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Rank Dependent Utility for Arbitrary Consequence Spaces

by

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Abstract

Quiggin's anticipated utility, sometimes called rank dependent utility, generalizes von Neumann-Morgenstern expected utility to accommodate Allais type violations of preference judgments. His theory and the subsequent axiomatic refinements presume that the underlying consequence spaces are rich, so that certainty equivalents of gambles exist. This paper developed an axiomatic characterization of rank dependent utility for arbitrary consequence spaces, so that certainty equivalents of gambles do not necessarily exist.

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1 Introduction

In the last decade growing research activities has been made to generalize expected utility theories of von Neumann and Morgenstern (1953) and Savage (1954). Under various frameworks numerous axiom systems and their numerical representations have been developed (see surveys by Fishburn (1988) and Machina (1987)). In decisions under risk the anticipated utility theory by Quiggin (1982) and independently by Yaari (1987) initiated much research interest. Their theory is sometimes called a *rank dependent utility (RDU) theory*, since expected utility of a gamble is given by the expectation of a utility function with respect to a transformed (de)cumulative probability distribution of that gamble.

There have been made many subsequent refinements and generalizations of Quiggin's approach, i.e., *the rank dependent approach.* They include Chateauneuf (1990), Chew (1989), Chew, Karni, and Safra (1987), Green and Jullien (1988), Hilton (1988), Luce (1988), Luce and Fishburn (1991), Nakamura (1992), Quiggin (1985), Quiggin and Wakker (1992), Segal (1987, 1989, 1993) and others. However, those rank dependent approaches do not fully generalize von Neumann-Morgenstern expected utility in the sense that the underlying consequence spaces must be rich, so that certainty equivalents of all gambles exist.

In decisions under uncertainty, a rank dependent approach was initiated by Schmeidler (1984, 1989). We say that his approach is a *Choquet expected utility (CEU) approach.* He generalized subjective expected utility of Anscombe and Aumann (1963) to allow for non-additive probabilities over finite states, and adopted Choquet (1953-54) integration. Since then, the CEU approach motivated various axiomatic characterizations to generalize subjective expected utility theories. They include Gilboa (1987), Nakamura (1990, 1992) and Wakker (1989a, b, c, 1991, 1993). Wakker (1990) argued that CEU model is more general than RDU model. Except Gilboa's axiomatization, other CEU approaches do not apply to a full characterization of rank dependent utility, since similar richness of the consequence spaces is presumed. Although in Gilboa's approach the consequence space is arbitrary in the sense that it includes at least three consequences that are not mutually indifferent, his axiomatization leads to a *bounded* utility function and is complicated to translate into an axiomatization of rank dependent utility. Therefore we need a new approach to fully characterize rank dependent utility.

The present paper develops such an axiomatic characteriztion of rank dependent utility in a general set-up such that the consequent space includes at least three consequences that are not mutually indifferent, so that certainty equivalents of gambles do not necessarily exist. Quiggin and others' rank dependent approaches presume the existence of both certainty and probability equivalents for gambles. We drop the former and retain the latter. Our axiomatization for probability measures with bounded supports is based on the weak multi-symmetric structure developed in Nakamura (1992). We also apply the truncation continuity axiom introduced by Wakker (1989b, c) to obtain a rank dependent utility representation for unbounded probability measures. Since a utility function on the consequence set X need not be bounded when X is infinite, our axiomatization generalizes the unbounded expected utility representation by Wakker (1993).

The paper is organized as follows. Section 2 states a rank dependent utility representation for arbitrary consequence spaces. Then Section 3 discusses axioms and presents three representation theorems. In Section 4, we prove a rank dependent utility representation when the consequence space is finite. Then Section 5 extends the result in Section 4 to all step probability measures. Section 6 proves a representation for bounded and unbounded probability measures. Section 7 concludes the paper.

2 Rank Dependent Utility

Let X be a set of consequences. By $\mathcal{B}$, we denote a Boolean algebra on X, i.e., $\mathcal{B}$ contains $\emptyset$ (empty set) and X, and is closed under finite unions and complementation. We shall assume throughout the paper that $\mathcal{B}$ contains the singleton subset $\{x\}$ for each $x \in X$. Let A^c denote the complement of a subset A of X. A finitely additive probability measure is a nonnegative real valued function p on $\mathcal{B}$ such that $p(X) = 1$ and $p(Y \cup Z) = p(Y) + p(Z)$ whenever $Y, Z \in \mathcal{B}$ are disjoint. By $\mathcal{P}$, we denote a set of finitely additive probability measures on $\mathcal{B}$. Let $\preceq$ on $\mathcal{P}$ be the binary preference relation with $\sim$ and $\prec$ defined in the usual way: for $p, q \in \mathcal{P}$, $p \sim q$ if $p \preceq q$ and $q \preceq p$; $p \prec q$ if $p \preceq q$ and not$(q \preceq p)$. A *one-point* probability measure is a probability measure $p \in \mathcal{B}$ such that $p(\{x\}) = 1$ for some $x \in X$. Each $x \in X$ is identified with a one-point probability measure p that has $p(\{x\}) = 1$, so that $x \preceq y$ means that $p \preceq q$ when $p(\{x\}) = q(\{y\}) = 1$. We write $p \preceq q \preceq r$ when $p \preceq q$ and $q \preceq r$; $p \prec q \prec r$ when $p \prec q$ and $q \prec r$. When A and B are subsets of X, we write $A \preceq B$ when $x \preceq y$ for all $x \in A$ and all $y \in B$; $A \prec B$ when $x \prec y$ for all $x \in A$ and $y \in B$.

We say that $(\mathcal{P}, \prec)$ has a rank dependent utility (RDU) representation when there exists a mapping V from $\mathcal{P}$ into $[-\infty, +\infty]$ such that for all $p, q \in \mathcal{P}$, $p \preceq q$ if and only if $V(p) \leq V(q)$, and V is given as follows: for all $p \in \mathcal{P}$,

$$\begin{aligned} V(p) &= \int_0^{+\infty} (\Psi(1) - \Psi(p(\{x \in X : u(x) \leq \tau\})))d\tau \\ &\quad - \int_{-\infty}^{0} (\Psi(p(\{x \in X : u(x) \leq \tau\})) - \Psi(0))d\tau, \end{aligned} \tag{1}$$

where u is a real valued function on X and Ψ is a strictly increasing and continuous real valued function on $[0, 1]$. Moreover, u and Ψ are unique up to positive linear transformations. When Ψ is a linear function on $[0, 1]$, $V(p)$ in (1) is reduced to expected value of u with respect to p. We note that u on X could be unbounded, while $V(p)$ in (1) is well defined.

The existing axiomatizations for (1) presume the richness of X. For example, X may be a real interval or, more generally, a connected separable topological space. Nakamura (1992) assumed that X is dense, i.e., for $x, y \in X$, if $x \prec y$, then $x \prec z \prec y$ for some $z \in X$. Our axiomatization allows for any set X, finite or infinite. Since our RDU representation deals with a set of finitely additive probability measures, we shall establish the representation in four steps. The first step considers a finite X. The representation for *simple* probability measures will be developed in the second step. Then *step* probability measures defined later in this section will be dealt with in the third step. Finally, (1) will be established for a set of finitely additive probability measures that includes all step probability measures.

Following Fishburn (1982, Chapter 3), a nonempty subset A of X is said to be a *preference interval* if $z \in A$ whenever $x, y \in A$, $x \preceq z$, and $z \preceq y$. By Γ, we denote the set of all preference intervals. We say that a preference interval A is *bounded* if $a \preceq A \preceq b$ for some $a, b \in X$. Let Γ^b be the set of all bounded preference intervals. Certain preference intervals will be denoted as follows.

$$\begin{aligned}(-\infty, a] &= \{x \in X : x \preceq a\},\\ [a, +\infty) &= \{x \in X : a \preceq x\},\\ [a, b] &= \{x \in X : a \preceq x \preceq b\}.\end{aligned}$$

Preference intervals such as $(-\infty, a), (a, +\infty), [a, b)$, and so forth are similarly defined. A *partition* of a preference interval A is a finite sequence of preference intervals that are mutually disjoint and whose union equals A. Thus if preference intervals B and C are in the partition, then either $B \prec C$ or $C \prec B$.

We say that $p \in \mathcal{P}$ is a *step* probability measure if $p(A) = 1$ for some $A \in \Gamma^b$, and there exist a partition $\{A_1, \ldots, A_n\}$ of A and real numbers $\alpha_1, \ldots, \alpha_n$ such that for all $i = 1, \ldots, n$,

$$p((-\infty, x]) = \alpha_i \text{ whenever } x \in A_i.$$

Equivalently, there are real numbers $\beta_1, \ldots, \beta_n$ such that for all $i = 1, \ldots, n$,

$$p([x, +\infty)) = \beta_i \text{ whenever } x \in A_i.$$

If $A_1 \prec \ldots \prec A_n$, then for $k = 1, \ldots, n-1$, $0 \le \alpha_k \le p(\cup_{i=1}^k A_i) \le \alpha_{k+1} \le 1$ and $0 \le \beta_{n-k+1} \le p(\cup_{i=1}^k A_{n-i+1}) \le \beta_{n-k} \le 1$. In particular, p is said to be a *one-step* probability measure if $n = 1$, and $\alpha_1 = 1$ or $\beta_1 = 1$. In other words, p is a one-step probability measure if $p(A) = 1$ for some $A \in \Gamma^b$, and $p((-\infty, x]) = 1$ for all $x \in A$ or $p([x, +\infty)) = 1$ for all $x \in A$. Every one-point probability measure is a one-step probability measure. Let $\mathcal{S}$ be the set of all one-step probability measures, so $\mathcal{S} \supseteq X$. Every step probability measure p is a convex combination of a finite number of one-step probability measures, i.e., $p = \sum_{i=1}^{n} p_i$ for $p_i \in \mathcal{S}$ and $i = 1, \ldots, n$. In particular, when p is a simple probability measure, $\sum_{x \in Y} p(\{x\}) = 1$ for a finite $Y \subseteq X$.

We define

$$\begin{aligned} \mathcal{S}^+ &= \{p \in \mathcal{S} : p(A) = 1 \text{ for some } A \in \Gamma^b \text{ and } p((-\infty, x]) = 1 \text{ for all } x \in A\}, \\ \mathcal{S}^- &= \{p \in \mathcal{S} : p(A) = 1 \text{ for some } A \in \Gamma^b \text{ and } p([x, +\infty)) = 1 \text{ for all } x \in A\}. \end{aligned}$$

Thus $\mathcal{S} = \mathcal{S}^+ \cup \mathcal{S}^-$. Suppose that $\mathcal{S} \subseteq \mathcal{P}$ and $(\mathcal{P}, \prec)$ has the RDU representation. Thus for all $p \in \mathcal{S}$, if $p(A) = 1$ for some $A \in \Gamma^b$, then

$$V(p) = \begin{cases} \inf_{x \in A} u(x) & \text{when } p \in \mathcal{S}^+ \\ \sup_{x \in A} u(x) & \text{when } p \in \mathcal{S}^- \end{cases}$$

If p is a simple probability measure, then $V(p) = u(x)$ for some $x \in X$. However, we may have that $V(p) = u(x)$ for no $x \in X$ when $p \in \mathcal{S}$.

When p is a step probability measure, (1) is given as follows. There exists a finite increasing sequence of real numbers, $\tau_1 < \ldots < \tau_n$, such that for some nonnegative numbers, $0 \leq \alpha_1 \leq \ldots \leq \alpha_{n-1} \leq 1$,

$$p(\{x \in X : u(x) \leq \tau\}) = \begin{cases} 0 & \text{if } \tau < \tau_1 \\ \alpha_i & \text{if } \tau_i < \tau < \tau_{i+1} \text{ and } 1 \leq i < n \\ 1 & \text{if } \tau_n < \tau \end{cases}$$

Then we have

$$V(p) = \sum_{i=1}^{n} (\Psi(\alpha_i) - \Psi(\alpha_{i-1}))\tau_i, \tag{2}$$

where $\alpha_0 = 0$ and $\alpha_n = 1$. We note that for each $i = 1, \ldots, n$, there does not necessarily exist a consequence x_i such that $\tau_i = u(x_i)$. When p is a simple probability measure, $\sum_{i=1}^{n} p(\{x_i\}) = 1$ for a finite subset $\{x_1, \ldots, x_n\} \subseteq X$ with $x_1 \preceq \ldots \preceq x_n$. Then we have

$$V(p) = \sum_{i=1}^{n} (\Psi(\alpha_i) - \Psi(\alpha_{i-1}))u(x_i), \tag{3}$$

where $\alpha_0 = 0$ and $\alpha_i = p(\{x_1, \ldots, x_i\})$ for $i = 1, \ldots, n$.

We note that the RDU representations (1)–(3) do not necessarily imply that each p has a certainty equivalent $x_p \in X$, i.e., $V(p) = V(x_p)$. Our approach to axiomatize (1)–(3) is based on the existence of a probability equivalent for a bounded $p \in \mathcal{P}$ such that $p([a,b]) = 1$ for some $a, b \in X$. Let $a, b \in X$ with $a \prec b$ be fixed. Define $\mathcal{P}_{ab} = \{p \in \mathcal{P} : p([a,b]) = 1\}$. For $p \in \mathcal{P}_{ab}$, a probability equivalent of p, denoted $\sigma(p)$, is a probability number $0 \leq \lambda \leq 1$ such that $p \sim \lambda a + (1-\lambda)b$. By (1), $\sigma(p)$ on $\mathcal{P}_{ab}$ must be unique. With no loss of generality, let $u(a) = 0$ and $u(b) = 1$. Then (1) gives that $V(p) = \Psi(1) - \Psi(\sigma(p))$, so for all $p, q \in \mathcal{P}_{ab}$, $p \preceq q$ if and only if $\Psi(\sigma(q)) \leq \Psi(\sigma(p))$. Thus $\Psi(\sigma)$ is regarded as a disutility function on $\mathcal{P}_{ab}$. Our approach is to first construct Ψ on $\mathcal{P}_{ab}$, and then to obtain u on X such that (1) holds for all bounded $p \in \mathcal{P}$. Details will be stated in the following sections.

3 Axioms and Theorems

This section presents necessary and sufficient axioms for the RDU representation (3) and then extends (3) to all step probability measures to obtain the representation (2). Finally, we cover more general probability measures for (1). To axiomatize the representations (2) and (3), a structural assumption for $\mathcal{B}$ and X is given as follows.

Assumption 1 *$\mathcal{B}$ is a Boolean algebra on a nonempty set X and contains all preference intervals. There are $a, b, c \in X$ such that $a \prec b \prec c$.*

Let $\Gamma^* = \{A \in \Gamma : x \in A \text{ whenever } y \in A \text{ and } x \preceq y\}$ be the set of all preference intervals that are unbounded below. Thus if $A \in \Gamma^*$ and $A^c \neq \emptyset$, then A^c is a preference interval that is unbounded above, i.e., $x \in A^c$ whenever $y \in A^c$ and $y \preceq x$. In Sarin and Wakker (1992), those special preference intervals are called cumulative consequence sets. We use five axioms for the representation (3). Three of them, which are understood as applying to all $p, q, r \in \mathcal{P}$, are stated as follows:

Axiom 1 *$\preceq$ on $\mathcal{P}$ is a weak order.*

Axiom 2 *If* $a \preceq p$ *and* $r \preceq b$ *for some* $a, b \in X$, *and if* $p \preceq q \preceq r$, *then* $q \sim \alpha p + (1-\alpha)r$ *for some* $0 \leq \alpha \leq 1$.

Axiom 3 *If* $p(A) \leq q(A)$ *for all* $A \in \Gamma^*$, *then* $q \preceq p$.

A weak order means by definition that it is transitive and complete, i.e., for $p, q \in \mathcal{P}$, $p \preceq q$ or $q \preceq p$. Axiom 2 is a continuity axiom which says that if p and r are bounded in preferences, and if p is not preferred to q and q is not preferred to r, then q is indifferent to some convex combination of p and r. Boundedness with respect to preferences is required, since $V(p) = -\infty$ or $V(r) = +\infty$ may imply that α in the axiom does not exist or that $V(\alpha p + (1-\alpha)r)$ is not well defined. Axiom 3 is a dominance axiom that is similar to the cumulative dominance proposed by Sarin and Wakker (1992). We note that those three axioms apply to any probabillity measures in $\mathcal{P}$.

A finite increasing sequence of consequences in X is denoted by $\langle x_1, \ldots, x_n \rangle$ for distinct $x_i \in X$, $i = 1, \ldots, n$, where $x_1 \preceq \ldots \preceq x_n$. Let Π be the set of all such finite increasing sequences of consequences. For $\pi \in \Pi$, let $\mid \pi \mid$ denote the number of elements of π. For $\pi, \pi_1, \pi_2 \in \Pi$, we shall write $x \in \pi$ when $x \in X$ is in π, and $\pi_1 \subseteq \pi_2$ when all $x \in \pi_1$ are in π_2. Let Π^* be the set of all sequences in Π such that if $\pi \in \Pi^*$, then $a \prec b \prec c$ for some $a, b, c \in \pi$.

Given $\pi = \langle x_1, \ldots, x_{n+1} \rangle$ and $0 \leq \alpha_i \leq 1$ for $i = 1, \ldots, n$, let $p_\pi(\alpha_1 \ldots \alpha_n)$ denote a simple probability measure p such that $p(\{x_1, \ldots, x_i\}) = \alpha_i$ for $1 \leq i \leq n+1$, where $\alpha_{n+1} = 1$. Thus $\alpha_1 \leq \ldots \leq \alpha_n$, so they are cumulative probabilities. We note that a simple probability measure p has many different forms of $p_\pi(\alpha_1 \ldots \alpha_n)$ representations as shown in the following examples: $p_{\pi_1}(\alpha \ldots \alpha) = p_{\pi_2}(\alpha)$ when $\pi_1 = \langle x_1, \ldots, x_{n+1} \rangle$ and $\pi_2 = \langle x_1, x_{n+1} \rangle$; $p_{\pi_1}(\alpha\beta\beta) = p_{\pi_2}(\alpha\beta)$ when $\pi_1 = \langle x, y, z, w \rangle$ and $\pi_2 = \langle x, y, w \rangle$. However, each $p_\pi(\alpha_1 \ldots \alpha_n)$ specifies a unique simple probability measure by definition.

Throughout the rest of the paper, let $\mathcal{P}^*$ be a convex set of simple probability measures that contains every one-point measure. Also let $\mathcal{P}^{**}$ be a convex set of step probability measures that contains every one-step measure. Convexity of a set $\mathcal{P}$ means that $\lambda p +$

$(1-\lambda)q \in \mathcal{P}$ whenever $0 \leq \lambda \leq 1$ and $p, q \in \mathcal{P}$.

The next two axioms are concerned with all simple probability measures, which applies to all $p, p_1, p_2, q_1, q_2, r_1, r_2 \in \mathcal{P}^*$, all $x, y \in X$, all $\pi \in \Pi^*$ with $|\pi| = 3$, and all $0 < \lambda < 1$.

Axiom 4 *If $x \prec y$, then $\lambda x + (1-\lambda)p \prec \lambda y + (1-\lambda)p$.*

Axiom 5 *If $p_i((-\infty, a]) \leq q_i((-\infty, a])$ for all $a \in X$ and $i = 1, 2$, and if $p_1 \sim p_2$, $q_1 \sim q_2$, and $p_\pi(p_i((-\infty, a])q_i((-\infty, a])) \sim p_\pi(r_i((-\infty, a])r_i((-\infty, a]))$ for all $a \in X$ and $i = 1, 2$, then $r_1 \sim r_2$.*

A familiar independence axiom in the expected utility theory is that for all $p, q, r \in \mathcal{P}$ and all $0 < \lambda < 1$, if $q \prec r$, then $\lambda q + (1-\lambda)p \prec \lambda r + (1-\lambda)p$. Axiom 4 replaces q and r by $x \in X$ and $y \in X$, respectively, and requires that p be in $\mathcal{P}^*$. Axiom 5 is concerned with relaxation of another type of independence axiom, i.e., for all $p_1, p_2, q_1, q_2 \in \mathcal{P}^*$, if $p_1 \sim p_2$ and $q_1 \sim q_2$, then $\lambda p_1 + (1-\lambda)q_1 \sim \lambda p_2 + (1-\lambda)q_2$ for all $0 < \lambda < 1$. To see that this implies Axiom 5, suppose that for $\pi \in \Pi$ and $0 < \lambda < 1$, and for $i = 1, 2$ and $j = 1, \ldots, n$,

$$\begin{aligned} p_i &= p_\pi(\alpha_{i1} \ldots \alpha_{in}), \\ q_i &= p_\pi(\beta_{i1} \ldots \beta_{in}), \\ r_i(\lambda) &= p_\pi(\gamma_{i1}(\lambda) \ldots \gamma_{in}(\lambda)), \end{aligned}$$

and $\gamma_{ij}(\lambda) = \lambda\alpha_{ij} + (1-\lambda)\beta_{ij}$, and that $p_1 \sim p_2$ and $q_1 \sim q_2$. Then by the independence axiom, for all $0 < \lambda < 1$, $r_1(\lambda) \sim r_2(\lambda)$. Given $\pi^* \in \Pi^*$ with $|\pi^*| = 3$, it follows from Axioms 1–3 that there exist a unique $0 < \lambda^* < 1$ such that

$$p_{\pi^*}(\alpha_{ij}\beta_{ij}) \sim p_{\pi^*}(\gamma_{ij}(\lambda^*)\gamma_{ij}(\lambda^*)),$$

Since $r_1(\lambda^*) \sim r_2(\lambda^*)$, Axiom 5 follows.

With axioms 1–5, we obtain the following representation theorem for all simple probabilitiy measures. The proof is deferred to Section 5.

Theorem 1 *Suppose that $\mathcal{P}^* \subseteq \mathcal{P}$ and Assumption 1 holds. Then Axioms 1–5 hold if and only if $(\mathcal{P}^*, \prec)$ has the RDU representation.*

To obtain the representation (2), one more axiom is required. The following axiom applies to all $p_1, \ldots, p_n, q \in \mathcal{P}^{**}$ and all nonnegative numbers $\alpha_1, \ldots, \alpha_n$ with $\sum_{i=1}^n \alpha_i = 1$:

Axiom 6 *If $\{A_1, \ldots, A_n\}$ is a partition of some bounded preference interval and $p_i(A_i) = 1$ for $i = 1, \ldots, n$, then $q \preceq \sum_{i=1}^n \alpha_i p_i$ if $q \prec \sum_{i=1}^n \alpha_i x_i$ for all $x_i \in A_i$ and $i = 1, \ldots, n$, and $\sum_{i=1}^n \alpha_i p_i \preceq q$ if $\sum_{i=1}^n \alpha_i x_i \prec q$ for all $x_i \in A_i$ and $i = 1, \ldots, n$.*

The first part of Axiom 6 says that if p_i is certain to yield a consequence in A_i and if every simple probability measure that yields exactly one consequence x_i in each A_i with probability α_i is strctly preferred to q, then the step probability measure that yields step probability measure p_i with probaility α_i is weakly preferred to q. The second part has a similar interpretation. Adding Axiom 6 to Theorem 1, we obtain the following representation theorem for all step probability measures. The proof is deferred to Section 5.

Theorem 2 *Suppose that $\mathcal{P}^{**} \subseteq \mathcal{P}$ and Assumption 1 holds. Then Axioms 1–6 hold if and only if $(\mathcal{P}^{**}, \prec)$ has the RDU representation.*

In the sequel, we extend the representations of Theorems 1 and 2 to more general probability measures in $\mathcal{P}$. To this end we need to add a *truncation continuity* axiom introduced by Wakker (1989a, b). When all preference intervals are contained in $\mathcal{B}$, we define *upper* and *lower truncations* of probability measures as introduced by Wakker (1989a, b). Given $a \in X$, an upper truncation of $p \in \mathcal{P}$, denoted p^a, is a probability measure on $\mathcal{B}$ such that

$$
\begin{aligned}
p^a(Y) &= p(Y) \text{ for all } Y \in \Gamma \text{ with } Y \subseteq (-\infty, a), \\
p^a(\{a\}) &= p([a, +\infty)),
\end{aligned}
$$

and a lower truncation of $p \in \mathcal{P}$, denoted p_a, is a probability measure on $\mathcal{B}$ such that

$$\begin{aligned} p_a(Y) &= p(Y) \text{ for all } Y \in \Gamma \text{ with } Y \subseteq (a, +\infty), \\ p_a(\{a\}) &= p((-\infty, a]). \end{aligned}$$

To obtain the representation (1), we need the following structural assumption for $\mathcal{P}$.

Assumption 2 *$\mathcal{P}$ contains all upper and lower truncations of probability measures in $\mathcal{P}$.*

Then one more axiom is required for the RDU representation (1). The following axiom, which is understood as applying to all $p, q \in \mathcal{P}$, is Wakker's truncation continuity axiom.

Axiom 7 *If $p \prec q$, then $p_a \prec q$ and $p \prec q^b$ for some $a, b \in X$.*

The RDU representation (1) for general probability measures is given as follows. The proof is deferred to Section 6.

Theorem 3 *Suppose that $\mathcal{P}^{**} \subseteq \mathcal{P}$ and Assumptions 1 and 2 hold. Then Axioms 1–7 hold if and only if $(\mathcal{P}, \prec)$ has the RDU representation.*

4 A Weak Multisymmetric Structure

This section shows that the set of all simple probability measures with a support $\pi \in \Pi^*$ has a weak multisymmetric structure developed in Nakamura (1992), whose numerical structure leads to a rank dependent utility representation when X is finite. Throughout the section we assume that Assumption 1 and Axioms 1–5 hold.

The first two lemmas are concerned with monotonicity properties with respect to probability numbers.

Lemma 1 *If $\pi = \langle x, y \rangle$ and $x \prec y$, then $\alpha \leq \beta$ iff $p_\pi(\beta) \preceq p_\pi(\alpha)$.*

Proof. Suppose that the hypotheses of the lemma hold. If $\alpha \leq \beta$, then the desired result easily follows from Axiom 3. Assume next that $p_\pi(\beta) \preceq p_\pi(\alpha)$. Suppose on the contrary that $\beta < \alpha$. Then we derive a contradiction. Since $x \prec y$, Axiom 4 implies that

$\lambda x + (1-\lambda)q \prec \lambda y + (1-\lambda)q$ for all $q \in \mathcal{P}^*$ and all $0 < \lambda < 1$. Thus we take $\lambda = \alpha - \beta$ and $q = \gamma x + (1-\gamma)y$, where $\gamma = \beta/(1-\alpha+\beta)$. Then

$$\begin{aligned} \lambda x + (1-\lambda)q &= \alpha x + (1-\alpha)y = p_\pi(\alpha), \\ \lambda y + (1-\lambda)q &= \beta x + (1-\beta)y = p_\pi(\beta), \end{aligned}$$

so that $p_\pi(\alpha) \prec p_\pi(\beta)$, a contradiction. Hence, we must have $\alpha \leq \beta$. □

Lemma 2 *Suppose that* $\pi = \langle x, y, z \rangle$. *Then we have*
(1) if $x \prec y$, $\alpha \leq \gamma$, *and* $\beta \leq \gamma$, *then* $\alpha \leq \beta$ *iff* $p_\pi(\beta\gamma) \preceq p_\pi(\alpha\gamma)$,
(2) if $y \prec z$, $\gamma \leq \alpha$, *and* $\gamma \leq \beta$, *then* $\alpha \leq \beta$ *iff* $p_\pi(\gamma\beta) \preceq p_\pi(\gamma\alpha)$.

Proof of Lemma 2. We show (1). The proof of (2) is similar. Suppose that the hypotheses of the lemma hold. First we assume that $\alpha \leq \beta$. If $\alpha = \beta$, then $p_\pi(\beta\gamma) = p_\pi(\alpha\gamma)$, so $p_\pi(\beta\gamma) \preceq p_\pi(\alpha\gamma)$. Thus we assume $\alpha < \beta$. Then

$$\begin{aligned} p_\pi(\beta\gamma) &= (\beta-\alpha)x + (1-\beta+\alpha)p_\pi(\alpha'\beta'), \\ p_\pi(\alpha\gamma) &= (\beta-\alpha)y + (1-\beta+\alpha)p_\pi(\alpha'\beta'), \end{aligned}$$

where $\alpha' = \alpha/(1-\beta+\alpha)$ and $\beta' = (\alpha-\beta+\gamma)/(1-\beta+\alpha)$. Since $x \prec y$, it follows from Axiom 4 that $p_\pi(\beta\gamma) \prec p_\pi(\alpha\gamma)$.

Assume next that $p_\pi(\beta\gamma) \preceq p_\pi(\alpha\gamma)$. If $\beta < \alpha$, then it follows from a similar argument of the preceding paragraph that $p_\pi(\alpha\gamma) \prec p_\pi(\beta\gamma)$. This is a contradiction. Hence, we must have $\alpha \leq \beta$. □

If $\pi = \langle x_1, \ldots, x_{n+1} \rangle$ and $x_1 \prec x_{n+1}$, then by Axiom 3, $x_1 \preceq p_\pi(\alpha_1 \ldots \alpha_n) \preceq x_{n+1}$. Then Axiom 2 implies that $p_\pi(\alpha_1 \ldots \alpha_n) \sim \lambda x_1 + (1-\lambda)x_{n+1}$ for some $0 \leq \lambda \leq 1$. It follows from Lemma 1 that λ must be unique. In other words, since $\lambda x_1 + (1-\lambda)x_{n+1} = p_\pi(\lambda \ldots \lambda)$, there exist a unique probability equivalent $0 \leq \lambda \leq 1$ such that $p_\pi(\alpha_1 \ldots \alpha_n) \sim p_\pi(\lambda \ldots \lambda)$. We shall denote such a λ by $\sigma_\pi(\alpha_1 \ldots \alpha_n)$.

Lemma 3 *Suppose that* $\pi_1 = \langle x_1, \ldots, x_{n+1} \rangle$ *and* $\pi_2 = \langle y_1, \ldots, y_{m+1} \rangle$. *If* $x_1 \sim y_1$, $x_{n+1} \sim y_{m+1}$, *and* $x_1 \prec x_{n+1}$, *then*

$$p_{\pi_1}(\alpha_1 \ldots \alpha_n) \preceq p_{\pi_2}(\beta_1 \ldots \beta_m) \text{ iff } \sigma_{\pi_2}(\beta_1 \ldots \beta_m) \leq \sigma_{\pi_1}(\alpha_1 \ldots \alpha_n).$$

Proof. This follows from Axioms 1–4, Lemma 1, and the definitions of p_π and σ_π. □

In what follows, we denote the closed unit interval $[0,1]$ by I. Let $I^n = I \times \ldots \times I$ (n times) and $I^n_\uparrow = \{(\alpha_1, \ldots, \alpha_n) \in I^n : \alpha_1 \leq \ldots \leq \alpha_n\}$. When $\pi \in \Pi^*$ and $n = |\pi| - 1$, σ_π can be regarded as an n-ary operation that maps $I^n_\uparrow$ into I. Given $\pi \in \Pi^*$ with $|\pi| = n+1$, we shall denote $p^k_\pi(\alpha\beta) = p_\pi(\alpha_1 \ldots \alpha_n)$ and $\sigma^k_\pi(\alpha\beta) = \sigma_\pi(\alpha_1 \ldots \alpha_n)$ for $1 \leq k \leq n$ when $\alpha_i = \alpha$ for $i = 1, \ldots, k$, and $\alpha_i = \beta$ for $i = k+1, \ldots, n$. Note that $\sigma^n_\pi(\alpha\beta) = \sigma_\pi(\alpha \ldots \alpha)$. We say that k is *left-inessential* if for all $\alpha, \beta, \gamma \in I$, $\sigma^k_\pi(\alpha\gamma) = \sigma^k_\pi(\beta\gamma)$ whenever $\alpha \leq \beta \leq \gamma$, and *right-inessential* if for all $\alpha, \beta, \gamma \in I$, $\sigma^k_\pi(\alpha\beta) = \sigma^k_\pi(\alpha\gamma)$ whenever $\alpha \leq \beta \leq \gamma$. Note that n is not left-inessential but right-inessential. When k is not left (right)-inessential, we say that k is *left (right)-essential.* When k is left and right-essential, we say that k is *essential.* When there is an essential k, we say that σ_π is *essential.* Left and right essentialities are characterized by the following lemma.

Lemma 4 *Suppose that* $\pi = \langle x_1, \ldots, x_{n+1} \rangle \in \Pi^*$. *Then for* $1 \leq k < n$,

(1) k *is left-essential if and only if* $x_1 \prec x_{k+1}$.

(2) k *is right-essential if and only if* $x_{k+1} \prec x_{n+1}$.

Proof. We show (1). The proof of (2) is similar. Given k, let $\pi_1 = \langle x_1, x_{k+1}, x_{n+1} \rangle$, $\pi_2 = \langle x_1, x_{n+1} \rangle$, and $\pi_3 = \langle x_{k+1}, x_{n+1} \rangle$. We note that $p_{\pi_1}(\alpha\alpha) = p_{\pi_2}(\alpha)$, $p_{\pi_1}(0\alpha) = p_{\pi_3}(\alpha)$, and $p^k_\pi(\alpha\beta) = p_{\pi_1}(\alpha\beta)$ for all $\alpha, \beta \in I$.

Suppose first that k is left-essential. By Axiom 3, $p_{\pi_1}(\gamma\gamma) \preceq p_{\pi_1}(\alpha\gamma) \preceq p_{\pi_1}(0\gamma)$, so $p_{\pi_2}(\gamma) \preceq p_{\pi_1}(\alpha\gamma) \preceq p_{\pi_3}(\gamma)$. If $x_1 \sim x_{k+1}$, then by Axiom 3, $p_{\pi_2}(\gamma) \sim p_{\pi_3}(\gamma)$. Thus by Axiom 1, $p_{\pi_1}(\alpha\gamma) \sim p_{\pi_1}(\beta\gamma)$ for all $\alpha, \beta, \gamma \in I$, so $p^k_\pi(\alpha\gamma) \sim p^k_\pi(\beta\gamma)$. Therefore, Lemma 3 implies that $\sigma^k_\pi(\alpha\gamma) = \sigma^k_\pi(\beta\gamma)$, so k is left-inessential. This is a contradiction. Since $x_1 \preceq x_{k+1}$, we must have $x_1 \prec x_{k+1}$.

Suppose next that $x_1 \prec x_{k+1}$. If $\alpha < \beta \leq \gamma$, then by Lemma 2, $p_{\pi_1}(\beta\gamma) \prec p_{\pi_1}(\alpha\gamma)$. Thus by Lemma 3, $\sigma_\pi^k(\alpha\gamma) < \sigma_\pi^k(\beta\gamma)$, so that k is left-essential. □

When I_1 and I_2 are subsets of I, $I_1 \leq I_2$ means that $\alpha \leq \beta$ for all $\alpha \in I_1$ and all $\beta \in I_2$. Let K be any set of consecutive integers. Given $\pi \in \Pi^*$ and an essential k, we define a *standard sequence* as a set $\{\alpha_i : \alpha_i \in I, i \in K\}$ for which there exist $\alpha, \beta \in I$ such that $\alpha \neq \beta$, either $\{\alpha, \beta\} \leq \{\alpha_i\}$ and $\sigma_\pi^k(\alpha\alpha_i) = \sigma_\pi^k(\beta\alpha_{i+1})$ for all $i, i+1 \in K$, or $\{\alpha_i\} \leq \{\alpha, \beta\}$ and $\sigma_\pi^k(\alpha_i\alpha) = \sigma_\pi^k(\alpha_{i+1}\beta)$ for all $i, i+1 \in K$. We say that a standard sequence $\{\alpha_i : i \in K\}$ is *strictly bounded* when $\alpha < \alpha_i < \beta$ for all $i \in K$ and some real numbers α and β.

For $\pi \in \Pi^*$ with $|\pi| = n+1$, the triple $(\leq, \sigma_\pi, I_\uparrow^n)$ that satisfies B1–B6 in the following proposition is said to be a *weak multisymmetric structure.* The proof of the proposition appears at the end of the section.

Proposition 1 *Suppose that $\pi \in \Pi^*$ and $|\pi| = n+1$. Then the following six axioms hold: for all $\alpha, \beta, \gamma, \delta, \alpha_i, \beta_i \in I$, $i = 1, \ldots, n$, and $k = 1, \ldots, n$,*

B1. *$\leq$ on I is a weak order,*

B2. *if $\{\alpha, \beta\} \leq \gamma$ and $\sigma_\pi^k(\alpha\gamma) \leq \delta \leq \sigma_\pi^k(\beta\gamma)$, then $\delta = \sigma_\pi^k(\lambda\gamma)$ for some $\lambda \in I$; if $\gamma \leq \{\alpha, \beta\}$ and $\sigma_\pi^k(\gamma\alpha) \leq \delta \leq \sigma_\pi^k(\gamma\beta)$, then $\delta = \sigma_\pi^k(\gamma\lambda)$ for some $\lambda \in I$,*

B3. *if k is left-essential and $\{\alpha, \beta\} \leq \gamma$, then $\alpha \leq \beta$ iff $\sigma_\pi^k(\alpha\gamma) \leq \sigma_\pi^k(\beta\gamma)$; if k is right-essential and $\gamma \leq \{\alpha, \beta\}$, then $\alpha \leq \beta$ iff $\sigma_\pi^k(\gamma\alpha) \leq \sigma_\pi^k(\gamma\beta)$,*

B4. *Every strictly bounded standard sequence is finite,*

B5. *if $\alpha_1 \leq \ldots \leq \alpha_n$, $\beta_1 \leq \ldots \leq \beta_n$, and $\alpha_i \leq \beta_i$ for $i = 1, \ldots, n$, then*

$$\sigma_\pi(\alpha_1 \ldots \alpha_n) \leq \sigma_\pi(\beta_1 \ldots \beta_n),$$

B6. *if $\alpha_1 \leq \ldots \leq \alpha_n$, $\beta_1 \leq \ldots \leq \beta_n$, and $\alpha_i \leq \beta_i$ for $i = 1, \ldots, n$, then*

$$\sigma_\pi^k(\sigma_\pi(\alpha_1 \ldots \alpha_n)\sigma_\pi(\beta_1 \ldots \beta_n)) = \sigma_\pi(\sigma_\pi^k(\alpha_1\beta_1) \ldots \sigma_\pi^k(\alpha_n\beta_n)).$$

The numerical representation of the weak multisymmetric structure, $(\leq, \sigma_\pi, I_\uparrow^n)$, is given as follows. The proof is deferred to the end of the section.

Proposition 2 *For all $\pi \in \Pi^*$ with $|\pi| = n+1$, there exist real numbers $\lambda_i(\pi) \in I$ for $i = 1, \ldots, n$ with $\sum_{i=1}^n \lambda_i(\pi) = 1$, and a continuous real valued function Ψ on I such that $\lambda_j(\pi) > 0$ and $\lambda_k(\pi) > 0$ for some distinct j, k, and for all $\alpha, \beta, \alpha_1, \ldots, \alpha_n \in I$,*

(1) $\alpha \leq \beta$ iff $\Psi(\alpha) \leq \Psi(\beta)$,

(2) if $\alpha_1 \leq \ldots \leq \alpha_n$, then $\Psi(\sigma_\pi(\alpha_1 \ldots \alpha_n)) = \sum_{i=1}^n \lambda_i(\pi)\Psi(\alpha_i)$,

(3) $\lambda_i(\pi)$ for $i = 1, \ldots, n$ are unique and Ψ is unique up to a positive linear transformation.

To see that Proposition 2 gives the RDU representation for a finite X, we suppose that $X = \{x_1, \ldots, x_{n+1}\}$ and $\pi = \langle x_1, \ldots, x_{n+1}\rangle \in \Pi^*$. Then $\mathcal{P} = \{p_\pi(\alpha_1 \ldots \alpha_n) : \alpha_1 \leq \ldots \leq \alpha_n \text{ and } \alpha_i \in I \text{ for } i = 1, \ldots, n\}$. Assign any numbers $u(x_1)$ and $u(x_{n+1})$ with $u(x_1) < u(x_{n+1})$ to x_1 and x_{n+1}. Given $\lambda_i(\pi) \in I$ for $i = 1, \ldots, n$ obtained in Proposition 2, for $k = 1, \ldots, n-1$, we define

$$u(x_{k+1}) = u(x_1) - (u(x_1) - u(x_{n+1})) \sum_{i=1}^{k} \lambda_i(\pi).$$

Thus $\lambda_k(\pi) = (u(x_k) - u(x_{k+1}))/(u(x_1) - u(x_{n+1}))$ for $k = 1, \ldots, n$. Since $\lambda_k(\pi)$ for $k = 1, \ldots, n$ are unique, u on X is unique up to a positive linear transformation. The RDU representation easily follows from Lemma 3 and Proposition 2. Hence we obtain Theorem 1 for a finite X.

Proof of Proposition 1. Suppose that $\pi = \langle x_1, \ldots, x_{n+1}\rangle \in \Pi^*$. For $k = 1, \ldots, n$, we shall write $\sigma_k(\alpha\beta) = \sigma_\pi(\alpha_1 \ldots \alpha_n)$ and $p_k(\alpha\beta) = p_\pi(\alpha_1 \ldots \alpha_n)$ when $\alpha_i = \alpha$ for $i = 1, \ldots, k$ and $\alpha_i = \beta$ for $i = k+1, \ldots, n$. B1 follows, since $\leq$ on I is a linear order.

We show the first part of B2. The second part of B2 similarly follows. Suppose that $\{\alpha, \beta\} \leq \gamma$, and $\sigma_k(\alpha\gamma) \leq \delta \leq \sigma_k(\beta\gamma)$. We are to prove that $\delta = \sigma_k(\lambda\gamma)$ for some $\lambda \in I$. By Lemma 3 and the definition of σ_π, $p_k(\beta\gamma) \preceq p_k(\delta\delta) \preceq p_k(\alpha\gamma)$. It follows from Axiom 2 that $p_k(\delta\delta) \sim \theta p_k(\beta\gamma) + (1-\theta)p_k(\alpha\gamma)$ for some $\theta \in I$. Let $\lambda = \theta\beta + (1-\theta)\alpha$, so $\lambda \leq \gamma$. Since $p_k(\lambda\gamma) = \theta p_k(\beta\gamma) + (1-\theta)p_k(\alpha\gamma)$, $p_k(\delta\delta) \sim p_k(\lambda\gamma)$. Thus by definition, $\delta = \sigma_k(\lambda\gamma)$.

To show B3, suppose that k is left-essential. If $k = n$, B3 easily follows. Thus assume $1 \leq k < n$. Let $\pi_1 = \langle x_1, x_{k+1}, x_{n+1}\rangle$. By Lemma 4(1), $x_1 \prec x_{k+1} \preceq x_{n+1}$. If

$\{\alpha, \beta\} \leq \gamma$, then by Lemma 2, $\alpha \leq \beta$ iff $p_{\pi_1}(\beta\gamma) \preceq p_{\pi_1}(\alpha\gamma)$. Hence by Lemma 3, $\alpha \leq \beta$ iff $\sigma_k(\alpha\gamma) \leq \sigma_k(\beta\gamma)$, so the first part of B3 follows. A similar proof applies to get the second part of B3.

To show B4, let $\{\alpha_i : i \in K\}$ be a standard sequence. Since $0 \leq \alpha_i \leq 1$ for all $i \in K$, every standard sequence is strictly bounded. Suppose that for $\alpha, \beta \in I$, $\beta < \alpha$, $\{\alpha, \beta\} \leq \alpha_i$, and $\sigma_k(\alpha\alpha_i) = \sigma_k(\beta\alpha_{i+1})$ for all $i, i+1 \in K$ and an essential k. The proofs for the other cases are similar. Then B3 implies that $\alpha_i < \alpha_{i+1}$ for all $i \in K$, so that $\{\alpha_i\}$ is a bounded and strictly increasing sequence in I. Assume that $\{\alpha_i\}$ is infinite. Let α' be the least upper bound of $\{\alpha_i\}$. Noting that $\beta < \alpha$, $\alpha_i < \alpha'$, and $\sigma_k(\alpha\alpha_i) = \sigma_k(\beta\alpha_{i+1})$ for all $i \in K$, it follows from B3 that $\sigma_k(\alpha\alpha_i) < \sigma_k(\beta\alpha') < \sigma_k(\alpha\alpha')$ for all $i \in K$. Thus by B2 and B3, $\sigma_k(\beta\alpha') = \sigma_k(\alpha\beta')$ for some $\beta' \in I$ with $\alpha_i < \beta' < \alpha'$. On the other hand, since α' is the least upper bound of $\{\alpha_i\}$, there exists an $\alpha'' \in \{\alpha_i\}$ such that $\beta' < \alpha'' < \alpha'$. Therefore, by B3, $\sigma_k(\alpha\beta') < \sigma_k(\alpha\alpha'')$. Since $\alpha'' \in \{\alpha_i\}$, we obtain $\sigma_k(\alpha\alpha'') < \sigma_k(\beta\alpha')$. Thus $\sigma_k(\alpha\beta') < \sigma_k(\beta\alpha')$. This contradicts $\sigma_k(\beta\alpha') = \sigma_k(\alpha\beta')$. Hence $\{\alpha_i\}$ must be finite.

B5 follows from Axiom 3 and Lemma 3. If $k = n$, B6 follows from the definition of σ_π. Therefore, to show B6, we assume $1 \leq k < n$. Let $\pi_k = \langle x_1, x_{k+1}, x_{n+1}\rangle$. Suppose that $\alpha_1 \leq \ldots \leq \alpha_n$, $\beta_1 \leq \ldots \leq \beta_n$, and $\alpha_i \leq \beta_i$ for $i = 1, \ldots, n$. Let $\alpha = \sigma_\pi(\alpha_1 \ldots \alpha_n)$, $\beta = \sigma_\pi(\beta_1 \ldots \beta_n)$, $\gamma_i = \sigma_k(\alpha_i\beta_i)$ for $i = 1, \ldots, n$, and $\gamma = \sigma_k(\alpha\beta)$. Then by definition,

$$
\begin{aligned}
p_\pi(\alpha_1 \ldots \alpha_n) &\sim p_\pi(\alpha \ldots \alpha), \\
p_\pi(\beta_1 \ldots \beta_n) &\sim p_\pi(\beta \ldots \beta), \\
p_{\pi_k}(\alpha_i\beta_i) &\sim p_{\pi_k}(\gamma_i\gamma_i), \\
p_{\pi_k}(\alpha\beta) &\sim p_{\pi_k}(\gamma\gamma).
\end{aligned}
$$

Noting that $\gamma_1 \leq \ldots \leq \gamma_n$, Axiom 5 implies that $p_\pi(\gamma_1 \ldots \gamma_n) \sim p_\pi(\gamma \ldots \gamma)$, so $\gamma = \sigma_\pi(\gamma_1 \ldots \gamma_n)$. Hence, $\sigma_k(\alpha\beta) = \sigma_\pi(\gamma_1 \ldots \gamma_n)$, so that B6 immediately follows. □

Proof of Proposition 2. Suppose that $\pi = \langle x_1, \ldots, x_{n+1}\rangle \in \Pi^*$. By the definition of Π^*, $x_1 \prec x_{j+1} \prec x_{n+1}$ for some $1 \leq j < n$. Thus Lemma 4 implies that j is essential, so σ_π is essential. We note by definition that σ_π is idempotent, i.e., $\sigma_\pi(\alpha\dots\alpha) = \alpha$ for all $\alpha \in I$. Since by Proposition 1, $(\leq, \sigma_\pi, I_\uparrow^n)$ is a weak multisymmetric structure with σ_π essential and idempotent, it follows from Theorem 1 in Nakamura (1992) that there are real numbers $\lambda_i(\pi) \in I$ for $i = 1, \dots, n$ with $\sum_{i=1}^n \lambda_i(\pi) = 1$, and a real valued function Ψ_π on I such that $\lambda_j(\pi) > 0$ and $\lambda_k(\pi) > 0$ for some distinct j, k, and for all $\alpha, \beta, \alpha_1, \dots, \alpha_n \in I$,

$$\alpha \leq \beta \text{ iff } \Psi_\pi(\alpha) \leq \Psi_\pi(\beta),$$
$$\Psi_\pi(\sigma_\pi(\alpha_1 \dots \alpha_n)) = \sum_{i=1}^n \lambda_i(\pi)\Psi_\pi(\alpha_i).$$

Moreover, $\lambda_i(\pi)$ for $i = 1, \dots, n$ are unique and Ψ_π is unique up to a positive linear transformation.

For $\pi_1, \pi_2 \in \Pi^*$ with $|\pi_1| = n+1$ and $|\pi_2| = m+1$, it follows from a similar proof of B6 in Proposition 1 that for essential $1 \leq k \leq n$ and essential $1 \leq j \leq m$,

$$\sigma_{\pi_1}^k(\sigma_{\pi_2}^j(\alpha\beta)\sigma_{\pi_2}^j(\gamma\delta)) = \sigma_{\pi_2}^j(\sigma_{\pi_1}^k(\alpha\gamma)\sigma_{\pi_1}^k(\beta\delta)),$$

which is a weak isometry condition defined in Nakamura (1992). Thus Proposition 1 in Nakamura (1992) implies that $\Psi_{\pi_1} = \Psi_{\pi_2}$. Let $\Psi = \Psi_\pi$ for all $\pi \in \Pi^*$, so (1), (2) and (3) follow.

It remains to show that Ψ is continuous. Suppose that Ψ is not left continuous. Let $\{\alpha_i\}$ be any strictly increasing sequence in I such that $\lim_{i\to\infty} \alpha_i = \alpha^*$. Then $\lim_{i\to\infty} \Psi(\alpha_i) < \Psi(\alpha^*)$, since Ψ is strictly increasing. Let $\pi = \langle x, y, z\rangle \in \Pi^*$. Since $\alpha_i < \alpha_{i+1} < \alpha^*$, it follows from the definition of σ_π and B3 in Proposition 1 that $\sigma_\pi(\alpha_i\alpha^*) < \sigma_\pi(\alpha_{i+1}\alpha^*)$ and $\alpha_i < \sigma_\pi(\alpha_i\alpha^*) < \alpha^*$. Thus $\lim_{i\to\infty} \sigma_\pi(\alpha_i\alpha^*) = \alpha^*$, so $\lim_{i\to\infty} \Psi(\sigma_\pi(\alpha_i\alpha^*)) = \lim_{i\to\infty} \Psi(\alpha_i)$. By (2),

$$\Psi(\sigma_\pi(\alpha_i\alpha^*)) = \lambda_1(\pi)\Psi(\alpha_i) + (1 - \lambda_1(\pi))\Psi(\alpha^*).$$

Taking the limit, we obtain

$$\lim_{i\to\infty} \Psi(\alpha_i) = \lambda_1(\pi) \lim_{i\to\infty} \Psi(\alpha_i) + (1 - \lambda_1(\pi))\Psi(\alpha^*),$$

so that $\lambda_1(\pi) = 1$. Since $\pi \in \Pi^*$, $0 < \lambda_1(\pi) < 1$. This is a contradiction. Therefore, Ψ is left continuous. Right continuity of Ψ similarly follows. Hence Ψ is continuous. □

5 Proofs of Theorems 1 and 2

Throughout the section we assume that Assumption 1 holds. The necessity of Axioms 1–4 easily follows. Fisrt we show the necessity of Axiom 5 for the RDU representation of Theorem 1. Suppose that the RDU representation of Theorem 1 holds. Also, we suppose that the hypotheses of Axiom 5 hold. Then we let $\pi = \langle x, y, z\rangle$. With no loss of generality, we assume that $p_i, q_i, r_i \in \mathcal{P}^*$ for $i = 1, 2$ have a common support $\pi^* = \langle x_1, \ldots, x_{n+1}\rangle$. Thus for $i = 1, 2$, let

$$\begin{aligned} p_i &= p_{\pi^*}(\alpha_{i1} \ldots \alpha_{in}), \\ q_i &= p_{\pi^*}(\beta_{i1} \ldots \beta_{in}), \\ r_i &= p_{\pi^*}(\gamma_{i1} \ldots \gamma_{in}). \end{aligned}$$

Since $p_1 \sim p_2$ and $q_1 \sim q_2$, the RDU representation gives

$$\begin{aligned} \sum_{j=1}^{n} \Psi(\alpha_{1j})(u(x_{j+1}) - u(x_j)) &= \sum_{j=1}^{n} \Psi(\alpha_{2j})(u(x_{j+1}) - u(x_j)), \\ \sum_{j=1}^{n} \Psi(\beta_{1j})(u(x_{j+1}) - u(x_j)) &= \sum_{j=1}^{n} \Psi(\beta_{2j})(u(x_{j+1}) - u(x_j)), \end{aligned}$$

which merge into the following:

$$\begin{aligned} &\sum_{j=1}^{n}(\Psi(\alpha_{1j})u_{xy} + \Psi(\beta_{1j})u_{yz})(u(x_{j+1}) - u(x_j)) \\ &= \sum_{j=1}^{n}(\Psi(\alpha_{2j})u_{xy} + \Psi(\beta_{2j})u_{yz})(u(x_{j+1}) - u(x_j)), \end{aligned}$$

where $u_{xy} = u(x) - u(y)$ and $u_{yz} = u(y) - u(z)$.

Since $p_\pi(\alpha_{ij}\beta_{ij}) \sim p_\pi(\gamma_{ij}\gamma_{ij})$ for $i = 1, 2$ and $j = 1, \ldots, n$, the RDU representation gives

$$\Psi(\alpha_{ij})u_{xy} + \Psi(\beta_{ij})u_{yz} = \Psi(\gamma_{ij})u_{xz},$$

where $u_{xz} = u(x) - u(z)$. Therefore, the last equation in the preceding paragraph is rearranged to give

$$\sum_{j=1}^{n} \Psi(\gamma_{1j})(u(x_{j+1}) - u(x_j)) = \sum_{j=1}^{n} \Psi(\gamma_{2j})(u(x_{j+1}) - u(x_j)).$$

This implies that $r_1 \sim r_2$, so Axiom 5 holds.

Next we show the sufficiency of Axioms 1–5 for the RDU representation of Theorem 1. Suppose that Axioms 1–5 hold. For $a, b \in X$ with $a \prec b$, define

$$X_{ab} = \{x \in X : a \preceq x \preceq b\}.$$

For all $\pi \in \Pi^*$, let Ψ and $\lambda_i(\pi)$ for $i = 1, \ldots, |\pi| - 1$ be obtained in Proposition 2. Define a real valued function λ_{ab} on X_{ab} as follows: for $\pi = \langle a, x, b\rangle$,

$$\lambda_{ab}(x) = \begin{cases} \lambda_1(\pi) & \text{if } a \prec x \prec b, \\ 0 & \text{if } x \sim a, \\ 1 & \text{if } x \sim b. \end{cases}$$

Note by Proposition 2 that if $\pi \in \Pi^*$, then $0 < \lambda_1(\pi) < 1$.

Before providing the sufficiency proof of the theorem, we need the following properties of λ_{ab}.

Lemma 5 *(1) For all $x, y \in X_{ab}$, $x \preceq y$ iff $\lambda_{ab}(x) \leq \lambda_{ab}(y)$.*

(2) For all $\pi = \langle x_1, \ldots, x_{n+1}\rangle \in \Pi^*$, if $x_1 \sim a$ and $x_{n+1} \sim b$, then for all $\alpha_1, \ldots, \alpha_n \in I$ with $\alpha_1 \leq \ldots \leq \alpha_n$,

$$\Psi(\sigma_\pi(\alpha_1 \ldots \alpha_n)) = \sum_{i=1}^{n} (\lambda_{ab}(x_{i+1}) - \lambda_{ab}(x_i))\Psi(\alpha_i).$$

(3) For all $x, y, z \in X_{ab}$, if $y \in X_{xz}$, then

$$\lambda_{xz}(y) = \frac{\lambda_{ab}(y) - \lambda_{ab}(x)}{\lambda_{ab}(z) - \lambda_{ab}(x)}.$$

Proof. (1) Suppose that $x, y \in X_{ab}$ and $a \prec \{x, y\} \prec b$. Let $\pi_x = \langle a, x, b\rangle$ and $\pi_y = \langle a, y, b\rangle$. Fix $\alpha, \beta \in I$ with $\alpha < \beta$. If $x \sim y$, then by Axiom 3, $p_{\pi_x}(\alpha\beta) \sim p_{\pi_y}(\alpha\beta)$. If $x \prec y$, then by Axiom 4, for all $\lambda, \gamma \in I$,

$$\lambda x + (1 - \lambda)p_{\pi_x}(\gamma\gamma) \prec \lambda y + (1 - \lambda)p_{\pi_x}(\gamma\gamma).$$

Let $\lambda = \beta - \alpha$ and $\gamma = \alpha/(1-\beta+\alpha)$. Then we have

$$\begin{aligned} p_{\pi_x}(\alpha\beta) &= \lambda x + (1-\lambda)p_{\pi_x}(\gamma\gamma), \\ p_{\pi_y}(\alpha\beta) &= \lambda y + (1-\lambda)p_{\pi_x}(\gamma\gamma), \end{aligned}$$

so that $p_{\pi_x}(\alpha\beta) \prec p_{\pi_y}(\alpha\beta)$. Hence $x \preceq y$ implies that $p_{\pi_x}(\alpha\beta) \preceq p_{\pi_y}(\alpha\beta)$. The converse claim easily follows, so that $x \preceq y$ if and only if $p_{\pi_x}(\alpha\beta) \preceq p_{\pi_y}(\alpha\beta)$.

Note that if $\alpha < \beta$, then $\Psi(\alpha) < \Psi(\beta)$. Then it follows from Lemma 3 and Proposition 2 that for $\alpha, \beta \in I$ with $\alpha < \beta$,

$$\begin{aligned} x \preceq y \quad &\text{iff} \quad p_{\pi_x}(\alpha\beta) \preceq p_{\pi_y}(\alpha\beta) \\ &\text{iff} \quad \sigma_{\pi_y}(\alpha\beta) \le \sigma_{\pi_x}(\alpha\beta) \\ &\text{iff} \quad \lambda_{ab}(y)\Psi(\alpha) + (1-\lambda_{ab}(y))\Psi(\beta) \le \lambda_{ab}(x)\Psi(\alpha) + (1-\lambda_{ab}(x))\Psi(\beta) \\ &\text{iff} \quad \lambda_{ab}(x) \le \lambda_{ab}(y). \end{aligned}$$

Since $0 < \lambda_{ab}(x) < 1$ when $a \prec x \prec b$, the definition of λ_{ab} gives that for all $x, y \in X_{ab}$, $x \preceq y$ iff $\lambda_{ab}(x) \le \lambda_{ab}(y)$.

(2) Suppose that $a \sim x_1$, $b \sim x_{n+1}$, and $\pi = \langle x_1, \ldots, x_{n+1}\rangle \in \Pi^*$. It follows from Proposition 2 that

$$\Psi(\sigma_\pi(\alpha_1 \ldots \alpha_n)) = \sum_{i=1}^{n} \lambda_i(\pi)\Psi(\alpha_i).$$

We are to show that $\lambda_i(\pi) = \lambda_{ab}(x_{i+1}) - \lambda_{ab}(x_i)$ for $i = 1, \ldots, n$. For $1 \le k \le n$, let $\sigma_k(\alpha\beta) = \sigma_\pi(\alpha_1 \ldots \alpha_n)$ and $p_k(\alpha\beta) = p_\pi(\alpha_1 \ldots \alpha_n)$ when $\alpha_i = \alpha$ for $i = 1, \ldots, k$ and $\alpha_i = \beta$ for $i = k+1, \ldots, n$. Then

$$\Psi(\sigma_k(\alpha\beta)) = \left(\sum_{i=1}^{k} \lambda_i(\pi)\right)\Psi(\alpha) + \left(\sum_{i=k+1}^{n} \lambda_i(\pi)\right)\Psi(\beta).$$

For $1 \le k < n$, let $\pi_k = \langle a, x_{k+1}, b\rangle$ and $\pi_k^* = \langle x_1, x_{k+1}, x_{n+1}\rangle$. Then by Axiom 3 and Lemma 3, $\sigma_{\pi_k}(\alpha\beta) = \sigma_{\pi_k^*}(\alpha\beta)$. Since $p_k(\alpha\beta) = p_{\pi_k^*}(\alpha\beta)$, $\sigma_k(\alpha\beta) = \sigma_{\pi_k^*}(\alpha\beta)$. Thus $\sigma_k(\alpha\beta) = \sigma_{\pi_k}(\alpha\beta)$ for $1 \le k < n$. It follows from Proposition 2 that if $a \prec x_{k+1} \prec b$ for

$1 \leq k < n$, then

$$\begin{aligned}\Psi(\sigma_k(\alpha\beta)) &= \Psi(\sigma_{\pi_k}(\alpha\beta)) \\ &= \lambda_{ab}(x_{k+1})\Psi(\alpha) + (1 - \lambda_{ab}(x_{k+1}))\Psi(\beta).\end{aligned}$$

Note by Axiom 3 and Lemma 3 that $\sigma_{\pi_k}(\alpha\beta) = \beta$ if $a \sim x_{k+1}$, and $\sigma_{\pi_k}(\alpha\beta) = \alpha$ if $x_{k+1} \sim b$. Thus by the definition of λ_{ab}, the above equation holds for all $1 \leq k < n$. Hence, $\lambda_{ab}(x_{k+1}) = \sum_{i=1}^{k} \lambda_i(\pi)$ for $1 \leq k < n$, which are solved with respect to $\lambda_i(\pi)$ for $i = 1, \ldots, n-1$ to give $\lambda_i(\pi) = \lambda_{ab}(x_{i+1}) - \lambda_{ab}(x_i)$ for $i = 1, \ldots, n-1$. Since $\sum_{i=1}^{n} \lambda_i(\pi) = 1$, $\lambda_n(\pi) = 1 - \lambda_{ab}(x_n)$.

(3) Suppose that $x, y, z \in X_{ab}$ and $y \in X_{xz}$. If either $y \sim x$ or $y \sim z$, then the desired result follows from (1) and the definition of $\lambda_{xz}(y)$. Thus assume that $x \prec y \prec z$. Suppose that $a \neq x$ and $b \neq z$. When $a = x$ or $b = z$, the proof is similar. Let $\pi_1 = \langle a, x, y, z, b\rangle$ and $\pi_2 = \langle x, y, z\rangle$. Then $\pi_1, \pi_2 \in \Pi^*$. It follows from Proposition 2 that for all $\alpha, \beta, \gamma, \delta \in I$,

$$\begin{aligned}&\sigma_{\pi_2}(\alpha\beta) \leq \sigma_{\pi_2}(\gamma\delta) \\ &\quad \text{iff} \quad \lambda_{xz}(y)\Psi(\alpha) + (1 - \lambda_{xz}(y))\Psi(\beta) \leq \lambda_{xz}(y)\Psi(\gamma) + (1 - \lambda_{xz}(y))\Psi(\delta).\end{aligned}$$

Since $p_{\pi_2}(\alpha\beta) = p_{\pi_1}(0\alpha\beta 1)$, (2) and Lemma 3 imply

$$\begin{aligned}\sigma_{\pi_2}(\alpha\beta) \leq \sigma_{\pi_2}(\gamma\delta) \quad &\text{iff} \quad p_{\pi_2}(\gamma\delta) \preceq p_{\pi_2}(\alpha\beta) \\ &\text{iff} \quad p_{\pi_1}(0\gamma\delta 1) \preceq p_{\pi_1}(0\alpha\beta 1) \\ &\text{iff} \quad \sigma_{\pi_1}(0\alpha\beta 1) \leq \sigma_{\pi_1}(0\gamma\delta 1) \\ &\text{iff} \quad \frac{\lambda_{ab}(y) - \lambda_{ab}(x)}{\lambda_{ab}(z) - \lambda_{ab}(x)}\Psi(\alpha) + \left(1 - \frac{\lambda_{ab}(y) - \lambda_{ab}(x)}{\lambda_{ab}(z) - \lambda_{ab}(x)}\right)\Psi(\beta) \\ &\qquad \leq \frac{\lambda_{ab}(y) - \lambda_{ab}(x)}{\lambda_{ab}(z) - \lambda_{ab}(x)}\Psi(\gamma) + \left(1 - \frac{\lambda_{ab}(y) - \lambda_{ab}(x)}{\lambda_{ab}(z) - \lambda_{ab}(x)}\right)\Psi(\delta).\end{aligned}$$

Hence by Proposition 2(3), we obtain the desired result. □

Sufficiency Proof of Theorem 1. Given $a, b \in X$ with $a \prec b$, assign any numbers $u_{ab}(a)$ and $u_{ab}(b)$ with $u_{ab}(a) < u_{ab}(b)$ to a and b. Then define a real valued function u_{ab}

on X as follows:

$$u_{ab}(x) = \begin{cases} u_{ab}(a) - \lambda_{ab}(x)(u_{ab}(a) - u_{ab}(b)) & \text{if } a \preceq x \preceq b, \\ \dfrac{u_{ab}(a) - \lambda_{xb}(a)u_{ab}(b)}{1 - \lambda_{xb}(a)} & \text{if } x \prec a, \\ \dfrac{u_{ab}(b) - u_{ab}(a)(1 - \lambda_{ax}(b))}{\lambda_{ax}(b)} & \text{if } b \prec x, \end{cases}$$

Given u_{ab} and u_{cd} on X as defined above, we show that u_{cd} is a positive linear transformation of u_{ab}. To do this, it suffices to show that when $c \preceq a \prec b \preceq d$, $u_{cd} = \alpha u_{ab} + \beta$ on X_{cd} for some numbers $\alpha > 0$ and β.

Assume first that $a \prec x \prec b$. By Lemma 5(3), we obtain

$$\lambda_{ab}(x) = \frac{\lambda_{cd}(x) - \lambda_{cd}(a)}{\lambda_{cd}(b) - \lambda_{cd}(a)}.$$

Then $\lambda_{cd}(x) = (u_{cd}(c) - u_{cd}(x))/(u_{cd}(c) - u_{cd}(d))$ and $\lambda_{ab}(x) = (u_{ab}(a) - u_{ab}(x))/(u_{ab}(a) - u_{ab}(b))$ by definition, so substituting those for the above, we get

$$\frac{u_{ab}(a) - u_{ab}(x)}{u_{ab}(a) - u_{ab}(b)} = \frac{u_{cd}(a) - u_{cd}(x)}{u_{cd}(a) - u_{cd}(b)}.$$

Assume next that $c \preceq x \prec a$. By Lemma 5(3), we obtain

$$\lambda_{xb}(a) = \frac{\lambda_{cd}(a) - \lambda_{cd}(x)}{\lambda_{cd}(b) - \lambda_{cd}(x)}.$$

Then $\lambda_{cd}(x) = (u_{cd}(c) - u_{cd}(x))/(u_{cd}(c) - u_{cd}(d))$ and $\lambda_{xb}(a) = (u_{ab}(a) - u_{ab}(x))/(u_{ab}(b) - u_{ab}(x))$ by definition, so substituting those for the above, we get

$$\frac{u_{ab}(a) - u_{ab}(x)}{u_{ab}(b) - u_{ab}(x)} = \frac{u_{cd}(x) - u_{cd}(a)}{u_{cd}(x) - u_{cd}(b)}.$$

Rearrangements of the last equations in the preceding two paragraphs give

$$u_{cd}(x) = \frac{u_{cd}(b) - u_{cd}(a)}{u_{ab}(b) - u_{ab}(a)} u_{ab}(x) + \frac{u_{ab}(b)u_{cd}(a) - u_{ab}(a)u_{cd}(b)}{u_{ab}(b) - u_{ab}(a)}.$$

This also follows when $b \prec x \preceq d$. Hence the desired result obtains.

Under appropriate positive linear transformations we can take $u = u_{ab}$ on X for all $a, b \in X$ with $a \prec b$. We note that u is unique up to a positive linear transformation and for all $x, y, z \in X$ with $x \prec y \prec z$,

$$\lambda_{xz}(y) = \frac{u(x) - u(y)}{u(x) - u(z)}.$$

For $\pi \in \Pi$, there is a $\pi^* \in \Pi^*$ such that $\pi \subseteq \pi^*$. Let $\pi^* = \langle x_1, \ldots x_n \rangle$ and $\pi = \langle x_{i_1}, \ldots x_{i_{m+1}} \rangle$. Then $p_\pi(\alpha_1 \ldots \alpha_m) = p_{\pi^*}(\alpha_1^* \ldots \alpha_n^*)$ when

$$\alpha_j^* = \begin{cases} 0 & \text{for } i \leq j < i_1, \\ \alpha_k & \text{for } i_k \leq j < i_{k+1}, \\ 1 & \text{for } i_{m+1} \leq j \leq n. \end{cases}$$

Then define V on $\mathcal{P}^*$ as follows:

$$V(p_\pi(\alpha_1 \ldots \alpha_m)) = (u(x_1) - u(x_{n+1}))\Psi(\sigma_{\pi^*}(\alpha_1^* \ldots \alpha_n^*)) - u(x_1)\Psi(0) + u(x_{n+1})\Psi(1).$$

Let $a = x_1$, $b = x_{n+1}$, $\alpha_0 = 0$, and $\alpha_{m+1} = 1$. It follows from Lemma 5(2) and the preceding paragraph that

$$\begin{aligned} \Psi(\sigma_{\pi^*}(\alpha_1^* \ldots \alpha_n^*)) &= \sum_{i=1}^{n} (\lambda_{ab}(x_{i+1}) - \lambda_{ab}(x_i))\Psi(\alpha_i^*) \\ &= \sum_{i=1}^{n} \frac{u(x_i) - u(x_{i+1})}{u(a) - u(b)} \Psi(\alpha_i^*) \\ &= \frac{1}{u(a) - u(b)} \left(\sum_{k=1}^{m+1} (\Psi(\alpha_k) - \Psi(\alpha_{k-1}))u(x_{i_k}) + u(a)\Psi(0) - u(b)\Psi(1) \right). \end{aligned}$$

Therefore,

$$V(p_\pi(\alpha_1 \ldots \alpha_m)) = \sum_{i=1}^{m+1} (\Psi(\alpha_k) - \Psi(\alpha_{k-1}))u(x_{i_k}),$$

which does not depend on the choice of π^* as long as $\pi \subseteq \pi^*$ and $\pi^* \in \Pi^*$.

Given $\pi_1, \pi_2 \in \Pi$, there is a $\pi^* \in \Pi^*$ such that $\pi_1 \subseteq \pi^*$ and $\pi_2 \subseteq \pi^*$. Let $p_{\pi_1}(\alpha_1 \ldots \alpha_k) = p_{\pi^*}(\alpha_1^* \ldots \alpha_n^*)$ and $p_{\pi_2}(\beta_1 \ldots \beta_m) = p_{\pi^*}(\beta_1^* \ldots \beta_n^*)$. Then it follows from Lemma 3, Proposition 2, and the definition of V that

$$\begin{aligned} p_{\pi_1}(\alpha_1 \ldots \alpha_k) \preceq p_{\pi_2}(\beta_1 \ldots \beta_m) \quad &\text{iff} \quad p_{\pi^*}(\alpha_1^* \ldots \alpha_n^*) \preceq p_{\pi^*}(\beta_1^* \ldots \beta_n^*) \\ &\text{iff} \quad \sigma_{\pi^*}(\beta_1^* \ldots \beta_n^*) \leq \sigma_{\pi^*}(\alpha_1^* \ldots \alpha_n^*) \\ &\text{iff} \quad \Psi(\sigma_{\pi^*}(\beta_1^* \ldots \beta_n^*)) \leq \Psi(\sigma_{\pi^*}(\alpha_1^* \ldots \alpha_n^*)) \\ &\text{iff} \quad V(p_{\pi_1}(\alpha_1 \ldots \alpha_k)) \leq V(p_{\pi_2}(\beta_1 \ldots \beta_m)). \end{aligned}$$

Hence the representaion of the theorem obtains. □

Sufficiency Proof of Theorem 2. Suppose that Axioms 1–6 hold. Since $\mathcal{P}^* \subseteq \mathcal{P}^{**}$,

$(\mathcal{P}^*, \prec)$ has the RDU representation. Let V be a real valued function on $\mathcal{P}^*$ such that for all $p, q \in \mathcal{P}^*$, $p \prec q$ iff $V(p) < V(q)$. Then by Theorem 1, there are real valued functions u on X and Ψ on I such that

$$V(p_\pi(\alpha_1 \dots \alpha_n)) = \sum_{i=1}^{n} (\Psi(\alpha_i) - \Psi(\alpha_{i-1}))u(x_i),$$

where $\alpha_0 = 0$ and $\alpha_n = 1$.

Given $p \in \mathcal{P}^{**} \setminus \mathcal{P}^*$, there are $a, b \in X$ such that $a \prec b$ and $p([a, b]) = 1$. By Axiom 3, $a \preceq p \preceq b$, so by Axiom 2, $p \sim \lambda a + (1 - \lambda)b$ for some $0 \le \lambda \le 1$. Thus we assign a real number $V(p)$ to p as follows.

$$V(p) = (\Psi(\lambda) - \Psi(0))u(a) + (\Psi(1) - \Psi(\lambda))u(b).$$

It easily follows from Axiom 1 and Theorem 1 that for all $q, r \in \mathcal{P}^{**}$, $q \prec r$ iff $V(q) < V(r)$.

In what follows, we fix $p \in \mathcal{P}^{**} \setminus \mathcal{P}^*$ with $a \prec b$ and $p([a, b]) = 1$. Since p is a step probability measure, let $\{A_1, \dots, A_{n+1}\}$ be a partition of $[a, b]$ such that $A_1 \prec \dots \prec A_{n+1}$ and for some real numbers, $\alpha_1, \dots, \alpha_{n+1}$,

$$p((-\infty, x]) = \alpha_i \text{ whenever } x \in A_i.$$

Since $b \in A_{n+1}$, $\alpha_{n+1} = 1$. Let $\tau_{2i-1} = \inf_{x \in A_i} u(x)$ and $\tau_{2i} = \sup_{x \in A_i} u(x)$ for $i = 1, \dots, n+1$. Also, let $\beta_0 = 0$, $\beta_{2i-1} = \alpha_i$ for $i = 1, \dots, n+1$ and $\beta_{2i} = p(\cup_{k=1}^{i} A_k)$ for $i = 1, \dots, n$. Then we are to show that

$$V(p) = \sum_{i=1}^{2n+1} (\Psi(\beta_i) - \Psi(\beta_{i-1}))\tau_i.$$

For $i = 1, \dots, 2n+1$, define $\gamma_i = \beta_i - \beta_{i-1}$. Let $\Pi_p^+ = \{\langle x_1, \dots, x_{n+1} \rangle \in \Pi : x_i \in A_i \text{ for } i = 1, \dots, n+1\}$ and $\Pi_p^- = \{\langle a, x_1, \dots, x_n \rangle \in \Pi : x_i \in A_i \text{ for } i = 1, \dots, n\}$. For $k = 1, \dots, n+1$, let p_k^+ denote a one-step probability measure such that $p_k^+(A_k) = 1$ and $p_k^+((-\infty, x]) = 1$ for all $x \in A_k$. For $k = 1, \dots, n$, let p_k^- denote a one-step probability measure such that $p_k^-(A_k) = 1$ and $p_k^-([x, +\infty)) = 1$ for all $x \in A_k$. We note that $p_1^+ = a$, $p_k^+ \in \mathcal{S}^+$, and $p_k^- \in \mathcal{S}^-$. Then we have

$$p = \sum_{k=1}^{n+1} \gamma_{2k-1} p_k^+ + \sum_{k=1}^{n} \gamma_{2k} p_k^-.$$

We have three cases to examine:

Case 1. $p(\cup_{i=1}^k A_i) = \alpha_k$ for $k = 1, \ldots, n$,

Case 2. $p(\cup_{i=1}^k A_i) = \alpha_{k+1}$ for $k = 1, \ldots, n$,

Case 3. $\alpha_k \le p(\cup_{i=1}^k A_i) \le \alpha_{k+1}$ for $k = 1, \ldots, n$.

Case 1. Suppose that $p(\cup_{i=1}^k A_i) = \alpha_k$ for $k = 1, \ldots, n$. Then $p = \sum_{k=1}^{n+1} \gamma_{2k-1} p_k^+$. Given $\pi \in \Pi_p^+$, Theorem 1 gives

$$V(p_\pi(\alpha_1 \ldots \alpha_n)) = \sum_{i=1}^{n+1} (\Psi(\alpha_i) - \Psi(\alpha_{i-1}))u(x_i),$$

where $\alpha_0 = 0$. Since $\tau_{2i-1} = \inf_{x \in A_i} u(x)$ for $i = 1, \ldots, n+1$, we have

$$\inf_{\pi \in \Pi_p^+} V(p_\pi(\alpha_1 \ldots \alpha_n)) = \sum_{i=1}^{n+1} (\Psi(\alpha_i) - \Psi(\alpha_{i-1}))\tau_{2i-1}.$$

Since Ψ is strictly increasing and continuous and $u(a) \le \inf_{\pi \in \Pi_p^+} V(p_\pi(\alpha_1 \ldots \alpha_n)) \le u(b)$, there is a $0 \le \lambda \le 1$ such that

$$(\Psi(\lambda) - \Psi(0))u(a) + (\Psi(1) - \Psi(\lambda))u(b) = \inf_{\pi \in \Pi_p^+} V(p_\pi(\alpha_1 \ldots \alpha_n)),$$

so $\lambda a + (1-\lambda)b \prec p_\pi(\alpha_1 \ldots \alpha_n) = \sum_{i=1}^{n+1} \gamma_{2i-1} x_i$. Since $p_i^+(A_i) = 1$ and $x_i \in A_i$ for all $i = 1, \ldots, n+1$, it follows from Axiom 6 that $\lambda a + (1-\lambda)b \preceq \sum_{i=1}^{n+1} \gamma_{2i-1} p_i^+ = p$. Thus

$$\sum_{i=1}^{n+1} (\Psi(\alpha_i) - \Psi(\alpha_{i-1})))\tau_{2i-1} \le V(p).$$

On the other hand, Axiom 3 implies that $p \preceq p_\pi(\alpha_1 \ldots \alpha_n)$ for all $\pi \in \Pi_p^+$. Thus we have

$$V(p) \le \sum_{i=1}^{n+1} (\Psi(\alpha_i) - \Psi(\alpha_{i-1})))\tau_{2i-1}.$$

Hence the desired result obtains.

Case 2. Suppose that $p(\cup_{i=1}^k A_i) = \alpha_{k+1}$ for $k = 1, \ldots, n$. Then $p = \gamma_1 a + \sum_{k=1}^n \gamma_{2k} p_k^-$. Given $\pi \in \Pi_p^-$, Theorem 1 gives

$$V(p_\pi(\alpha_1 \ldots \alpha_n)) = \sum_{i=1}^{n+1} (\Psi(\alpha_i) - \Psi(\alpha_{i-1}))u(x_{i-1}),$$

where $x_0 = a$. Since $\tau_{2i} = \sup_{x \in A_i} u(x)$ for $i = 1, \ldots, n$. Then

$$\sup_{\pi \in \Pi_p^-} V(p_\pi(\alpha_1 \ldots \alpha_n)) = \sum_{i=1}^{n+1} (\Psi(\alpha_i) - \Psi(\alpha_{i-1}))\tau_{2(i-1)},$$

where $\tau_0 = u(a)$.

Since $u(a) \leq \sup_{\pi \in \Pi_p^-} V(p_\pi(\alpha_1 \ldots \alpha_n)) \leq u(b)$ and Ψ is strictly increasing and continuous, there is a $0 \leq \lambda \leq 1$ such that

$$(\Psi(\lambda) - \Psi(0))u(a) + (\Psi(1) - \Psi(\lambda))u(b) = \sup_{\pi \in \Pi_p^-} V(p_\pi(\alpha_1 \ldots \alpha_n)),$$

so $p_\pi(\alpha_1 \ldots \alpha_n) = \gamma_1 a + \sum_{i=1}^n \gamma_{2i} x_i \prec \lambda a + (1 - \lambda)b$. Since $p_i^-(A_i) = 1$ and $x_i \in A_i$ for all $i = 1, \ldots, n$, it follows from Axiom 6 that $p = \gamma_1 a + \sum_{i=1}^n \gamma_{2i} p_i^- \preceq \lambda a + (1 - \lambda)b$. Thus

$$V(p) \leq \sum_{i=1}^{n+1} (\Psi(\alpha_i) - \Psi(\alpha_{i-1})))\tau_{2(i-1)}.$$

On the other hand, Axiom 3 implies that $p_\pi(\alpha_1 \ldots \alpha_n) \preceq p$ for all $\pi \in \Pi_p^-$. Thus we have

$$\sum_{i=1}^{n+1} (\Psi(\alpha_i) - \Psi(\alpha_{i-1})))\tau_{2(i-1)} \leq V(p).$$

Hence the desired result obtains.

Case 3. Take $x_i \in A_i$ for $i = 1, \ldots, n+1$. Then by Axiom 3, $p_k^+ \preceq x_k \preceq p_k^-$ for $k = 1, \ldots, n+1$, where $p_{n+1}^- = b$. For $\pi = \langle x_1, \ldots, x_{n+1} \rangle \in \Pi_p^+$, let

$$p_\pi^+ = \sum_{k=1}^{n+1} \gamma_{2k-1} p_k^+ + \sum_{k=1}^{n} \gamma_{2k} x_k,$$

$$p_\pi^- = \sum_{k=1}^{n+1} \gamma_{2k-1} x_k + \sum_{k=1}^{n} \gamma_{2k} p_k^-.$$

Since $x_k \in \mathcal{S}^+ \cap \mathcal{S}^-$, we have $p_\pi^+, p_\pi^- \in \mathcal{P}^{**}$. By Axiom 3, $p_\pi^+ \preceq p \preceq p_\pi^-$, so that $V(p_\pi^+) \leq V(p) \leq V(p_\pi^-)$. It follows from Cases 1 and 2 that

$$\begin{aligned} V(p_\pi^+) &= \sum_{k=1}^{n+1} (\Psi(\beta_{2k-1}) - \Psi(\beta_{2k-2}))\tau_{2k-1} + \sum_{k=1}^{n} (\Psi(\beta_{2k}) - \Psi(\beta_{2k-1}))u(x_k), \\ V(p_\pi^-) &= \sum_{k=1}^{n+1} (\Psi(\beta_{2k-1}) - \Psi(\beta_{2k-2}))u(x_k) + \sum_{k=1}^{n} (\Psi(\beta_{2k}) - \Psi(\beta_{2k-1}))\tau_{2k}. \end{aligned}$$

Since $\tau_{2k-1} = \inf_{x_k \in A_k} u(x_k)$ and $\tau_{2k} = \sup_{x_k \in A_k} u(x_k)$, we have

$$\sup_{\pi \in \Pi_p^+} V(p_\pi^+) = \inf_{\pi \in \Pi_p^+} V(p_\pi^-).$$

Hence the desired result obtains. □

6 Proof of Theorem 3

Suppose that $\mathcal{P}^{**} \subseteq \mathcal{P}$ and Assumptions 1 and 2 hold. The necessity proof easily follows, so we give the sufficiency proof. We shall assume that Axioms 1–7. Since $\mathcal{P}^{**} \subseteq \mathcal{P}$, the representation of Theorem 2 holds. Let u and Ψ be obtained in Theorem 2. The uniqueness part of the theorem follows from Theorem 2. Since Ψ is unique up to a positive linear transformation, with no loss of generality we assume that $\Psi(0) = 0$ and $\Psi(1) = 1$.

We prove the sufficiency of the axioms in three steps. We define two subsets of $\mathcal{P}$ as follows.

$$\begin{aligned} \mathcal{P}^0 &= \{p \in \mathcal{P} : p([a,b]) = 1 \text{ for some } a, b \in X\}, \\ \mathcal{P}^b &= \{p \in \mathcal{P} : a \preceq p \preceq b \text{ for some } a, b \in X\}. \end{aligned}$$

By definitions, $\mathcal{P}^* \subseteq \mathcal{P}^{**} \subseteq \mathcal{P}^0 \subseteq \mathcal{P}^b \subseteq \mathcal{P}$. The first step is concerned with the set $\mathcal{P}^0$ of probability measures with bounded supports. The second step extends the first step to the set $\mathcal{P}^b$ of probability measures that are bounded in preferences. Finally the third step covers all $p \in \mathcal{P}$. Since $\Psi(0) = 0$ and $\Psi(1) = 1$, define

$$\begin{aligned} E_\Psi(u,p) &= \int_0^{+\infty} (1 - \Psi(p(\{x \in X : u(x) \le \tau\})))d\tau \\ &\quad - \int_{-\infty}^0 \Psi(p(\{x \in X : u(x) \le \tau\}))d\tau. \end{aligned}$$

In the following three steps, we are to show that $E_\Psi(u,p)$ for all $p \in \mathcal{P}$ are well defined and for all $p, q \in \mathcal{P}$, $p \preceq q$ iff $E_\Psi(u,p) \le E_\Psi(u,q)$. By I, we denote the closed unit interval

[0, 1].

Step 1 (all $p \in \mathcal{P}^0$). Suppose $p \in \mathcal{P}^0$. Then there are $a, b \in X$ such that $p([a, b]) = 1$. By Axiom 3, $a \preceq p \preceq b$, so by Axiom 2, $p \sim \lambda a + (1-\lambda)b$ for some $\lambda \in I$. Then define $V(p) = \Psi(\lambda)u(a) + (1 - \Psi(\lambda))u(b)$. Thus by Axiom 1, for all $p, q \in \mathcal{P}^0$, $p \preceq q$ iff $V(p) \leq V(q)$. Since p has a bounded support, $E_\Psi(u, p)$ is well defined.

We are to show that $V(p) = E_\Psi(u,p)$ for all $p \in \mathcal{P}^0$. Suppose that $p([a,b]) = 1$. Since u is unique up to a positive linear transformation, we assume with no loss of generality that $u(a) = 0$ and $u(b) = 1$. For $\tau \in I$, let $A_\tau = \{x \in X : u(x) \leq \tau\}$. We note that A_τ for all $\tau \in I$ are in Γ^*. Then it suffices to show that

$$V(p) = \int_0^1 (1 - \Psi(p(A_\tau)))d\tau.$$

To do this we construct two sequences of step probability measures that converge uniformly to p from above and below, respectively.

For all $\delta \in I$, let $\sigma(\delta) = \sup\{\tau \in I : \Psi(p(A_\tau)) \leq \delta\}$. Given $\tau \in I$, let p_τ^+ denote a one-step probability measure in $\mathcal{S}^+$ such that $p_\tau^+(A_\tau^c) = 1$ and $p_\tau^+((-\infty, x])) = 1$ for all $x \in A_\tau^c$. Also, let p_τ^- denote a one-step probability measure in $\mathcal{S}^-$ such that $p_\tau^-(A_\tau) = 1$ and $p([x, +\infty)) = 1$ for all $x \in A_\tau$.

For all positive intergers n, let p_{0n} and p_{nn} be one-point probability measures such that $p_{0n}(\{a\}) = 1$ and $p_{nn}(\{b\}) = 1$. Given n, we define $n-1$ one-step probability measures, $p_{1n}, \ldots, p_{n-1,n} \in \mathcal{S}^- \cup \mathcal{S}^+$, as follows: for $1 \leq k < n$,

$$p_{kn} = \begin{cases} p^-_{\sigma(\frac{k}{n})} & \text{when } \Psi(p(A_{\sigma(\frac{k}{n})})) \leq \frac{k}{n}, \\ p^+_{\sigma(\frac{k}{n})} & \text{when } \Psi(p(A_{\sigma(\frac{k}{n})})) > \frac{k}{n}. \end{cases}$$

Since Ψ on I is strictly increasing and continuous, $\Psi^{-1}(\tau)$ for all $\tau \in I$ exist. Let $\alpha_k = \Psi^{-1}(\frac{k}{n})$ for $k = 1, \ldots, n-1$, and $\beta_k = \alpha_k - \alpha_{k-1}$ for $k = 1, \ldots, n$, where $\alpha_0 = 0$. Then given n, we define two step probability measures, p^n and q^n, as follows.

$$\begin{aligned} p^n &= \sum_{k=1}^{n} \beta_k p_{kn}, \\ q^n &= \sum_{k=0}^{n-1} \beta_{k+1} p_{kn}. \end{aligned}$$

It follows from Axiom 3 that for all positive integers n, $q^n \preceq q^{n+1} \preceq p \preceq p^{n+1} \preceq p^n$, so $\{p^n\}$ and $\{q^n\}$ are two sequences of step probability measures that converge uniformly to p from above and below, respectively.

By Theorem 2,

$$E_\Psi(u, p^n) = \frac{1}{n}\sum_{k=1}^{n} \sigma(\frac{k}{n}),$$
$$E_\Psi(u, q^n) = \frac{1}{n}\sum_{k=1}^{n-1} \sigma(\frac{k}{n}).$$

Then we have

$$E_\Psi(u, q^n) \leq V(p) \leq E_\Psi(u, p^n).$$

Since $E_\Psi(u,p^n) - E_\Psi(u,q^n) = \frac{1}{n}$, which vanishes as n gets large, $V(p) = E_\Psi(u,p) = \sup E_\Psi(u,q^n) = \inf E_\Psi(u,p^n)$. Therefore, for all $p, q \in \mathcal{P}^0$, $p \preceq q$ iff $E_\Psi(u,p) \leq E_\Psi(u,q)$.

Step 2 (all $p \in \mathcal{P}^b$). Suppose that $a \preceq p \preceq b$ for some $a, b \in X$. Let $\pi = \langle a, b\rangle$. By Axiom 2, $p \sim p_\pi(\lambda)$ for some $\lambda \in I$. Then define $V(p) = \Psi(\lambda)u(a) + (1-\Psi(\lambda))u(b)$. Thus by Axiom 1, for all $p, q \in \mathcal{P}^b$, $p \preceq q$ iff $V(p) \leq V(q)$. We are to show that $V(p) = E_\Psi(u,p)$. We have two cases to examine: p is either unbounded above or unbounded below; p is unbounded from both sides.

Case 1 (either unbounded above or unbounded below). Suppose that p is unbounded above but bounded below. Then $p(\{x \in X : a \preceq x\}) = 1$ for some $a \in X$. When p is unbounded below but bounded above, the proof is similar. By Axiom 3, $p^x \preceq p$ for all $x \in X$, so $\sup_{x\in X} V(p^x) \leq V(p)$. We note that $E_\Psi(u,p)$ is well defined. Since p^x for all $x \in X$ are in $\mathcal{P}^0$, it follows from Step 1 and the definition of $E_\Psi(u,p)$ that

$$\sup_{x\in X} V(p^x) = \sup_{x\in X} E_\Psi(u,p^x) = E_\Psi(u,p).$$

Therefore, $E_\Psi(u,p) \leq V(p)$. We show that the equality holds. Assume that $a \preceq p \preceq b$ and $E_\Psi(u,p) < V(p)$. With no loss of generality, let $u(a) = 0$ and $u(b) = 1$. Let $\pi = \langle a, b\rangle$ and $p \sim p_\pi(\lambda)$. Then $E_\Psi(u,p) < 1 - \Psi(\lambda)$. Since Ψ is strictly increasing and continuous, there is a $\lambda^* \in I$ such that $E_\Psi(u,p) < 1 - \Psi(\lambda^*) < 1 - \Psi(\lambda)$. Thus $p_\pi(\lambda^*) \prec p$. By

Axiom 7, $p_\pi(\lambda^*) \prec p^c$ for some $c \in X$. Since $E_\Psi(u,p^c) \leq E_\Psi(u,p)$, $E_\Psi(u,p^c) < 1-\Psi(\lambda^*)$. Noting that p^c and $p_\pi(\lambda^*)$ are in $\mathcal{P}^0$, it follows from Step 1 that $p^c \prec p_\pi(\lambda^*)$. This is a contradiction. Hence $V(p) = E_\Psi(u,p)$.

Case 2 (unbounded from both sides). Suppose that $a \preceq p \preceq b$. If $E_\Psi(u,p)$ is well defined, then a similar proof of Case 1 applies to obtain that $V(p) = E_\Psi(u,p)$. Thus it suffices to verify that $E_\Psi(u,p)$ is well defined. Suppose that $E_\Psi(u,p)$ is undefined, so that

$$\int_0^{+\infty}(1-\Psi(p(\{x\in X: u(x)\leq\tau\})))d\tau = \int_{-\infty}^0 \Psi(p(\{x\in X: u(x)\leq\tau\}))d\tau = +\infty.$$

Since p is unbounded below, there is a $c \in X$ such that $c \prec a$. Thus $c \prec p$. By Axiom 7, $c \prec p^d$ for some $d \in X$. Since p^d is bounded above, Case 1 implies that $V(c) < V(p^d) = E_\Psi(u,p^d) = -\infty$. This is a contradiction. Hence $E_\Psi(u,p)$ is well defined.

Step 3 (all $p \in \mathcal{P}$). Suppose that p is not contained in $\mathcal{P}^b$. Then there are no $a, b \in X$ such that $a \preceq p \preceq b$. Thus for all $x \in X$, either $x \prec p$ or $p \prec x$. A similar proof of Case 2 in Step 2 gives that $E_\Psi(u,p)$ is well defined. It suffices to show that if we define $V(p) = E_\Psi(u,p)$, then for all $p, q \in \mathcal{P}$, $p \preceq q$ iff $V(p) \leq V(q)$. To do this, it suffices to verify the following claim.

Claim 1 *(1) If for all $x \in X$, either $x \prec \{p,q\}$ or $\{p,q\} \prec x$, then $p \sim q$ and $E_\Psi(u,p) = E_\Psi(u,q)$.*
(2) If $x \prec p$ for all $x \in X$, then $E_\Psi(u,p) \geq \sup_{x\in X} u(x)$.
(3) If $p \prec x$ for all $x \in X$, then $E_\Psi(u,p) \leq \inf_{x\in X} u(x)$.

Proof. (1) Suppose that $x \prec \{p,q\}$ for all $x \in X$. When $\{p,q\} \prec x$ for all $x \in X$, the proof is similar. First we show that $p \sim q$. Assume $p \prec q$. Then by Axiom 7, $p \prec q^a$ for some $a \in X$. By Axiom 3, $q^a \preceq a$, so by Axiom 1, $p \prec a$. This is a contradiction. Hence $q \preceq p$. If $q \prec p$, then similarly we obtain a contradiction, so $p \preceq q$. Hence $p \sim q$.

Next we show that $E_\Psi(u,p) = E_\Psi(u,q)$. Assume that $E_\Psi(u,p) < E_\Psi(u,q)$. Since $E_\Psi(u,p) = \sup_{x\in X} E(u,p^x)$ and $E_\Psi(u,q) = \sup_{x\in X} E(u,q^x)$, there is an $a \in X$ such that

$E_\Psi(u, p^x) < E_\Psi(u, q^a)$ for all $x \in X$. Since p^x and q^a are bounded above, it follows from Axiom 3 and the hypothesis of the claim that $y \preceq p^x \preceq x$ and $z \preceq q^a \preceq a$ for some $y, z \in X$. Thus Step 2 implies that $p^x \prec q^a$ for all $x \in X$. Since q is unbounded above, $a \prec b$ for some $b \in X$. Then we have $q^a \prec p$. By the hypothesis of the claim, $b \prec q$, so $q^a \prec q$. Since $p \sim q$, $q^a \prec p$. By Axiom 7, $q^a \prec p^c$ for some $c \in X$. This is a contradiction. When $E_\Psi(u, q) < E_\Psi(u, p)$, we obtain a similar contradiction. Hence $E_\Psi(u, p) = E_\Psi(u, q)$.

(2) Suppose that $x \prec p$ for all $x \in X$. Assume $\sup_{x \in X} u(x) > E_\Psi(u, p)$. Then there is an $a \in X$ such that $u(a) > E_\Psi(u, p)$. Since $a \prec p$, Axiom 7 implies that $a \prec p^b$ for some $b \in X$. By Axiom 3, $a \prec p^b \prec b$, so Step 2 gives $u(a) < E_\Psi(u, p^b)$. Since $E_\Psi(u, p)$ is well defined, $E_\Psi(u, p^b) \leq E_\Psi(u, p)$. Thus $u(a) < E_\Psi(u, p)$, a contradiction. Hence $E_\Psi(u, p) \geq \sup_{x \in X} u(x)$.

(3) Similar to (2). □

7 Conclusions

The purpose of this paper has been to provide an axiomatic characterization of rank dependent utility for arbitrary consequence spaces. First we established a RDU representation for simple probability measures, where the consequence space includes at least three elements that are not mutually indifferent. To obtain a representaion for finitely additive probability measures, we introduced step probability measures and established a RDU representation for those measures. Finally we applied Wakker's truncation continuity axiom to obtain the representation for finitely additive probability measures, where a utility function need not be bounded.

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