3):

(1) For each $v \in V$
  - (1a) if $F(-v) = \emptyset$, add a selfloop a at v such that $\partial a = -2v$,
  - (1b) if $F(+v) = \emptyset$, add a selfloop a at v such that $\partial a = 2v$.

(2) For each distinct $v, w \in V$
  - (2a) if $w \in F(+v)^+$, add an arc a such that $\partial a = v - w$,
  - (2b) if $w \in F(+v)^-$, add an arc a such that $\partial a = v + w$,
  - (2c) if $w \in F(-v)^+$, add an arc a such that $\partial a = -v - w$,
  - (2d) if $w \in F(-v)^-$, add an arc a such that $\partial a = -v + w$.

(3) For any two selfloops a_1 and a_2 that are incident to distinct vertices, add an arc a_3 such that $2\partial a_3 = \partial a_1 + \partial a_2$.

During the construction of the arc set A, if an arc to be added has already been constructed, then we skip the operation. It should be noted that if we do not require condition (iv) for the signed poset, or more precisely if we remove such an arc a_3 appearing in condition (iv), then we do not need procedure (3) given above.

To show that the bidirected graph $G(\mathcal{F})$ constructed above is a signed poset, we need some lemmas.

Lemma 3.1: *For any distinct $v, w \in V$*

(*a*) *if $w \in F(+v)^+$, then $v \notin F(+w)^+$,*
(*b*) *if $w \in F(+v)^-$, then $v \notin F(-w)^+$,*
(*c*) *if $w \in F(-v)^+$, then $v \notin F(+w)^-$,*
(*d*) *if $w \in F(-v)^-$, then $v \notin F(-w)^-$.*

(Proof) We show (a) (the proofs of the other cases are similar).

Suppose, on the contrary, that $w \in F(+v)^+$ and $v \in F(+w)^+$. Since $\mathcal{F}$ is simple, there is some $(X, Y) \in \mathcal{F}$ such that (1) $v \in X \cup Y$ and $w \notin X \cup Y$, or (2) $v \notin X \cup Y$ and $w \in X \cup Y$. In Case (1), if $v \in X$, then $(F(+v) \sqcap (X, Y))^+$ contains v but not w, which contradicts the minimality of $F(+v)$; and if $v \in Y$, then $(F(+w) \sqcup (X, Y))^+$ contains w but not v, which contradicts the minimality of $F(+w)$. Case (2) can be treated similarly as Case (1). □

For $(X_i, Y_i) \in \mathcal{F}$ $(i = 1, 2)$ we write $(X_1, Y_1) \sqsubseteq (X_2, Y_2)$ if $X_1 \subseteq X_2$ and $Y_1 \subseteq Y_2$. If we have $(X_1, Y_1) \sqsubseteq (X_2, Y_2)$ and $(X_1, Y_1) \neq (X_2, Y_2)$, we write $(X_1, Y_1) \sqsubset (X_2, Y_2)$.

Lemma 3.2: *For any distinct $v, w \in V$*

(*a*) *if $w \in F(+v)^+$, then $F(+w) \sqsubset F(+v)$,*
(*b*) *if $w \in F(+v)^-$, then $F(-w) \sqsubset F(+v)$,*
(*c*) *if $w \in F(-v)^+$, then $F(+w) \sqsubset F(-v)$,*
(*d*) *if $w \in F(-v)^-$, then $F(-w) \sqsubset F(-v)$.*

(Proof) If the inclusion $\sqsubset$ is replaced by the inclusion $\sqsubseteq$ with equality, then each assertion easily follows from the definition (the minimality) of $F(\cdot)$. The strict inclusion $\sqsubset$ is due to Lemma 3.1. □

Lemma 3.3: *For any distinct $v, w \in V$*

(*a*) *if $w \in F(+v)^+$ and $F(-w) \neq \emptyset$, then $v \in F(-w)^-$,*
(*b*) *if $w \in F(+v)^-$ and $F(+w) \neq \emptyset$, then $v \in F(+w)^-$,*
(*c*) *if $w \in F(-v)^+$ and $F(-w) \neq \emptyset$, then $v \in F(-w)^+$,*
(*d*) *if $w \in F(-v)^-$ and $F(+w) \neq \emptyset$, then $v \in F(+w)^+$.*

(Proof) We show (a) (the proofs of the other cases are similar).

Suppose, on the contrary, that $w \in F(+v)^+$, $F(-w) \neq \emptyset$. and $v \notin F(-w)^-$. Then we have $v \in (F(+v) \sqcup F(-w))^+$ and $w \notin (F(+v) \sqcup F(-w))^+$, which contradicts the minimality of $F(+v)$. □

Lemma 3.3 partly corresponds to Proposition 4.6 in [6].

Lemma 3.4: *For any distinct $v, w \in V$*

(*a*) *if $w \in F(+v)^+$ and $F(-v) = \emptyset$, then $F(-w) = \emptyset$,*
(*b*) *if $w \in F(+v)^-$ and $F(-v) = \emptyset$, then $F(+w) = \emptyset$,*
(*c*) *if $w \in F(-v)^+$ and $F(+v) = \emptyset$, then $F(-w) = \emptyset$,*
(*d*) *if $w \in F(-v)^-$ and $F(+v) = \emptyset$, then $F(+w) = \emptyset$.*

(Proof) We show (a) (the proofs of the other cases are similar).

Suppose, on the contrary, that $w \in F(+v)^+$, $F(-v) = \emptyset$ and $F(-w) \neq \emptyset$. Then from Lemma 3.3 we have $v \in F(-w)^-$,which contradicts the assumption that $F(-v) = \emptyset$. □

From Lemmas 3.1∼3.4 we have the following.

Theorem 3.5: *The bidirected graph $G(\mathcal{F}) = (V, A; \partial)$ defined above is a signed poset.*

(Proof) Let us check conditions (i)∼(iii) in the definition of a signed poset.

(i) This follows from Lemma 3.1 for non-selfloops and from the remark, given after the definition of $F(\cdot)$, for selfloops.

(ii) First, consider the case when $\partial a_1 + \partial a_2 = \pm v \pm w$ for some distinct $v, w \in V$. We treat only the case when $\partial a_1 + \partial a_2 = v + w$ since the other cases are treated similarly. Then we have $v - u \in \partial A$ and $u + w \in \partial A$. This means that we have one of the following (a)∼(f): (a) $v \in F(-u)^-, u \in F(+w)^-$; (b) $u \in F(+v)^+, w \in F(+u)^-$; (c) $v \in F(-u)^-, w \in F(+u)^-$; (d) $u \in F(+v)^+, u \in F(+w)^-$; (e) $F(+v) = \emptyset, F(-u) = \emptyset, w \in F(+u)^-$; (f) $F(+w) = \emptyset, F(+u) = \emptyset, v \in F(-u)^-$. Both (a) and (b) imply $v + w \in \partial A$ from Lemma 3.2. In Case (c), if both $F(+v) = \emptyset$ and $F(+w) = \emptyset$, then we have $v + w \in \partial A$ due to procedure (3). Moreover, if $F(+w) \neq \emptyset$ (or $F(+v) \neq \emptyset$), then Case (c) is reduced to Case (a) (or Case (b)) from Lemma 3.3. Case (d) is reduced to Case (a) (and Case (b)) from Lemma 3.3 since $F(-u) \neq \emptyset$ (and $F(+u) \neq \emptyset$). In Case (e) (or Case (f)) we have $F(+w) = \emptyset$ (or $F(+v) = \emptyset$) due to Lemma 3.4, so that we have $v + w \in \partial A$ by procedure (3).

Next, let us consider the case when $\partial a_1 + \partial a_2 = \pm 2v$ for some $v \in V$. We treat only the case when $\partial a_1 + \partial a_2 = 2v$ since the other case is treated similarly. Then we have $v + w \in \partial A$ and $v - w \in \partial A$ for some $w \in V$. If there is a selfloop a with $\partial a = 2v$, then we are finished. So, suppose that arcs a_1, a_2 with $\partial a_1 = v - w$ and $\partial a_2 = v + w$ are not constructed by procedure (3). Then we have (I) $v \in F(+w)^-$ or $w \in F(+v)^-$ and (II) $v \in F(-w)^-$ or $w \in F(+v)^+$. If we have $F(+v) \neq \emptyset$, from Lemma 3.3 (I) implies $w \in F(+v)^-$ and (II) implies $w \in F(+v)^+$, a contradiction. Hence, $F(+v) = \emptyset$.

(iii) This follows from Lemma 3.4.

(iv) This is due to the definition of $G(\mathcal{F})$. □

We now denote the signed poset $G(\mathcal{F})$ by $\mathcal{P}(\mathcal{F}) = (V, A; \partial)$.

Lemma 3.6: *Let $(X, Y) \in 3^V$ be an ideal of the signed poset $\mathcal{P}(\mathcal{F})$. Then we have*

$$\emptyset \neq F(+v) \sqsubseteq (X, Y) \qquad (v \in X) \tag{3.4}$$

$$\emptyset \neq F(-v) \sqsubseteq (X, Y) \qquad (v \in Y) \tag{3.5}$$

(Proof) Since (X, Y) is an ideal of $\mathcal{P}(\mathcal{F})$, relations (3.4) and (3.5) follow from the definition of $\mathcal{P}(\mathcal{F})$. □

Now, we show the following theorem.

Theorem 3.7: *The set of all the ideals of the signed poset $\mathcal{P}(\mathcal{F})$ coincides with the given $\mathcal{F}$.*

(Proof) Suppose that $(X, Y) \in 3^V$ is an ideal of $\mathcal{P}(\mathcal{F})$. From Lemma 3.6 we have

$$(X, Y) = (\sqcup_{v \in X} F(+v)) \sqcup (\sqcup_{v \in Y} F(-v)). \tag{3.6}$$

Hence, $(X, Y) \in \mathcal{F}$.

Conversely, suppose $(X, Y) \in \mathcal{F}$. Then we have

$$\emptyset \neq F(+v) \sqsubseteq (X, Y) \qquad (v \in X), \tag{3.7}$$

$$\emptyset \neq F(-v) \sqsubseteq (X, Y) \qquad (v \in Y). \tag{3.8}$$

If (X, Y) is not an ideal of $\mathcal{P}(\mathcal{F})$, then we have the following (I) or (II):

(I) For some $v \in X$ we have one of the following three:
- (a) There is a selfloop a such that $\partial a = 2v$.
- (b) There are a non-selfloop arc a and a vertex $w \notin Y$ such that $\partial a = v + w$.
- (c) There are a non-selfloop arc a and a vertex $w \notin X$ such that $\partial a = v - w$.

(II) For some $v \in Y$ we have one of the following three:
- (a) There is a selfloop a such that $\partial a = -2v$.
- (b) There are a non-selfloop arc a and a vertex $w \notin X$ such that $\partial a = -v - w$.
- (c) There are a non-selfloop arc a and a vertex $w \notin Y$ such that $\partial a = -v + w$.

Case (I-a) is impossible since $F(+v) \neq \emptyset$ for $v \in X$. In Case (I-b), we have $w \in F(+v)^-$ or $v \in F(+w)^-$. But $w \in F(+v)^-$ is impossible from (3.7) since $w \notin Y$. So, we must have $v \in F(+w)^-$, which implies $w \in F(+v)^- \subseteq Y$ (due to Lemma 3.3 and (3.7)), a contradiction. Similarly as Case (I-b), Case (I-c) also leads us to a contradiction. Case (II) can be treated similarly as Case (I).

Consequently, (X, Y) must be an ideal of $\mathcal{P}(\mathcal{F})$. □

Theorem 3.7 asserts that $\mathcal{P}(\cdot)$ defines a one-to-one mapping from the set of simple spanning $\sqcup, \sqcap$-closed families, on V, containing $(\emptyset, \emptyset)$ to the set of signed posets on V. We show that the mapping is also onto.

For a signed poset $\mathcal{P} = (V, A; \partial)$ and a vertex $v \in V$, when $2v \notin \partial A$, define

$$I(+v) = (\{w \mid v - w \in \partial A\} \cup \{v\}, \{w \mid v + w \in \partial A\}) \tag{3.9}$$

and when $-2v \notin \partial A$, define

$$I(-v) = (\{w \mid -v - w \in \partial A\}, \{w \mid -v + w \in \partial A\} \cup \{v\}). \tag{3.10}$$

Also, if $2v \in \partial A$ (or $-2v \in \partial A$), we define $I(+v) = \emptyset$ (or $I(-v) = \emptyset$). We can easily see that $I(+v)$ and $I(-v)$, if nonempty, are ideals of $\mathcal{P}$. We call $I(+v)$ the *positive principal ideal* at v and $I(-v)$ the *negative principal ideal* at v of $\mathcal{P}$. (In fact, for a simple and spanning $\sqcup, \sqcap$-closed family $\mathcal{F}$ on V and its corresponding signed poset $\mathcal{P}$ on V we have $I(+v) = F(+v)$ and $I(-v) = F(-v)$ for $v \in V$.)

Lemma 3.8: *Let $\mathcal{I}(\mathcal{P})$ be the set of all the ideals of a signed poset $\mathcal{P}$ on V. Then $\mathcal{I}(\mathcal{P})$ is a simple and spanning $\sqcup, \sqcap$-closed family on V with $(\emptyset, \emptyset) \in \mathcal{I}(\mathcal{P})$.*

(Proof) First, we show that $\mathcal{I}(\mathcal{P})$ is $\sqcup, \sqcap$-closed. Let (X_i, Y_i) $(i = 1, 2)$ be ideals of $\mathcal{P}$. Let us consider their intersection $(X_1, Y_1) \sqcap (X_2, Y_2)$. For any $a \in A$, if

$$(\partial^+ a \cap (X_1 \cap X_2)) \cup (\partial^- a \cap (Y_1 \cap Y_2)) \neq \emptyset, \tag{3.11}$$

then we have

$$(\partial^+ a \cap Y_1) \cup (\partial^- a \cap X_1) \neq \emptyset, \quad (\partial^+ a \cap Y_2) \cup (\partial^- a \cap X_2) \neq \emptyset \tag{3.12}$$

since (X_i, Y_i) $(i = 1, 2)$ are ideals. Since we do not have $\partial^- a \cap X_1 \neq \emptyset$ and $\partial^+ a \cap Y_2 \neq \emptyset$ (or $\partial^+ a \cap Y_1 \neq \emptyset$ and $\partial^- a \cap X_2 \neq \emptyset$) due to (3.11), it follows from (3.12) that we have $\partial^+ a \cap (Y_1 \cap Y_2) \neq \emptyset$ or $\partial^- a \cap (X_1 \cap X_2) \neq \emptyset$, i.e.,

$$(\partial^+ a \cap (Y_1 \cap Y_2)) \cup (\partial^- a \cap (X_1 \cap X_2)) \neq \emptyset. \tag{3.13}$$

Hence, $(X_1, Y_1) \sqcap (X_2, Y_2)$ is an ideal of $\mathcal{P}$. Let us now consider the reduced union $(X_1, Y_1) \sqcup (X_2, Y_2)$. Suppose that for an arc $a \in A$

$$(\partial^+ a \cap ((X_1 \cup X_2) - (Y_1 \cup Y_2))) \cup (\partial^- a \cap ((Y_1 \cup Y_2) - (X_1 \cup X_2))) \neq \emptyset. \tag{3.14}$$

Then we have $(\partial^+ a \cap Y_1) \cup (\partial^- a \cap X_1) \neq \emptyset$ or $(\partial^+ a \cap Y_2) \cup (\partial^- a \cap X_2) \neq \emptyset$, i.e.,

$$(\partial^- a \cap (X_1 \cup X_2)) \cup (\partial^+ a \cap (Y_1 \cup Y_2)) \neq \emptyset, \tag{3.15}$$

since (X_i, Y_i) $(i = 1, 2)$ are ideals. If $\partial^+ a \cap ((X_1 \cup X_2) \cap (Y_1 \cup Y_2)) \neq \emptyset$ (or $\partial^- a \cap ((X_1 \cup X_2) \cap (Y_1 \cup Y_2)) \neq \emptyset$), we must have $\partial^- a \cap (Y_1 \cup Y_2) \neq \emptyset$ (or $\partial^+ a \cap (X_1 \cup X_2) \neq \emptyset$) due to (3.14) and since (X_i, Y_i) $(i = 1, 2)$ are ideals. However, this together with (3.14) contradicts the assumption that (X_i, Y_i) $(i = 1, 2)$ are ideals. Hence, from (3.15) we have

$$(\partial^- a \cap ((X_1 \cup X_2) - (Y_1 \cup Y_2))) \cup (\partial^+ a \cap ((Y_1 \cup Y_2) - (X_1 \cup X_2))) \neq \emptyset. \quad (3.16)$$

Therefore, $(X_1, Y_1) \sqcup (X_2, Y_2)$ is an ideal.

Next, we show that $\mathcal{I}(\mathcal{P})$ is spanning. By the definition of the signed poset $\mathcal{P} = (V, A; \partial)$, for any $v \in V$ we have $2v \notin \partial A$ or $-2v \notin \partial A$. Therefore, there exists an ideal $I(+v)$ or $I(-v)$ for any $v \in V$ and hence $\mathcal{I}(\mathcal{P})$ is spanning since $\mathcal{I}(\mathcal{P})$ is $\sqcup, \sqcap$-closed.

Finally, we show that $\mathcal{I}(\mathcal{P})$ is simple. For any distinct $v, w \in V$, suppose $2v \notin \partial A$ without loss of generality. If $w \notin I(+v)^+ \cup I(+v)^-$, then we are done and if $w \in I(+v)^+ \cup I(+v)^-$, then $(I(+v)^+ - \{v\}, I(+v)^-)$ is a desired ideal that separates v and w.

We conclude the proof by noting that $(\emptyset, \emptyset)$ is also an ideal of $\mathcal{P}$. □

Lemma 3.9: *For two signed posets $\mathcal{P} = (V, A; \partial)$ and $\mathcal{P}' = (V, A'; \partial')$ on V, if $\partial A \neq \partial' A'$, then $\mathcal{I}(\mathcal{P}) \neq \mathcal{I}(\mathcal{P}')$.*

(Proof) If there exists some $v \in V$ such that $2v \in \partial A - \partial' A'$ (or $-2v \in \partial A - \partial' A'$), then the positive (or negative) principal ideal at v of $\mathcal{P}'$ is not contained in $\mathcal{I}(\mathcal{P})$. So, suppose that $\mathcal{P}$ and $\mathcal{P}'$ have the same set of selfloops. If there exist distinct $v, w \in V$ such that $v - w \in \partial A - \partial' A'$, then $2v \notin \partial' A'$ or $-2w \notin \partial' A'$. If $2v \notin \partial' A'$ (or $-2w \notin \partial' A'$), then for the positive (or negative) principal ideal W at v (or w) of $\mathcal{P}'$ we have $w \notin W^+$ (or $v \notin W^-$). Hence, W is not an ideal of $\mathcal{P}$. Other cases are treated similarly. □

From Theorem 3.7, Lemma 3.8 and Lemma 3.9 we have the main theorem.

Theorem 3.10: *There exists a one-to-one correspondence between the set of all the simple and spanning $\sqcup, \sqcap$-closed families $\mathcal{F} \subseteq 3^V$ on V with $(\emptyset, \emptyset) \in \mathcal{F}$ and the set of all the signed posets on V such that each such $\mathcal{F}$ is the set of all the ideals of the corresponding signed poset $\mathcal{P}$. In fact, such a one-to-one correspondence is obtained by making each $\mathcal{F}$ correspond to $\mathcal{P}(\mathcal{F})$.* □

This is a strengthening of the signed Birkhoff theorem of Reiner [6].

4. General $\sqcup, \sqcap$-closed Families

In this section we discuss the representations of non-simple or non-spanning $\sqcup, \sqcap$-closed families.

Consider any $\sqcup, \sqcap$-closed family $\mathcal{F} \subseteq 3^V$.

Lemma 4.1: *Every maximal element* $(X, Y) \in \mathcal{F}$ *has the same set* $X \cup Y$, *where the order among* $\mathcal{F}$ *is with respect to* $\sqsubseteq$.
(Proof) Since

$$(X_1 \cup (X_2 - Y_1), Y_1 \cup (Y_2 - X_1)) = ((X_1, Y_1) \sqcup (X_2, Y_2)) \sqcup (X_1, Y_1), \tag{4.1}$$

we have

$$(X_1 \cup (X_2 - Y_1), Y_1 \cup (Y_2 - Y_1)) \in \mathcal{F} \tag{4.2}$$

for any $(X_i, Y_i) \in \mathcal{F}$ $(i = 1, 2)$. Also, note that we have

$$(X_1, Y_1) \sqsubseteq (X_1 \cup (X_2 - Y_1), Y_1 \cup (Y_2 - X_1)), \tag{4.3}$$

$$X_1 \cup (X_2 - Y_1) \cup Y_1 \cup (Y_2 - X_1) = X_1 \cup X_2 \cup Y_1 \cup Y_2. \tag{4.4}$$

Therefore, the present lemma follows from (4.1)~(4.4). □

Due to Lemma 4.1, for any maximal $(X, Y) \in \mathcal{F}$ let us call $X \cup Y$ the *support* of $\mathcal{F}$.

We thus see that for a non-spanning $\sqcup, \sqcap$-closed family $\mathcal{F} \subseteq 3^V$ on V we can restrict the underlying set V to its support. Therefore, without loss of generality we can assume that $\mathcal{F}$ is spanning.

Now, consider any spanning $\sqcup, \sqcap$-closed family $\mathcal{F} \subseteq 3^V$. Define an equivalence relation $\sim$ on V as follows. For any $v, w \in V$ we have $v \sim w$ if and only if for each $(X, Y) \in \mathcal{F}$ either $v, w \in X \cup Y$ or $v, w \notin X \cup Y$. The equivalence classes associated with the equivalence relation $\sim$ give a partition $\Pi(\mathcal{F})$ of V. By the definition of the equivalence relation we see that each component $K \in \Pi(\mathcal{F})$ is divided into two sets K_1 and K_2 (either but not both possibly empty) such that for each $(X, Y) \in \mathcal{F}$ with $K \subseteq X \cup Y$ we have either

(1) $K_1 \subseteq X$ and $K_2 \subseteq Y$, or

(2) $K_1 \subseteq Y$ and $K_2 \subseteq X$.

Therefore, we should consider $\Pi(\mathcal{F})$ as a *double* partition, where each component $K \in \Pi(\mathcal{F})$ is further partitioned into two sets K_1 and K_2.

Choose a representative v_K from each component $K \in \Pi(\mathcal{F})$ and define

$$\hat{\mathcal{F}} = \{(\cup\{v_K \mid K \in \Pi(\mathcal{F}), v_K \in X\}, \cup\{v_K \mid K \in \Pi(\mathcal{F}), v_K \in Y\}) \mid (X, Y) \in \mathcal{F}\}. \tag{4.5}$$

We can easily show that $\hat{\mathcal{F}}$ is a simple and spanning $\sqcup, \sqcap$-closed family on $\hat{V} = \{v_K | K \in \Pi(\mathcal{F})\}$. We call $\hat{\mathcal{F}}$ a *simplification* of $\mathcal{F}$. It should be noted that the pair of the simplification $\hat{\mathcal{F}}$ and the double partition $\Pi(\mathcal{F})$ has the complete information to reconstruct the original $\mathcal{F}$ with $(\emptyset, \emptyset) \in \mathcal{F}$. That is, each $(\hat{X}, \hat{Y}) \in \hat{\mathcal{F}}$ is made correspond to $(X, Y) \in \mathcal{F}$ in such a way that

$$X = X_{11} \cup X_{12} \cup X_{21} \cup X_{22}, \tag{4.6}$$

$$Y = Y_{11} \cup Y_{12} \cup Y_{21} \cup Y_{22}, \tag{4.7}$$

where

$$\begin{aligned}
X_{11} &= \bigcup\{K_1 \mid v_K \in \hat{X}, v_K \in K_1, K \in \Pi(\mathcal{F})\}, &(4.8)\\
X_{12} &= \bigcup\{K_1 \mid v_K \in \hat{Y}, v_K \in K_2, K \in \Pi(\mathcal{F})\}, &(4.9)\\
X_{21} &= \bigcup\{K_2 \mid v_K \in \hat{Y}, v_K \in K_1, K \in \Pi(\mathcal{F})\}, &(4.10)\\
X_{22} &= \bigcup\{K_2 \mid v_K \in \hat{X}, v_K \in K_2, K \in \Pi(\mathcal{F})\}, &(4.11)\\
Y_{11} &= \bigcup\{K_1 \mid v_K \in \hat{Y}, v_K \in K_1, K \in \Pi(\mathcal{F})\}, &(4.12)\\
Y_{12} &= \bigcup\{K_1 \mid v_K \in \hat{X}, v_K \in K_2, K \in \Pi(\mathcal{F})\}, &(4.13)\\
Y_{21} &= \bigcup\{K_2 \mid v_K \in \hat{X}, v_K \in K_1, K \in \Pi(\mathcal{F})\}, &(4.14)\\
Y_{22} &= \bigcup\{K_2 \mid v_K \in \hat{Y}, v_K \in K_2, K \in \Pi(\mathcal{F})\}. &(4.15)
\end{aligned}$$

Finally, it should be noted that if a $\sqcup, \sqcap$-closed family $\mathcal{F}$ on V does not contain $(\emptyset, \emptyset)$, then for the minimum element (X_0, Y_0) of $\mathcal{F}$ define

$$\mathcal{F}' = \{(X - X_0, Y - Y_0) \mid (X, Y) \in \mathcal{F}\}. \qquad (4.16)$$

Then $\mathcal{F}'$ is a $\sqcup, \sqcap$-closed family on V with $(\emptyset, \emptyset) \in \mathcal{F}'$. Also, note that $\mathcal{F} \cup \{(\emptyset, \emptyset)\}$ is a simple and spanning $\sqcup, \sqcap$-closed family on V that contains $(\emptyset, \emptyset)$ as well.

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