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A Greedy Algorithm for Minimizing a Separable Convex Function over a Finite Jump System

by

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Abstract

We present a greedy algorithm for minimizing a separable convex function over a finite jump system of A. Bouchet and W. H. Cunningham. The algorithm starts with an arbitrary feasible solution and a current feasible solution incrementally moves toward an optimal one in a greedy way. We also show that the greedy algorithm terminates after changing an initial feasible solution at most

$$\sum_{e\in E}\{f(\{e\},\emptyset)+f(\emptyset,\{e\})\}$$

times, where f is the *bisubmodular function* such that the convex hull of the feasible solution set of the given finite jump system coincides with the polyhedron given by

$$\{x \mid x\in \mathbf{R}^E,\ \forall (X,Y)\in 3^E : x(X)-x(Y)\le f(X,Y)\}.$$

1. Introduction

A. Bouchet and W. H. Cunningham [2] have introduced a concept of jump system. A jump system is a pair $(E,\mathcal{F})$ of a finite set E and a set $\mathcal{F}$ of integral points in $\mathbf{Z}^E$ satisfying an exchange axiom. The convex hull $\mathrm{Co}(\mathcal{F})$ of $\mathcal{F}$ is a polypseudomatroid ([2]), i.e., for a given finite jump system $(E,\mathcal{F})$ there exists a *bisubmodular function* $f : 3^E \to \mathbf{Z}$ such that

$$\mathrm{Co}(\mathcal{F})=\{x \mid x\in \mathbf{R}^E,\ \forall (X,Y)\in 3^E : x(X)-x(Y)\le f(X,Y)\} \qquad (1.1)$$

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(a precise definition of bisubmodular function will be given later).

Recently, we presented a greedy algorithm for minimizing a separable convex function over an *integral bisubmodular polyhedron*, which is the set of integral points of a polypseudomatroid [1]. The concept of polypseudomatroid was introduced by R. Chandrasekaran and S. N. Kabadi [3]. Our algorithm starts with an arbitrary initial feasible point and repeats coordinate-wise augmentations and/or exchanges in a greedy way. We show in this paper that the algorithm given in [1] also works over a finite jump system. Moreover, we examine the behavior of the greedy algorithm and show that the greedy algorithm terminates after changing an initial feasible solution at most

$$\sum_{e\in E}\{f(\{e\},\emptyset)+f(\emptyset,\{e\})\} \tag{1.2}$$

times.

2. Definitions

Let E be a nonempty finite set. Denote by 3^E the set of all the ordered pairs (X, Y) of disjoint subsets X and Y of E. Let $f : 3^E \to \mathbf{Z}$ be a function from 3^E to the set $\mathbf{Z}$ of integers such that $f(\emptyset,\emptyset) = 0$ and for each $(X_i, Y_i) \in 3^E$ (i=1,2)

$$\begin{aligned} &f(X_1,Y_1)+f(X_2,Y_2)\\ &\quad \geq f((X_1\cup X_2)-(Y_1\cup Y_2),(Y_1\cup Y_2)-(X_1\cup X_2))+f(X_1\cap X_2,Y_1\cap Y_2). \end{aligned} \tag{2.1}$$

We call such an f a *bisubmodular function*, which was first considered by Chandrasekaran and Kabadi [3]. Define a polyhedron

$$\mathrm{P}_*(f)=\{x\mid x\in\mathbf{Z}^E,\ \forall(X,Y)\in 3^E : x(X)-x(Y)\leq f(X,Y)\} \tag{2.2}$$

associated with f, where $x(X) = \sum_{e\in X} x(e)$ for any $X \subseteq E$ and $x \in \mathbf{Z}^E$. We call the polyhedron $\mathrm{P}_*(f)$ an *integral bisubmodular polyhedron.*

Bouchet and Cunningham [2] introduced a concept of *jump system.* A *step* is a $\{0,\pm1\}$-vector with a unique nonzero component. Denote by S the set of all the steps in $\mathbf{Z}^E$. For any $x, y \in \mathbf{Z}^E$ a *step s from x to y* is a step such that $\sum_{e\in E}|x(e)+s(e)-y(e)| = \sum_{e\in E}|x(e)-y(e)|-1$. We denote by $\mathrm{St}(x, y)$ the set of all the steps from x to y. A *jump system* on a nonempty finite set E is a pair $(E,\mathcal{F})$ of E and a nonempty $\mathcal{F}\subseteq\mathbf{Z}^E$ which satisfies the 2-*step axiom*:

(2-SA) For any $x, y \in \mathcal{F}$ and $s \in \mathrm{St}(x, y)$ with $x + s \notin \mathcal{F}$ there exists $t \in \mathrm{St}(x + s, y)$ such that

$$x+s+t\in\mathcal{F}. \tag{2.3}$$

We see from a result in [2] that an integral bisubmodular polyhedron satisfies the axiom (2-SA).

3. A Greedy Algorithm

Let $w : \mathbf{R}^E \to \mathbf{R}$ be a separable convex function given by

$$w(x) = \sum_{e \in E} w_e(x(e)), \tag{3.1}$$

where for each $e \in E$ w_e is a convex function on $\mathbf{R}$. Consider a discrete optimization problem described as

$$\begin{aligned} \mathbf{P}\text{: Minimize} \quad & \sum_{e \in E} w_e(x(e)) \\ \text{subject to} \quad & x \in \mathcal{F}, \end{aligned} \tag{3.2}$$

where $(E, \mathcal{F})$ is a finite jump system, i.e., a jump system with a finite $\mathcal{F}$. We describe a greedy algorithm for solving the above problem **P**. The validity is shown in the next section.

A greedy algorithm

Input: a vector $x \in \mathcal{F}$.

Output: an optimal solution x of Problem **P**.

Step 1: If neither of the following two conditions is satisfied, then stop (x is an optimal solution).
(1) There exists a step $s \in S$ such that $x + s \in \mathcal{F}$ and $w(x + s) < w(x)$.
(2) There exist steps $s, t \in S$ such that $x + s \notin \mathcal{F}, x + s + t \in \mathcal{F}$ and $w(x + s + t) < w(x)$.

Step 2: Compute

$$w_1 = \min\{w(x + s) \mid \text{step } s \text{ satisfying Condition (1) in Step 1}\}, \tag{3.3}$$

$$w_2 = \min\{w(x + s) \mid \text{steps } s, t \text{ satisfying Condition (2) in Step 1}\}, \tag{3.4}$$

where the minimum over the empty set is defined to be $+\infty$.
Put $\hat{w} := \min\{w_1, w_2\}$.
If we have $\hat{w} = w_1$, let $\hat{s}$ be the step s that attains the minimum of (3.3), put $x := x + \hat{s}$ and go to Step 1.
If $\hat{w} \neq w_1$, let $\hat{s}$ and $\hat{t}$ be the steps s and t that attain the minimum of (3.4), put $x := x + \hat{s} + \hat{t}$ and go to Step 1.

(End)

It should be noted that in (3.4) $w(x+s)$ but not $w(x+s+t)$ is minimized and that each step s in the above algorithm is chosen in a greedy way.

Denote by x^k the current x obtained after the kth execution of Step 2 of the greedy algorithm. During the execution of the greedy algorithm, if the current x^k is not an optimal solution, then x^k is changed into either $x^{k+1} := x^k + \hat{s}$ or $x^{k+1} := x^k + \hat{s} + \hat{t}$ in Step 2. Denote such steps $\hat{s}$ and $\hat{t}$ by s^k and t^k. Then, we have

Remark 3.1: For any $s \in S$ such that $x^k + s \in \mathcal{F}$,

$$w(x^k) - w(x^k + s^k) \geq w(x^k) - w(x^k + s). \tag{3.5}$$

□

Remark 3.2: For any $s, t \in S$ such that $x^k + s \notin \mathcal{F}$, $x^k + s + t \in \mathcal{F}$ and $w(x^k) > w(x^k + s + t)$ we have

$$w(x^k) - w(x^k + s^k) \geq w(x^k) - w(x^k + s). \tag{3.6}$$

□

4. Validity of the Greedy Algorithm

In this section we prove the validity of the greedy algorithm. It should be noted that the algorithm terminates in finitely many steps since $\mathcal{F}$ is finite and the value of the objective function is reduced every time Step 2 is executed. For each step $s \in S$ let $\mathrm{e}(s)$ be the element e of E such that $s(e) = 1$ or -1.

Theorem 4.1: *The greedy algorithm described in Section 3 finds an optimal solution of Problem* **P**.

Proof: Let x be the solution found by the greedy algorithm when it terminates.

Claim: Suppose that x is not an optimal solution of Problem **P**. Then, there exists an optimal solution $x^*(\neq x)$ that satisfies the following three conditions:

(i) If $s \in \mathrm{St}(x^*, x)$ and $x^* + s \in \mathcal{F}$, then $w(x^*) < w(x^* + s)$.

(ii) If $s \in \mathrm{St}(x^*, x)$, $t \in \mathrm{St}(x^* + s, x)$, $x^* + s \notin \mathcal{F}$ and $x^* + s + t \in \mathcal{F}$, then $w(x^*) < w(x^* + s + t)$.

(iii) There exists some $s \in \mathrm{St}(x^*, x)$ such that $w(x^*) < w(x^* + s)$.

(Proof of Claim) We can easily find an optimal solution x^* that satisfies (i) and (ii) (see the similar argument in the proof of Theorem 4.1 in [1]).

We will show that this x^* also satisfies (iii). On the contrary, suppose that x^* does not satisfy (iii), i.e., for any $s \in \mathrm{St}(x^*, x)$

$$w(x^* + s) \leq w(x^*). \tag{4.1}$$

We will prove that this leads to a contradiction.

Since $\mathcal{F}$ satisfies (2-SA) and x^* satisfies (i), for any $s \in \mathrm{St}(x^*, x)$ we have $x^*+s \notin \mathcal{F}$ and there exists $s' \in \mathrm{St}(x^*+s, x)$ such that $x^*+s+s' \in \mathcal{F}$. Let s be an element of $\mathrm{St}(x^*, x)$ that satisfies

$$w(x^*) - w(x^*+s) = \max_{t \in \mathrm{St}(x^*,x)} \{w(x^*) - w(x^*+t)\}, \tag{4.2}$$

and choose $s' \in \mathrm{St}(x^*+s, x)$ such that $x^*+s+s' \in \mathcal{F}$.

Let us consider the following Case a and Case b.

[Case a]: $s \neq s'$.
As in the proof of the claim in Theorem 4.1 in [1], we can show that

$$w(x^*+s+s') \leq w(x^*). \tag{4.3}$$

This contradicts the fact that x^* satisfies (ii).

[Case b]: $s = s'$.
In this case,

$$s \in \mathrm{St}(x^*+s, x), \tag{4.4}$$

$$x^*+s \notin \mathcal{F}, \tag{4.5}$$

$$x^*+2s (= x^*+s+s') \in \mathcal{F}. \tag{4.6}$$

From (ii), (4.1) and (4.4)~(4.6), we must have

$$w(x^*+s) < w(x^*+2s) \tag{4.7}$$

and from (4.4)

$$-s \in \mathrm{St}(x, x^*+s) \subseteq \mathrm{St}(x, x^*). \tag{4.8}$$

It follows from (4.7), (4.8) and the separable convexity of w that

$$w(x-s) < w(x). \tag{4.9}$$

Since x is the solution found by the greedy algorithm when it terminates, we have $x-s \notin \mathcal{F}$. So, from (4.8) and (2-SA), there exists $-t \in \mathrm{St}(x-s, x^*) \subseteq \mathrm{St}(x, x^*)$ such that

$$x-s-t \in \mathcal{F}. \tag{4.10}$$

From the separable convexity of w we have

$$w(x-t) - w(x) \leq w(x^*) - w(x^*+t). \tag{4.11}$$

Case b is divided into Case b-1 and Case b-2.

[Case b-1]: $s \neq t$.
In this case, from the separable convexity of w we have

$$w(x) - w(x - s - t) = (w(x) - w(x - s)) + (w(x) - w(x - t)). \qquad (4.12)$$

Since (4.4), it follows from (ii) and the separable convexity of w that

$$\begin{aligned} w(x^*) - w(x^* + s) &< w(x^* + 2s) - w(x^* + s) \\ &\leq w(x) - w(x - s). \end{aligned} \qquad (4.13)$$

Hence, from (4.2), (4.11), (4.12) and (4.13),

$$\begin{aligned} 0 &\leq (w(x^*) - w(x^* + s)) - (w(x^*) - w(x^* + t)) \\ &< (w(x) - w(x - s)) + (w(x) - w(x - t)) \\ &= w(x) - w(x - s - t). \end{aligned} \qquad (4.14)$$

This contradicts the assumption that x is the solution found by the greedy algorithm when it terminates.

[Case b-2]: $s = t$.
It follows from (4.10) that

$$x - 2s \in \mathcal{F}. \qquad (4.15)$$

Let e be e(s). Without loss of generality we can suppose $s(e) = 1$. Then,

$$(x - 2s)(e) \geq x^*(e). \qquad (4.16)$$

If $(x - 2s)(e) = x^*(e)$, then from the assumption, (4.6) and (ii) we have

$$\begin{aligned} w(x) - w(x - 2s) &= w(x^* + 2s) - w(x^*) \\ &> 0. \end{aligned} \qquad (4.17)$$

This contradicts the assumption that x is the solution found by the greedy algorithm when it terminates. Hence, $(x - 2s)(e) > x^*(e)$. However, from the assumption, (4.7), (4.9) and the separable convexity of w we have

$$w(x - 2s) < w(x - s) < w(x), \qquad (4.18)$$

which is again a contradiction.
(The end of the proof of Claim)

Now, suppose that x is not an optimal solution of Problem **P**. Then, we can choose an optimal solution $x^*(\neq x)$ that satisfies the conditions (i)$\sim$(iii) of the above claim.

Let s be an element of $\mathrm{St}(x^*, x)$ that satisfies

$$w(x^* + s) - w(x^*) = \max_{t \in \mathrm{St}(x^*, x)} \{w(x^* + t) - w(x^*)\}. \tag{4.19}$$

As in the proof of Theorem 4.1 in [1], we have

$$w(x^*) < w(x^* + s), \tag{4.20}$$

$$w(x) - w(x - s) > 0, \tag{4.21}$$

$$w(x - s - t) \geq w(x), \tag{4.22}$$

where $-t \in \mathrm{St}(x - s, x^*)$ such that $x - s - t \in \mathcal{F}$.

Let us consider the following Case 1 and Case 2.

[Case 1]: $s = t$.
From (4.22) we have

$$w(x - 2s) \geq w(x). \tag{4.23}$$

Hence, (4.21) and (4.23) imply

$$w(x - 2s) > w(x - s). \tag{4.24}$$

Since $-s \in \mathrm{St}(x - s, x^*)$, it follows from (4.24) and the separable convexity of w that

$$w(x^*) - w(x^* + s) \geq w(x - 2s) - w(x - s) > 0, \tag{4.25}$$

which contradicts (4.20).

[Case 2]: $s \neq t$.
As in the proof of Theorem 4.1 in [1], we can show

$$w(x^*) - w(x^* + t) \geq w(x - t) - w(x) > 0, \tag{4.26}$$

and there exists $t' \in \mathrm{St}(x^* + t, x)$ such that $x^* + t + t' \in \mathcal{F}$.

Case 2 is divided into Case 2-1 and Case 2-2.

[Case 2-1]: $t \neq t'$.
As in the proof of Theorem 4.1 in [1], we have $t' \in \mathrm{St}(x^*, x)$ and

$$w(x^* + t') - w(x^*) > w(x^* + s) - w(x^*). \tag{4.27}$$

This contradicts (4.19).

[Case 2-2]: $t = t'$.
From (ii) of the above claim we have

$$w(x^* + 2t) > w(x^*). \tag{4.28}$$

Hence, from (4.26) we have

$$w(x^* + 2t) > w(x^* + t). \tag{4.29}$$

Since $t = t' \in \mathrm{St}(x^* + t, x)$, it follows from the separable convexity of w that

$$w(x) - w(x - t) \geq w(x^* + 2t) - w(x^* + t) > 0, \tag{4.30}$$

which contradicts (4.26). This completes Case 2.

From the arguments for [**Case 1**] and [**Case 2**] x must be an optimal solution of Problem **P**. □

A vector $x \in \mathcal{F}$ is called a *local optimal solution* of Problem **P** if the following two hold:

(L1) For any $s \in S$ such that $x + s \in \mathcal{F}$, we have $w(x) \leq w(x + s)$.
(L2) For any $s, t \in S$ such that $x + s \notin \mathcal{F}$ and $x + s + t \in \mathcal{F}$, we have $w(x) \leq w(x + s + t)$.

From Theorem 4.1 we have the following.

Corollary 4.2: *Every local optimal solution of Problem* **P** *is also an optimal solution.* □

5. Properties of the Greedy Algorithm

During the execution of the greedy algorithm the current x^k is changed into either $x^{k+1} := x^k + \hat{s}$ or $x^{k+1} := x^k + \hat{s} + \hat{t}$ ($x^k + \hat{s} \notin \mathcal{F}$), if not optimal, in Step 2. Recall that such steps $\hat{s}$ and $\hat{t}$ are denoted by s^k and t^k. Then, we have

Lemma 5.1:

$$w(x^k) > w(x^k + s^k). \tag{5.1}$$

Proof: If $x^{k+1} = x^k + s^k$, then (5.1) is trivial by the definition of Step 2. Suppose $x^{k+1} = x^k + s^k + t^k$.

Let us consider the following two cases.

[**Case 1**]: $x^k + t^k \in \mathcal{F}$.
In this case,

$$t^k \neq s^k, -s^k. \tag{5.2}$$

By the definition of s^k in Step 2 we must have

$$w(x^k + t^k) > w(x^k + s^k), \tag{5.3}$$

$$w(x^k) > w(x^k + s^k + t^k). \tag{5.4}$$

It follows from (5.2)∼(5.4) that

$$\begin{aligned} 0 &> w(x^k + s^k + t^k) - w(x^k) \\ &= \{w(x^k + s^k) - w(x^k)\} + \{w(x^k + t^k) - w(x^k)\}, \\ &> 2\{w(x^k + s^k) - w(x^k)\}. \end{aligned} \tag{5.5}$$

[Case 2]: $x^k + t^k \notin \mathcal{F}$.
Then, from (3.4) we have

$$w(x^k + s^k) \leq w(x^k + t^k). \tag{5.6}$$

Therefore, since $w(x^{k+1}) < w(x^k)$, we can easily prove (5.1) whether $s^k \neq t^k$ or not.
□

Lemma 5.2: *If* $x^{k+1} = x^k + s^k + t^k$, *then*

$$w(x^k + s^k) \leq w(x^k + t^k). \tag{5.7}$$

Proof: Easy. □

Lemma 5.3: *If*

$$\mathrm{e}(s^{k-1}) \neq \mathrm{e}(s^k), \tag{5.8}$$

$$x^{k-1} + s^k \notin \mathcal{F}, \tag{5.9}$$

$$x^{k-1} + s^k + s^{k-1} \in \mathcal{F} \tag{5.10}$$

and

$$w(x^k) - w(x^k + s^k) = w(x^{k-1}) - w(x^{k-1} + s^k), \tag{5.11}$$

then

$$w(x^k) - w(x^k + s^k) \leq w(x^{k-1}) - w(x^{k-1} + s^{k-1}). \tag{5.12}$$

Proof: From Lemma 5.1 we have

$$w(x^{k-1} + s^{k-1}) < w(x^{k-1}), \tag{5.13}$$

$$w(x^k + s^k) < w(x^k). \tag{5.14}$$

Relations (5.8), (5.11), (5.13) and (5.14) imply

$$\begin{aligned} & w(x^{k-1} + s^k + s^{k-1}) - w(x^{k-1}) \\ = \ & \{w(x^{k-1} + s^k) - w(x^{k-1})\} + \{w(x^{k-1} + s^{k-1}) - w(x^{k-1})\} \\ < \ & 0. \end{aligned} \tag{5.15}$$

From Remark 3.2, (5.9), (5.10) and (5.15), we have

$$w(x^{k-1}) - w(x^{k-1} + s^k) \leq w(x^{k-1}) - w(x^{k-1} + s^{k-1}). \tag{5.16}$$

It follows from (5.11) and (5.16) that

$$\begin{aligned} w(x^k) - w(x^k + s^k) &= w(x^{k-1}) - w(x^{k-1} + s^k) \\ &\leq w(x^{k-1}) - w(x^{k-1} + s^{k-1}). \end{aligned} \tag{5.17}$$

□

Lemma 5.4: *Suppose that x^{k-1} is changed into x^k and further x^k into x^{k+1} by the greedy algorithm. If* $\mathrm{e}(s^{k-1}) = \mathrm{e}(s^k)$ *holds, then*

$$w(x^{k-1}) - w(x^{k-1} + s^{k-1}) \geq w(x^k) - w(x^k + s^k). \tag{5.18}$$

Proof: Suppose $s^k = -s^{k-1}$ and let $e = \mathrm{e}(s^{k-1})$. Then, from Lemma 5.1 we have

$$w_e((x^{k-1} + s^{k-1})(e)) < w_e(x^{k-1}(e)), \tag{5.19}$$

$$w_e((x^k - s^{k-1})(e)) < w_e(x^k(e)). \tag{5.20}$$

Furthermore, from the greedy algorithm we have

$$|x^k(e) - x^{k-1}(e)| \leq 2. \tag{5.21}$$

From (5.19) $\sim$ (5.21) and the convexity of w_e we obtain

$$x^k = x^{k-1} + 2s^{k-1}. \tag{5.22}$$

Therefore,

$$\begin{aligned} w(x^k) - w(x^k + s^k) &= w(x^k) - w(x^k - s^{k-1}) \\ &= w(x^k) - w(x^{k-1} + s^{k-1}) \\ &< w(x^{k-1}) - w(x^{k-1} + s^{k-1}) \end{aligned} \tag{5.23}$$

as required.

Also, if $s^k = s^{k-1}$, we have (5.18) from $s^{k-1} \in \mathrm{St}(x^{k-1}, x^k)$ and the separable convexity of w. □

Lemma 5.5: *Suppose that x^{k-1} is changed into $x^k = x^{k-1} + s^{k-1}$ and further x^k into $x^{k+1} = x^k + s^k$ by the greedy algorithm. Then,* (5.8) *implies* (5.18).

Proof: From (5.8) and the separability of w we have

$$w(x^k) - w(x^k + s^k) = w(x^{k-1}) - w(x^{k-1} + s^k). \tag{5.24}$$

If $x^{k-1} + s^k \in \mathcal{F}$, then from Remark 3.1 and (5.24) we have (5.18). Otherwise, since $x^{k+1} = x^{k-1} + s^{k-1} + s^k$, we have

$$x^{k-1} + s^k + s^{k-1} \in \mathcal{F}. \tag{5.25}$$

Hence, from (5.24) and Lemma 5.3 we have (5.18). □

Lemma 5.6: *Suppose that x^{k-1} is changed into $x^k = x^{k-1} + s^{k-1}$ and further x^k into $x^{k+1} = x^k + s^k + t^k$ ($x^k + s^k \notin \mathcal{F}$) by the greedy algorithm. Then,* (5.8) *implies* (5.18).

Proof: From (5.8) we have (5.24).

First, suppose $t^k = s^{k-1}$. Then, we have $x^{k+1} = x^{k-1} + s^k + 2s^{k-1}$. Hence, $s^k \in \mathrm{St}(x^{k-1}, x^{k+1})$ and $s^{k-1} \in \mathrm{St}(x^{k-1} + s^k, x^{k+1}) = \{s^{k-1}\}$. So, if $x^{k-1} + s^k \notin \mathcal{F}$, then from (2-SA) we have $x^{k-1} + s^k + s^{k-1} \in \mathcal{F}$, which contradicts $x^k + s^k \notin \mathcal{F}$. Therefore, we have $x^{k-1} + s^k \in \mathcal{F}$ and (5.18) follows from Remark 3.1 and (5.24).

Next, suppose $t^k = -s^{k-1}$. Then, $x^{k+1} = x^k + s^k - s^{k-1} = x^{k-1} + s^k \in \mathcal{F}$ and (5.18) follows from Remark 3.1 and (5.24).

Therefore, we can suppose $\mathrm{e}(s^{k-1}) \neq \mathrm{e}(t^k)$. Let us consider the following two cases (i) and (ii).

(i): $x^{k-1} + s^k \in \mathcal{F}$.
In this case, Remark 3.1 and (5.24) give (5.18).

(ii): $x^{k-1} + s^k \notin \mathcal{F}$.
Since $s^k \in \mathrm{St}(x^{k-1}, x^{k+1})$ and $\mathrm{St}(x^{k-1} + s^k, x^{k+1}) = \{s^{k-1}, t^k\}$, because of (2-SA)

$x^{k-1} + s^k + s^{k-1} \notin \mathcal{F}$ implies $x^{k-1} + s^k + t^k \in \mathcal{F}$. Since $\mathrm{e}(s^{k-1}) \neq \mathrm{e}(s^k)$ and $\mathrm{e}(s^{k-1}) \neq \mathrm{e}(t^k)$, by the greedy algorithm and the separability of w we have

$$w(x^{k-1} + s^k + t^k) - w(x^{k-1}) = w(x^k + s^k + t^k) - w(x^k) < 0. \tag{5.26}$$

Consequently, (5.18) holds due to Remark 3.2 and (5.24). □

Lemma 5.7: *Suppose that x^{k-1} is changed into $x^k = x^{k-1}+s^{k-1}+t^{k-1}$ ($x^{k-1}+s^{k-1} \notin \mathcal{F}$) and further x^k into $x^{k+1} = x^k + s^k$ by the greedy algorithm. Then,* (5.8) *implies* (5.18).

Proof: First, suppose $s^k = -t^{k-1}$. Then, we have $x^{k+1} = x^{k-1} + s^{k-1}$. However, the greedy algorithm implies $x^{k-1} + s^{k-1} \notin \mathcal{F}$, which contradicts the fact $x^{k+1} \in \mathcal{F}$. Hence, we have $s^k \neq -t^{k-1}$.

Next, suppose $s^k = t^{k-1}$. Since $\mathrm{e}(s^{k-1}) \neq \mathrm{e}(s^k)$ (i.e., $\mathrm{e}(s^{k-1}) \neq \mathrm{e}(t^{k-1})$), from Lemma 5.2 and the separable convexity of w we have

$$\begin{aligned}
(0 \quad <) \quad & w(x^k) - w(x^k + s^k) \\
\leq \quad & w(x^k - s^k) - w(x^k) \\
= \quad & w(x^k - t^{k-1}) - w(x^k) \\
= \quad & w(x^{k-1} + s^{k-1}) - w(x^{k-1} + s^{k-1} + t^{k-1}) \\
= \quad & w(x^{k-1}) - w(x^{k-1} + t^{k-1}) \\
\leq \quad & w(x^{k-1}) - w(x^{k-1} + s^{k-1})
\end{aligned} \tag{5.27}$$

as desired.

Therefore, we can also suppose $\mathrm{e}(t^{k-1}) \neq \mathrm{e}(s^k)$. Then, we have (5.24). Let us consider the following two cases (i) and (ii).

(i): $x^{k-1} + s^k \in \mathcal{F}$.
In this case, from Remark 3.1 we have (5.18).

(ii): $x^{k-1} + s^k \notin \mathcal{F}$.
Since $s^k \in \mathrm{St}(x^{k-1}, x^{k+1})$ and $\mathrm{St}(x^{k-1} + s^k, x^{k+1}) = \{s^{k-1}, t^{k-1}\}$, from (2-SA) we have the following two subcases.

(ii-1): $x^{k-1} + s^k + s^{k-1} \in \mathcal{F}$.
In this case, (5.24) and Lemma 5.3 yield (5.18).

(ii-2): $x^{k-1} + s^k + t^{k-1} \in \mathcal{F}$.
If $t^{k-1} = s^{k-1}$, then by the same argument in **(ii-1)** we have (5.18). Therefore, we can suppose $\mathrm{e}(t^{k-1}) \neq \mathrm{e}(s^{k-1})$. If $w(x^{k-1} + s^k + t^{k-1}) < w(x^{k-1})$, then (5.18) follows from Remark 3.2. Since $\mathrm{e}(s^{k-1}) \neq \mathrm{e}(s^k)$ and $\mathrm{e}(t^{k-1}) \neq \mathrm{e}(s^k)$, if $w(x^{k-1} + s^k + t^{k-1}) \geq w(x^{k-1})$, then from (5.24) and the separable convexity of w,

$$\begin{aligned}
w(x^k) - w(x^k + s^k) \quad &= \quad w(x^{k-1}) - w(x^{k-1} + s^k) \\
&\leq \quad w(x^{k-1} + t^{k-1}) - w(x^{k-1}).
\end{aligned} \tag{5.28}$$

Since $w(x^{k-1}+s^{k-1}+t^{k-1}) = w(x^k) < w(x^{k-1})$ and $\mathrm{e}(t^{k-1}) \neq \mathrm{e}(s^{k-1})$, we have

$$w(x^{k-1}+t^{k-1}) - w(x^{k-1}) < w(x^{k-1}) - w(x^{k-1}+s^{k-1}). \tag{5.29}$$

From (5.28) and (5.29), we have (5.18). □

Lemma 5.8: *Suppose that x^{k-1} is changed into $x^k = x^{k-1}+s^{k-1}+t^{k-1}$ ($x^{k-1}+s^{k-1} \notin \mathcal{F}$) and further x^k into $x^{k+1} = x^k + s^k + t^k$ ($x^k + s^k \notin \mathcal{F}$) by the greedy algorithm. Then,* (5.8) *implies* (5.18).

Proof: If $s^k = t^{k-1}$, then by the same argument in Lemma 5.7 we have (5.18).
Moreover, if $s^k = -t^{k-1}$, it follows from the greedy algorithm that

$$\begin{aligned} w(x^k) - w(x^k+s^k) &= w(x^k) - w(x^k - t^{k-1}) \\ &= w(x^{k-1}+s^{k-1}+t^{k-1}) - w(x^{k-1}+s^{k-1}) \\ &< w(x^{k-1}) - w(x^{k-1}+s^{k-1}) \end{aligned} \tag{5.30}$$

as is desired.

Hence, we can also suppose $\mathrm{e}(t^{k-1}) \neq \mathrm{e}(s^k)$, which gives (5.24). Let us consider the following two cases (i) and (ii).

(i): $x^{k-1} + s^k \in \mathcal{F}$.
In this case, from Remark 3.1 and (5.24) we have (5.18).

(ii): $x^{k-1} + s^k \notin \mathcal{F}$.
Since $s^k \in \mathrm{St}(x^{k-1}, x^{k+1})$ and $\mathrm{St}(x^{k-1}+s^k, x^{k+1}) \subseteq \{s^{k-1}, t^{k-1}, t^k\}$, from (2-SA) we have the following three subcases.

(ii-1): $x^{k-1} + s^k + s^{k-1} \in \mathcal{F}$.
In this case, from (5.24) and Lemma 5.3 we have (5.18).

(ii-2): $x^{k-1} + s^k + t^{k-1} \in \mathcal{F}$.
In this case, by the same argument in **(ii-2)** of Lemma 5.7 we have (5.18).

(ii-3): $x^{k-1} + s^k + t^k \in \mathcal{F}$.
If $t^k = s^{k-1}$, then by the same argument in **(ii-1)** we have (5.18). If $t^k = t^{k-1}$, then by the same argument in **(ii-2)** we have (5.18). If $t^k = -s^{k-1}$, then $x^{k+1} = x^k + s^k - s^{k-1} = x^{k-1} + s^k + t^{k-1}$. Hence, by the same argument in **(ii-2)** we have (5.18). If $t^k = -t^{k-1}$, then $x^{k+1} = x^k + s^k - t^{k-1} = x^{k-1} + s^k + s^{k-1}$. Hence, by the same argument in **(ii-1)** we have (5.18). Therefore, we can also suppose that $\mathrm{e}(t^k) \neq \mathrm{e}(s^{k-1})$ and $\mathrm{e}(t^k) \neq \mathrm{e}(t^{k-1})$. Then, by the same argument in **(ii)** of Lemma 5.6 we have (5.18). □

From Lemmas 5.4 ~ 5.8 we have the following theorem.

Theorem 5.9: *If x^{k-1} is changed into x^k and further x^k into x^{k+1} successively by the greedy algorithm, then*

$$w(x^{k-1}) - w(x^{k-1} + s^{k-1}) \geq w(x^k) - w(x^k + s^k). \tag{5.31}$$

□

Based on the results of Bouchet and Cunningham [2], we have the following theorem.

Theorem 5.10 ([2]): *For a finite jump system $(E, \mathcal{F})$ the convex hull of $\mathcal{F}$ coincides with the convex hull of an integral bisubmodular polyhedron in $\mathbf{R}^E$.* □

It follows from Theorem 5.10 that for a given finite jump system $(E, \mathcal{F})$ there exists a *bisubmodular function* f such that the convex hull of $\mathcal{F}$ is given by

$$\mathrm{Co}(\mathcal{F}) = \{x \mid x \in \mathbf{R}^E, \forall (X, Y) \in 3^E : x(X) - x(Y) \leq f(X, Y)\}. \tag{5.32}$$

By the use of such a *bisubmodular function* f we obtain the following theorem.

Theorem 5.11: *The greedy algorithm executes Step 2 at most*

$$\sum_{e \in E} \Big\{ f(\{e\}, \emptyset) + f(\emptyset, \{e\}) \Big\} \tag{5.33}$$

times.

Proof: Suppose that starting with an initial solution x^0, the algorithm terminates with x^l. Recall that x^k is changed into $x^{k+1} := x^k + s^k$ or $x^{k+1} := x^k + s^k + t^k$ for $0 \leq k < l$. Let us consider the following sequence of ordered pair

$$c^k = (x^k(\mathrm{e}(s^k)), s^k) \ (0 \leq k < l). \tag{5.34}$$

For example, if $s^k = -\chi_{e'}$, then $c^k = (x^k(e'), -\chi_{e'})$, where χ_e is the unit vector with $\chi_e(e) = 1$ and $\chi_e(e') = 0$ $(e' \in E - \{e\})$. Denote by C the set of all the c^k $(0 \leq k < l)$, i.e.,

$$C = \{c^k \mid 0 \leq k < l\}. \tag{5.35}$$

Suppose $(\alpha, s) \in C$. It follows from the greedy algorithm and the separable convexity of w that

(C1) the pair of α and $\mathrm{e}(s)$ uniquely determines s, which is either $\chi_{\mathrm{e}(s)}$ or $-\chi_{\mathrm{e}(s)}$,
(C2) $-f(\emptyset, \{\mathrm{e}(s)\}) \leq \alpha \leq f(\{\mathrm{e}(s)\}, \emptyset)$,
(C3) α is not a minimizer of $w_{\mathrm{e}(s)}$ on the interval $[-f(\emptyset, \{\mathrm{e}(s)\}), f(\{\mathrm{e}(s)\}, \emptyset)]$.

Hence, we have

$$|C| \leq \sum_{e \in E} \left\{ f(\{e\}, \emptyset) + f(\emptyset, \{e\}) \right\}. \tag{5.36}$$

Suppose that for some $c = (\alpha, s) \in C$ and some j, h with $0 \leq j < h < l$, we have

$$\alpha = x^j(\mathrm{e}(s)) = x^h(\mathrm{e}(s)), \tag{5.37}$$

$$s = s^j = s^h. \tag{5.38}$$

Put $e = \mathrm{e}(s)$ and suppose $s = \chi_e$ without loss of generality. We will show a contradiction. By the present assumption,

(1) when x^j is changed into x^{j+1}, $\alpha = x^j(e)$ is changed into $x^j(e)+1$ (or $x^j(e)+2$), and

(2) when x^h is changed into x^{h+1}, $\alpha = x^h(e)$ is changed into $x^h(e)+1$ (or $x^h(e)+2$).

Hence, for some k with $j < k < h$, when x^k is changed into x^{k+1}, we have

(3) $x^k(e) = \alpha + 1$ is changed into $x^{k+1}(e) = \alpha$,

or

(3′) $x^k(e) = \alpha + 2$ is changed into $x^{k+1}(e) = \alpha$,

or

(3″) $x^k(e) = \alpha + 1$ is changed into $x^{k+1}(e) = \alpha - 1$.

From Lemma 5.1 and the separable convexity of w we have

$$w_e(\alpha + 1) < w_e(\alpha). \tag{5.39}$$

Let us consider the following three cases.

[**Case 1**]: (3) holds.
In this case, (5.39) implies

$$x^{k+1} = x^k + s^k - \chi_e = x^k + s^k - s. \tag{5.40}$$

Also, from the greedy algorithm we have

$$w(x^k + s^k - \chi_e) < w(x^k), \tag{5.41}$$

i.e.,

$$w(x^k - \chi_e) - w(x^k) < w(x^k) - w(x^k + s^k). \tag{5.42}$$

(Note that $e \neq \mathrm{e}(s^k)$.) Furthermore, from (1) and (3), we have

$$\begin{aligned} w(x^j) - w(x^j + s^j) &= w_e(x^j(e)) - w_e(x^j(e) + 1) \\ &= w_e(\alpha) - w_e(\alpha + 1) \\ &= w(x^k - \chi_e) - w(x^k). \end{aligned} \tag{5.43}$$

Therefore, from (5.42) and (5.43) we have

$$w(x^j) - w(x^j + s^j) < w(x^k) - w(x^k + s^k). \tag{5.44}$$

This contradicts Theorem 5.9.

[Case 2]: (3′) holds.
(3′) implies

$$x^{k+1} = x^k - 2\chi_e \tag{5.45}$$

and

$$s^k = -\chi_e. \tag{5.46}$$

Also, we have

$$w_e(\alpha) < w_e(\alpha + 2). \tag{5.47}$$

It follows from (1), (3′), (5.46) and (5.47) that

$$\begin{aligned} w(x^j) - w(x^j + s^j) &= w_e(x^j(e)) - w_e(x^j(e) + 1) \\ &= w_e(\alpha) - w_e(\alpha + 1) \\ &< w_e(\alpha + 2) - w_e(\alpha + 1) \\ &= w(x^k) - w(x^k - \chi_e) \\ &= w(x^k) - w(x^k + s^k). \end{aligned} \tag{5.48}$$

This also contradicts Theorem 5.9.

[Case 3]: (3″) holds.
It follows from (5.39) and the separable convexity of w we have

$$w_e(\alpha) < w_e(\alpha - 1). \tag{5.49}$$

Then, we have

$$w_e(\alpha + 1) < w_e(\alpha - 1) \tag{5.50}$$

and from (3″),

$$w(x^k) < w(x^{k+1}). \tag{5.51}$$

This contradicts the definition of the greedy algorithm.

From the above argument, for any $j < h \leq l$ we have $c^j \neq c^h$. Therefore, from (5.36) the present theorem holds. □

References

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