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Economics of City Sizes in a Hierarchical Inter-urban System

by

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ECONOMICS OF CITY SIZES IN A HIERARCHICAL INTER-URBAN SYSTEM *

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[Abstract] This paper presents an open model of a simple hierarchical inter-urban system, from which a number of spatial structures, such as monocentric and multicentric patterns, can be endogenously derived. It is shown that the city size at some city-rank in the urban system will depend on not only the population demand-supply relationship of its own rank but also on that of other ranks as well. It is also shown that the market equilibrium of city sizes in monocentric and multicentric patterns would be inconsistent with the social optimum.

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Key words: Hierarchy, city size, equilibrium, optimum

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1. Introduction

In urban Japan, as the tremendous suburbanization keeps on in progress, the Tokyo metropolitan area becomes more and more large and complex. During the last decade, many urban policies and development projects have been proposed for the peripheral cities around Tokyo, aiming at the consolidation of multicentered decentralization of the area. In the practical planning of these peripheral cities, such as Yokohama, Chiba and Omiya, a demographic indicator is frequently used to measure the self-reliableness of their urban economies. It is called *Shujuhi* in Japanese, meaning the rate of employees working in the city to those residing in the same city. At present, the *Shujuhi* rates of most of the peripheral cities around Tokyo range from 0.7 to 0.9. This implies that among a hundred of employees residing in those cities, 10 to 30 persons are commuting to Tokyo for work. Therefore, in the planning of those cities, the future levels of their *Shujuhi* rates tend to be planed as one unit, that is, the city should and will have sufficient employment opportunities for its residents. The idea behind such kind of city planning, in terms of full employment that city population equals employee numbers, can be equivalently interpreted as that in the existing metropolitan area, the peripheral cities are too small in city size whereas the central city too large. Concerning this kind of important propositions, unfortunately, very few attempts have been made on the theoretical investigation and rigorous justification among the city planners and project consultants. In this sense, in the planning of a city's future, it is required to refer to urban economics about the city size developed in the recent years.

In modern urban economics, the city size is considered as the outcome of the city's aggregate production and its total spending. The analysis of city sizes involving both the production and cost sides began with Alonso's work (1971). The state of art of the economic theory concerning the city size can be summarized as follows. In the production-side or the population demand-side, there have been introduced a number of causes of city formation, such as scale economy [Mills (1967) and Dixit (1973)], localization economy [Henderson (1974) and Arnott (1979)], urbanization economy [Kanemoto (1980) and Abdel-Rahman (1990)], and recently product variety [Abdel-Rahman (1988) and Abdel-Rahman and Fujita (1993)]. On the other hand, in the cost-side or the population supply-side, a monocentric city is always supposed in which residents are supposed to commute to the central business district (CBD), where the production takes place [Henderson (1974), Kanemoto (1980), Abdel-Rahman (1988), and Abdel-Rahman and Fujita (1990)]. It has been shown that by the market principle, the equilibrium of city sizes will result from the trade-off between the aforementioned urban agglomeration economies, and the total costs that include urban opportunity cost, residents' consumption, and commuting cost. The question whether such market-resulted city sizes are too large or too small is investigated by comparing the equilibrium solution with the social optimum, which is derived from the maximization of the surplus of the city's aggregate production and total costs. [See Fujita (1989) for a recent survey] The main conclusion of the comparison is that, when the externalities in urban agglomeration economies are positive, the market city size will be smaller than the optimum; but when the externalities are negative, the equilibrium city size will become too large [Henderson (1974), Kanemoto (1980) and Abdel-Rahman (1990)].

The economic theory of city sizes developed so far is based on the monocentric urban model, in which the urban space considered is sufficiently vast so that cities will never overlap. So it becomes very doubtful that such a theory can be applied without any modification to the real world like the Tokyo metropolitan area, where most cities border each other and most residents in the peripheral cities (say Yokohama) are commuting to the central city (Tokyo). In view of this point, the introduction of the space factor (e.g., the location of cities) and the hierarchical aspect (e.g., the central city and peripheral cities) into the existing city size theory will be very important and interesting.

The purpose of this paper is to develop an open model of a hierarchical inter-urban system and to investigate principles of determining the city size in such an urban economy. Like Zheng (1990), which is basically a closed model, we shall consider a simple hierarchy with two ranks of cities in a one-dimensional region. At the center of the region there is a port that provides imported goods for two regional industries. The first industry possesses only one firm that is located at the center. The second industry has two identical establishments located symmetrically on both sides of the center. Each industry supplies a consumption good to the national (or international) market and is assumed to communicate with all the residents (including agricultural residents) in the region. There exist two types of urban households, each of which consumes the two consumption goods and a fixed amount of land, providing workers for a business firm. The first industry and urban households form the first-rank (or central) city, while the second industry and urban households constitute the second-rank (or peripheral) cities. There are three possible spatial patterns obtainable from the model: a monocentric pattern, where the second-rank firms (firms form the second-rank city) locate very near to or just at th regional center; a multicentric pattern, where the second-rank firms locate away from the center but the first- and second-rank cities border each other; and a separate pattern, where the second-rank firms locate so distantly that an agricultural area exists between the first- and second-rank cities. The last case, in fact, is just what has been considered in the traditional city size theory reviewed right now. By comparative statics analysis, we shall show that, like the determination of the spatial structure, city sizes will also depend upon the cost rate of transportation to communication. City sizes of different ranks are interdependent in the sense that the city size of one city rank is determined by not only its own income level but also that of other city ranks as well. Furthermore, we shall also calculate the optimal solution for the city size and show that the (in)consistency between the equilibrium and optimum city sizes will rely on what spatial structure the urban space is of. Particularly, since the externalities in production are not considered, the market city sizes of the separate pattern is found to be the same as the optimum. In the multicentric pattern, however, the market city sizes would be larger or smaller than the optimum, which depends on the scale of cities relative to the rest of the system. And in the monocentric pattern, the market city size of the central city tends to be too large whereas that of the peripheral city too small, by comparing to the optimum.

The paper is organized as follows. In Section 2 we shall propose an open model for the simple hierarchical inter-urban system, and show the equilibrium solution. In Sections 3 and 4, comparative statics of the equilibrium and the optimum solution will be demonstrated, respectively. In Section 5, we shall conclude the paper and suggest some further extension directions of the work.

2. Equilibrium city sizes

2.1. The model

2.1.1. The hierarchy of cities

The hierarchy of cities considered here is similar to that in Zheng (1990). Suppose there is a long strip of homogeneous agricultural land. Its width is one unit of distance and its length, let to be $2B$ units, is sufficiently long. The central point of the narrow strip is considered as the origin, so its two ends read $-B$ and B, respectively. For simplicity, we think an urban system of two-rank cities on this one-dimensional space. In the first rank, there is only one city (called as the first-rank city or the central city) that contains a business firm and a class of households supplying workers to the firm. In the second rank, two identical cities (called as the second-rank cities or the peripheral cities) exist symmetrically about the center; each has a firm and a class of households as employees of the firm. The first-rank city is assumed to have located at the center, while the location of the second-rank cities is to be determined endogenously from the model.

2.1.2. Business firms

It is assumed that there is a port at the regional center that provides the imported goods for the business firms. The firm in the two-rank cities produce two different consumption goods by using imported goods and labor as inputs.

Since in the real business world, firms frequently interact with the local residents for their marketing, advertising and recruiting, we assume that the firms considered here need to communicate with all the residents in the region and that the communication bears a cost that has to be deduced from their profit. This setting, in fact, implies a spatial externality between firms and households, which seems to be hardly internalized by the market mechanism or government intervention. More specifically, it is assumed that the cost of communication between firms and households is proportional to the distance between them. So, for a firm at z, the total cost of communication with the residents living in segment $[f_1, f_2]$ can be given by

$$cT(z) = \int_{f_1}^{f_2} h(x)|x - z|\, dx, \tag{1}$$

where c is the communication cost per unit distance, $h(x)$ the density of households at x, and $T(z)$ the total communication distance for the firm at z.

For simplicity, the firms are assumed to have fixed-coefficient production technology. That is, if, for the first-rank firm, Q_1, N_1, and S_1 denote the amounts of the product, labor, and imported goods, we have $Q_1 = \min\ \{a_1N_1,\ b_1S_1\}$, where a_1 and b_1 are positive parameters. And its profit can be written as

$$\begin{aligned}\pi_1 &= p_1Q_1 - w_1N_1 - S_1 - cT_1 \\ &= p_1a_1N_1 - w_1N_1 - \frac{a_1}{b_1}N_1 - cT_1,\end{aligned} \tag{2}$$

where p_1 is the price of the product, w_1 is the wage, and the price of imported goods is one unit as the *numeraire*. T_1 is th total communication distance that can be expressed as

$$T_1 = \int_{-B}^{B} h(x)|x|\, dx, \tag{3}$$

Similarly, for firms in the second-rank cities, let Q_2, N_2, and S_2 denote the amounts of the product, labor, and imported goods, respectively, its production function can be written as $Q_2 = \min\ \{a_2N_2,\ b_2S_2\}$, and its profit as

$$\begin{aligned}\pi_2 &= p_2Q_2 - w_2N_2 - (1+kz)S_2 - cT_2(z) \\ &= p_2a_2N_2 - w_2N_2 - (1+kz)\frac{a_2}{b_2}N_2 - cT_2(z),\end{aligned} \tag{4}$$

where a_2 and b_2 are positive parameters, p_2 is the product price, w_2 is the wage, z is the distance from the firm's location to the center, and k is the transportation cost of imported goods. $T_2(z)$ is the total communication distance given as

$$T_2(z) = \int_0^B h(x)|x - z|\, dx, \qquad (5)$$

which implies the communication scope for the firm on the right side of the center. The profit function for the firm on the left side follows by a similar argument.

2.1.3.Households

Households considered in the linear region are divided into three classes. The first type of household consumes L_1 fixed units of land, C_{11} units of goods as the first-rank firm supplies, and C_{12} units of goods as the second-rank firms produce. The consumption goods are supplied from the national (or international) market but not necessarily directly from the firms of the region. The utility function of each household is

$$U_1 = C_{11}^{\alpha} C_{12}^{\beta} L_1, \qquad (6)$$

where α and β are positive parameters satisfying $\alpha + \beta = 1$. Each household of this type is supposed to provide one worker for the first-rank firm, from which wages are gained as the only source of income. The budget of a household at x is given by

$$w_1 - t|x| = p_1 C_{11} + p_2 C_{12} + R_1(x) L_1, \qquad (7)$$

where t is the commuting cost per unit distance and $R_1(x)$ is the land rent at x.

The second type of household is those supplying workers for the second-rank firms. Denote L_2, C_{21}, and C_{22} as the fixed units of land, the units of the two goods as the first- and second-rank firms supply, respectively, the utility function can be written as

$$U_2 = C_{21}^{\alpha} C_{22}^{\beta} L_2, \qquad (8)$$

of which the only difference from (6) is the amount of land consumed. The budget of a household of this type at x is given as

$$w_2 - t|x - z| = p_1 C_{21} + p_2 C_{22} + R_2(x) L_2, \qquad (9)$$

where $R_2(x)$ is the land rent at x.

The third type of household is those engaged in agriculture. It is assumed that each household of this type consumes only L_3 fixed units of land and sell their agricultural good at a price of R_A per unit land used.

Finally, concerning land preference of the three types of households, we assume that $L_1 < L_2 < L_3$, for simplicity.

2.2. Market equilibrium

Suppose the inter-urban system considered here is a small open economy. The wage levels, w_1 and w_2, and the prices of the products, p_1 and p_2, can be treated as exogenously given from the national (or international) labor and product markets. On the household-side, the utility levels of urban households equals that of the rest of the economy, which are also exogenously fixed. The numbers of the two types of urban households, denoted as N_1 and $2N_2$, are to be determined endogenously from the model. As to the number of the agricultural households, denoted as $2N_3$, if we assume the number is sufficiently large so that any space without urban activities in the linear region will be occupied by this type of household, N_3 can be easily calculated from the following relation

$$2B = N_1L_1 + 2N_2L_2 + 2N_2L_3. \tag{10}$$

First of all, let us consider the behavior of business firms. Because of fixed-coefficient technology and the small open hypothesis, when the populations, N_1 and N_2, are determined from the demand-supply relation, with the residential distribution being known, the profits of the two-rank firms can then be calculated by (2) and (4). The only unknown is the location of the second-rank firm, denoted as z, which, from (4), can be solved by

$$\frac{\partial \pi_2}{\partial z} = -k\frac{a_2}{b_2}N_2 - c\frac{\partial T_2(z)}{\partial z} = 0. \tag{11}$$

The equilibrium conditions in the labor market are

$$N_1 = \int_{X_1}\frac{dx}{L_1}, \tag{12}$$

$$N_2 = \int_{X_2}\frac{dx}{L_2}, \tag{13}$$

where X_1 and X_2 represent the areas where the two types of households are located.

Now turning to the behavior of two urban households, we find that under their budget constraints, maximization of utility with respect to the consumption of goods will yield the following bid-rent functions

$$R_1(x) = \frac{1}{L_1}\left[w_1 - tx - \frac{u_1}{L_1}\left(\frac{p_1}{\alpha}\right)^{\alpha}\left(\frac{p_2}{\beta}\right)^{\beta}\right], \tag{14}$$

$$R_2(x) = \frac{1}{L_2}\left[w_2 - t|x-z| - \frac{u_2}{L_2}\left(\frac{p_1}{\alpha}\right)^{\alpha}\left(\frac{p_2}{\beta}\right)^{\beta}\right], \tag{15}$$

where u_1 and u_2 are the utility levels of the two households exogenously given from the rest of the economy.

Suppose the land market is perfectly competitive and agricultural rent equals R_A. According to Alonso's (1964) bid-rent theory, the spatial distribution of residents can be determined by the land-rent profile, denoted as $\Phi(x)$, which is the upper-envelope of all households' bid-rent curves. That is

$$\Phi(x) = \max\{R_1(x),\ R_2(x),\ R_A\}, \tag{16}$$

$$\Phi(x) = R_1(x), \quad \text{if } h(x) = \frac{1}{L_1}, \tag{17}$$

$$\Phi(x) = R_2(x), \quad \text{if } h(x) = \frac{1}{L_2}, \tag{18}$$

$$\Phi(x) = R_A, \quad \text{if } h(x) = \frac{1}{L_3}, \tag{19}$$

In short, the inter-urban system involving the following variables as N_1, N_2, z, and the boundaries of cities can be solved from Equations (11)-(19).

To solve for the aforementioned variables, we need to specify the possible spatial configurations of the inter-urban system. In view of the shapes of land rent curves $R_1(x)$, $R_2(x)$, and R_A, and their spatial combination, there turn out to be three possible spatial patterns to appear in the linear region.

(i) Separate pattern: the first- and second-rank cities are spatially separate so that an agricultural area exists between them. (See Fig.1)

(ii) Multicentric pattern: the first- and second-rank cities border each other, and the center of each city is in its own area. (See Fig.2)

(iii) Monocentric pattern: the first- and second-rank cities border each other but all city centers are located within the first-rank city's area. (See Fig.3)

In the following, we shall show the equilibrium conditions of city sizes for these three spatial patterns, respectively.

2.2.1. Separate pattern

As shown by Fig.1, for the spatial symmetry of the structure, only the right side of the regional center is discussed here. Let f_1 denote the boundary of the first-rank city, f_2 and f_3 those of the second-rankcity, we have

$$f_1 = \frac{1}{2} N_1 L_1, \tag{20}$$

$$f_3 - z = z - f_2 = \frac{1}{2} N_2 L_2. \tag{21}$$

Fig.1 is about here

On the other hand, at f_1 and f_3 there exist $R_1(f_1) = R_A$ and $R_2(f_3 - z) = R_A$. By using (14) and (15) we get

$$w_1 - \frac{t}{2} N_1 L_1 - R_A L_1 - D_1 = 0, \tag{22}$$

$$w_2 - \frac{t}{2} N_2 L_2 - R_A L_2 - D_2 = 0, \tag{23}$$

where D_1 and D_2 are positive constants defined as

$$D_i \equiv \frac{u_i}{L_i} \left(\frac{p_1}{\alpha} \right)^{\alpha} \left(\frac{p_2}{\beta} \right)^{\beta}, \qquad i = 1,\ 2. \tag{24}$$

For such a spatial pattern, the total communication distance of the second-rank firm is expressed as follows

$$T_2(z) = \int_0^{f_1} \frac{z - x}{L_1} dx + \int_{f_1}^{f_2} \frac{z - x}{L_3} dx + \int_{f_2}^{f_3} \frac{|x - z|}{L_2} dx + \int_{f_3}^{B} \frac{x - z}{L_3} dx, \tag{25}$$

from which we derive

$$\frac{\partial T_2(z)}{\partial z} = \frac{f_1}{L_1} + \frac{1}{L_2}(2z - f_2 - f_3) + \frac{1}{L_3}(f_2 + f_3 - f_1 - B). \tag{26}$$

Substituting (26) into (11) and using (20)-(21), we have

$$z = -\frac{a_2}{2b_2}\left(\frac{k}{c}\right) N_2 L_3 - \frac{1}{4} N_1 (L_3 - L_1) + \frac{1}{2} B. \tag{27}$$

So, from (22), (23) and (27) we can solve for N_1, N_2 and z.

The necessary condition for this spatial pattern is $f_2 > f_1$, which, by using (20)-(21), means

$$z > \frac{1}{2}(N_1 L_1 + N_2 L_2). \tag{28}$$

By using (22), (23) and (27), it can be shown that the necessary and sufficient conditions for the equilibrium of the separate pattern are the following three inequalities (see Appendix A for the derivation)

$$w_1 - R_A L_1 - D_1 > 0, \tag{29}$$

$$w_2 - R_A L_2 - D_2 > 0, \tag{30}$$

$$(L_1 + L_3) \frac{w_1 - R_A L_1 - D_1}{t L_1} + 2\left[\frac{a_2}{b_2}\left(\frac{k}{c}\right) L_3 + L_2\right] \frac{w_2 - R_A L_2 - D_2}{t L_2} < B. \tag{31}$$

2.2.2.Multicentric pattern

Note that in this pattern the first- and second-rank cities border each other. (See Fig.2) The boundaries can be expressed as

$$f_1 = f_2 = \frac{1}{2}N_1L_1, \tag{32}$$

$$f_3 = \frac{1}{2}N_1L_1 + N_2L_2. \tag{33}$$

Fig. 2 is about here

Replacing (32) and (33) into the bid-rent functions of (14) and (15), we have

$$w_1 - \frac{t}{2}N_1L_1 - R_1(f_1)L_1 - D_1 = 0, \tag{34}$$

$$w_2 - t(\frac{1}{2}N_1L_1 + N_2L_2 - z) - R_2(f_3 - z)L_2 - D_2 = 0. \tag{35}$$

Since at f_1 and f_3 there exist $R_1(f_1) = R_2(z\text{-}f_1)$ and $R_2(f_3\text{-}z) = R_A$, we get

$$\begin{aligned} R_1(f_1) &= R_2(z - f_1) \\ &= \frac{1}{L_2}\left[w_2 - t(z - f_1) - D_2\right] \\ &= \frac{1}{L_2}\left[w_2 - t(f_3 - z) - D_2 + t(f_3 - z) - t(z - f_1)\right] \\ &= R_2(f_3 - z) + \frac{t}{L_2}(f_1 + f_3 - 2z) \\ &= R_A + \frac{t}{L_2}(N_1L_1 + N_2L_2 - 2z). \end{aligned} \tag{36}$$

Thus, (34) and (35) can be rewritten as

$$w_1 - \frac{t}{2}N_1L_1 - \frac{tL_1}{L_2}(N_1L_1 + N_2L_2 - 2z) - R_AL_1 - D_1 = 0, \tag{37}$$

$$w_2 - t(\frac{1}{2}N_1L_1 + N_2L_2 - z) - R_AL_2 - D_2 = 0. \tag{38}$$

In this case, the total communication distance of the second-rank firm and its first derivative have the following expressions

$$T_2(z) = \int_0^{f_1}\frac{z - x}{L_1}dx + \int_{f_1}^{f_3}\frac{|x - z|}{L_2}dx + \int_{f_3}^{B}\frac{x - z}{L_3}dx, \tag{39}$$

$$\frac{\partial T_2(z)}{\partial z} = \frac{f_1}{L_1} + \frac{1}{L_2}(2z - f_1 - f_3) + \frac{1}{L_3}(B - f_3). \tag{40}$$

So, from (11) we obtain

$$z = -\frac{a_2}{2b_2}\left(\frac{k}{c}\right)N_2L_2 + \frac{1}{2}(N_1L_1 + N_2L_2) - \frac{1}{4}N_1L_2 + \frac{L_2}{2L_3}(B - \frac{1}{2}N_1L_1 - N_2L_2). \tag{41}$$

By solving (37), (38) and (41) we can get the solution of N_1, N_2 and z.

To see the necessary and sufficient conditions for this case, we note that the second-rank firm locates within its own area, or $z > f_1$, and that the second-rank city's area tends to be asymmetric to the location of the second-rank firm, i.e., $(z-f_1) \le (f_3-z)$. That is, there must exist the following inequalities

$$z > \frac{1}{2}N_1L_1, \tag{42}$$

$$z \le \frac{1}{2}(N_1L_1 + N_2L_2). \tag{43}$$

Similar to the separate pattern, we can show that the necessary and sufficient conditions for the equilibrium of the multicentric pattern are as follows (see Appendix A)

$$\frac{1}{L_1}(w_1 - R_AL_1 - D_1) > \frac{1}{2L_2}(w_2 - R_AL_2 - D_2), \tag{44}$$

$$w_2 - R_AL_2 - D_2 > 0, \tag{45}$$

$$(L_1 + L_3)\frac{w_1 - R_AL_1 - D_1}{tL_1} + 2\left[\frac{a_2}{b_2}\left(\frac{k}{c}\right)L_3 + L_2\right]\frac{w_2 - R_AL_2 - D_2}{tL_2} \ge B, \tag{46}$$

$$(L_1 + L_3)\frac{w_1 - R_AL_1 - D_1}{tL_1} + \left[\frac{a_2}{b_2}\left(\frac{k}{c}\right)L_3 + L_2 - \frac{L_1 + 3L_3}{2}\right]\frac{w_2 - R_AL_2 - D_2}{tL_2} < B. \tag{47}$$

2.2.3. Monocentric pattern

As Fig.3 shows, in this pattern the first- and second-rank cities share one border, we can express the urban boundaries as

$$f_1 = f_2 = \frac{1}{2}N_1L_1, \tag{48}$$

$$f_3 = \frac{1}{2}N_1L_1 + N_2L_2. \tag{49}$$

Fig.3 is about hear

The bid-rent functions at these boundaries can be written as

$$w_1 - \frac{t}{2}N_1L_1 - R_1(f_1)L_1 - D_1 = 0, \tag{50}$$

$$w_2 - t(\frac{1}{2}N_1L_1 + N_2L_2 - z) - R_2(f_3 - z)L_2 - D_2 = 0. \tag{51}$$

Note that $R_1(f_1) = R_2(f_1\text{-}z)$ and $R_2(f_3\text{-}z) = R_A$, we have

$$\begin{aligned} R_1(f_1) &= R_2(f_1 - z) \\ &= \frac{1}{L_2}[w_2 - t(f_1 - z) - D_2] \\ &= \frac{1}{L_2}[w_2 - t(f_3 - z) - D_2 + t(f_3 - z) - t(f_1 - z)] \\ &= R_2(f_3 - z) + \frac{t}{L_2}(f_3 - f_1) \end{aligned}$$

$$= R_A + tN_2. \tag{52}$$

Thus, we can rewrite (50) and (51) as

$$w_1 - \frac{t}{2}N_1L_1 - tN_2L_1 - R_AL_1 - D_1 = 0, \tag{53}$$

$$w_2 - t(\frac{1}{2}N_1L_1 + N_2L_2 - z) - R_AL_2 - D_2 = 0. \tag{54}$$

In this pattern, the total communication distance of the second-rank firm and its first derivative can be expressed as

$$T_2(z) = \int_0^{f_1} \frac{|x-z|}{L_1} dx + \int_{f1}^{f_3} \frac{x-z}{L_2} dx + \int_{f_3}^{B} \frac{x-z}{L_3} dx, \tag{55}$$

$$\frac{\partial T_2(z)}{\partial z} = \frac{1_1}{L_1}(2z - f_1) - \frac{1}{L_2}(f_1 - f_3) - \frac{1}{L_3}(B - f_3). \tag{56}$$

From (11) and (56), we obtain

$$z = -\frac{a_2}{2b_2}\left(\frac{k}{c}\right)N_2L_1 + \frac{1}{4}N_1L_1 + \frac{1}{2}N_2L_1 + \frac{L_1}{2L_3}(B - \frac{1}{2}N_1L_1 - N_2L_2). \tag{57}$$

Equations (53), (54) and (57) will give the equilibrium solution for N_1, N_2 and z.

The necessary conditions for the spatial pattern is that the second-rank firm locates within the first-rank city's area, i.e., $z < f_1$, or

$$z < \frac{1}{2}N_1L_1. \tag{58}$$

The necessary and sufficient conditions for the equilibrium of the monocentric pattern can be derived as follows

$$\frac{1}{L_1}(w_1 - R_AL_1 - D_1) > \frac{1}{L_2}(w_2 - R_AL_2 - D_2), \tag{59}$$

$$L_2(w_2 - R_AL_2 - D_2) > w_1 - R_AL_1 - D_1 > 0. \tag{60}$$

$$L_2(L_1 + L_3)\frac{w_1 - R_AL_1 - D_1}{t(L_2 - L_1)} + \left[\frac{a_2}{b_2}\left(\frac{k}{c}\right)L_3 + L_2 - L_3 - \frac{(L_1 + L_3)L_2^2}{L_2 - L_1}\right]\frac{w_2 - R_AL_2 - D_2}{tL_2} > B. \tag{61}$$

To sum up, in the previous analysis we have shown that *when the wage income levels (w_1 and w_2) and the cost rate of transportation to communication (k/c) satisfy certain conditions, we would have the equilibrium of separate pattern, multicentric pattern, and monocentric pattern to appear in the hierarchical inter-urban system.* However, unlike the results obtained in Zheng (1990), the correspondence between the spatial structure and the income levels or the cost rate, say as the cost rate of transportation to communication is increasing to some extent the monocentricity will definitely appear, is no longer so clear. This is because that in the present model, the economy is set to be open and the population can be endogenously adjusted to various levels of income and cost rate, so even lower income levels and cost rate would possibly result in the concentrated monocentric or multicentric structures.

3. Comparative statics

In this section, we shall conduct a comparative statics analysis to show the relationships among variables and parameters of the equilibrium in the hierarchical inter-urban system. In particular, our emphasis will be placed on the impacts of changes in income and in the cost rate of transportation to communication on the spatial structure and city sizes, the relation of city sizes and spatial concentration, and the interdependence between city sizes of different ranks.

3.1. Separate pattern

As is shown in the last section, the urban system of this spatial pattern is described by Equations (22), (23) and (27). Comparative statics analysis (see Appendix B) gives the following results

$$\frac{dN_1}{dw_1} > 0, \tag{62}$$

$$\frac{dN_2}{dw_2} > 0, \tag{63}$$

$$\frac{\partial z}{\partial w_1} < 0, \qquad \frac{\partial z}{\partial w_2} < 0, \qquad \frac{\partial z}{\partial \left(k/c\right)} < 0. \tag{64}$$

It becomes clear that in this pattern, the city size of some city rank is directly related with the income level of its own city rank, independent from that of other city ranks. This is because that cities in this case are spatial separate from each other, so a change in one city will not affect other cities. Traditional city size theory reviewed in Section 1 is mainly concerned with this kind of cities.

3.2. Multicentric pattern

Concerning Equations (37), (38) and (41), which describe the urban system of the multicentric pattern, by comparative statics we have (see Appendix B)

$$\frac{\partial N_1}{\partial w_1} > 0, \qquad \frac{\partial N_1}{\partial w_2} < 0, \qquad \frac{\partial N_1}{\partial \left(k/c\right)} < 0, \tag{65}$$

$$\frac{\partial N_2}{\partial w_1} < 0, \qquad \frac{\partial N_2}{\partial w_2} > 0, \qquad \frac{\partial N_2}{\partial \left(k/c\right)} < 0, \tag{66}$$

$$\frac{\partial z}{\partial w_1} >< 0, \qquad \frac{\partial z}{\partial w_2} >< 0, \qquad \frac{\partial z}{\partial \left(k/c\right)} < 0. \tag{67}$$

So, in this pattern, the city size of some city rank depends not only on the income level of its own rank, but also on that of other city ranks, and on the cost rate of transportation to communication in the region as well. The reason for this result is that cities of different ranks in this case are spatially interdependent. If the income level of a city in question increases, it will pull people to migrate into the city. But, if the income of other cities is raised, their population will increase with the urban space being expanded. Thus the city in question will shrink and its size will be decreased to some extent.

To see the relationship among N_1, N_2 and z, we need to assume a short run period during which some of the variables can be considered as constants. Firstly, let us fix the location of the second-rank firm, i.e., the variable z is supposed to be constant. From Equations (37) and (38), through a simple comparative statics analysis, we can easily obtain

$$\frac{dN_1}{dz} > 0, \qquad \frac{dN_2}{dz} > 0. \tag{68}$$

That is, the more the spatial structure is concentrated and the larger the city will be.This characteristics is very important in showing the difference between the equilibrium and optimal solutions in the next section.

Second, besides z we also fix N_2, the population of the second-rank city. Then from (37) we easily get

$$\frac{dN_1}{dN_2} < 0. \tag{69}$$

If we fix the population of the first-rank city, N_1, instead of N_2. Equation (38) will give

$$\frac{dN_2}{dN_1} < 0. \tag{70}$$

(69) and (70) imply the relation between city sizes of different ranks in the short run that the distance between them is fixed. That is, the increasing (decreasing) of the city size at one city rank will result in the decreasing (increasing) in the city size of other ranks.

3.3. Monocentric pattern

The urban system of this spatial pattern is described by Equations (53), (54) and (57). Through comparative statics analysis (see Appendix B) we have the following results

$$\frac{\partial N_1}{\partial w_1} > 0, \qquad \frac{\partial N_1}{\partial w_2} < 0, \qquad \frac{\partial N_1}{\partial \left(k/c\right)} > 0, \tag{71}$$

$$\frac{\partial N_2}{\partial w_1} < 0, \qquad \frac{\partial N_2}{\partial w_2} > 0, \qquad \frac{\partial N_2}{\partial \left(k/c\right)} < 0, \tag{72}$$

$$\frac{\partial z}{\partial w_1} > 0, \qquad \frac{\partial z}{\partial w_2} < 0, \qquad \frac{\partial z}{\partial \left(k/c\right)} < 0. \tag{73}$$

Similar to the multicentric pattern, we can conclude that the city size of some city rank depends on the income levels of different city ranks and on the transportation-to-communication cost rate. As for the difference from the multicentric pattern, we find that the impacts of changes in k/c on city sizes are different. This stems from the difference in the spatial characteristics of the two patterns.

As a further result of the difference, the relationship between the extent of spatial concentration and the city size also differs from that in the multicentric pattern. In fact, if we fix the location of the second-rank firm, from (53) and (54) we have

$$\frac{dN_1}{dz} < 0, \qquad \frac{dN_2}{dz} > 0, \tag{74}$$

which are different from (68) of the multicentric pattern. So, in the monocentric pattern, when the spatial structure becomes a little much concentrated (i.e., the value of z decreases), the size of the first-rank (or central) city will increase whereas that of the second-rank (or peripheral)city will decrease.

The policy implication from this analytic result is meaningful. For a city planner, before putting an urban deconcentration-promoting policy into practice, it is needed to confirm the characteristics of the existing spatial structure of the area in question. If it is a monocentric pattern as defined in this paper, the deconcentration-promoting policy will be effective in that it would increase the size of peripheral cities and decrease that of the central city. But, if it happens to be the multicentric pattern, such a policy becomes no longer appropriate because it may even make the central city much larger.

4. Optimal city sizes

In this section, we shall present an optimal solution of city sizes for the inter-urban system and compare it with the equilibrium obtained previously. Optimality here is defined as such an allocation of regional resources that the regional net revenue (surplus), which includes the profits of business firms and the total differential land rent, is maximized.

Noting that the spatial structure of the urban system is symmetrical, and that on the right-hand side of the regional center, the fringe of the first-rank city is denoted as f_1 and those of the second-rank city as f_2 and f_3, we can express the optimality as the following maximization problem

$$\max \phi = p_1 Q_1 - w_1 N_1 - S_1 - cT_1$$

$$+2[p_2Q_2 - w_2N_2 - (1+kz)S_2 - cT_2(z)]$$
$$+2\int_0^{f_1}\frac{1}{L_1}(w_1 - tx - p_1C_{11} - p_2C_{12} - R_AL_1)\,dx$$
$$+2\int_{f_2}^{f_3}\frac{1}{L_2}(w_2 - t|x-z| - p_1C_{21} - p_2C_{22} - R_AL_2)\,dx, \tag{75}$$

with respect to C_{11}, C_{12}, C_{21}, C_{22}, w_1, w_2, f_1, f_2, f_3, and z. The constraints for the maximization are

$$Q_1 = \min\{a_1N_1,\ b_1S_1\}, \tag{76}$$
$$Q_2 = \min\{a_2N_2,\ b_2S_2\}, \tag{77}$$
$$u_1 = C_{11}^{\alpha}C_{12}^{\beta}L_1, \tag{78}$$
$$u_2 = C_{21}^{\alpha}C_{22}^{\beta}L_2, \tag{79}$$

where u_1 and u_2 are the target utility levels arbitrarily chosen. The first-order conditions can be calculated as follows

$$C_{i1} = \frac{u_i}{L_1}\left(\frac{\alpha p_2}{\beta p_1}\right)^{\beta}, \quad i=1,\ 2, \tag{80}$$
$$C_{i2} = \frac{u_i}{L_1}\left(\frac{\beta p_1}{\alpha p_2}\right)^{\alpha}, \quad i=1,\ 2, \tag{81}$$
$$f_1 = \frac{1}{2}N_1L_1, \tag{82}$$
$$f_3 - f_2 = N_2L_2, \tag{83}$$
$$w_1 - tf_1 - p_1C_{11} - p_2C_{12} - R_AL_1 = 0, \tag{84}$$
$$w_2 - t|f_2 - z| - p_1C_{21} - p_2C_{22} - R_AL_2 = 0, \tag{85}$$
$$w_2 - t|f_3 - z| - p_1C_{21} - p_2C_{22} - R_AL_2 = 0, \tag{86}$$
$$-k\frac{a_2}{b_2}N_2 - c\frac{\partial T_2(z)}{\partial z} - \frac{\partial}{\partial z}\int_{f_2}^{f_3}\frac{t}{L_2}|x-z|\,dx = 0. \tag{87}$$

It should be noted that (87) differs from (11) of the equilibrium conditions in the term concerning the commuting cost, which would cause the inconsistency between the equilibrium and optimal city sizes.

In the following, we shall show the optimal solution and the difference between the optimum and market equilibrium for the three spatial structures respectively..

4.1 Separate pattern

In this pattern, as shown in Section 2, we have

$$z - f_2 = f_3 - z = \frac{1}{2}N_1L_1. \tag{88}$$

So the first-order conditions (80)-(87) can be rewritten as

$$w_1 - \frac{t}{2}N_1L_1 - R_AL_1 - D_1 = 0, \tag{89}$$

$$w_2 - \frac{t}{2}N_2L_2 - R_AL_2 - D_2 = 0, \tag{90}$$

$$z = -\frac{a_2}{2b_2}\left(\frac{k}{c}\right)N_2L_3 - \frac{1}{4}N_1(L_3 - L_1) + \frac{1}{2}B. \tag{91}$$

where D_i ($i = 1, 2$) have the same form as defined in (24).

By comparing (89), (90) and (91) with (22), (23) and (27), we find that they are completely equivalent to each other. That is, for the value of (u_1, u_2) chosen to be equal to that of the rest of the economy, from the optimal conditions we can have $w_1^*(u_1, u_2)$ and $w_2^*(u_1, u_2)$, which are the same as given in the equilibrium conditions. As for the other variables, the optimum and the equilibrium will yield just the same solution. So, the market city sizes are consistent with the optimal ones in the separate pattern.

4.2. Multicentric pattern

For this pattern, urban fringes are represented as

$$f_1 = f_2 = \frac{1}{2}N_1L_1, \tag{92}$$

$$f_3 = \frac{1}{2}N_1L_1 + N_2L_2. \tag{93}$$

From the first-order conditions of the optimality, we get

$$w_1 - \frac{t}{2}N_1L_1 - \frac{tL_1}{L_2}(N_1L_1 + N_2L_2 - 2z) - R_AL_1 - D_1 = 0, \tag{94}$$

$$w_2 - t(\frac{1}{2}N_1L_1 + N_2L_2 - z) - R_AL_2 - D_2 = 0, \tag{95}$$

$$z = \frac{c}{c+t}\left[-\frac{a_2}{2b_2}\left(\frac{k}{c}\right)N_2L_2 - \frac{1}{4}N_1L_2 + \frac{L_2}{2L_3}(B - \frac{1}{2}N_1L_1 - N_2L_2)\right] + \frac{1}{2}(N_1L_1 + N_2L_2). \tag{96}$$

Comparison of these three conditions with the equilibrium ones, (37), (38) and (41), shows that the only difference between them is the conditions concerning z.

To see the inconsistency in the city size between the optimum and the equilibrium, let us denote X^{opt} and X^{eq} as the optimal and equilibrium solutions for variable X, respectively. Let $z^{eq} = z^{opt}$. By using (41) and (96) and rearranging we have

$$\frac{L_1 + L_3}{2}N_1 + \left[\frac{a_2}{b_2}\left(\frac{k}{c}\right)L_3 + L_2\right]N_2 = B, \tag{97}$$

which implies a straight line in the N_1-N_2 plane. (See Fig.4) For any (N_1, N_2) on the line, the optimal and equilibrium conditions are just the same, so we have the same optimal and equilibrium city sizes.

Fig.4 is about here

For any (N_1, N_2) above the line, or satisfying the following inequality

$$\frac{L_1+L_3}{2}N_1+\left[\frac{a_2}{b_2}\left(\frac{k}{c}\right)L_3+L_2\right]N_2 > B, \tag{98}$$

we have

$$z^{eq} < z^{opt}. \tag{99}$$

On the other hand, from (37) and (38) [or (94) and (95), equivalently] there exist the following relation [see also (68)]

$$\frac{dN_1}{dz} > 0, \qquad \frac{dN_2}{dz} > 0. \tag{100}$$

Thus, (99) implies

$$N_i^{eq} < N_i^{opt}, \qquad i = 1,\ 2. \tag{101}$$

Similarly, if there is a set of (N_1, N_2) that is below the line, i.e., (N_1, N_2) satisfies

$$\frac{L_1+L_3}{2}N_1+\left[\frac{a_2}{b_2}\left(\frac{k}{c}\right)L_3+L_2\right]N_2 < B, \tag{102}$$

there turns out to be

$$z^{eq} > z^{opt}. \tag{103}$$

By using (100) we have

$$N_i^{eq} > N_i^{opt}, \qquad i = 1,2. \tag{104}$$

From the above analysis, we can conclude that in the multicentric pattern, when the central and peripheral cities are both relatively small, the equilibrium spatial structure tends to be less concentrated than the optimum, and the market city sizes would be larger than the optimum; but when the city sizes become relative large, the equilibrium spatial structure turns to be much concentrated and the market city sizes would be smaller than the optimum.

The inconsistency between the market and optimal solutions can be explained in the following way. When the city sizes are small, the rest of the region turn out to be relatively large. The optimality, which also takes the commuting cost of the residents into account, would tend to give a rather smaller solution of z than the market equilibrium one. On the other hand, in this pattern the more it is spatial concentrated (i.e., z is small) and the less city sizes will be. So, a larger equilibrium z will result in larger market city sizes. When the city sizes are large, the rest of the region becomes relatively small. By taking the commuting cost of the urban residents into account, the optimal solution of z

is possible to be larger than the equilibrium. Hence the market city sizes would become smaller than the optimum.

4.3. Monocentric pattern

In this pattern, urban fringes are the same as (92) and (93). The first-order conditions of the optimality will give

$$w_1 - \frac{t}{2}N_1L_1 - tN_2L_1 - R_AL_1 - D_1 = 0, \tag{105}$$

$$w_2 - t(\frac{1}{2}N_1L_1 + N_2L_2 - z) - R_AL_2 - D_2 = 0, \tag{106}$$

$$z = -\frac{a_2}{2b_2}\left(\frac{k}{c}\right)N_2L_1 + \frac{1}{4}N_1L_1 + \frac{c+t}{2c}N_2L_1 + \frac{L_1}{2L_3}(B - \frac{1}{2}N_1L_1 - N_2L_2). \tag{107}$$

By comparing these with the equilibrium conditions (53), (54) and (57), It can be found that like the multicentric pattern, only the condition concerning z differs in the optimum and the equilibrium.

For any positive (N_1, N_2), from (57) and (107) we have

$$z^{eq} - z^{opt} = -\frac{t}{2c}N_2L_1 < 0. \tag{108}$$

On the other hand, by analyzing (53) and (54) [or (105) and (106)] we get [see also (74)]

$$\frac{dN_1}{dz} < 0, \qquad \frac{dN_2}{dz} > 0. \tag{109}$$

So, (108) will give the following relations (see Fig.5)

$$N_1^{eq} > N_1^{opt}, \quad N_2^{eq} < N_2^{opt}. \tag{110}$$

Fig.5 is about here

That is, in the monocentric pattern, by comparing to the optimum, the spatial structure is much concentrated, and the size of the central city tends to be too large whereas that of the peripheral cities too small.

From the analysis of this section, we can conclude that *the (in)consistency between the market and optimal city sizes is due to what kind of spatial structure the urban system in question is. In the separate pattern, the market and optimal solutions are the same. But in the multicentric or monocentric patterns, the market city sizes tend to be different from the optimum.*

5. Concluding remarks

We have presented an open model of a simple hierarchical inter-urban system in which principle of determining city sizes are investigated. It was shown that being relative to the characteristics of urban spatial structure, the city size depends on the cost rate of transportation to communication in the urban system.It was also made clear that in the multicentric or monocentric patterns, the city size at different city ranks are interdependent in the sense that the city size of some city rank is determined not only by the income level of its own rank but also that of other ranks as well.

By comparing the equilibrium and optimal solutions of the system, we found that the (in)consistency between them relies on the spatial characteristics of the urban system. When it is a separate pattern, the equilibrium is consistent with the optimum. But, as it happens to be a multicentric or monocentric patterns, inconsistency would be given rise to the two solutions. In particular, in the multicentric pattern, by comparing to the optimum, when city sizes are relatively small, the market city sizes tend to be larger; but when city sizes become lager, the market principle would result in too small city sizes. In the monocentric pattern, the market city size of the central city would be too large whereas that of the peripheral city too small than the optimum.

The most important policy implication from the above analytic results is that before the judgment of whether a city is too large or too small, confirmation of its spatial structure is very important. For a hierarchical inter-urban system of the monocentric structure, as the urban space under the market principle tends to be much concentrated than the optimum, policies promoting urban deconcentration, such as a proper zoning ordinance or a location subsidy, are expected. But, for an urban system of the multicentric structure, as the (in)consistency of the market and optimal solutions depends on the scale the cities, we need at first to identify whether the cities are small or large by comparing to the rest of the region, and then consider the reasonable policy measures to reduce the possible inconsistency of the market solution with the optimum.

In this paper, in order to show the importance of space factor in th determination of city sizes, we ignored some other causes in city formation, such as externalities and variety in production. The real city is, in fact, an outcome from the combination of all these causes. So, in the future a more comprehensive study, which includes most of these factors, is needed to be conducted. As another possibility to enrich the present work, we need to carry out an empirical study for the real urban world. In this direction, Zheng (1991) has presented a case study on the spatial structure of the Tokyo metropolitan area. Now it is required to empirically study the city sizes of the area and provide a solid theoretical basis for the investigation of city sizes and the related urban indicators, such as the *Shujuhi* rate mentioned in the beginning of this paper.

Appendix A. Necessary and sufficient conditions for the equilibrium

A.1. Separate pattern

First of all, by using (27) the necessary condition (28) can be expressed as

$$\frac{L_1+L_3}{2}N_1+\left[\frac{a_2}{b_2}\left(\frac{k}{c}\right)L_3+L_2\right]N_2<B. \tag{A.1}$$

Note that (22) and (23) give the solutions of N_1 and N_2 as

$$N_1=\frac{2}{tL_1}(w_1-R_AL_1-D_1), \tag{A.2}$$

$$N_2=\frac{2}{tL_2}(w_2-R_AL_2-D_2). \tag{A.3}$$

For N_1 and N_2 to be positive, w_1 and w_2 must satisfy

$$w_1-R_AL_1-D_1>0, \tag{A.4}$$

$$w_2-R_AL_2-D_2>0. \tag{A.5}$$

Next, by using (A.2) and (A.3), (A.1) can be rewritten as

$$(L_1+L_3)\frac{w_1-R_AL_1-D_1}{tL_1}+2\left[\frac{a_2}{b_2}\left(\frac{k}{c}\right)L_3+L_2\right]\frac{w_2-R_AL_2-D_2}{tL_2}<B, \tag{A.6}$$

So, (A.4), (A.5) and (A.6) constitute the necessary and sufficient conditions for the equilibrium of the separate pattern.

A.2. Multicentric pattern

The necessary conditions (42) and (43), by using (41), can be expressed as

$$\frac{L_1+L_3}{2}N_1+\left[\frac{a_2}{b_2}\left(\frac{k}{c}\right)L_3+L_2-L_3\right]N_2<B, \tag{A.7}$$

$$\frac{L_1+L_3}{2}N_1+\left[\frac{a_2}{b_2}\left(\frac{k}{c}\right)L_3+L_2\right]N_2\geq B, \tag{A.8}$$

respectively.

On the other hand, by using (42), from (37) and (38) we have

$$w_1-\frac{t}{2}N_1L_1-tN_2L_1-R_AL_1-D_1=\frac{tL_1}{2L_2}(\frac{1}{2}N_1L_1-z)<0, \tag{A.9}$$

$$w_2-tN_2L_2-R_AL_2-D_2=t(\frac{1}{2}N_1L_1-z)<0, \tag{A.10}$$

Similarly, by using (43), from (37) and (38) we get

$$w_1-\frac{t}{2}N_1L_1-R_AL_1-D_1=\frac{tL_1}{2L_2}(\frac{1}{2}N_1L_1+\frac{1}{2}N_2L_2-z)\geq 0, \tag{A.11}$$

$$w_{21} - \frac{t}{2} N_2 L_2 - R_A L_2 - D_2 = t(\frac{1}{2} N_1 L_1 + \frac{1}{2} N_2 L_2 - z) \geq 0, \quad (A.12)$$

which will give

$$N_1 \leq \frac{2}{tL_1}(w_1 - R_A L_1 - D_1), \quad (A.13)$$

$$N_2 \leq \frac{2}{tL_2}(w_2 - R_A L_2 - D_2). \quad (A.14)$$

By using (A.14), from (A.9) we can obtain

$$N_1 > \frac{2}{tL_1}(w_1 - R_A L_1 - D_1) - 2N_2$$

$$> \frac{2}{tL_1}(w_1 - R_A L_1 - D_1) - \frac{1}{tL_2}(w_2 - R_A L_2 - D_2). \quad (A.15)$$

And (A.10) can be rewritten as

$$N_2 > \frac{1}{tL_2}(w_2 - R_A L_2 - D_2). \quad (A.16)$$

Inequalities (A.13)-(A.16) imply the possible value ranges of N_1 and N_2. To ensure that N_1 and N_2 are positive, there must exist

$$\frac{1}{L_1}(w_1 - R_A L_1 - D_1) > \frac{1}{2L_2}(w_2 - R_A L_2 - D_2), \quad (A.17)$$

$$w_2 - R_A L_2 - D_2 > 0. \quad (A.18)$$

Next, substitution of (A.13) and (A.14) into (A.8) will give

$$(L_1 + L_3)\frac{w_1 - R_A L_1 - D_1}{tL_1} + 2\left[\frac{a_2}{b_2}\left(\frac{k}{c}\right)L_3 + L_2\right]\frac{w_2 - R_A L_2 - D_2}{tL_2} \geq B, \quad (A.19)$$

Similarly, by replacing (A.15) and (A.16) into (A.7) and rearranging we can get

$$(L_1 + L_3)\frac{w_1 - R_A L_1 - D_1}{tL_1} + \left[\frac{a_2}{b_2}\left(\frac{k}{c}\right)L_3 + L_2 - \frac{L_1 + 3L_3}{2}\right]\frac{w_2 - R_A L_2 - D_2}{tL_2} < B. \quad (A.20)$$

In short, (A.17)-(A.20) are the necessary and sufficient conditions for the multicentric pattern.

A.3. Monocentric pattern

By using (57), the necessary condition (58) can be rewritten as

$$\frac{L_1 + L_3}{2} N_1 + \left[\frac{a_2}{b_2}\left(\frac{k}{c}\right)L_3 + L_2 - L_3\right]N_2 > B, \quad (A.21)$$

To see the value range of N_1 and N_2, by using (58), from (54) we have

$$w_2 - tN_2 L_2 - R_A L_2 - D_2 = t(\frac{1}{2} N_1 L_1 - z) > 0, \quad (A.22)$$

which can be rewritten as

$$N_2 < \frac{1}{tL_2}(w_2 - R_A L_2 - D_2). \tag{A.23}$$

Since in this case the second-rank firm is possible to locate at the regional center, we have $z \geq 0$. Thus, from (54) it can be shown that

$$w_2 - \frac{t}{2}N_1 L_1 - tN_2 L_2 - R_A L_2 - D_2 = -tz \leq 0, \tag{A.24}$$

which gives

$$N_2 \geq \frac{1}{tL_2}(w_2 - R_A L_2 - D_2) - \frac{L_1}{2L_2}N_1. \tag{A.25}$$

By using (A.23) and (A.25), from (53) we can obtain the value range of N_1 and as follows

$$N_1 \leq \frac{2L_2}{t(L_2 - L_1)}\left(\frac{w_1 - R_A L_1 - D_1}{L_1} - \frac{w_2 - R_A L_2 - D_2}{L_2}\right), \tag{A.26}$$

$$N_1 > \frac{2}{t}\left(\frac{w_1 - R_A L_1 - D_1}{L_1} - \frac{w_2 - R_A L_2 - D_2}{L_2}\right). \tag{A.27}$$

Replacing (A.23) into (A.25) will give

$$N_2 \geq (L_2 - L_1)[L_2(w_2 - R_A L_2 - D_2) - (w_1 - R_A L_1 - D_1)]. \tag{A.28}$$

To ensure that N_1 and N_2 are positive, from (A.27) and (A.28) we need to have the following conditions

$$\frac{1}{L_1}(w_1 - R_A L_1 - D_1) > \frac{1}{L_2}(w_2 - R_A L_2 - D_2), \tag{A.29}$$

$$L_2(w_2 - R_A L_2 - D_2) > w_1 - R_A L_1 - D_1 > 0. \tag{A.30}$$

Finally, by combining (A.23) and (A.26) with (A.21) we can obtain

$$L_2(L_1 + L_3)\frac{w_1 - R_A L_1 - D_1}{t(L_2 - L_1)} + \left[\frac{a_2}{b_2}\left(\frac{k}{c}\right)L_3 + L_2 - L_3 - \frac{(L_1 + L_3)L_2^2}{L_2 - L_1}\right]\frac{w_2 - R_A L_2 - D_2}{tL_2} > B. \tag{A.31}$$

The inequalities (A.29)-(A.31) are in fact the necessary and sufficient conditions for the equilibrium of the monocentric pattern.

Appendix B. Comparative statics

B.1. Separate pattern

In the equation system of (22), (23) and (27), N_1, N_2 and z are endogenous variables while w_1, w_2 and k/c are exogenous parameters in question, Taking total differentials and putting the result into matrix from, we have

$$\begin{bmatrix} -\frac{tL_1}{2} & 0 & 0 \\ 0 & -\frac{tL_2}{2} & 0 \\ \frac{1}{4}(L_3 - L_1) & \frac{a_2}{2b_2}\left(\frac{k}{c}\right)L_3 & 1 \end{bmatrix} \begin{bmatrix} dN_1 \\ dN_2 \\ dz \end{bmatrix} = \begin{bmatrix} -dw_1 \\ -dw_2 \\ -\frac{a_2}{2b_2} N_2 L_3 d\left(k/c\right) \end{bmatrix}, \tag{A.32}$$

in which the matrix of the system, denoted as A, is nonsingular since $|A| = (1/4)t^2L_1L_2 > 0$. For such a system of linear equations, Cramer's rule gives

$$dN_1 = \frac{tL_2}{2|A|} dw_1, \tag{A.33}$$

$$dN_2 = \frac{tL_1}{2|A|} dw_2, \tag{A.34}$$

$$dz = \frac{1}{|A|}\left[-\frac{1}{8}tL_2(L_3 - L_1)dw_1 - \frac{a_2}{4b_2}\left(\frac{k}{c}\right)tL_1L_3dw_2 - \frac{a_2}{8b_2}t^2L_1L_2L_3N_2d\left(k/c\right)\right], \tag{A.35}$$

which imply the following relations

$$\frac{dN_1}{dw_1} > 0, \tag{A.36}$$

$$\frac{dN_2}{dw_2} > 0, \tag{A.37}$$

$$\frac{\partial z}{\partial w_1} < 0, \qquad \frac{\partial z}{\partial w_2} < 0, \qquad \frac{\partial z}{\partial\left(k/c\right)} < 0. \tag{A.38}$$

B.2. Multicentric pattern

Similarly, concerning the system of equations (37), (38) and (41), we take total differentials and get

$$\begin{bmatrix} -\frac{tL_1}{2} - \frac{tL_1^2}{L_2} & -tL_1 & \frac{2tL_1}{L_2} \\ -\frac{tL_1}{2} & -tL_2 & t \\ -\frac{L_1}{2} + \frac{L_1L_2}{4L_3} + \frac{L_2}{4} & \frac{a_2}{2b_2}\left(\frac{k}{c}\right)L_2 - \frac{L_2}{2} + \frac{L_2^2}{2L_3} & 1 \end{bmatrix} \begin{bmatrix} dN_1 \\ dN_2 \\ dz \end{bmatrix} = \begin{bmatrix} -dw_1 \\ -dw_2 \\ -\frac{a_2}{2b_2} N_2 L_2 d\left(k/c\right) \end{bmatrix}, \tag{A.39}$$

Denote the matrix of the system as A. Its determinant gives

$$|A| = \frac{t^2L_1L_2}{4}\left[\frac{a_2}{b_2}\left(\frac{k}{c}\right) + 2 + \frac{L_2}{L_3} + \frac{L_1}{L_3}\right] > 0. \tag{A.40}$$

By Cramer's rule we obtain

$$dN_1 = \frac{1}{|A|}\left\{\frac{tL_2}{2}\left[\frac{a_2}{b_2}\left(\frac{k}{c}\right) + 1 + \frac{L_2}{L_3}\right]dw_1 - tL_1\left[\frac{a_2}{b_2}\left(\frac{k}{c}\right) + \frac{L_2}{L_3}\right]dw_2 - \frac{a_2}{2b_2}t^2L_1L_2N_2d\left(k/c\right)\right\}, \tag{A.41}$$

$$dN_2 = \frac{1}{|A|}\left[-\frac{tL_2}{4}\left(\frac{L_1}{L_3}+1\right)dw_1 + tL_1\left(\frac{L_1}{2L_3}+1\right)dw_2 - \frac{a_2}{4b_2}t^2 L_1 L_2 N_2 d\left(k/c\right)\right], \qquad (A.42)$$

$$dz = \frac{1}{|A|}\left\{\frac{tL_2}{4}\left[\frac{a_2}{b_2}\left(\frac{k}{c}\right)+L_1-L_2\right]dw_1 - \frac{tL_1}{4}\left[\frac{a_2}{b_2}\left(\frac{k}{c}\right)(L_2+2L_1)+\frac{L_1L_2}{L_3}+\frac{L_2^2}{L_3}-2L_2\right]dw_2\right.$$
$$\left.-\frac{a_2}{4b_2}t^2 L_1 L_2 N_2 (L_1+L_2) d\left(k/c\right)\right\}. \qquad (A.43)$$

So, we can have

$$\frac{\partial N_1}{\partial w_1} > 0, \qquad \frac{\partial N_1}{\partial w_2} < 0, \qquad \frac{\partial N_1}{\partial\left(k/c\right)} < 0, \qquad (A.44)$$

$$\frac{\partial N_2}{\partial w_1} < 0, \qquad \frac{\partial N_2}{\partial w_2} > 0, \qquad \frac{\partial N_2}{\partial\left(k/c\right)} < 0, \qquad (A.45)$$

$$\frac{\partial z}{\partial w_1} >< 0, \qquad \frac{\partial z}{\partial w_2} >< 0, \qquad \frac{\partial z}{\partial\left(k/c\right)} < 0. \qquad (A.46)$$

B.3. Monocentric pattern

By taking total differentials of the system of equations (53), (54) and (57), we have

$$\begin{bmatrix} -\frac{tL_1}{2} & -tL_1 & 0 \\ -\frac{tL_1}{2} & -tL_2 & t \\ -\frac{L_1}{4}+\frac{L_1^2}{4L_3} & \frac{a_2}{2b_2}\left(\frac{k}{c}\right)L_1 - \frac{L_1}{2}+\frac{L_1L_2}{2L_3} & 1 \end{bmatrix}\begin{bmatrix} dN_1 \\ dN_2 \\ dz \end{bmatrix} = \begin{bmatrix} -dw_1 \\ -dw_2 \\ -\frac{a_2}{2b_2}N_2L_1 d\left(k/c\right) \end{bmatrix}, \qquad (A.47)$$

The determinant of the system's matrix, denoted as | *A* |, gives

$$|A| = \frac{t^2L_1}{4}\left[\frac{a_2}{b_2}\left(\frac{k}{c}\right)L_1 + (L_2-L_1)\left(2+\frac{L_1}{L_3}\right)\right] > 0. \qquad (A.48)$$

Using Cramer's rule, we get

$$dN_1 = \frac{1}{|A|}\left\{\frac{t}{2}\left[\frac{a_2}{b_2}\left(\frac{k}{c}\right)L_1 + \frac{L_1L_2}{L_3}+2L_2-L_1\right]dw_1 - tL_1 dw_2 + \frac{a_2}{2b_2}t^2L_1^2N_2 d\left(k/c\right)\right\}, (A.49)$$

$$dN_2 = \frac{1}{|A|}\left[-\frac{tL_1}{4}\left(\frac{L_1}{L_3}+1\right)dw_1 + \frac{tL_1}{2}dw_2 - \frac{a_2}{4b_2}t^2L_1^2N_2 d\left(k/c\right)\right], \qquad (A.50)$$

$$dz = \frac{1}{|A|}\left\{\frac{tL_1}{4}\left[\frac{a_2}{b_2}\left(\frac{k}{c}\right)L_1 + L_2 - L_1\right]dw_1 - \frac{tL_1^2}{4L_3}\left[\frac{a_2}{b_2}\left(\frac{k}{c}\right)L_3 + L_2 - L_1\right]dw_2\right.$$
$$\left.-\frac{a_2}{4b_2}t^2L_1^2N_2(L_2-L_1)d\left(k/c\right)\right\}. \qquad (A.51)$$

which give the following results

$$\frac{\partial N_1}{\partial w_1} > 0, \qquad \frac{\partial N_1}{\partial w_2} < 0, \qquad \frac{\partial N_1}{\partial \left(k/c\right)} > 0, \tag{A.52}$$

$$\frac{\partial N_2}{\partial w_1} < 0, \qquad \frac{\partial N_2}{\partial w_2} > 0, \qquad \frac{\partial N_2}{\partial \left(k/c\right)} < 0, \tag{A.53}$$

$$\frac{\partial z}{\partial w_1} > 0, \qquad \frac{\partial z}{\partial w_2} < 0, \qquad \frac{\partial z}{\partial \left(k/c\right)} < 0. \tag{A.54}$$

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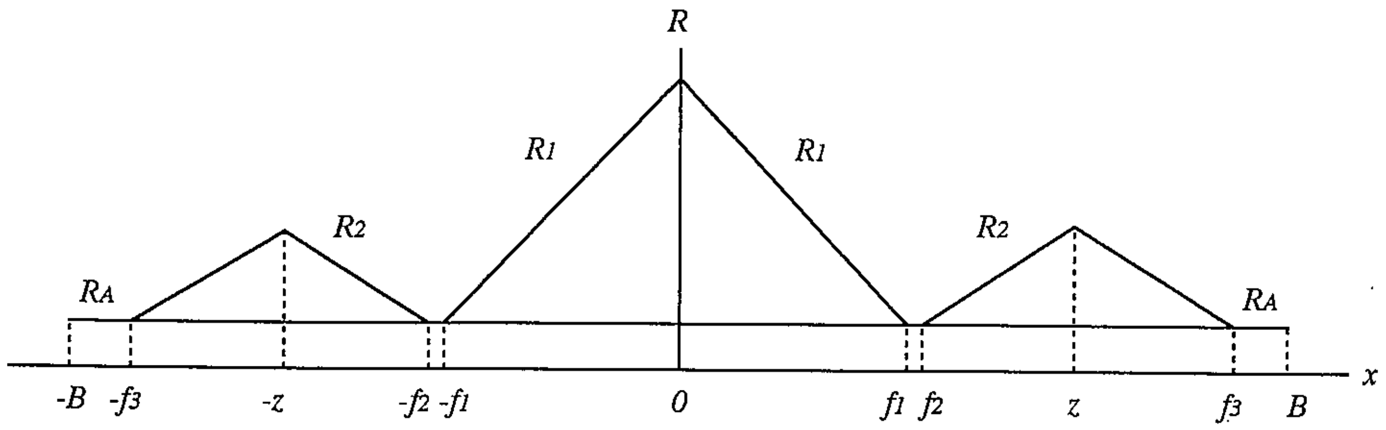

Fig.1 Separate pattern

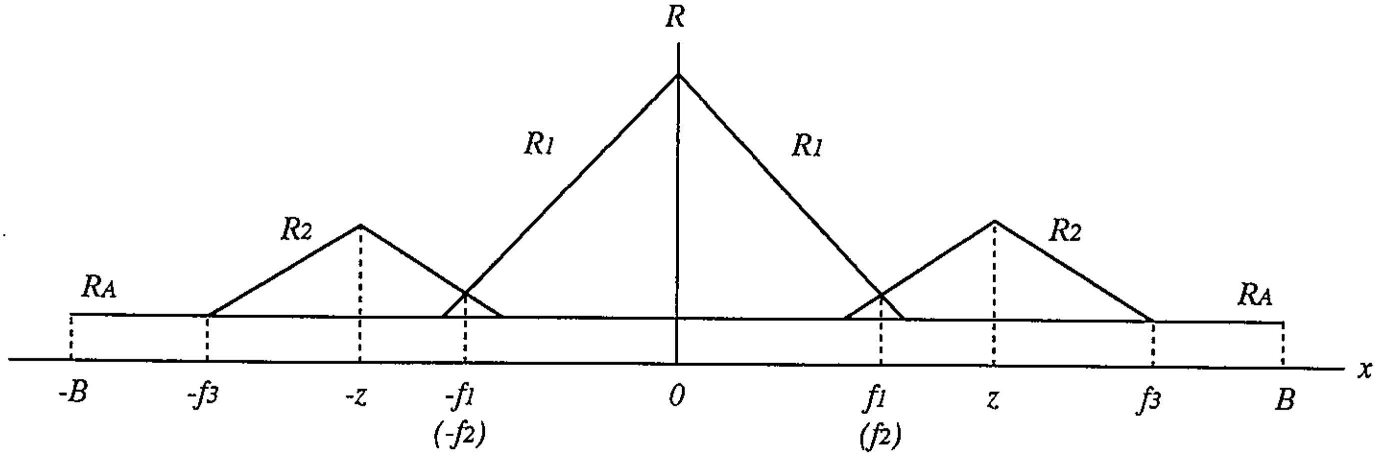

Fig.2 Multicentric pattern

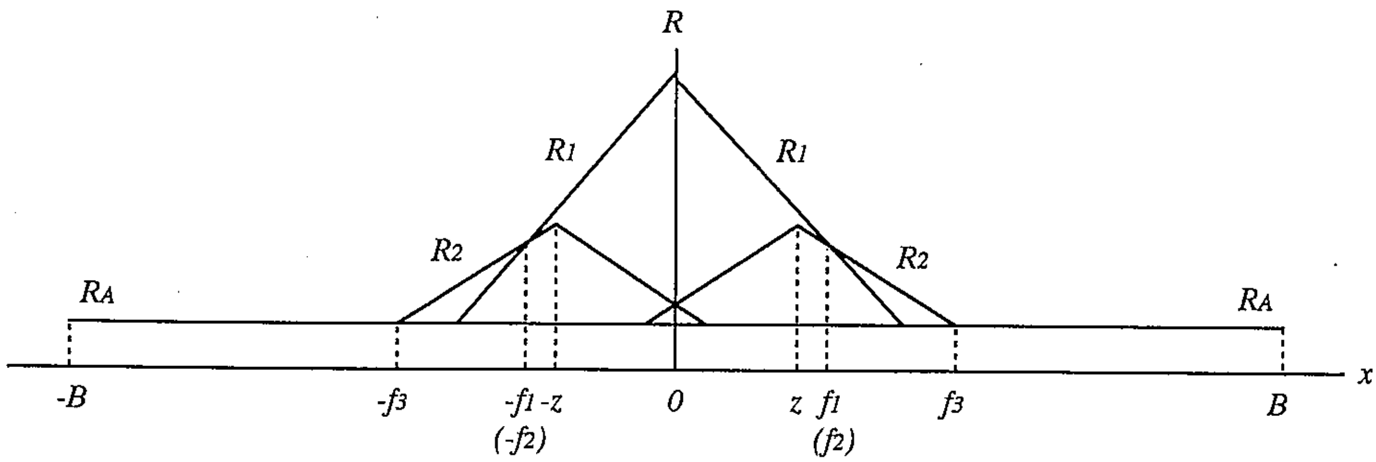

Fig.3 Monocentric pattern

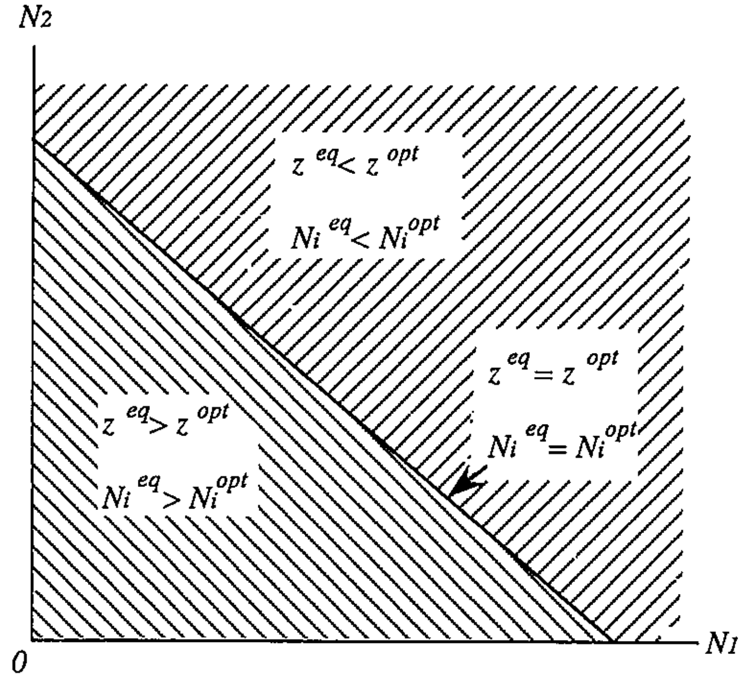

Fig.4 Optimum and equilibrium in multicentric pattern (i=1, 2)

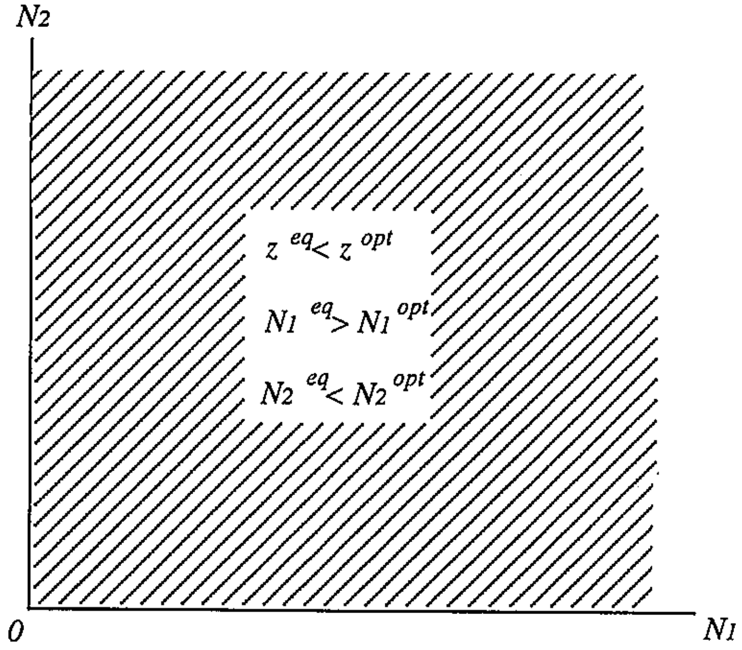

Fig.5 Optimum and equilibrium in monocentric pattern