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Multimarket Contact, Imperfect Monitoring and Implicit Collusion

by

Hitoshi Matsushima

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Hitoshi Matsushima*

Institute of Socio-Economic Planning,

University of Tsukuba

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ABSTRACT

We will investigate infinitely repeated games with discounting, where players as diversified firms contact with each other in multiple industries. With imperfect monitoring the effects of multimarket contact are very significant on facilitating implicit collusion. As the degree of multimarket contact increases, the optimal sequential equilibrium outcomes approach full collusion. This permissive result holds on almost the same condition as the perfect monitoring case. The high degree of multimarket contact successfully eliminates all the obstacles of imperfect monitoring, and full collusion can be approximated by a sequential equilibrium outcome even in the low discount factor case.

KEYWORDS: infinitely repeated games, imperfect monitoring, multimarket contact, the Folk theorem, low discount factor.

1. INTRODUCTION

We will investigate infinitely repeated games with discounting, where players as diversified firms contact with each other in multiple industries. We will argue that with imperfect monitoring the effects of multimarket contact are very significant on facilitating implicit collusion. As the degree of multimarket contact increases, i.e., as the number of industries in which the diversified firms contact with each other increases, the optimal sequential equilibrium outcomes approach full collusion. This permissive result holds on almost the same condition as the perfect monitoring case. The high degree of multimarket contact successfully eliminates all the obstacles of imperfect monitoring, and full collusion can be approximated by a sequential equilibrium outcome even in the low discount factor case.

We already have many articles on the study of infinitely repeated games with imperfect monitoring, showing that the imperfection of monitoring prevents the collusive and self-enforcing agreement.[1] Green and Porter (1984) formulated repeated oligopoly in which the unobservable demand shock promotes "secret price cutting", and therefore, makes it difficult to establish the mutual friendship among the rival firms.

Several authors have attempted to calm this difficulty. They have considered either the no discounting case or the limit of the discounting case into no discounting (Radner (1986), Matsushima (1989), Fudenberg, Levine and Maskin (1989), Abreu, Milgrom and Pearce (1991), and others). In this paper, we depart from their approaches in the following two senses. First, each player is faced with multiple repeated games, i.e., multiple repeated industries. A player's choice in an industry is allowed to depend not only on the history in this industry but also on the histories in the other industries. Second, a discount factor is fixed less than unity, and the fixed discount factor is not necessarily required to be close to unity.

It has long been argued whether multimarket contact gives any real impacts on the level of competitiveness in the relevant industries. The traditional economists have emphasized the effects of scope economy and M & A on enhancing efficiency. On the other hand, Edwards (1955) presented the stimulating intuition that the diversified firms contacting with each other are more diplomatic than the single-product firms. Edwards explained: if a diversified firm deviates from their collusive agreement locally in a particular industry, then the rivals will penalize it not only in this industry but also globally in all industries.

Bernheim and Winston (1990) have made the first attempt to formulate multimarket contact in the context of infinitely repeated games with perfect monitoring. They criticized Edwards' intuition on the ground that local deviation is irrelevant to the incentive constraint, i.e., only global deviation in all industries is relevant. Multimarket contact serves to pool the incentive constraints and relax binding incentive constraints by shifting the slack enforcement power in the collusive markets to the competitive markets. Their arguments are not neccesarily affirmative. In particular, when the game structures are identical in all industries, multimarket contact simply multiplies both of the costs and benefits from deviation, producing no effective slack enforcement power.

The arguments in Bernheim and Winston (1990) crucially depend on the assumption of perfect monitoring. Once we turn our attention to the imperfect monitoring case, we must change their predictions drastically.

Global deviation influences the probability structures in so many industries. Given the sufficiently high degree of multimarket contact, and by using the law of large numbers, we can find the event of "the punishment region" which almost surely occurs if a player deviates globally, but which hardly does if no player deviates at all. On the other hand, local deviation influences the probability structures in only a few industries, which makes only a little change on the probability of reaching the punishment region. Nevertheless it can be shown that with imperfect monitoring the gain from local deviation will be overcomed by the enormous future loss from the subsequent global punishment in all industries. We would like to regard the latter point as the logical foundation of Edwards' intuition.

The organization of this paper is as follows. In Section 2, we will explain our logical core in a simple example of prisoners' dilemma. This is essentially the same example as the one in Radner, Myerson and Maskin (1986), which induces the uniform inefficiency without multimarket contact. We will show that with multimarket contact full collusion is attained approximately if the fixed discount factor is more than $\frac{1}{3}$. We will compare our possibility theorem with the Folk theorems addressed by Radner (1986), Abreu, Milgrom and Pearce (1991), and others, to fiqure out the originality of our basic idea. In Section 3, we will generalize the model and present the complete proof of our main theorem.

2. EXAMPLE

Consider the following two-person prisoners' dilemma. Each player i = 1,2 chooses either "c (cooperation)" or "d (defection)". We assume imperfect monitoring. Each player can not observe the opponent's action, but commonly observe a public signal which is a random variable conditioning on both players' actions. The realized public signal is either "g (good)" or "b (bad)". The probability of getting signal "g" is

$\frac{1}{4}$	if both players choose "c",
$\frac{1}{2}$	if some chooses "c" and the other chooses "d",
$\frac{3}{4}$	if both choose "d".

Each player's realized payoff is

5	if she chooses "c" and observes "g",
- 7	if she chooses "c" and observes "b",
- 3	if she chooses "d" and observes "g",
9	if she chooses "d" and observes "b".

Hence, each player's expected payoff is

2	if both players choose "c",
3	if she chooses "d" and the opponent does "c",
- 1	if she chooses "c" and the opponent does "d",
0	if both players choose "d".

Obviously, action "d" is dominant, and (d,d) is the unique Nash equilibrium.

In the context of duopolistic industry, action "c" is regarded as a small amount of supply, and "d" a large amount of supply. Signal "g" is regarded as the high price, and "b" the low price.

In the perfect monitoring case, the efficient outcome (2,2) is attained as a sequential equilibrium outcome in the infinitely repeated game if and only if

$$\delta \geq \frac{1}{3}. \tag{1}$$

(Consider the trigger strategy which chooses "c" in period 1, chooses "c" in any period whenever (c,c) has been chosen in every previous period, and continues to choose "d" forever once a player choose "d". The trigger strategy profile induces the efficient outcome (2,2). From the standard analysis, this trigger strategy profile is a sequential equilibrium if and only if inequality (1) holds. Since the one-shot Nash equilibrium outcome (0,0) is the minimax point, inequality (1) is also necessary for the attainability of (2,2) as a sequential equilibrium outcome.)

In the imperfect monitoring case, as Green and Porter (1984) have explained, it is impossible to attain (2,2) as a sequential equilibrium irrespective of the discount factor. As Radner, Myerson and Maskin (1986) have explained, it is also impossible to attain an outcome (v_1, v_2) irrespective of the discount factor whenever $v_1 + v_2 \geq 2$. Hence, full collusion (2,2) can not be attained even in the approximate sense, i.e., as a limit equilibrium with respect to the discount factor.

2.1. MULTIMARKET CONTACT

We consider the situation in which players supply multiple products and contact with each other at a time in multiple industries. There are m independent industries which are described by the same structure as the above prisoners' dilemma. The component game of our repeated play is defined by the combination of m independent but identical prisoners' dilemmas.

In each period, each player i = 1,2 chooses either "c" or "d" in every industry, that is, chooses

$$(a_i^{[1]},a_i^{[2]},\dots,a_i^{[m]})$$

as her choice of actions in the component game. Here $a_i^{[h]}$ is player i's supply in the h-th industry which is either "c" or "d". Since $a_i^{[1]},\dots,a_i^{[m]}$ are independently chosen, $\{c,d\}^m$ is regarded as the set of possible choices for player i in our component game.

In each period, players commonly observe m signals

$$(\omega^{[1]},\omega^{[2]},\dots,\omega^{[m]}),$$

where $\omega^{[h]}$ is the signal observed in the h-th industry which is either "g" or "b". Signal $\omega^{[h]}$ depends on $(a_1^{[h]},a_2^{[h]})$ only, and is independently drawn according to the probability structure specified above.

Each player's choice of action in one industry can condition not only on the history in this industry but also on the histories in the other industries. The influence of multimarket contact on the attainability of implicit collusion will be due to this mutual history dependence.

2.2. THE FOLK THEOREM

We will show that as the degree of the multimarket contact increases, that is, as m increases, the obstacles of imperfect monitoring are successfully eliminated. Let ε be a positive real number close to zero, and define the review strategy in the following way:

(i) Choose "c" in every industry in period 1.

(ii) Choose "c" in every industry in a period t if in every previous period $\tau \leqq t - 1$,

the number of industries in which signal "b" is observed in period τ is less than, or equal to, $(\frac{1}{4} + \varepsilon)m$.

(iii) Continue to choose "d" in every industry after a period t if there exists a previous period $\tau \leqq t - 1$ such that

the number of industries in which signal "b" is observed in period τ is more than $(\frac{1}{4} + \varepsilon)m$.

The interpretation is the following. Players examine the statistical test in each period where

they pass the test if the relative frequency of getting signal "b" in this period is less than, or equal to, $\frac{1}{4} + \varepsilon$, whereas

they fail the test if it is more than $\frac{1}{4} + \varepsilon$.

Players continue to choose (c,c) as long as they continue to pass the statistical tests. Once they fail the test the collusive agreement will be broken out, and the infinite repetition of the one-shot Nash equilibrium (d,d) will follow.

Given that m is sufficiently large, the review strategy proifle approximately induces (2,2) as the expected normalized payoff vector per industry. It is clear from the law of large numbers that if players choose (c,c) in all industries, the resultant relative frequency of getting signal "b" is almost surely around $\frac{1}{4}$. This means that players almost surely pass the tests, and continue to choose (c,c) over a long time. Hence, the review strategy profile approximately induces (2,2). Note that this property holds irrespective of the discount factor.

We will show below that given that m is large enough, the review strategy proifle is a sequential equilibrium if

$$\delta > \frac{1}{3}, \tag{2}$$

where $\delta \varepsilon [0,1)$ is the discount factor. We must note that inequality (2) is almost equivalent to inequality (1), i.e., the necessary and sufficient condition in the perfect monitoring case. It is clear from the standard analysis that all we have to do is to check the once-and-for-all deviation only. We will separately investigate the following two cases:

global deviation: a player chooses "d" in so many industries.

local deviation: a player chooses "d" in only a few industries.

The extreme case of global deviation is that a player chooses "d" in all industries at a time. The extreme case of local deviation is that a player chooses "d" in only a single industry. In this section, we will investigate these extreme cases only. The basic ideas can be applied to the general case with no major change. The complete proof will be given in Section 3.

Consider the first case, where the deviant chooses "d" in all industries. It is clear from the law of large numbers that the resultant relative frequency of getting signal "b" is almost surely around $\frac{1}{2}$, where remember that $\frac{1}{2}$ is the probability of getting signal "b" when a single player chooses "d". Hence, almost surely, this relative frequency is more than $\frac{1}{4} + \varepsilon$, and therefore, players fail the statistical test. This means that almost surely the global deviant will be punished, and the normalized expected future loss is almost equivalent to, but slightly less than, $2\delta m$, i.e., the future loss in the perfect monitoring case. Given that m is sufficiently large, the sufficient condition not to deviate globally in all industries is

$$(1 - \delta)m < 2\delta m, \text{ that is, inequality (2),}$$

where $(1 - \delta)m$ is the normalization of the gain from global deviation.

Next, consider the second case, where the deviant chooses "d" in a single industry, say, in the m-th industry only. Local deviation gives only a little change on the probability of failing the statistical test. This will make it difficult to prevent local deviation. On the other hand, the gain from local deviation is almost negligible as compared to the possible global future punishment in all industries. This will make it easy to prevent local deviation. We will show below that the latter positive aspect can overcome the former negative aspect.

Define a positive integer y^* by

$$\frac{y^*}{m} < \frac{1}{4} - \varepsilon \leq \frac{y^* + 1}{m}.$$

Let h denote the number of industries other than the m-th industry in which signal "b" is observed. If h is less than y^*, players will certainly pass the test regardless of the result in the m-th industry. If h is more than y^*, they will certainly fail the test regardless of the result in the m-th industry. Hence, the result in the m-th industry is relevant to the statistical test only if h is equivalent to y^*. Given that $h = y^*$, the conditional probability of failing the test is $\frac{1}{4}$ if (c,c) is chosen in the m-th industry, whereas

the conditional probability of failing the test is $\frac{1}{2}$ if either (c,d) or (d,c) is chosen in the m-th industry.

Let f denote the probability that $h = y^*$. Hence, the increase of the probability of failing the test is

$$\frac{f}{2} - \frac{f}{4} = \frac{f}{4},$$

and therefore, the normalized expected future loss is

$$\frac{f}{4}\ (2\delta m) = (fm)\ \frac{\delta}{2}.$$

We must note that f(m - 1) is the mean rate of change of the probability distribution on the relative frequency of getting signal "g" from the first industry through the (m-1)-th industry in the interval $(\frac{y^* - 1}{m - 1}, \frac{y^*}{m - 1}]$. It is clear from the law of large numbers that the mean rates of change are entirely high around the average $\frac{1}{4}$, and therefore, f(m - 1), or fm, is very large. Hence, by choosing m sufficiently large, we can make the normalized expected loss, $(fm)\ \frac{\delta}{2}$, larger than the normalization of the gain from local deviation, $1 - \delta$.

From these observations, we can prove that given that the degree of multimarket contact m is sufficiently high, full collusion (2,2) is approximated by a sequential equilibrium outcome per industry.

2.3. DISCUSSION

As we have already mentioned, our example of prisoners' dilemma is essentially the same as the one in Radner, Myerson, and Maskin (1986), which induces the uniform inefficiency without multimarket contact. Moreover, we must note that our example is symmetrical. According to the argument in Bernheim and Winston (1990), multimarket contact typically gives no influence on implicit collusion in symmetrial games provided that perfect monitoring is assumed. This is in contrast with our permissive result with imperfect monitoring.

The basic idea of review strategy is originated by Radner (1986). Differently from our construction, Radner organized the statistical test through successive m periods, which makes m binomial trials one at a time, not simultaneously but successively. Radner considered the no discounting case according to the limit of means criterion, deriving the general Folk theorem.

Several authors have criticized the use of the limit of means criterion because it unreasonably weakens the best response restriction. Radner, Myerson and Maskin (1986) have made it clear that Radner's permissive result crucially depends on this shortcomming of the limit of means criterion. By requiring the strict sense of best response property instead, they have derived the uniform inefficiency with respect to the discount factor.

Abreu, Milgrom and Pearce (1986) argued that this uniform inefficiency is caused by players' accumulation of information during the statistical test. In a Radner's review strategy profile, players successively observe the realized signals before completing the statistical test. They gradually accumulate information on whether they will pass the test or not. Before reaching the m-th period, i.e., the final round of the statistical test, they can almost surely be convinced of whether they will pass it or not, which weakens their incentives to behave honestly in the final round. This trifle triggers off the overall deviation, inducing the anti-Folk theorem. Based on these observations, Abreu, Milgrom and Pearce modified the information structure in a rather artificial way: Players can observe no signal during the statistical test, and observe all the realized signals at once at the end of the m-th period. By using this delay of information release, they have succeeded to derive the Folk theorem in our example.

The Folk theorem by Abreu, Milgrom and Pearce is the limit theorem with respect to the discount factor, like any other Folk theorem with discounting (Matsushima (1989), Fudenberg, Levine and Maskin (1989), and others). They had to choose the discount factor close to unity for the following two reasons. First, the punishment must occur after the long lapse of m periods. Second, they did not use the idea of the relative frequency of getting signal "b". They instead considered, as the punishment region, the event that signal "b" was observed in every period, and concerned its relative likelihood. Clearly, this event rarely occurs even in the global deviation case. These aspects lead to the need of the punishment as large as possible, and therefore, to the choice of the discount factor as close to unity as possible.

3. GENERALIZATION

In this section we will present the general model and the complete proof of our main theorem.

3.1. THE MODEL

Let $N = \{1,\dots,n\}$ denote the player set. We fix an m-tuple of n-person noncooperative games $(G^{[h]})_{h=1}^{m}$, where $G^{[h]} = (A_i^{[h]}, \pi_i^{[h]})_{i\varepsilon N}$ is called a local game, $A_i^{[h]}$ is the finite set of actions for player i, $A^{[h]} = \times_{i\varepsilon N} A_i^{[h]}$, and $\pi_i^{[h]}:A^{[h]} \to R$ is player i's payoff function in local game h.

Each player can not observe the other players' actions, but commonly observe signal $\omega^{[h]}$ which is a stochastic variable conditioning on the action profile $a^{[h]}\varepsilon A^{[h]}$ only. Let $\Omega^{[h]}$ denote the finite set of possible signals $\omega^{[h]}$, and let $p^{[h]}(\omega^{[h]}|a^{[h]})$ denote the probability of getting signal $\omega^{[h]}$ when action profile $a^{[h]}$ is chosen. We will regard $\pi_i^{[h]}(a^{[h]})$ as the expected payoff for player i, that is,

$$\pi_i^{[h]}(a^{[h]}) = \sum_{\omega^{[h]}\varepsilon\Omega^{[h]}} \rho_i^{[h]}(a_i^{[h]},\omega^{[h]})p^{[h]}(\omega^{[h]}|a^{[h]}),$$

where $\rho_i^{[h]}(a_i^{[h]},\omega^{[h]})$ is the realized payoff for player i when she chooses $a_i^{[h]}$ and observes signal $\omega^{[h]}$. We must note that it does not depend on the other players' actions.

We will consider the situation in which players simultaneously play all the local games as the global game $\Gamma = (S_i, u_i)_{i \in N}$, where $S_i = \times_{h=1}^{m} A_i^{[h]}$, $S = \times_{i \in N} S_i$, S_i is player i's action set, and $u_i : S \to R$ is player i's payoff function defined in such a way that for every $s = (a^{[1]}, \ldots, a^{[m]}) \in S$, $u_i(s)$ is the average of the expected payoffs, that is,

$$u_i(s) = \frac{1}{m} \sum_{h=1}^{m} \pi_i^{[h]}(a^{[h]}).$$

Let $s^* = (a^{[1]*}, \ldots, a^{[m]*}) \in S$ be a Nash equilibrium in Γ.[2]

The play of global game Γ is repeated infinitely, which is described by an infinitely repeated game with discounting $\Gamma^{\infty}(\delta)$, where $\delta \in [0,1)$ is the discount factor. We will allow public randomization: In each period after playing local game h, players commonly observe another signal $\phi^{[h]}$ which is a random variable independent of their actions, and is independently drawn according to the uniform distribution over the closed interval [0,1].

Let $\Omega = \times_{h=1}^{m} \Omega^{[h]}$ and $\Phi = \times_{h=1}^{m} \Phi^{[h]}$. A strategy for player i is an infinite sequence $\sigma_i = (\sigma_i(t))_{t=1}^{\infty}$, where $\sigma_i(1) \in S_i$, and for every $t = 2,3,\ldots$,

$$\sigma_i(t) : S_i^{t-1} \times \Omega^{t-1} \times \Phi^{t-1} \to S_i .$$

Denote $s_i(\tau) = (a_i^{[1]}(\tau), \ldots, a_i^{[m]}(\tau))$, $\omega(\tau) = (\omega^{[1]}(\tau), \ldots, \omega^{[m]}(\tau))$, $\Phi(\tau) = (\phi^{[1]}(\tau), \ldots, \phi^{[m]}(\tau))$, $s(t) = (s_i(t))_{i \in N}$, $s_i^t = (s_i(1), \ldots, s_i(t))$, $\omega^t = (\omega(1), \ldots, \omega(t))$, and $\Phi^t = (\Phi(1), \ldots, \Phi(t))$. $s_i(\tau)$, $\omega(\tau)$, and $\Phi(\tau)$ are player i's action, the informative signal profile, and the profile of public randomization in period τ, respectively. $\sigma_i(t)(s_i^{t-1},\omega^{t-1},\phi^{t-1})$ is the action chosen by player i according to strategy σ_i in period t given the previous history $(s_i^{t-1},\omega^{t-1},\phi^{t-1})$. Player i's normalized value associated with an infinite sequence of action profiles (s(1),s(2),...) is defined by

$$(1 - \delta)\sum_{t=1}^{\infty}\delta^{t-1}u_i(s(t)).$$

Player i's expected normalized value given strategy profile σ is defined by

$$v_i(\sigma,\delta) = (1 - \delta)\sum_{t=1}^{\infty}\delta^{t-1}E[u_i(s(t))|\sigma],$$

where $E[\cdot|\sigma]$ is the expectation with respect to the probability measure on histories induced by strategy profile σ. We will use sequential equilibrium as the solution concept (see Kreps and Wilson (1982)).

3.2. THE FOLK THEOREM

We will present sufficient conditions under which our Folk theorem holds. We fix an action profile $\hat{s} = (\hat{a}^{[1]},\ldots,\hat{a}^{[m]})$, and also fix two positive real numbers $q > 0$ and $w > 0$, arbitrarily.

C.1: There exist functions $\mu^{[h]}:\Omega^{[h]} \to [0,1]$, $h = 1,\ldots,m$, such that for every $h = 1,\ldots,m$,

$$\sum_{\omega^{[h]}\in\Omega^{[h]}}\mu^{[h]}(\omega^{[h]})p^{[h]}(\omega^{[h]}|\hat{a}^{[h]})$$

$$= \sum_{\omega^{[1]} \varepsilon \Omega^{[1]}} \mu^{[1]}(\omega^{[1]}) p^{[1]}(\omega^{[1]} | \hat{a}^{[1]}),$$

and for every $i \varepsilon N$ and every $a_i^{[h]} \varepsilon A_i^{[h]} / \{\hat{a}_i^{[h]}\}$,

$$\sum_{\omega^{[h]} \varepsilon \Omega^{[h]}} \mu^{[h]}(\omega^{[h]}) p^{[h]}(\omega^{[h]} | \hat{a}^{[h]}) + q$$

$$< \sum_{\omega^{[h]} \varepsilon \Omega^{[h]}} \mu^{[h]}(\omega^{[h]}) p^{[h]}(\omega^{[h]} | \hat{a}^{[h]} / a_i^{[h]}).$$

It is clear from the Fan's theorem that, by ignoring the positiveness of q, C.1 is equivalent to the condition that player i's deviation from $\hat{a}_i^{[h]}$ to any other mixed action always influences the probability distribution of $\omega^{[h]}$.

C.2: For every $i \varepsilon N$ and every $s_i \varepsilon S_i$,

$$u_i(\hat{s}/s_i) - u_i(\hat{s}) + w < \frac{\delta}{1 - \delta} \{u_i(\hat{s}) - u_i(s^*)\}.$$

It is clear that, by ignoring the strictness of the inequalities, C.2 is the sufficient and necessary condition on which with perfect monitoring $u(\hat{s})$ is sustained as a sequential equilibrium outcome with Nash reversion. We can regard C.2 as the extended idea in Bernheim and Winston (1990).

The example in Section 2 satisfies C.1 and C.2, where $\delta > \frac{1}{3}$, $s^* = ((d,d),\ldots,(d,d))$, $\hat{s} = ((c,c),\ldots,(c,c))$, $q < \frac{1}{4}$, $w < \frac{2\delta}{1 - \delta} - 1$, and

$$\mu^{[h]}(g) = 0 \text{ and } \mu^{[h]}(b) = 1 \text{ for all } h = 1,\ldots,m.$$

THEOREM: Suppose that C.1 and C.2 hold. Fix a positive real number $\psi > 0$ arbitrarily. Given that m is sufficiently large, there exists a sequential equilibrium $\hat{\sigma} = (\hat{\sigma}_i)_{i\in N}$ in $\Gamma^\infty(\delta)$ such that for every $i\in N$,

$$v_i(\hat{\sigma},\delta) \geq u_i(\hat{s}) - \psi.$$

PROOF: Without loss of generality, we will confine our attention to $\psi > 0$ close to zero enough to satisfy that for every $i\in N$,

$$u_i(\hat{s}/s_i) - u_i(\hat{s}) + w < \frac{\delta}{1-\delta}\{u_i(\hat{s}) - u_i(s^*) - \psi\}. \qquad (3)$$

(Note that inequalities (3) are equivalent to C.1 given that $\psi = 0$.)

Consider z homogeneous and independent binomial trials in which each trial results in "1" with probability p, and "0" with probability 1 - p, which is represented by (z,p). For every h = 0,...,z, let f(h;z,p) denote the probability that h is the sum of the results of (z,p), i.e.,

$$f(h;z,p) = \frac{z!}{h!(z-h)!}p^h(1-p)^{z-h}.$$

Define the probability distribution by $F(h;z,p) = \sum_{k=0}^{h} f(k;z,p)$. Define

$$f(h;z_1,z_2,p_1,p_2) = \sum_{k=\max[0,h-z_2]}^{\min[h,z_1]} \frac{z_1!z_2!}{k!(z_1-k)!(h-k)!(z_2-h+k)!}$$

$$\cdot p_1^k(1-p_1)^{z_1-k}p_2^{h-k}(1-p_2)^{z_2-h+k},$$

and define $F(h;z_1,z_2,p_1,p_2) = \sum_{k=0}^{h} f(k;z_1,z_2,p_1,p_2)$. $f(h;z_1,z_2,p_1,p_2)$ is the probability that h is the sum of the results of (z_1,p_1) and (z_2,p_2).

CLAIM: Fix four real numbers $p_1 \varepsilon [0,1]$, $p_2 \varepsilon (p_1,1]$, $\alpha > 0$, and $d > 0$ arbitrarily. There exists a positive integer $\hat{z}$ which satisfies the following property: For every integer $z \geqq \hat{z}$, there exists an integer y^* such that

$$F(y^*;z,0,p_1,p_2) \geqq 1 - \alpha, \quad f(y^*;z,0,p_1,p_2) \geqq \frac{d}{z},$$

and for every pair of nonnegative integers (z_1,z_2) with $z = z_1 + z_2$, either

$$F(y^*;z_1,z_2,p_1,p_2) \leqq \alpha, \text{ or } f(y^*;z,0,p_1,p_2) \geqq \frac{d}{z}.$$

The proof of the claim will be given in Appendix. The implication of the latter part is that, by choosing α close to zero and d large enough, for every (z_1,z_2) with $z_1 + z_2 = z$, either

the associated relative frequency of getting "1" is almost surely more than $\frac{y^*}{z}$, or

the means rate of change around $\frac{y^*}{z}$, $zf(y^*;z,0,p_1,p_2)$, is large,

where $\frac{y^*}{z}$ will be around p_1. As we will clarify later, the former case corresponds to global deviation, and the latter case local deviation.

Let $\mu^{[h]}$, $h = 1,\ldots,m$, be the functions introduced in C.1. We specify

$$p_1 = \sum_{\omega^{[1]} \varepsilon \Omega^{[1]}} \mu^{[1]}(\omega^{[1]}) p^{[1]}(\omega^{[1]} | \hat{a}^{[1]}), \text{ and}$$

$$p_2 = p_1 + q.$$

Note from C.1 that for every $h = 1,\ldots,m$,

$$\sum_{\omega^{[h]} \varepsilon \Omega^{[h]}} \mu^{[h]}(\omega^{[h]}) p^{[h]}(\omega^{[h]} | \hat{a}^{[h]}) = p_1,$$

and for every $i \varepsilon N$ and every $a_i^{[h]} \varepsilon A_i^{[h]} / \{\hat{a}_i^{[h]}\}$,

$$\sum_{\omega^{[h]} \varepsilon \Omega^{[h]}} \mu^{[h]}(\omega^{[h]}) p^{[h]}(\omega^{[h]} | \hat{a}^{[h]} / a_i^{[h]}) > p_2.$$

Choose $\alpha > 0$ close to zero such that for every $i \varepsilon N$,

$$\psi \geq \frac{\delta(\alpha + \varepsilon)}{1 - \delta(1 - \alpha - \varepsilon)} \{u_i(\hat{s}) - u_i(s^*)\}, \text{ and} \tag{4}$$

$$u_i(\hat{s}/s_i) - u_i(\hat{s})$$

$$< (1 - 2\alpha - \varepsilon) \frac{\delta}{1 - \delta} \{u_i(\hat{s}) - u_i(s^*) - \psi\}, \tag{5}$$

where ε is a positive real number close to zero. Inequality (5) is derived from inequality (3). Choose $d > 0$ large enough to satisfy that for every $h = 1,\ldots,m$, every $i \varepsilon N$ and every $a_i^{[h]} \varepsilon A_i^{[h]}$,

$$\pi_i^{[h]}(\hat{a}^{[h]} / a_i^{[h]}) - \pi_i^{[h]}(\hat{a}^{[h]}) < qdb, \tag{6}$$

where b is a positive real number such that for every $i \varepsilon N$,

$$0 < b < \delta\{u_i(\hat{s}) - u_i(s^*) - \psi\}. \tag{7}$$

Here we will choose b close to $\delta\{u_i(\hat{s}) - u_i(s^*) - \Psi\}$ to satisfy that for every $i \varepsilon N$,

$$u_i(\hat{s}/s_i) - u_i(\hat{s}) < (1 - 2\alpha - \varepsilon) \frac{b}{1 - \delta}. \tag{8}$$

(Inequality (8) is derived from inequality (5).) Throughout this proof, we will suppose that m is large enough to satisfy that

$$m - 1 \geqq \hat{z},$$

and for every $h \varepsilon \{0,\ldots,m-1\}$ and every (z_1, z_2) with $z_1 + z_2 = m - 1$,

$$F(h;z_1+1,z_2,p_1,p_2) \geqq F(h;z_1,z_2,p_1,p_2) - \varepsilon, \text{ and}$$

$$F(h;z_1,z_2+1,p_1,p_2) \geqq F(h;z_1,z_2,p_1,p_2) - \varepsilon. \tag{9}$$

(Inequality (9) is derived from the fact that $f(h;z_1,z_2,p_1,p_2)$ can be close to zero.) Finally, we will choose y^* as the integer specified in the claim given that z is replaced by m - 1.

Based on these specifications, we define the review strategy profile $\hat{\sigma}$ as follows: Define the punishment region as the subset of $\Omega \times \Phi$

$$W = \{(\omega,\phi) \mid 0 \leq \phi^{[h]} \leq \mu^{[h]}(\omega^{[h]}) \text{ for at least } y^* + 1 \text{ local games}\},$$

where $\omega = (\omega^{[1]},\ldots,\omega^{[m]}) \varepsilon \Omega$ and $\phi = (\phi^{[1]},\ldots,\phi^{[m]}) \varepsilon \Phi$. The interpretation of W is that the result in local game h where the inequalities in the definition of W hold is regarded as the "bad" result, and at least $y^* + 1$ "bad" results are obtained. We define

(iv) $\hat{\sigma}(1) = \hat{s}$,

for every $t \geqq 2$,

(v) if for every $\tau \varepsilon \{1,\ldots,t-1\}$ $(\omega(\tau),\Phi(\tau))$ is not an element of the punishment region W, then

$$\hat{\sigma}(t)(s_i^{t-1},\omega^{t-1},\phi^{t-1}) = \hat{s}, \text{ and}$$

(vi) if $(\omega(\tau),\Phi(\tau))$ is an element of W for some $\tau \varepsilon \{1,\ldots,t-1\}$, then

$$\hat{\sigma}(t)(s_i^{t-1},\omega^{t-1},\phi^{t-1}) = s^*.$$

Denote $F = F(y^*;m,0,p_1,p_2)$. One gets that for every $i \varepsilon N$,

$$v_i(\hat{\sigma},\delta) = u_i(\hat{s}) - \frac{\delta(1 - F)}{1 - \delta F}\{u_i(\hat{s}) - u_i(s^*)\}.$$

From the former part of the claim and inequalities (9),

$$F \geqq 1 - \alpha - \varepsilon, \tag{10}$$

which, together with inequalities (4), implies that for every $i \varepsilon N$,

$$v_i(\hat{\sigma},\delta) \geqq u_i(\hat{s}) - \frac{\delta(\alpha + \varepsilon)}{1 - \delta(1 - \alpha - \varepsilon)}\{u_i(\hat{s}) - u_i(s^*)\}$$

$$\geqq u_i(\hat{s}) - \psi. \tag{11}$$

We will show below that $\hat{\sigma}$ is a sequential equilibrium in $\Gamma^\infty(\delta)$. All we have to do is to check any strategy σ_i such that player i chooses $\sigma_i(1) = s_i \neq \hat{s}_i$ in period 1 and afterwards conforms to $\hat{\sigma}_i$. Define

$$x = \# \{h \varepsilon \{1,\ldots,m\} | \ a_i^{[h]} \neq \hat{a}_i^{[h]}\},$$

where $s_i = (a_i^{[1]},\ldots,a_i^{[m]})$. x is the number of local games in which player i deviates from $\hat{s}_i$ in period 1 given that she conforms to σ_i. Since the probability that $0 \leq \phi^{[h]}(1) \leq \mu^{[h]}(\omega^{[h]}(1))$ when player i chooses $a_i^{[h]} \neq \hat{a}_i$ is more than p_2, the probability of reaching the punishment region W is at least $1 - F(y^*;m-x,x,p_1,p_2)$. From inequalities (7) and (11), and from the standard calculation, one gets that for every $i \in N$,

$$v_i(\hat{\sigma},\delta) - v_i(\hat{\sigma}/\sigma_i,\delta) > (1-\delta)\{u_i(\hat{s}) - u_i(\hat{s}/s_i)\}$$

$$+ b\{F - F(y^*;m-x,x,p_1,p_2)\}. \qquad (12)$$

Suppose that $F(y^*;m-x,x-1,p_1,p_2) \leq \alpha$, which corresponds to the global deviation case. From inequalities (10),

$$F - F(y^*;m-x,x,p_1,p_2) \geq 1 - 2\alpha - \varepsilon,$$

which, together with inequalities (8), implies that for every $i \in N$,

$$(1-\delta)\{u_i(\hat{s}) - u_i(\hat{s}/s_i)\} + b\{F - F(y^*;m-x,x,p_1,p_2)\}.$$

$$\geq (1-\delta)\{u_i(\hat{s}) - u_i(\hat{s}/s_i)\} + b(1 - 2\alpha - \varepsilon) > 0. \qquad (13)$$

From inequalities (12) and (13), one gets that for every $i \in N$,

$$v_i(\hat{\sigma},\delta) > v_i(\hat{\sigma}/\sigma_i,\delta).$$

Next, suppose that $f(y^*;m-x,x-1,p_1,p_2) \geq \frac{d}{m-1}$, which corresponds to the local deviation case. Notice

$$F(y^*;m-x+1,x-1,p_1,p_2) - F(y^*;m-x,x,p_1,p_2)$$

$$= (p_2 - p_1)f(y^*;m-x,x-1,p_1,p_2) \geq \frac{qd}{m-1}. \qquad (14)$$

Choose an integer $r \in \{1,\dots,m\}$ such that $a_i^{[r]} \neq \hat{a}_i^{[r]}$. We define another strategy σ_i' such that player i chooses $\tilde{\sigma}_i(1) = \tilde{s}_i$ in period 1 and afterwards conforms to $\hat{\sigma}_i$, where $\tilde{s}_i = (\tilde{a}_i^{[1]},\dots,\tilde{a}_i^{[m]})$, $\tilde{a}_i^{[r]} = \hat{a}_i^{[r]}$, and $\tilde{a}_i^{[h]} = a_i^{[h]}$ for all $h \neq r$. Similarly to (12), one gets that for every $i \in N$,

$$v_i(\hat{\sigma},\delta) - v_i(\hat{\sigma}/\tilde{\sigma}_i,\delta) > (1-\delta)\{u_i(\hat{s}) - u_i(\hat{s}/\tilde{s}_i)\}$$
$$+ b\{F - F(y^*;m-x+1,x-1,p_1,p_2)\}.$$

Note that $x - 1$ is the number of local games where player i deviates from $\hat{s}_i$ given that she conforms to $\tilde{\sigma}_i$. From inequalities (6) and (14), one gets

$$(1-\delta)\{u_i(\hat{s}) - u_i(\hat{s}/s_i)\} + b\{F - F(y^*;m-x,x,p_1,p_2)\}$$
$$\leq (1-\delta)\{u_i(\hat{s}) - u_i(\hat{s}/\tilde{s}_i)\} + b\{F - F(y^*;m-x+1,x-1,p_1,p_2)\}$$
$$+ \frac{1}{m}\{\pi_i^{[r]}(\hat{a}^{[r]}/\tilde{a}_i^{[r]}) - \pi_i^{[r]}(\hat{a}^{[r]})\} - \frac{qdb}{m-1}$$
$$< (1-\delta)\{u_i(\hat{s}) - u_i(\hat{s}/\tilde{s}_i)\}$$
$$+ b\{F - F(y^*;m-x+1,x-1,p_1,p_2)\}. \qquad (15)$$

Suppose that $v_i(\hat{\sigma},\delta) < v_i(\hat{\sigma}/\sigma_i,\delta)$. Then, from inequality (12),

$$(1-\delta)\{u_i(\hat{s}) - u_i(\hat{s}/s_i)\} + b\{F - F(y^*;m-x,x,p_1,p_2)\} < 0.$$

This inequality, together with inequality (15), implies

$$(1 - \delta)\{u_i(\hat{s}) - u_i(\hat{s}/\tilde{s}_i)\}$$

$$+ b\{F - F(y^*;m-x+1,x-1,p_1,p_2)\} < 0,$$

that is, the same inequality holds also for $\tilde{s}_i$. By repeating this argument, one gets that there exists a strategy $\bar{\sigma}_i$ which satisfies

$$F(y^*;m-x,x-1,\bar{p}_1,\bar{p}_2) \leqq \alpha, \text{ and}$$

$$(1 - \delta)\{u_i(\hat{s}) - u_i(\hat{s}/\bar{s}_i)\} + b\{F - F(y^*;m-x,x,\bar{p}_1,\bar{p}_2)\} < 0,$$

where $\bar{s}_i = \bar{\sigma}_i(1)$, $\bar{\sigma}_i$ is the same strategy as $\hat{\sigma}_i$ after period 2, and $\bar{x}$ is the number of local games in which player i deviates from $\hat{s}_i$ given that she conforms to $\bar{\sigma}_i$. Since $\bar{\sigma}_i$ corresponds to global deviation, this inequality contradicts inequality (13). Hence, $v_i(\delta,\hat{\sigma}) \geqq v_i(\delta,\hat{\sigma}/\sigma_i)$ must hold.

From the latter part of the claim, we have proven the theorem. **Q.E.D.**

3.3. FURTHER REMARKS

In our general model it might be the case that $|A_i^{[h]}| = 1$ in many local games, i.e., a player i is influential in only a few local games. On the other hand, C.2 requires that each player's utility have the external effects from many local games. Moreover, our construction of equilibrium strategy is based on the assumption that each player can observe the signals realized in any local game irrespective of whether she is an actual participant or not. From these observations, one can conclude that the publicity of signals, together with the existence of global externality, induces our permissive result. The situation of multimarket contact is regarded as a special case satisfying these properties in a realistic manner.

We have assumed that there exists no macro shock through all industries. The companion paper by Matsushima (in preparation) investigates multiple industries in which the unobservable macro shock exists. We can again derive the Folk theorem with fixed discount factor, where firms must make parallel diversifications only partially, and instead establish the global information network in order to estimate the realizations of random macro shock so accurately.

FOOTNOTES

1 I would recommend readers unfamiliar with the literature of repeated games to read the excellent survey by Pearce (1992).

2 Only for simplicity we assume the existence of pure strategy Nash equilibrium. The existence of mixed strategy Nash equilibrium is enough for our argument.

REFERENCES

ABREU, D., P. MILGROM, and D. PEARCE (1991): "Information and Timing in Repeated Partnerships", Econometrica 56, 383-396.

BERNHEIM, D. and M. WINSTON (1990): "Multimarket Contact and Collusive Behavior", Rand Journal of Economics 21, 1-25.

EDWARDS, C. (1955): "Conglomerate Bigness as a Source of Power, in Business Concentration and Price Policy, NBER Conference Report, Princeton University Press.

FUDENBERG, D., D. LEVINE, and E. MASKIN (1989): "The Folk Theorem with Imperfect Public Information", mimeo.

GREEN, E. and R. PORTER (1984): "Noncooperative Collusion under Imperfect Price Information", Econometrica 52, 87-100.

KREPS, D. and R. WILSON (1982): "Sequential Equilibria," Econometrica 50, 863-94.

MATSUSHIMA, H. (1989): "Efficiency in Repeated Games with Imperfect Monitoring", Journal of Economic Theory 48, 428-442.

MATSUSHIMA, H. (in preparation): "Unobservable Macro Shock and Implicit Collusion in Multiple Industries."

PEARCE, D. (1992): Repeated Games: Cooperation and Rationality", in Advances in Economic Theory: Sixth World Congress, ed. by J.-J. Laffont, Cambridge: Cambridge University Press.

RADNER, R. (1986): "Repeated Partnership Games with Imperfect Monitoring and No Discounting", Review of Economic Studies 53, 43-58.

RADNER, R., R. MYERSON and E. MASKIN (1986): An Example of a Repeated Partnership Game with Discounting and with Uniformly Inefficient Equilibria, Review of Economic Studies 53, 59-70.

APPENDIX: PROOF OF THE CLAIM

We define

$$\text{mode}(z,p) = \min\ [\ \underset{h=1,\ldots,z}{\text{argmax}}\ f(h;z,p)],\ \text{and}$$

$$\text{mode}(z_1,z_2,p_1,p_2) = \min\ [\ \underset{h=1,\ldots,z_1+z_2}{\text{argmax}}\ f(h;z_1,z_2,p_1,p_2)].$$

LEMMA A-1: Fix (z_1,p_1) and (z_2,p_2) arbitrarily. If $0 \leqq h \leqq h' \leqq \text{mode}(z_1,z_2,p_1,p_2)$, then

$$f(h;z_1,z_2,p_1,p_2) \leqq f(h';z_1,z_2,p_1,p_2).$$

If $\text{mode}(z_1,z_2,p_1,p_2) \leqq h \leqq h' \leqq z_1 + z_2$, then

$$f(h;z_1,z_2,p_1,p_2) \geqq f(h';z_1,z_2,p_1,p_2).$$

PROOF: Clearly, the properties in Lemma A-1 hold whenever $z_2 = 0$.

Suppose that $z_2 \geqq 1$ and the properties in Lemma A-1 hold for (z_1,z_2-1). We will show that these properties also hold for (z_1,z_2). First, consider any integer h such that $0 \leqq h \leqq \text{mode}(z_1,z_2-1,p_1,p_2) - 1$. Notice

$$f(h-1;z_1,z_2-1,p_1,p_2) \leqq f(h;z_1,z_2-1,p_1,p_2)$$
$$\leqq f(h+1;z_1,z_2-1,p_1,p_2),$$

where we assume $f(-1;z_1,z_2-1,p_1,p_2) = 0$. Hence,

$$f(h;z_1,z_2,p_1,p_2)$$
$$= p_2 f(h-1;z_1,z_2-1,p_1,p_2) + (1 - p_2) f(h;z_1,z_2-1,p_1,p_2)$$
$$\leqq p_2 f(h;z_1,z_2-1,p_1,p_2) + (1 - p_2) f(h+1;z_1,z_2-1,p_1,p_2)$$

$$= f(h+1; z_1, z_2, p_1, p_2).$$

Next, consider any integer h such that $\text{mode}(z_1, z_2-1, p_1, p_2) + 1 \leq h \leq z_1 + z_2 - 1$. Notice

$$f(h-1; z_1, z_2-1, p_1, p_2) \geq f(h; z_1, z_2-1, p_1, p_2)$$

$$\geq f(h+1; z_1, z_2-1, p_1, p_2),$$

where we assume $f(z_1+z_2; z_1, z_2-1, p_1, p_2) = 0$. Hence,

$$f(h; z_1, z_2, p_1, p_2)$$

$$= p_2 f(h-1; z_1, z_2-1, p_1, p_2) + (1 - p_2) f(h; z_1, z_2-1, p_1, p_2)$$

$$\geq p_2 f(h; z_1, z_2-1, p_1, p_2) + (1 - p_2) f(h+1; z_1, z_2-1, p_1, p_2)$$

$$= f(h+1; z_1, z_2, p_1, p_2).$$

From these observations, we have proven Lemma A-1. **Q.E.D.**

LEMMA A-2: Fix $p_1 \varepsilon [0,1]$, $p_2 \varepsilon [0,1]$, $\eta > 0$, and $\xi > 0$ arbitrarily. There exists a positive integer $\hat{z}$ such that for every pair of nonnegative integers (z_1, z_2) with $z_1 + z_2 = z \geq \hat{z}$,

$$\sum_{h: |h - z_1 p_1 - z_2 p_2| < z\xi} f(h; z_1, z_2, p_1, p_2) \geq 1 - \eta.$$

PROOF: The law of large numbers says that for every integer $z > 0$,

$$\sum_{h: |h - zp| < z\xi} f(h; z, p) \geq 1 - \frac{p(1 - p)}{z\xi^2}.$$

Fix two positive integers k and k' where $k' \geq 2$ and $(k' - 1)\xi > 1$. Consider (z_1, p_1) and (z_2, p_2) with $z_1 + z_2 = z \geq kk'$. First, suppose that z_1

$> k$ and $z_2 < k$. Note that the sum of the results of (z_1, p_1) and (z_2, p_2) is in $[z_1p_1 + z_2p_2 - z\xi,\ z_1p_1 + z_2p_2 + z\xi]$ whenever the sum of the results of (z_1, p_1) is in $[z_1p_1 - (z\xi - z_2),\ z_1p_1 + (z\xi - z_2)]$. Moreover, note that

$$z\xi - z_2 \geqq z_1(\xi - \frac{1}{k' - 1}).$$

Hence,

$$\sum_{h:|h-z_1p_1-z_2p_2|<z\xi} f(h;z_1,z_2,p_1,p_2)$$

$$\geqq \sum_{h:|h-z_1p_1|<z\xi-z_2} f(h;z_1,p_1)$$

$$\geqq \sum_{h:|h-z_1p_1|<z_1(\xi-\frac{1}{k'-1})} f(h;z_1,p_1)$$

$$\geqq 1 - \frac{p_1(1 - p_1)}{k(\xi - \frac{1}{k' - 1})^2}. \qquad (16)$$

Next, suppose that $z_1 < k$ and $z_2 > k$. Similarly, one gets

$$\sum_{h:|h-z_1p_1-z_2p_2|<z\xi} f(h;z_1,z_2,p_1,p_2)$$

$$\geqq 1 - \frac{p_2(1 - p_2)}{k(\xi - \frac{1}{k' - 1})^2}. \qquad (17)$$

Finally, suppose that $z_1 \geqq k$ and $z_2 \geqq k$. Then

$$\sum_{h:|h-z_1p_1-z_2p_2|<z\xi} f(h;z_1,z_2,p_1,p_2)$$

$$\geqq \sum_{h:|h-z_1p_1|<z_1\xi} f(h;z_1,p_1) \cdot \sum_{h:|h-z_2p_2|<z_2\xi} f(h;z_2,p_2)$$

$$\geqq (1 - \frac{p_1(1 - p_1)}{k\xi^2})(1 - \frac{p_2(1 - p_2)}{k\xi^2}). \qquad (18)$$

By choosing k sufficiently large, all of the right hand sides of inequalities (16), (17) and (18) are more than $1 - \eta$. **Q.E.D.**

LEMMA A-3: *Fix* $p_1 \varepsilon [0,1]$, $p_2 \varepsilon (p_1,1]$, $\eta > 0$, $\xi > 0$, *and* $d > 0$ *arbitrarily, where* η *and* ξ *are small enough to satisfy* $\eta + 2d\xi < \frac{1}{2}$. *There exists a positive integer* $\hat{z}$ *which satisfies the following properties: For every pair of nonnegative integers* (z_1,z_2) *with* $z_1 + z_2 = z \geqq \hat{z}$, *there exist positive integers* $\underline{y} = \underline{y}(z_1,z_2)$ *and* $\bar{y} = \bar{y}(z_1,z_2)$ *such that*

$$\underline{y} < \bar{y},\ \bar{y} \geqq \bar{y}(z,0),$$

$$|z_1p_1 + z_2p_2 - \underline{y}| \leqq z\xi,\ |z_1p_1 + z_2p_2 - \bar{y}| < z\xi,$$

$$f(\underline{y};z_1,z_2,p_1,p_2) \geqq \frac{d}{z},\ f(\bar{y};z_1,z_2,p_1,p_2) \geqq \frac{d}{z},$$

$$F(\underline{y};z_1,z_2,p_1,p_2) \leqq \eta + 2d\xi, \text{ and}$$

$$F(\bar{y};z_1,z_2,p_1,p_2) \geqq 1 - \eta - 2d\xi.$$

PROOF: Choose $\hat{z}$ large enough to satisfy the properties of Lemma A-2. For every (z_1,z_2) with $z_1 + z_2 = z \geqq \hat{z}$, choose as $\underline{y} = \underline{y}(z_1,z_2)$ the minimal integer k satisfying

$$|z_1p_1 + z_2p_2 - k| \leqq z\xi \text{ and } f(k;z_1,z_2,p_1,p_2) \geqq \frac{d}{z},$$

and choose as $\bar{y} = \bar{y}(z_1,z_2)$ the maximal integer k satisfying

$$|z_1p_1 - z_2p_2 - k| < z\xi \text{ and } f(k;z_1,z_2,p_1,p_2) \geqq \frac{d}{z}.$$

Obviously, $\underline{y} < \bar{y}$. From Lemmata A-1 and A-2, one gets

$$F(\underline{y};z_1,z_2,p_1,p_2)$$

$$\leqq 1 - \sum_{h:|h-z_1p_1-z_2p_2|<z\xi} f(h;z_1,z_2,p_1,p_2)$$

$$+ \sum_{h:z_1p_1+z_2p_2-z\xi<h\leqq\underline{y}} f(h;z_1,z_2,p_1,p_2)$$

$$\leqq \eta + (\underline{y} - z_1p_1 - z_2p_2 + z\xi)f(\underline{y};z_1,z_2,p_1,p_2)$$

$$\leqq \eta + 2d\xi, \text{ and}$$

$$F(\bar{y};z_1,z_2,p_1,p_2)$$

$$\geqq \sum_{h:|h-z_1p_1-z_2p_2|<z\xi} f(h;z_1,z_2,p_1,p_2)$$

$$- \sum_{h:\bar{y}<h<z_1p_1+z_2p_2+z\xi} f(h;z_1,z_2,p_1,p_2)$$

$$\geqq 1 - \eta - (z_1p_1 + z_2p_2 + z\xi - \bar{y})f(\bar{y};z_1,z_2,p_1,p_2)$$

$$\geqq 1 - \eta - 2d\xi.$$

Finally, we will show that $\bar{y} \geqq \bar{y}(z,0)$. Define

$$\tilde{y} = \min_{(z_1',z_2'):z_1+z_2=z} [\bar{y}(z_1',z_2')].$$

Obviously, $\bar{y} \geqq \tilde{y}$. We will replace $\bar{y}(z,0)$ by $\tilde{y}$. All the properties in Lemma A-3 still hold: Notice

$$F(\bar{y};z,0,p_1,p_2) \geqq F(\bar{y};z_1,z_2,p_1,p_2) \geqq 1 - \eta - 2d\xi$$

$$> \eta + 2d\xi \geqq F(\underline{y}(z,0);z,0,p_1,p_2),$$

which implies that $\underline{y}(z,0) < \tilde{y}$, $F(\tilde{y};z,0,p_1,p_2) \geqq 1 - \eta - 2d\xi$, and

$$|z_1p_1 + z_2p_2 - \tilde{y}| < z\xi.$$

Moreover, from Lemma A-1, one gets that $f(\tilde{y};z,0,p_1,p_2) \geqq \frac{d}{z}$. **Q.E.D.**

Choose η and ξ to satisfy $\alpha \geqq \eta + 2d\xi$. Choose $\hat{z}$ according to Lemma A-3. Fix any integer $z \geqq \hat{z}$ arbitrarily. Let $y^* = \tilde{y}$, where $\tilde{y}$ is the integer defined in the proof of Lemma A-3. By definition, it is clear that the first part of the claim holds. From Lemma A-3, if $\underline{y}(z_1,z_2) \geqq y^*$, then

$$F(y^*;z_1,z_2,p_1,p_2) \leqq F(\underline{y};z_1,z_2,p_1,p_2) \leqq \eta + 2d\xi \leqq \alpha.$$

From Lemma A-1 and the fact that $y^* \leqq \bar{y}$, if $\underline{y} < y^*$, then

$$f(y^*;z_1,z_2,p_1,p_2) \geqq \frac{d}{z}.$$

Hence, the latter part of the claim also holds, and therefore, we have proven the claim.