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Indefinability of the Common Knowledge Concept in Finitary Logics

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Abstract: This paper argues that the common knowledge concept cannot fully be defined by means of (nonlogical) axioms in finitary propositional epistemic logics. Its explicit (direct) definition needs an infinitary logic, but an implicit definition might be possible by means of (nonlogical) axioms in a finitary logic. We formulate requirements for an implicit definition of the common knowledge concept, and then show that these requirements cannot be fulfilled by any set of axioms. We use propositional epistemic logic (M) with the knowledge operators of two players to show this claim. The claim holds, however, for other finitary propositional epistemic logics such as (S4),(S5) and the finitary propositional versions of Kaneko-Nagashima's game logics. The claim is proved using the cut-elimination theorem for (M) in Gentzen-style sequent calculus. We prove other two theorems, which clarify the structure of the main theorem and broaden its applicability. We also discuss the relationship between our indefinability result and some existing results of defining common knowledge in formal logics.

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1. Introduction

In a social situation of decision making such as one to be considered in game theory and economics, we would often meet the common knowledge concept if we faithfully follow the argument that each player thinks about the situation, each thinks about the opponent's thinking about the situation, and so on. Of course, the concept of knowledge itself must be defined before the consideration of common knowledge. Although there are many mathematical models for describing knowledge, the approach in terms of symbolic logic would be ultimately unavoidable.[1] There are quite few approaches to knowledge in terms of symbolic logics. One often used is epistemic logic of Hintikka (1962) -- (M),(S4) or (S5) with the knowledge operators of two or more players.[2] In this paper, we consider the definability of common knowledge by means of (nonlogical) axioms in such logics.

The common knowledge concept is described as follows: for formula A,

$$A,\ Y_1(A),\ Y_1Y_2(A),\ Y_1Y_2Y_1(A),\ldots;\ \text{and}\ \ Y_2(A),\ Y_2Y_1(A),\ Y_2Y_1Y_2(A),\ldots \qquad (1.1)$$

Here $Y_i(A)$ $(i = 1,2)$ means that player i knows that A is true. These formulae mean that A is true, player i knows that A is true, player i knows that player j knows that A is true, and so on. The explicit (direct) definition of common knowledge in an object language is given by taking the conjunction of these formulae, but this needs an infinitary language. Nevertheless, it might be possible to make an implicit definition of common knowledge. For example, let C_0 be a given operator symbol, and consider the following formula:

$$C_0(A) \equiv A \wedge Y_1(C_0(A)) \wedge Y_2(C_0(A)), \qquad (1.2)$$

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[1]In game theory and related fields, a partition model is often used to discuss problems of knowledge and information. Aumann (1976) defined the concept of common knowledge using the partition model. Bacharach (1985) used some semantical model corresponding to (S5) and derived a partition model. Some other people constructed a probability model and also derived a partition model, e.g., Tan-Werlang (1988). See also Barwise (1988) for a general consideration of common knowledge.

[2]Exactly speaking, when we restrict epistemic logic (M) to one knowledge operator symbol, it is (M) in modal logic. This difference does not cause any difficulty in this paper. In this sense, we call epistemic logic (M) simply (M).

where $\equiv$ means logical equivalence. The intention is to characterize common knowledge by the property that if A is common knowledge, then A is true and both players know that A is common knowledge, and *vice versa*. Based on this formula, some methods to define the common knowledge concept are known, e.g., Halpern-Moses (1992) and Kaneko-Nagashima (1990). These definitions requires some extensions of finitary logics: Halpern-Moses (1992) extend (S4) by adding (1.2) as a logical axiom schema and an additional inference rule specific to common knowledge, and Kaneko-Nagashima (1990) extend a finitary logic to an infinitary one.

In this paper, we ask the question whether or not common knowledge can be defined by means of nonlogical axioms in finitary propositional epistemic logics. Actually we give an negative answer: the common knowledge concept is indefinable in finitary propositional epistemic logics by means of nonlogical axioms. This result will be proved for propositional epistemic logic (M). The same claim holds for various other propositional logics, e.g., (S4), (S5), and also the finitary propositional versions of game logics of Kaneko-Nagashima (1990,1991b).[3]

A central lemma to the indefinability result is proved, using the cut-elimination theorem obtained by Ohnishi-Matsumoto (1957) for (M) in Gentzen-style sequent calculus. Applying the lemma to (1.2), we obtain the result that even $Y_1Y_2(A)$ is not derived from (1.2) as a nonlogical axiom. This implies that (1.2) itself is far from the axiom determining common knowledge.

We also provide other two theorems, which clarify the structure of the indefinability theorem and broaden its applicability. One states that the requirement of unique determination implies that $C_0(A)$ is equivalent to some finitary formula B including no C_0, which is a modification of Beth's (1953) definability theorem. This explains why it is impossible to require $C_0(A)$ simultaneously to have the common knowledge property and to be determined uniquely. The other shows the applicability of our indefinability theorem to other logics obtained from (M) by adding logical axioms.

[3]In a predicate epistemic logic with some mathematical structure, e.g., number theory axioms, common knowledge could be defined (under appropriate assumptions on knowledge operators) without going to an infinitary logic. The purpose of this paper is to demarcate the indefinability of common knowledge from its definability. Therefore we do not discuss the indefinability question in a predicate epistemic logic.

Then we consider the definability results of Halpern-Moses (1992) and Kaneko-Nagashima (1990) .

2. Epistemic Logic (M) and the Indefinability Theorem

We start with the following basic symbols: propositional variables $p_0, p_1, \ldots$; logical connectives $\neg$ (not), $\supset$ (implies); operator symbols Y_1, Y_2, C_0, C_1; and parentheses (,). The *formulae* are defined inductively: 1) every propositional variable p is a formula; 2) if A,B are formulae, then $(A \supset B)$, $(\neg A)$, $Y_i(A)$ (i = 1,2), $C_0(A)$, and $C_1(A)$ are formulae; and 3) only those constructed by 1) and 2) are formulae. We will often abbreviate parentheses and will sometimes use different ones.

The intent of $Y_i(A)$ (i = 1,2) is that player i knows that A is true. By $C_0(A)$ or $C_1(A)$, we like to describe "A is common knowledge", but its definability by means of nonlogical axioms is now our problem. The reason why we prepare two operator symbols C_0 and C_1 is that the uniqueness requirement for definability is described by using these symbols.

Epistemic logic (M) is formulated as follows: for any formulae A,B, C and i = 1,2,

Logical Axioms: (L1): $A \supset (B \supset A)$; (L2): $(A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))$;

(L3): $(\neg A \supset \neg B) \supset ((\neg A \supset B) \supset A)$;

(MP_Y): $Y_i(A \supset B) \supset (Y_i(A) \supset Y_i(B))$; (V): $Y_i(A) \supset A$;

Inference Rules:

$$\frac{A \supset B \quad A}{B} \text{ (Modus Ponens)} \qquad \frac{A}{Y_i(A)} \text{ (Necessitation)}$$

A *proof* is a finite sequence of formulae with the property that each formula A in the sequence is either an instance of the logical axioms or is deduced with one of the inference rules from one or two formulae occurring before A in the sequence. We denoted by $\vdash A$ iff there is a proof whose last formula is A.

Epistemic logics (S4) and (S5) are defined, respectively, by adding logical axiom schema (PI) to (M) and by adding axiom schemas (PI) and (NI) to (M), respectively:

(PI): $Y_i(A) \supset Y_iY_i(A)$; and (NI) $\neg Y_i(A) \supset Y_i(\neg Y_i(A))$.[4]

Our main results hold for these logics and some other logics. We, first, describe the main result for epistemic logic (M), and state a generalization later

Before formulating our problem, let us see the description of common knowledge in our language. We define $K_i^n(A)$ for $i = 1,2$ and $n \geq 0$ inductively by

$$K_i^n(A) = A \text{ if } n = 0,\ i = 1,2; \text{ and } K_i^n(A) = Y_i(K_j^{n-1}(A)) \text{ if } n > 0,\ j = 1 \text{ or } 2,\ j \neq i. \quad (2.1)$$

These are the same as ones given in (1.1). We say that A is common knowledge when all of these $K_i^n(A)$'s hold. Here we have the definition of the common knowledge concept, but this definition is made outside our object language. When it is allowed to take infinite conjunctions, the explicit definition of common knowledge is given as the conjunction of all $K_i^n(A)$'s, but infinite conjunctions are not allowed in our finitary language. The possibility of making the definition of common knowledge in an implicit manner is our problem in this paper.

For our problem, we need a few more definitions. We write $A \wedge B$ for $\neg(A \supset \neg B)$. Let Γ be a (possibly infinite) set of formulae, and A a formula. Then we write $\Gamma \vdash A$ iff $\vdash B_0 \wedge \ldots \wedge B_m \supset A$ for some finite subset $\{B_0,\ldots,B_m\}$ of Γ. We say that Γ is *consistent* iff $\Gamma \vdash \neg B \wedge B$ does not hold for any formula B. We denote $(A \supset B) \wedge (B \supset A)$ by $A \equiv B$, and sometimes, $B_0 \wedge \ldots \wedge B_m$ by $\wedge \Delta$ when $\Delta = \{B_0,\ldots,B_m\}$. Given two sets Γ, Δ of formulae, we say that Δ is *deductively weaker than* Γ iff $\Gamma \vdash A$ for all A in Δ. We also use the notation Γ,A, and Γ,Δ to denote $\Gamma \cup \{A\}$ and $\Gamma \cup \Delta$.

Now we formulate our problem. Let $\Gamma(C_0;p_0)$ be a set of formulae, where some occurrences of C_0 and p_0 are specified in each formula in $\Gamma(C_1;p_0)$ for substitution. Let $\Gamma(C_1;p_0)$ be the set obtained from $\Gamma(C_0;p_0)$ by substituting C_1 for the specified occurrences of C_0 in each formula in $\Gamma(C_0;p_0)$. For any formulae A, we denote, by $\Gamma(C_0;A)$ and $\Gamma(C_1;A)$, the sets obtained from $\Gamma(C_0;p_0)$ and $\Gamma(C_1;p_0)$ by substituting A for the specified occurrences of p_0 in the formulae in these sets. We use the same convention for any set $\Delta(C_0;p_0)$ of formulae.

Our requirements for the definability of common knowledge are as follows:

[4] Axioms (V), (PI) and (NI) are often called, respectively, the veridicality, positive introspection and negative introspection axioms.

(R0) for any A, $\Gamma(C_0;A), C_0(A)$ is consistent whenever A itself is consistent;

(R1) for any A, $\Gamma(C_0;A), C_0(A) \vdash K_i^n(A)$ for all $n \geq 0$ and $i = 1,2$;

(R2) for any A, $\Gamma(C_0;A), \Gamma(C_1;A) \vdash C_0(A) \equiv C_1(A)$;

(R3) for any set $\Delta(C_1;p_0)$ of formulae, if $\Delta(C_0;p_0)$ is deductively weaker than $\Gamma(C_1;p_0)$ and satisfies (R2), then $\Delta(C_0;p_0)$ satisfies also (R1).

The set $\Gamma(C_0;A)$ of nonlogical axioms is intended to determine $C_0(A)$ to be the common knowledge of A. It is allowed for $\Gamma(C_1;p_0)$ to be an infinite set, since it may include axiom schemas. Without (R0), the problem would be meaningless. Requirement (R1) states that the set $\Gamma(C_0;A)$ gives enough contents to $C_0(A)$ to imply the common knowledge of A. Requirement (R2) means that the set $\Gamma(C_0;A)$ determines the meaning of $C_0(A)$ uniquely up to deducibility. Specifically, the unique determination of the meaning of $C_0(A)$ is formulated by the independence of the choice of a different symbol. Requirement (R3) states that if $\Delta(C_0;A)$ is deductively weaker than $\Gamma(C_0;A)$ and also determines the meaning of $C_0(A)$ uniquely, then $\Delta(C_0;A)$ should already have all information to yield the common knowledge of A. If $\Delta(C_0;A)$ satisfies (R2) but not (R1), then the meaning of $C_0(A)$ determined by $\Delta(C_0;A)$, *a fortiori*, by $\Gamma(C_0;A)$, is different from the common knowledge of A.

If there was a set $\Gamma(C_0;p_0)$ fulfilling the above four requirements, the common knowledge concept could be discussed virtually in epistemic logic (M), even though its explicit definition is not allowed. We have, however, the following theorem.

Theorem 1 (Indefinability Theorem). No set $\Gamma = \Gamma(C_0;p_0)$ fulfills requirements (R0) – (R3).

This theorem does not directly imply the same assertion for a weaker or stronger logic than (M), since even if some $\Gamma(C_0;p_0)$ fulfills (R0) – (R3) in a weaker or stronger logic, Γ might not be consistent in (M) or might not satisfy (R1) – (R3) in (M), respectively. Nevertheless, the same assertion holds for various logics, which will be discussed after stating a central lemma to the proof of Theorem 1 and also in Section 3.

To state the central lemma, we need the concept of Y-depth of a formula. First, we define Y_i-

depth $\delta_i(A)$ (i = 1,2) for any formula A by induction on the structure of formula A:

(i) for any propositional variable p, $\delta_i(p) = 0$; (ii) $\delta_i(A \supset B) = \max(\delta_i(A),\delta_i(B))$;

(iii) $\delta_i(\neg A) = \delta_i(A)$; (iv) $\delta_i(Y_j(A)) = 0$ $(i \neq j)$ and $\delta_i(Y_i(A)) = \max(\delta_i(A),\delta_j(A)+1)$ $(i \neq j)$;

(v) $\delta_i(C_0(A)) = \delta_i(C_1(A)) = \delta_i(A)$.

For example, $\delta_i(K_i^n(Y_j(A))) = \delta_j(Y_j(A)) + n - 1$ for even n and $\delta_i(K_i^n(Y_j(A))) = \delta_j(Y_j(A)) + n$ for odd n. We define the *Y-depth* $\delta(A)$ of a formula A by $\delta(A) = \max(\delta_1(A),\delta_2(A))$. Then it satisfies the following monotonicity property:

$$\text{if A is a subformula of B, then } \delta(A) \leq \delta(B), \qquad (2.2)$$

while $\delta_i(A)$ does not satisfy monotonicity since $\delta_i(Y_j(A)) = 0$ $(j \neq i)$. The following is the central lemma, which will be proved in Section 4.

Lemma. Let A,B be any formulae. If $\vdash B \supset K_i^n(A)$ and $\delta(B) < n$, then $\vdash \neg B$ or $\vdash A$.

Once the lemma is proved, Theorem 1 is proved as follows.

Proof of Theorem 1. Let A be a consistent and unprovable formula in (M). Suppose, on the contrary, that $\Gamma(C_0;p_0)$ fulfills (R0) – (R3) for any A. Then (R2) gives a finite subset $\Delta(C_0;A)$ of $\Gamma(C_0;A)$ to determine $C_0(A)$, i.e., $\Delta(C_0;A),\Delta(C_1;A) \vdash C_0(A) \equiv C_1(A)$, and (R3) implies that $\Delta(C_0;A)$ satisfies (R1), i.e., $\Delta(C_0;A),C_0(A) \vdash K_i^n(A)$ for any n. Since $\Delta(C_0;A)$ is a finite set, this is equivalent to $\vdash \wedge\Delta(C_0;A) \wedge C_0(A) \supset K_i^n(A)$. Take n larger than $\delta(\wedge\Delta(C_0;A) \wedge C_0(A))$. Applying the above lemma to $\vdash \wedge\Delta(C_0;A) \wedge C_0(A) \supset K_i^n(A)$, we have $\vdash \neg\left(\wedge\Delta(C_0;A) \wedge C_0(A)\right)$ or $\vdash A$. The first implies $\Delta(C_0;A),C_0(A) \vdash \neg A \wedge A$, and $\Gamma(C_0;A),C_0(A) \vdash \neg A \wedge A$ since $\Delta(C_0;A)$ is a subset of $\Gamma(C_0;A)$. This contradicts (R0). Since A is unprovable A in (M), the second is impossible. Thus any set $\Gamma(C_0;p_0)$ of formulae does not fulfill (R0) – (R3). □

The Lemma implies also that the formula $C_0(A) \equiv A \wedge Y_1(C_0(A)) \wedge Y_2(C_0(A))$ of (1.2) as a nonlogical axiom $\Gamma(C_0;A)$ yields neither $Y_1Y_2(A)$ or $Y_2Y_1(A)$. Thus this is far from common knowledge. We will discuss the derivation of common knowledge based on this formula in Section 3.

We prove the Lemma using the cut-elimination theorem for sequent calculus (M). The same method is applied to various logics such as (S4) and the finitary versions of GL_n $(1 \leq n \leq \omega)$ of

Kaneko-Nagashima (1990,1991b), since the cut-elimination theorem holds for them and the Lemma remains true for these logics. Another logic relevant here is epistemic logic (S5), for which the cut-elimination theorem does not hold (cf., Ohnishi-Matsumoto (1959)). Recently, however, Takano (1992) proved that any proof in (S5) is transformed into another proof with the same endsequent satisfying the subformula property. Using his result, we can prove the Lemma for epistemic logic (S5). Consequently, the above indefinability theorem holds for those epistemic logics.

The essential part of the above proof is the existence of a finite subset $\Delta(C_0;A)$ of $\Gamma(C_0;A)$ ensured by (R2) and (R3) with $\Delta(C_0;A), C_0(A) \vdash K_1^n(A)$ for any n. We can restrict $\Gamma(C_0;A)$ to the finite subset for the determination of $C_0(A)$ to be the common knowledge of A. The following theorem clarifies the role of (R2), which is a modification of Beth's (1953) definability theorem. It will be proved in Section 4.

Theorem 2. Let $\Gamma(C_0;p_0)$ be a set of formulae where no C_1 is contained and every occurrence of C_0 is specified for the substitution for C_1, and A any formula containing neither C_0 nor C_1. If (R2) holds, i.e., $\Gamma(C_0;A),\Gamma(C_1;A) \vdash C_0(A) \equiv C_1(A)$, there is a formula B such that (1) B contains neither C_0 nor C_1; (2) the propositional variables contained in B occur in $\Gamma(C_0;A)$; and (3) $\Gamma(C_0;A) \vdash C_0(A) \equiv B$.

This states that if $C_0(A)$ is determined uniquely, it must be described by some finitary formula B free from C_0 and C_1. This implies that B contain strictly less information than the common knowledge of A, since the common knowledge contains an infinite number of contents. As far as (R2) is required, $C_0(A)$ is determined uniquely but does not have enough information for common knowledge. Then (R3) requires the determined $C_0(A)$ should contain at least common knowledge; thus we obtain the indefinability result of Theorem 1.

3. Indefinability and Definability of Common Knowledge in Other Logics

This section considers the applicability of Theorem 1 to other logics obtained from (M) by adding some logical axioms and the relationship between Theorem 1 and the definability results of Halpern-Moses (1992) and Kaneko-Nagashima (1990).

3.1 Treatment of Additional Logical Axioms

Let Λ be a set of formulae. We denote the logic obtained from (M) adding Λ as logical axioms by $L(\Lambda)$. The deducibility relation in logic $L(\Lambda)$ is denoted by $\vdash_L$. Here $\Gamma \vdash_L \mathrm{A}$ is defined in the same as in (M), i.e., $\vdash_L \mathrm{B}_1 \wedge \ldots \wedge \mathrm{B}_m \supset \mathrm{A}$ for some $\mathrm{B}_1,\ldots,\mathrm{B}_m \in \Gamma$.

In epistemic logic (M), adding logical axioms is different from treating them as nonlogical axioms, because of Necessitation. When $\mathrm{A} \in \Lambda$ is added as a logical axiom, any formula of form $\mathrm{Y}_i\cdots\mathrm{Y}_i\mathrm{Y}_j\cdots\mathrm{Y}_j\mathrm{Y}_i\cdots\mathrm{Y}_k(\mathrm{A})$ (i,j,k = 1,2) is provable in (M) by Necessitation. We show that if all such formulae are added as nonlogical axioms, the deducibility $\vdash_L$ of $L(\Lambda)$ is represented by the deducibility $\vdash$ of (M). We denote the set of all formulae of form $\mathrm{Y}_i\cdots\mathrm{Y}_i\mathrm{Y}_j\cdots\mathrm{Y}_j\mathrm{Y}_i\cdots\mathrm{Y}_k(\mathrm{A})$ ($\mathrm{A} \in \Lambda$, i,j,k = 1,2) and $\mathbb{T} = p_1 \supset p_1$ by $\mathcal{K}(\Lambda)$. Then the following holds, which will be proved in Section 4.

Theorem 3. Let Γ be a set of formulae. Then $\Gamma \vdash_L \mathrm{A}$ if and only if $\mathcal{K}(\Lambda),\Gamma \vdash \mathrm{A}$.

Thus any assertion of logic $L(\Lambda)$ can be translated into one in (M). The following direct translation of Theorem 1 to logic $L(\Lambda)$ can be viewed as the indefinability theorem for logic $L(\Lambda)$.

Theorem 1_L. There is no set $\Gamma(C_0{:}p_0)$ such that $\mathcal{K}(\Lambda),\Gamma(C_0{:}p_0)$ fulfills (R0) – (R3) in (M) for any formula A, where no occurrences of C_0 and p_0 in $\mathcal{K}(\Lambda)$ are specified for substitution for C_1 and A.

Theorem 1_L is slightly different from asserting directly Theorem 1 for logic $L(\Lambda)$. By Theorem 3, $\mathcal{K}(\Lambda),\Gamma(C_0{:}p_0)$ fulfills (R0) – (R2) in (M) if and only if $\Gamma(C_0{:}p_0)$ fulfills (R0) – (R2) in $L(\Lambda)$, which tells that nothing is lost in these translations. The intent of (R3), however, is lost in the translation: if $\mathcal{K}(\Lambda),\Delta(C_0{:}p_0)$ is deductively weaker than $\mathcal{K}(\Lambda),\Gamma(C_0{:}p_0)$ and satisfies (R2), then $\mathcal{K}(\Lambda),\Delta(C_0{:}p_0)$ satisfies (R1). Each assertion is always associated with $\mathcal{K}(\Lambda)$. The point of (R3) is the restriction of $\Gamma(C_0{:}p_0)$ to the deductively weaker $\Delta(C_0{:}p_0)$: this restriction is not applied to $\mathcal{K}(\Lambda)$. Thus we cannot obtain the same assertion of Theorem 1 for logic $L(\Lambda)$ from Theorem 1_L.

Nevertheless, we can interpret Theorem 1_L as asserting the indefinability of common knowledge for logic $L(\Lambda)$. Requirement (R3) in Theorem 1_L is: if $\Delta(C_0;p_0)$ is deductively weaker than $\mathcal{K}(\Lambda),\Gamma(C_0{:}p_0)$ and $\Delta(C_0{:}\mathrm{A}),\Delta(C_1{:}\mathrm{A}) \vdash C_0(\mathrm{A}) \equiv C_1(\mathrm{A})$, then $\Delta(C_0{:}\mathrm{A}) \vdash \mathrm{K}_i^n(\mathrm{A})$ for all $n \geq 0$ and i = 1,2. If the meaning of $C_0(A)$ is uniquely determined by $\Delta(C_0{:}A)$, but if $\Delta(C_0{:}A)$ does not yield the common knowledge of A, the unique meaning of $C_0(A)$ is different from common knowledge. Thus (R3) in Theorem 1_L is a natural requirement for the definition of common knowledge in logic $L(\Lambda)$.

Theorem 1 itself can be proved directly in (S4), as was mentioned. To obtain the parallel result to Theorem 1_L from (S4), we can use the simpler $\mathcal{K}^*(\Lambda) = \{K_i^n(A)\colon A \in \Lambda,\ n \geq 0 \text{ and } i = 1,2\} \cup \{\top\}$ rather than the previous $\mathcal{K}(\Lambda)$. That is, we do not need the repetition of Y_i, since axiom (PI) takes care of the repetition of Y_i. Thus adding some logical axioms to (S4) is equivalent to adding the common knowledge of those formulae as nonlogical axioms.

3.2 Comparisons with Some Definability Results

Now we consider the definability results of Halpern-Moses (1992) and Kaneko-Nagashima (1990). Halpern-Moses discussed the definability of common knowledge adding formulae (1.2) and some inference rule to (S4) (and some others), and Kaneko-Nagashima used an infinitary logic to formulate them as nonlogical axioms. In those logics, requirements (R0) – (R3) are satisfied. We consider the relationship between their results and our indefinability result.

Halpern-Moses (1992) extend (S4) by adding (1.2): $C_0(A) \equiv A \wedge Y_1(C_0(A)) \wedge Y_2(C_0(A))$ as a logical axiom schema and the following inference rule:

$$\frac{B \equiv A \wedge Y_1(B) \wedge Y_2(B)}{B \supset C_0(A)} \qquad \mathrm{HM}(C_0)^5$$

Then they prove that in this logic, $C_0(A)$ is uniquely determined in a *semantical* sense.

To make a direct comparison between Halpern-Moses's result and our indefinability result, we add the axiom schema $C_1(A) \equiv A \wedge Y_1(C_1(A)) \wedge Y_2(C_1(A))$ and the inference rule $\mathrm{HM}(C_1)$ to their logic, where $\mathrm{HM}(C_1)$ is obtained from $\mathrm{HM}(C_0)$ by substituting C_1 for C_0. We denote the deducibility of this logic by $\vdash_{HM}$. In this logic, requirements (R0) – (R3) are satisfied by the empty set of formula. Requirement (R0), i.e., the consistency of $\vdash_{HM}$, is proved by showing that this logic is a

[5] Halpern-Moses (1992) used a slightly different formulae, i.e., $C_0(A) \supset Y_1(A \wedge C_0(A)) \wedge Y_2(A \wedge C_0(A))$, for (1.2) and the corresponding formula for (HM). These are equivalent formulations.

conservative extension of classical propositional logic. Requirement (R1), i.e., $C_0(A) \vdash_{HM} K_1^n(A) \wedge K_2^n(A)$ for all $n \geq 0$ and $i = 1,2$, can be proved from $C_0(A) \equiv A \wedge Y_1(C_0(A)) \wedge Y_2(C_0(A))$, without using $HM(C_0)$. Requirement (R2) is proved as follows: since $C_1(A) \equiv A \wedge Y_1(C_1(A)) \wedge Y_2(C_1(A))$ is an axiom, we have $\vdash_{HM} C_1(A) \supset C_0(A)$ by $HM(C_0)$, and by the symmetric argument, $\vdash_{HM} C_0(A) \supset C_1(A)$, which imply $\vdash_{HM} C_1(A) \equiv C_0(A)$. Thus (R2) is derived from inference rules $HM(C_0)$ and $HM(C_1)$. Since we consider no nonlogical axiom, (R3) automatically holds.

Since the Halpern-Moses logic has additional inference rules, Theorem 1_L is not directly applicable. Nevertheless, it may be helpful to translate it (partially) to that with nonlogical axioms. Adding (1.2) as an axiom schema to (S4) is equivalent to assuming nonlogical axioms $\mathcal{K}^*(A) = \{K_i^n\big(C_0(A) \equiv A \wedge Y_1(C_0(A)) \wedge Y_2(C_0(A))\big)$: A is a formula, $n \geq 0$ and $i = 1,2\} \cup \{\mathbb{T}\}$. Then

$$\mathcal{K}^*(A), C_0(A) \vdash K_i^n(A) \quad \text{for all } n \geq 0 \text{ and } i = 1,2. \qquad (3.1)$$

Thus $\mathcal{K}^*(A)$ satisfies requirement (R1). Nevertheless, $C_0(A)$ may be stronger than common knowledge, for example, $C_0(A) \equiv A \wedge Y_1(C_0(A)) \wedge Y_2(C_0(A))$ is provable when we substitute $\neg A \wedge A$ for $C_0(A)$. This means that the meaning of $C_0(A)$ is not yet uniquely determined. Here the inference rule $HM(C_0)$ plays the role of determining the unique meaning of $C_0(A)$.

If the upper formula $B \equiv A \wedge Y_1(B) \wedge Y_2(B)$ of $HM(C_0)$ is a theorem, any $K_i^n(B \equiv A \wedge Y_1(B) \wedge Y_2(B))$ is provable for any $n \geq 0$ and $i = 1,2$ by Necessitation. Then it follows that

$$\vdash B \supset K_i^n(A) \qquad \text{for all } n \geq 0 \text{ and } i = 1,2. \qquad (3.2)$$

That is, B also yields the common knowledge of A. Thus $HM(C_0)$ requires that $C_0(A)$ is deductively weakest among the formulae B having the information of the common knowledge of A, i.e. satisfying $B \equiv A \wedge Y_1(B) \wedge Y_2(B)$.

The counterpart of $HM(C_0)$ in the nonlogical framework is interpreted as the rule: if $K_i^n(B \equiv A \wedge Y_1(B) \wedge Y_2(B))$ are provable for all $n \geq 0$ and $i = 1,2$, then $B \supset C_0(A)$ is provable. Thus an infinitary aspect is hidden in the Halpern-Moses logic. It becomes clearer if we consider this inference rule in an infinitary logic.

In an infinitary version of (S4) or game logic GL_ω of Kaneko-Nagashima (1990), the above inference rule is formulated as an axiom schema:

$$\left(\wedge_{n<\omega}K_1^n(B \equiv A \wedge Y_1(B) \wedge Y_2(B))\right) \wedge \left(\wedge_{n<\omega}K_2^n(B \equiv A \wedge Y_1(B) \wedge Y_2(B))\right) \supset (B \supset C_0(A)), \quad (3.3)$$

Let $\Gamma(C_0;A)$ be the union of the set of all formulae of (3.3) and the above set $\mathcal{K}^*(A)$. Then Kaneko-Nagashima (1990) proved that

$$\Gamma(C_0;A) \vdash_\omega C_0(A) \equiv (\wedge_{n<\omega}K_1^n(A)) \wedge (\wedge_{n<\omega}K_2^n(A)), \quad (3.4)$$

where $\vdash_\omega$ is the deducibility relation in infinitary (S4) or game logic GL_ω.[6,7] Thus $\Gamma(C_0;A)$ determines $C_0(A)$ to be equivalent to the explicit definition of common knowledge $(\wedge_{n<\omega}K_1^n(A)) \wedge (\wedge_{n<\omega}K_2^n(A))$. This implies that the above $\Gamma(C_0;A)$ satisfies requirement (R2) in infinitary (S4) or GL_ω. In fact, $\Gamma(C_0;A)$ also satisfies (R0),(R1) and (R3). Thus, in the infinitary (S4) or game logic GL_ω, we do not need to extend the logic itself particularly to define common knowledge.

Since the Halpern-Moses logic is finitary, the common knowledge formula $(\wedge_{n<\omega}K_1^n(A)) \wedge (\wedge_{n<\omega}K_2^n(A))$ cannot be described, *a fortiori*, (3.4) is not available: their characterization of common knowledge is semantical. It remains open whether their semantical characterization has a syntactical counterpart.

4. Proofs of the Lemma , Theorem 2, and Theorem 3

4.1 Proof of the Lemma

We prove the Lemma using the cut-elimination theorem for (M) in sequent calculus, which is obtained in Ohnishi-Matsumoto (1957). First we reformulate epistemic logic (M) in sequent calculus.

Let Λ,Θ be finite sequences of formulae. We prepare another symbol $\rightarrow$ for the definition of

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[6] In these logics, infinitary conjunctions and disjunctions can be restricted to "constructive" infinite sets of formulae.

[7] Kaneko-Nagashima (1991a) derived, in a similar implicit manner, the solution concept called a final decision, which is the common knowledge of a Nash equilibrium strategy. This is done in the infinitary predicate game logic GL_ω. It can be proved in a manner similar to this paper that this cannot be done in finitary (predicate) logics.

sequent calculus. We call the expression $\Lambda \rightarrow \Theta$ a *sequent.* Epistemic logic (M) in sequent calculus is defined by one axiom schema and various inference rules.

Axiom Sequents: $A \rightarrow A$, where A is any formula.

Structural Rules:

$$\frac{\Lambda \rightarrow \Theta}{A,\Lambda \rightarrow \Theta}\ (t\rightarrow) \qquad \frac{\Lambda \rightarrow \Theta}{\Lambda \rightarrow \Theta,A}\ (\rightarrow t) \qquad \frac{A,A,\Lambda \rightarrow \Theta}{A,\Lambda \rightarrow \Theta}\ (c\rightarrow) \qquad \frac{\Lambda \rightarrow \Theta,A,A}{\Lambda \rightarrow \Theta,A}\ (\rightarrow c)$$

$$\frac{\Lambda,A,B,\Delta \rightarrow \Theta}{\Lambda,B,A,\Delta \rightarrow \Theta}\ (i\rightarrow) \qquad \frac{\Lambda \rightarrow \Theta,A,B,\Delta}{\Lambda \rightarrow \Theta,B,A,\Delta}\ (\rightarrow i) \qquad \frac{\Lambda \rightarrow \Theta,A \qquad A,\Gamma \rightarrow \Delta}{\Lambda,\Gamma \rightarrow \Theta, \Delta}\ (\text{Cut})$$

Operational Inferences:

$$\frac{\Lambda \rightarrow \Theta,A}{\neg A,\Lambda \rightarrow \Theta}\ (\neg\rightarrow) \qquad \frac{A,\Lambda \rightarrow \Theta}{\Lambda \rightarrow \Theta,\neg A}\ (\rightarrow\neg)$$

$$\frac{\Lambda \rightarrow \Theta,A \quad B,\Lambda \rightarrow \Theta}{A \supset B,\Lambda \rightarrow \Theta}\ (\supset\rightarrow) \qquad \frac{A,\Lambda \rightarrow \Theta,B}{\Lambda \rightarrow \Theta,A \supset B}\ (\rightarrow\supset)$$

Y-Inferences:

$$\frac{A,\Lambda \rightarrow \Theta}{Y(A),\Lambda \rightarrow \Theta}\ (Y\rightarrow) \qquad \frac{\Lambda \rightarrow A}{Y(\Lambda) \rightarrow Y(A)}\ (\rightarrow Y)$$

where Y denotes Y_i (i = 1,2) and $Y(\Lambda) = Y_i(A_0),\ldots,Y_i(A_k)$ $(\Lambda = A_0,\ldots,A_k)$.

A *proof* is a finite tree consisting of sequents whose leaves are axiom (initial) sequents and whose adjacent sequents are connected with above inference rules. If there is a proof whose root is sequent $\Lambda \rightarrow \Theta$, then $\Lambda \rightarrow \Theta$ is said to be *provable* in (M), which we denote by $\vdash \Lambda \rightarrow \Theta$.

The above formulation of (M) is equivalent to epistemic logic (M) in Hilbert-style in the sense that for any formula A, $\vdash A$ if and only if sequent $\rightarrow A$ is provable in the above sense. For details, see Ohnishi-Matsumoto (1957) and Gentzen (1935).

The key theorem to prove the Lemma is the *cut-elimination theorem* obtained by Ohnishi-Matsumoto (1957) for sequent calculus (M). It states that for any proof, there is a cut-free proof with the same endsequent. Although our epistemic logic (M) includes two knowledge operators, the cut-elimination theorem remains true.

In each inference of the above list, a formula (or formulae) changed in the lower sequent is called a *principal* formula and a formula (or formulae) to be changed in the upper sequent is called a *side formula.*

Proof of the Lemma: The lemma is stated in sequent calculus (M) as follows: if $\vdash B \rightarrow K_i^n(A)$ and $\delta(B) < n$, then $\vdash B \rightarrow$ or $\vdash \rightarrow A$. Suppose $\vdash B \rightarrow K_i^n(A)$ and $\delta(B) < n$. By the cut-elimination theorem, there is a cut-free proof $\mathcal{T}$ of $B \rightarrow K_i^n(A)$. Here we can assume that

any initial sequent in $\mathcal{T}$ contain no Y_i (i = 1,2) but in the scopes of the occurrences of C_0, C_1. (4.1)

This can be proved by adding the following steps to the initial sequents until every occurrence of Y_i (i = 1,2) but in the scopes of the occurrences of C_0 and C_1 is eliminated in the initial sequents:

$$\dfrac{\dfrac{A \rightarrow A}{\neg A, A \rightarrow}}{\neg A \rightarrow \neg A} \qquad \dfrac{\dfrac{A \rightarrow A \quad B \rightarrow B}{A \supset B, A \rightarrow B}}{A \supset B \rightarrow A \supset B} \qquad \dfrac{\dfrac{A \rightarrow A}{Y_i(A) \rightarrow A}}{Y_i(A) \rightarrow Y_i(A).}$$

Consider the ancestors of $K_i^n(A)$ of the endsequent in the proof $\mathcal{T}$ who have the form $K_j^m(A)$ (m > 0 and j = 1 or 2).[8] Let $\mathcal{T}_0$ be the part of $\mathcal{T}$ in which the ancestors $K_j^m(A)$ (m > 0 and j = 1 or 2) appear. Let $\mathcal{A}_0$ be the set of the ancestors in $\mathcal{T}_0$ (including $K_i^n(A)$ in the endsequent). Note that $\mathcal{A}_0$ is not a set of formulae but a set of occurrences of those formulae. Since every ancestor in $\mathcal{A}_0$ is of form $K_j^m(A)$ (m > 0 and j = 1 or 2), and since every initial sequent in $\mathcal{T}$ contains no Y_j (j = 1,2) by (4.1), we have the following:

every ancestor in $\mathcal{A}_0$ may be the principal or side formulae of only ($\rightarrow$Y) and ($\rightarrow$t); (4.2)

an ancestor in $\mathcal{A}_0$ is the principal formula of ($\rightarrow$t) only if it is in an uppermost sequent in $\mathcal{T}_0$; (4.3)

every uppermost sequent in $\mathcal{T}_0$ is the lower sequent of either ($\rightarrow$Y) or ($\rightarrow$t). (4.4)

[8] An ancestor is defined is as follows. Consider a particular occurrence of an inference and an occurrence of formula A in the lower sequent of the inference. If the occurrence of formula A is the principal formula of the inference, the side formulae are *immediate ancestors*, and otherwise, the directly corresponding occurrence of A in the upper sequent of the inference is an *immediate ancestor*. An occurrence of B is called an *ancestor* of A in a proof iff we reach a formula B by tracing upward immediate ancestors in each inference from the occurrence of A. The occurrence of A itself is a trivial ancestor.

By (4.4), it suffices to consider the following two cases:

(a): every uppermost sequent in $\mathcal{T}_0$ is the lower sequent of $(\rightarrow t)$;

(b): some uppermost sequent in $\mathcal{T}_0$ is the lower sequent of $(\rightarrow Y)$.

Consider case (a). Then we obtain a proof $\mathcal{T}^*$ of $B \rightarrow$ from the proof $\mathcal{T}$ of $B \rightarrow K_i^n(A)$ by eliminating all ancestors in $\mathcal{A}_0$ and the inference rules whose principal formulae are ancestors in $\mathcal{A}_0$. Thus $\vdash B \rightarrow$.

Next, consider case (b). We show the following:

(*) if a sequent in $\mathcal{T}_0$ is the lower sequent of $(\rightarrow Y)$, then the sequent has the form

$$Y_j(\Gamma) \rightarrow K_j^m(A) \text{ and } \delta(Y_j(\Gamma)) < m,$$

where $\delta(Y_j(\Gamma)) = \max\{\delta(Y_j(C)) \colon C \in \Gamma\}$. Once (*) is shown, we apply (*) to an occurrence of $(\rightarrow Y)$ given in (b). Then the lower sequent of this $(\rightarrow Y)$ has the form $Y_j(\Gamma) \rightarrow K_j^1(A)$. Since $\delta(Y_j(\Gamma)) < 1$ by (*), Γ is empty. Thus the lower sequent is $\rightarrow Y_j(A)$. The upper sequent of the same $(\rightarrow Y)$ is $\rightarrow A$, which implies $\vdash \rightarrow A$.

We prove (*) by upward induction from the root in $\mathcal{T}_0$. First, consider the occurrence η of $(\rightarrow Y)$ with the properties: (1) the lower sequent of η belongs to $\mathcal{T}_0$ and (2) no other $(\rightarrow Y)$ occurs below η. The part of $\mathcal{T}$ from η to the endsequent is as follows

$$\cfrac{\cfrac{Y_i(\Gamma) \rightarrow K_j^{n-1}(A)}{Y_i(\Gamma) \rightarrow K_i^n(A)}\ (\rightarrow Y)(\eta)}{\begin{array}{c} \ddots \ \vdots \ .\cdot^{\cdot} \\ \hline B \rightarrow K_i^n(A) \end{array}} \qquad \text{where } j \neq i.$$

If Γ is empty, $\delta(Y_j(\Gamma)) = 0 < n$. When Γ is nonempty, every formula in $Y_i(\Gamma)$ is a subformula of B, since $\mathcal{T}$ is a cut-free proof. This implies $\delta(Y_i(\Gamma)) \leq \delta(B) < n$ by the monotonicity of δ, i.e., (2.2).

Consider an occurrence η' of $(\rightarrow Y)$ whose lower sequent belongs to $\mathcal{T}_0$, and consider the uppermost occurrence η of $(\rightarrow Y)$ below η'. Make the induction hypothesis that (*) holds for η. If there are such occurrences, the part of $\mathcal{T}$ from η to η' is follows:

$$\frac{Y_j(\Gamma) \to K_k^m(A)}{Y_j(\Gamma) \to K_j^{m+1}(A)} \quad (\to Y)(\eta')$$

$$\ddots \quad \vdots \quad \cdot^{\cdot^{\cdot}}$$

$$\frac{Y_k(\Delta) \to K_j^{m+1}(A)}{Y_k(\Delta) \to K_k^{m+2}(A)} \quad (\to Y)(\eta) \qquad \text{where } j,\ k = 1,2 \text{ and } j \neq k.$$

If $Y_j(\Gamma)$ is empty, $\delta(Y_j(\Gamma)) = 0 < 1 \leq m$. Suppose that $Y_j(\Gamma)$ is nonempty. Since $\mathcal{T}$ is a cut-free proof, every formula $Y_j(D)$ of $Y_j(\Gamma)$ appears as a subformula of some element $Y_k(E)$ of $Y_k(\Delta)$, i.e., $Y_j(D)$ is a subformula of E. This implies $\delta(Y_j(D)) < \delta(Y_kY_j(D)) \leq \delta(Y_k(E))$ by the monotonicity of δ. From this, we have $\delta(Y_j(\Gamma)) < \delta(Y_k(\Delta))$, since Γ and Δ are finite. By the induction hypothesis, $\delta(Y_k(\Delta)) < m+2$, which together with $\delta(Y_j(\Gamma)) < \delta(Y_k(\Delta))$ implies $\delta(Y_j(\Gamma)) < m+1$. □

4.2 Proof of Theorem 2.

First we refer to an analog in (M) to Craig's (1957) interpolation theorem for a classical predicate logic: Let A,B be formulae with the properties: 1) some propositional variable occurs in both A,B; 2) neither C_0 nor C_1 occurs in both A and B; and 3) $\vdash A \supset B$. Then there is a formula Z with the following properties: (a) the propositional variables contained in Z occur both in A and B; (b) Z contains neither C_0 nor C_1; and (b) $\vdash A \supset Z$ and $\vdash Z \supset B$.

This theorem can be proved in a manner similar to Maehara's (1960) proof of Craig's theorem.

Let us return to the proof of Theorem 2. Since $\Gamma(C_0;A),\Gamma(C_1;A) \vdash C_0(A) \equiv C_1(A)$, there is a finite subset Δ of Γ such that $\Delta(C_0;A),\Delta(C_1;A) \vdash C_0(A) \equiv C_1(A)$. This implies $\vdash \Big(\wedge \Delta(C_0;A)\Big) \wedge C_0(A) \supset \Big(\wedge \Delta(C_1;A) \supset C_1(A)\Big)$. Regarding this formula as $A \supset B$ in the above theorem, we have a formula Z such that (a) it contains neither C_0 nor C_1; and (b) Z contains propositional variables only occurring in $\Delta(C_0;A)$; and (c) $\vdash \Big(\wedge \Delta(C_0;A) \wedge C_0(A)\Big) \supset Z$; and $\vdash Z \supset \Big(\wedge \Delta(C_1;A) \supset C_1(A)\Big)$. This Z satisfies the assertions (i) and (ii) of Theorem 2. Now we prove that it satisfies also the assertion (iii) of Theorem 2. It follows from (c) that $\vdash \wedge \Delta(C_0;A) \supset (C_0(A) \supset Z)$ and $\vdash \wedge \Delta(C_1;A) \supset (Z \supset C_1(A))$. In a proof of $\wedge \Delta(C_1;A) \supset (Z \supset C_1(A))$, we can replace every occurrence of C_1 by C_0 without affecting provability. Thus we obtain $\vdash \wedge\Delta(C_0;A) \supset (Z \supset C_0(A))$. Thus we have $\vdash \wedge\Delta(C_0;A) \supset Z \equiv C_0(A)$, i.e., $\Delta(C_0;A) \vdash Z \equiv C_0(A)$. □

4.3 Proof of Theorem 3

It suffices to prove the theorem for $\Gamma = \emptyset$. Indeed, suppose $\Gamma \vdash_L A$. Then $\vdash_L B_1 \wedge \ldots \wedge B_k \supset A$ for some $B_1,\ldots,B_k \in \Gamma$. The assertion with $\Gamma = \emptyset$ implies $\mathcal{K}(A) \vdash B_1 \wedge \ldots \wedge B_k \supset A$. This means that $\vdash C_1 \wedge \ldots \wedge C_n \supset (B_1 \wedge \ldots \wedge B_k \supset A)$, equivalently $\vdash (C_1 \wedge \ldots \wedge C_n) \wedge (B_1 \wedge \ldots \wedge B_k) \supset A$ for some $C_1,\ldots,C_n \in \mathcal{K}(A)$,. Thus $\mathcal{K}(A),\Gamma \vdash A$. The converse is proved by tracing this proof backward.

(Only-If): Suppose $\vdash_L A$. Then there is a proof $A_0,\ldots,A_l$ in L with $A_l = A$. By induction on this sequence, we show that for every $k = 1,\ldots,l$,

$$\vdash B_{k1} \wedge \ldots \wedge B_{kt_k} \supset A_k \quad \text{for some } B_{k1},\ldots,B_{kt_k} \in \mathcal{K}(A). \tag{4.5}$$

Suppose that A_k is a logical axiom of $L(A)$. Then A_k is either an instance of a logical axiom of (M) or an instance of A, in which case $\vdash \mathbb{T} \supset A_k$ or $\vdash A_k \supset A_k$. In either case, $\vdash B \supset A_k$ for some $B \in \mathcal{K}(A)$.

Now suppose that A_k is deduced from other formulae in the sequence by an inference rule, and make the induction hypothesis that (4.5) holds for any formula in the sequence before A_k.

Suppose that A_k is deduced from $A_m = C \supset A_k$ and $A_n = C$, where $m,n < k$. By the induction hypothesis, $\vdash B_{m1} \wedge \ldots \wedge B_{mt_m} \supset (C \supset A_k)$ and $\vdash B_{n1} \wedge \ldots \wedge B_{nt_n} \supset C$ for some $B_{m1},\ldots, B_{mt_m} \in \mathcal{K}(A)$ and $B_{n1},\ldots,B_{nt_n} \in \mathcal{K}(A)$. These imply $\vdash (B_{m1} \wedge \ldots \wedge B_{mt_m}) \wedge (B_{n1} \wedge \ldots \wedge B_{nt_n}) \supset (C \supset A_k) \wedge C$. Thus $\vdash (B_{m1} \wedge \ldots \wedge B_{mt_m}) \wedge (B_{n1} \wedge \ldots \wedge B_{nt_n}) \supset A_k$.

Suppose that A_k is deduced from A_m by Necessitation ($m < k$). Then A_k is of form $Y_i(A_m)$. The induction hypotheses states that $\vdash B_{m1} \wedge \ldots \wedge B_{mt_m} \supset A_m$ for some $B_{m1},\ldots,B_{mt_m} \in \mathcal{K}(A)$. By Necessitation, $\vdash Y_i(B_{m1} \wedge \ldots \wedge B_{mt_m} \supset A_m)$, and by ($MP_Y$) and (MP), $\vdash Y_i(B_{m1} \wedge \ldots \wedge B_{mt_m}) \supset Y_i(A_m)$. Since $\vdash Y_i(B_{m1}) \wedge \ldots \wedge Y_i(B_{mt_m}) \supset Y_i(B_{m1} \wedge \ldots \wedge B_{mt_m})$ in (M), we have $\vdash Y_i(B_{m1}) \wedge \ldots \wedge Y_i(B_{mt_m}) \supset Y_i(A_m)$. Since $B_{m1},\ldots,B_{mt_m}$ are formulae in $\mathcal{K}(A)$, every $Y_i(B_{ms})$ belongs also to $\mathcal{K}(A)$. Thus we have proved the assertion (4.5) for A_k.

(If): Suppose $\mathcal{K}(A) \vdash A$, i.e., $\vdash B_1 \wedge \ldots \wedge B_m \supset A$ for some $B_1,\ldots,B_m \in \mathcal{K}(A)$. Then $\vdash_L B_1 \wedge$

$\ldots \wedge B_m \supset A$. If $\vdash_L B$ for all $B \in \mathcal{K}(\Lambda)$, we have $\vdash_L A$. Since $\vdash_L \mathbb{T}$ and any formula $B \in \mathcal{K}(\Lambda)$ is expressed as $B = K_i^n(B')$ for some B' in Λ, we have $\vdash_L K_i^n(B')$ by n applications of Necessitation. □

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