LEXICOGRAPHICALLY OPTIMAL BASE OF A POLYMATROID WITH RESPECT TO A WEIGHT VECTOR *

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Abstract

Let (E,ρ) be a polymatroid with a ground set E and a rank function ρ. A base $\underset{\sim}{x} = (x(e))_{e\varepsilon E}$ of polymatroid (E,ρ) is called a lexicographically optimal base of (E,ρ) with respect to a weight vector $\underset{\sim}{w} = (w(e))_{e\varepsilon E}$ if the $|E|$-tuple of the numbers $x(e)/w(e)$ $(e \varepsilon E)$ arranged in order of increasing magnitude is lexicographically maximum among all $|E|$-tuples of numbers $y(e)/w(e)$ $(e \varepsilon E)$ arranged in the same manner for all bases $y = (y(e))_{e\varepsilon E}$ of (E,ρ). We give theorems which characterize the relationship between weight vectors and lexicographically optimal bases and point out that a lexicographically optimal base minimizes among all bases a quadratic objective function defined in terms of the associated weight vector. Also, we present an algorithm for finding the (unique) lexicographically optimal base with respect to a given weight vector. Furthermore, we consider the problem of determining the set of weight vectors with respect to which a given base is lexicographically optimal and provide an algorithm for solving it, which is useful for the sensitivity analysis of the optimal base with regard to the variation of the weight vector. The algorithms proposed in the present paper efficiently solve the problem, treated by N. Megiddo, of finding a lexicographically optimal flow in a network with multiple sources and sinks, which is a special case of the problem considered here.

1. Introduction

N. Megiddo [8] considered the problem of finding a lexicographically optimal flow in a network with multiple sources and sinks described as follows. Let $\underset{\sim}{N}$ be a network with a set S^+ of sources, a set S^- of sinks, a vertex set $V \supseteq S^+ \cup S^-$, an arc set A^* and a nonnegative capacity vector $\underset{\sim}{c}^* = (c^*(a))_{a \in A^*}$. For simplicity, suppose $S^+ \cap S^- = \emptyset$. A vector $\underset{\sim}{f} = (f(a))_{a \in A^*}$ is a flow in $\underset{\sim}{N}$ if

$$0 \leqq f(a) \leqq c^*(a) \qquad (a \in A^*), \tag{1.1}$$

$$\sum_{a \in \delta^+ v} f(a) - \sum_{a \in \delta^- v} f(a) = 0 \qquad (v \in V - S^+ \cup S^-) \tag{1.2}$$

$$s^+_{\underset{\sim}{f}}(v) \equiv \sum_{a \in \delta^+ v} f(a) - \sum_{a \in \delta^- v} f(a) \geqq 0 \qquad (v \in S^+), \tag{1.3}$$

$$s^-_{\underset{\sim}{f}}(v) \equiv \sum_{a \in \delta^- v} f(a) - \sum_{a \in \delta^+ v} f(a) \geqq 0 \qquad (v \in S^-), \tag{1.4}$$

where $\delta^+ v$ (resp. $\delta^- v$) is the set of all arcs having v as their initial (resp. terminal) vertex. Vector $\underset{\sim}{s}^+_{\underset{\sim}{f}} = (s^+_{\underset{\sim}{f}}(v))_{v \in S^+}$ and vector $\underset{\sim}{s}^-_{\underset{\sim}{f}} = (s^-_{\underset{\sim}{f}}(v))_{v \in S^-}$ are, respectively, called a supply vector and a demand vector. Let T be a function from the set of all real vectors of finite dimension to the set of all real sequences of finite length such that for any real vector $\underset{\sim}{x} = (x(e))_{e \in E}$ (E: an arbitrary finite set) $T(\underset{\sim}{x})$ is the sequence (or the $|E|$-tuple) of the numbers $x(e)$ $(e \in E)$ arranged in order of increasing magnitude. A flow $\underset{\sim}{f}^*$ in $\underset{\sim}{N}$ is called source-optimal (resp. sink-optimal) if $T(\underset{\sim}{s}^+_{\underset{\sim}{f}^*})$ (resp. $T(\underset{\sim}{s}^-_{\underset{\sim}{f}^*})$) is lexicographically maximum among all $T(\underset{\sim}{s}^+_{\underset{\sim}{f}})$ (resp. $T(\underset{\sim}{s}^-_{\underset{\sim}{f}})$) for all flows $\underset{\sim}{f}$ in N. A lexicographically optimal flow is a flow which is both sink-optimal and source-optimal.

The set of all supply vectors $\mathbf{s}_{\mathbf{f}}^{+}$ (resp. demand vectors $\mathbf{s}_{\mathbf{f}}^{-}$) coincides with the set of all independent vectors of a polymatroid defined on S^+ (resp. S^-), which is in fact shown in [8] (though not in terms of polymatroids). Since a lexicographically optimal flow is a maximum flow from S^+ to S^-, we can consider source-optimality and sink-optimality separately and source-optimality (resp. sink-optimality) is thus related only to the structure of the polymatroid defined on S^+ (resp. S^-). Therefore, it is natural to examine the optimal-flow problem of N. Megiddo from the point of view of polymatroids, which provides a deep insight into the problem, and some results obtained in polymatroid theory can be employed.

The present paper is concerned with the problem of finding a lexicographically optimal base of a polymatroid with respect to a weight vector, which is a generalization of the problem treated by N. Megiddo [8]. We show theorems which characterize a lexicographically optimal base with respect to a weight vector and point out that a lexicographically optimal base minimizes among all bases a quadratic objective function defined in terms of the associated weight vector. We present an algorithm for finding a lexicographically optimal base. The algorithm solves the optimal-flow problem of N. Megiddo more efficiently than that proposed in [9]. Furthermore, we consider the problem of determining the set of weight vectors with respect to which a given base is lexicographically optimal and provide an algorithm for solving it, which is useful for seeing whether or not an obtained optimal base remains optimal when the weight vector is altered.

2. Definitions and Preliminaries

Let E be a finite set and ρ be a function, from the set 2^E of all subsets of E into the set R_+ of all nonnegative real numbers, satisfying

$$\text{(i)} \quad \rho(\emptyset) = 0, \tag{2.1}$$

$$\text{(ii)} \quad \rho(A) \leqq \rho(B) \qquad (A \subseteq B \subseteq E), \tag{2.2}$$

$$\text{(iii)} \quad \rho(A) + \rho(B) \geqq \rho(A \cup B) + \rho(A \cap B) \qquad (A,\ B \subseteq E). \tag{2.3}$$

That is to say, the ρ is a monotone nondecreasing submodular function with $\rho(\emptyset) = 0$. The pair (E,ρ) is called a <u>polymatroid</u>, where E is called the <u>ground set</u> and ρ the <u>rank function</u> of the polymatroid.

We denote by R_+^E the set of all vectors $\underset{\sim}{x} = (x(e))_{e \in E}$ with nonnegative components. For any $\underset{\sim}{x} \in R_+^E$ we let

$$x(A) \equiv \sum_{e \in A} x(e) \qquad (A \subseteq E). \tag{2.4}$$

A vector $\underset{\sim}{x} \in R_+^E$ is called an <u>independent vector</u> of polymatroid (E,ρ) if

$$x(A) \leqq \rho(A) \qquad (A \subseteq E). \tag{2.5}$$

The following lemma shows a fundamental property of submodular functions.

Lemma 2.1 : Let $\underset{\sim}{x}$ be an independent vector of a polymatroid (E,ρ). For any $A, B \subseteq E$, if

$$x(A) = \rho(A), \qquad x(B) = \rho(B), \tag{2.6}$$

then

$$x(A \cup B) = \rho(A \cup B), \qquad x(A \cap B) = \rho(A \cap B). \tag{2.7}$$

(Proof) Easy. Q.E.D.

Let an order relation $\preceq$ be defined by

$$\underset{\sim}{x} \preceq \underset{\sim}{y} \iff x(e) \leqq y(e) \quad (e \in E) \tag{2.8}$$

for any $\underset{\sim}{x}$ and $\underset{\sim}{y}$ in R_+^E. A base of (E,ρ) is an independent vector of (E,ρ) which is maximal with respect to order relation $\preceq$.

For a vector $\underset{\sim}{v} \in R_+^E$, a reduction of polymatroid (E,ρ) with respect to $\underset{\sim}{v}$ is a polymatroid $(E,\rho_{\underset{\sim}{v}})$, where

$$\rho_{\underset{\sim}{v}}(A) = \min_{D \subseteq A}\{\rho(D) + v(A-D)\} \quad (A \subseteq E). \tag{2.9}$$

Proposition 2.1 : The set $P_{\rho_{\underset{\sim}{v}}}$ of all independent vectors of $(E,\rho_{\underset{\sim}{v}})$ is given by

$$P_{\rho_{\underset{\sim}{v}}} = \{\underset{\sim}{x} \mid \underset{\sim}{0} \preceq \underset{\sim}{x} \preceq \underset{\sim}{v};\ \underset{\sim}{x} \text{ is an independent vector of } (E,\rho)\}. \tag{2.10}$$

A base of $(E,\rho_{\underset{\sim}{v}})$ is called a base of $\underset{\sim}{v}$ with respect to polymatroid (E,ρ). For an independent vector $\underset{\sim}{x}$ of (E,ρ), a contraction of (E,ρ) with respect to $\underset{\sim}{x}$ is a polymatroid $(E,\rho^{\underset{\sim}{x}})$, where

$$\rho^{\underset{\sim}{x}}(A) = \min_{B \supseteq A}\{\rho(B) - x(B)\} \quad (A \subseteq E). \tag{2.11}$$

Proposition 2.2 : The set $P_{\rho^{\underset{\sim}{x}}}$ of all independent vectors of $(E,\rho^{\underset{\sim}{x}})$

is given by

$$P_{\rho \underset{\sim}{x}} = \{\underset{\sim}{y} \mid \underset{\sim}{y} \in R_+^E;\ \underset{\sim}{y} + \underset{\sim}{x} \text{ is an independent vector of } (E,\rho)\}. \tag{2.12}$$

In particular, a reduction of $\underset{\sim}{P} = (E,\rho)$ with respect to a vector $\underset{\sim}{v}$ such that for some $S \subseteq E$

$$v(e) = 0 \qquad (e \in E - S),$$

$$v(e) \geqq \rho(\{e\}) \qquad (e \in S)$$

is denoted by $\underset{\sim}{P} \cdot S = (E,\rho_S)$, where

$$\rho_S(A) = \rho(A \cap S) \qquad (A \subseteq E). \tag{2.13}$$

The reduction $\underset{\sim}{P} \cdot S$ is called a reduction of $\underset{\sim}{P}$ onto S. Moreover, a contraction of $\underset{\sim}{P} = (E,\rho)$ with respect to a base $\underset{\sim}{x}$ of $\underset{\sim}{P} \cdot (E-S)$ for some $S \subseteq E$ is denoted by $\underset{\sim}{P} \times S = (E,\rho^S)$, where

$$\rho^S(A) = \rho(A \cup (E-S)) - \rho(E-S) \qquad (A \subseteq E). \tag{2.14}$$

The contraction $\underset{\sim}{P} \times S$ is called a contraction of $\underset{\sim}{P}$ onto S. Note that $\underset{\sim}{P} \cdot S$ and $\underset{\sim}{P} \times S$ are determined by $\underset{\sim}{P}$ and S alone. A minor of $\underset{\sim}{P} = (E,\rho)$ is a polymatroid $(\underset{\sim}{P} \cdot S) \times S^*$ expressed in terms of subsets S and S^* of E such that $S^* \subseteq S$.

Let P_ρ be the set of all independent vectors of (E,ρ). The saturation function sat from P_ρ into 2^E is defined as follows. For any $u \in E$ let $\underset{\sim}{\chi}_u \in R_+^E$ be defined by

$$\chi_u(u) = 1, \qquad \chi_u(e) = 0 \quad (e \neq u).$$

Then, for an independent vector $\underset{\sim}{x}$, $\text{sat}(\underset{\sim}{x})$ is the set of all elements $u \in E$ such that for any $d > 0$ the vector $\underset{\sim}{y} \in R_+^E$ defined by

$$\underset{\sim}{y} = \underset{\sim}{x} + d\underset{\sim}{\chi}_u \tag{2.15}$$

is not an independent vector. In other words, $\mathrm{sat}(\underset{\sim}{x})$ is the set of those $u \in E$ for which $x(u)$ cannot be increased with $\underset{\sim}{x}$ remaining independent. The set $\mathrm{sat}(\underset{\sim}{x})$ is called the <u>saturated set with respect to the independent vector</u> $\underset{\sim}{x}$.

<u>Lemma 2.2</u> : Let $\underset{\sim}{x}$ be an independent vector of a polymatroid (E,ρ). Then the set $A_1 = \mathrm{sat}(\underset{\sim}{x})$ satisfies

$$x(A_1) = \rho(A_1). \tag{2.16}$$

Moreover, let $\mathcal{A}_1$ be defined by

$$\mathcal{A}_1 = \{A \mid A \subseteq E, x(A) = \rho(A)\}. \tag{2.17}$$

Then $\mathcal{A}_1$ is a distributive lattice with an order relation defined by set inclusion and $\mathrm{sat}(\underset{\sim}{x})$ is the maximum element of $\mathcal{A}_1$.

(Proof) Because of Lemma 2.1, $\mathcal{A}_1$ defined by (2.17) is a distributive lattice. Also, it follows from the definition of the saturation function that $e \in \mathrm{sat}(\underset{\sim}{x})$ if and only if $x(A) = \rho(A)$ for some $A \subseteq E$ such that $e \in A$. Therefore, $\mathrm{sat}(\underset{\sim}{x})$ is given by

$$\mathrm{sat}(x) = \bigcup\{A \mid A \subseteq E, x(A) = \rho(A)\} \tag{2.18}$$

which is the maximum element of $\mathcal{A}_1$. Q.E.D.

Furthermore, we define the <u>dependence function</u> dep from $P_\rho \times E$ into 2^E as follows. For an independent vector $\underset{\sim}{x}$, if $u \in \mathrm{sat}(\underset{\sim}{x})$, $\mathrm{dep}(\underset{\sim}{x},u)$ is the set of all elements $v \in E$ such that for some $d > 0$ the vector $\underset{\sim}{y} \in R_+^E$ defined by

$$\underset{\sim}{y} = \underset{\sim}{x} + d\underset{\sim}{\chi}_u - d\underset{\sim}{\chi}_v \tag{2.19}$$

is an independent vector. If $u \varepsilon E - \mathrm{sat}(\underset{\sim}{x})$, define

$$\mathrm{dep}(\underset{\sim}{x},u) = \emptyset. \tag{2.20}$$

Note that for any $u \varepsilon E$, if $\mathrm{dep}(\underset{\sim}{x},u) \neq \emptyset$,

$$u \varepsilon \mathrm{dep}(\underset{\sim}{x},u). \tag{2.21}$$

<u>Lemma 2.3</u> : Let $\underset{\sim}{x}$ be an independent vector of a polymatroid (E,ρ) such that $\mathrm{sat}(\underset{\sim}{x}) \neq \emptyset$. Then, for each $u \varepsilon \mathrm{sat}(\underset{\sim}{x})$, the set $A_2 = \mathrm{dep}(\underset{\sim}{x},u)$ satisfies

$$x(A_2) = \rho(A_2). \tag{2.22}$$

Moreover, let $\mathcal{A}_2$ be defined by

$$\mathcal{A}_2 = \{A \mid A \subseteq E, u \varepsilon A, x(A) = \rho(A)\}. \tag{2.23}$$

Then $\mathcal{A}_2$ is a distributive lattice with an order relation defined by set inclusion and $\mathrm{dep}(\underset{\sim}{x},u)$ is the minimum element of $\mathcal{A}_2$.

(Proof) Because of Lemma 2.1, $\mathcal{A}_2$ defined by (2.23) is a distributive lattice. Furthermore, we see from the definition of the dependence function that $e \varepsilon \mathrm{dep}(\underset{\sim}{x},u)$ if and only if $x(A) < \rho(A)$ for any $A \subseteq E$ such that $u \varepsilon A$ and $e \notin A$. Therefore, $\mathrm{dep}(\underset{\sim}{x},u)$ is given by

$$\mathrm{dep}(\underset{\sim}{x},u) = \cap\{A \mid A \subseteq E, u \varepsilon A, x(A) = \rho(A)\} \tag{2.24}$$

which is the minimum element of $\mathcal{A}_2$. Q.E.D.

Lemmas 2.2 and 2.3 characterize the saturation function and the dependence function in terms of the rank function ρ. The saturation function and the dependence function were first introduced in [3].

<u>Lemma 2.4</u> : Let $\underset{\sim}{x}$ be an independent vector of (E,ρ). Then, for

any subset S of E, the following two conditions are equivalent:

(i) $\rho(S) = x(S)$;

(ii) for any $e \in S$, $\emptyset \neq \mathrm{dep}(\underset{\sim}{x},e) \subseteq S$.

(Proof) Suppose (i) holds and $e \in S$. Then (ii) follows from Lemma 2.3. Conversely, if (ii) holds, then by (2.21) we have

$$\bigcup\{\mathrm{dep}(\underset{\sim}{x},e) \mid e \in S\} = S.$$

Therefore, (i) follows from Lemmas 2.1 and 2.3. Q.E.D.

For any real sequences $a = (a_1,a_2,\ldots,a_k)$ and $b = (b_1,b_2,\ldots,b_k)$ of the same length k, a is called <u>lexicographically greater than or equal to</u> b if for some $j \in \{1,2,\ldots,k\}$

$$a_i = b_i \quad (i = 1,\ 2,\ \ldots,\ j-1),$$

$$a_j > b_j$$

or

$$a_i = b_i \quad (i = 1,\ 2,\ \ldots,\ k).$$

A vector $\underset{\sim}{w} \in R_+^E$ such that $w(e) > 0\ (e \in E)$ is called a <u>weight vector</u>. For a vector $\underset{\sim}{x} \in R_+^E$, let us denote by $T(\underset{\sim}{x})$ the $|E|$-tuple (or sequence) of the numbers $x(e)\ (e \in E)$ arranged in order of increasing magnitude. Given a weight vector $\underset{\sim}{w}$, a base $\underset{\sim}{x}$ of (E,ρ) is called a <u>lexicographically optimal base with respect to the weight vector</u> $\underset{\sim}{w}$ if the $|E|$-tuple $T((x(e)/w(e))_{e\in E})$ is lexicographically maximum among all $|E|$-tuples $T((y(e)/w(e))_{e\in E})$ for all bases $\underset{\sim}{y}$ of (E,ρ).

For polymatroids, also see [2], [3], [4], [7] and [11].

3. Lexicographically Optimal Base

In this section, we consider the problem of finding a lexicographically optimal base of a polymatroid (E,ρ) with respect to a weight vector.

Let $\underset{\sim}{w}$ be a weight vector. For each $a \in R_+$, let $\underset{\sim}{v}_a \in R_+^E$ be defined by

$$\underset{\sim}{v}_a = (aw(e))_{e \in E} . \tag{3.1}$$

Also, let $\underset{\sim}{u}_a$ $(a \in R_+)$ be independent vectors of (E,ρ) satisfying

(i) for each $a \in R_+$, $\underset{\sim}{u}_a$ is a base of $\underset{\sim}{v}_a$, (3.2)

(ii) $\underset{\sim}{u}_a \preceq \underset{\sim}{u}_b$ $(0 \leqq a \leqq b)$. (3.3)

Such independent vectors $\underset{\sim}{u}_a$ $(a \in R_+)$ are uniquely determined by the weight vector $\underset{\sim}{w}$. Since the set of all independent vectors is bounded, there exist $c(e)$ $(e \in E)$ such that for each $e \in E$

$$u_a(e) = aw(e) \ (= v_a(e)) \qquad (0 \leqq a \leqq c(e)), \tag{3.4}$$

$$u_a(e) = c(e)w(e) \qquad (c(e) \leqq a). \tag{3.5}$$

Now, we show the following.

Theorem 3.1 : Suppose that $c(e)$ $(e \in E)$ are those defined by (3.1) $\sim$ (3.5). The vector $\underset{\sim}{x}^*$ defined by

$$\underset{\sim}{x}^* = (c(e)w(e))_{e \in E} \tag{3.6}$$

is the unique lexicographically optimal base of (E,ρ) with respect to a weight vector $\underset{\sim}{w}$.

(Proof) It is clear from the definition of $\underset{\sim}{x}^*$ that $\underset{\sim}{x}^*$ is a base of (E,ρ).

Let the distinct numbers of $c(e)$ $(e \in E)$ be given by c_i $(i=1, 2, \ldots, p)$ such that

$$c_1 < c_2 < \cdots < c_p. \qquad (3.7)$$

For each $i = 1, 2, \ldots, p$, define the set $S_i \subseteq E$ by

$$S_i = \{e \mid e \in E,\ c(e) \leqq c_i\}. \qquad (3.8)$$

Then we have

$$\emptyset \equiv S_0 \subsetneqq S_1 \subsetneqq \cdots \subsetneqq S_p = E \qquad (3.9)$$

and by the definition of $c(e)$ $(e \in E)$, Lemma 2.2 and (3.6)

$$\rho(S_i) = \sum_{e \in S_i} c(e)w(e) = x^*(S_i) \quad (i=0,1,\ldots,p). \qquad (3.10)$$

Let $\underset{\sim}{\overline{x}}$ be an arbitrary base such that $T((\overline{x}(e)/w(e))_{e \in E})$ is lexicographically greater than or equal to $T((x^*(e)/w(e))_{e \in E}) = T(c)$. Define a vector $\underset{\sim}{\overline{c}}$ by

$$\underset{\sim}{\overline{c}} = (\overline{x}(e)/w(e))_{e \in E}\ . \qquad (3.11)$$

We shall show that, if

$$\overline{x}(e) = x^*(e) \qquad (e \in S_i) \qquad (3.12)$$

for some $i = i_0 < p$, then (3.12) is also valid for $i = i_0 + 1$. Since (3.12) is trivially true for $i = 0$, we shall then get by induction

$$\overline{x}(e) = x^*(e) \qquad (e \in S_p = E),$$

which shows the optimality of the base $\underset{\sim}{x}^*$ and the uniqueness of the optimal base.

Now, suppose that (3.12) holds for some $i = i_0 < p$. Since $T(\bar{\underset{\sim}{c}})$ is lexicographically greater than or equal to $T(\underset{\sim}{c})$, we have from (3.7) and (3.8)

$$\bar{c}(e) \geqq c(e) = c_{i_0+1} \qquad (e \in S_{i_0+1} - S_{i_0}). \tag{3.13}$$

Consequently,

$$\bar{x}(e) \geqq x^*(e) \qquad (e \in S_{i_0+1} - S_{i_0}). \tag{3.14}$$

By (3.10),

$$x^*(S_{i_0+1}) = \rho(S_{i_0+1}). \tag{3.15}$$

It follows from (3.12), (3.14) and (3.15) that

$$\bar{x}(e) = x^*(e) \qquad (e \in S_{i_0+1}),$$

since $\bar{\underset{\sim}{x}}$ is an independent vector. This completes the proof of the theorem. Q.E.D.

A careful examination of the above proof of Theorem 3.1 yields the following.

Theorem 3.2 : Let $\underset{\sim}{x}$ be a base of (E,ρ) and $\underset{\sim}{w}$ be a weight vector. Define

$$c(e) = x(e)/w(e) \qquad (e \in E)$$

and let the distinct numbers of $c(e)$ $(e \in E)$ be given by $c_1 < c_2 < \cdots < c_p$. Furthermore, define $S_i \subseteq E$ $(i=1,2,\ldots,p)$ by

$$S_i = \{e \mid e \in E,\ c(e) \leqq c_i\} \qquad (i=1,2,\ldots,p).$$

Then the following three conditions are equivalent:

(i) $\underset{\sim}{x}$ is the lexicographically optimal base of (E,ρ) with respect to the weight vector $\underset{\sim}{w}$;

(ii) $x(S_i) = \rho(S_i) \quad (i=1,2,\ldots,p)$;

(iii) $\emptyset \neq \mathrm{dep}(\underset{\sim}{x},e) \subseteq S_i \quad (e \in S_i;\ i=1,2,\ldots,p)$.

(Proof) The proof of the equivalence of (i) and (ii) almost parallels that of Theorem 3.1. The equivalence of (ii) and (iii) follows from Lemma 2.4. Q.E.D.

Based on Theorems 3.1 and 3.2, we can propose an algorithm for finding the lexicographically optimal base of $\underset{\sim}{P} = (E,\rho)$ with respect to a weight vector $\underset{\sim}{w}$.

Algorithm for finding the lexicographically optimal base

1° Let L be a list such that its elements are the sets $\emptyset$ and E, $\emptyset$ is the head of L and E is the tail of L and next to $\emptyset$. Set

$$S^* := \emptyset,$$

$$\underset{\sim}{x}^* := \underset{\sim}{0} \quad \text{(a zero vector in } R_+^E).$$

2° Let S be the element next to S^* in list L. Put

$$d := \{\rho(S) - \rho(S^*)\}/w(S-S^*)$$

and let $\underset{\sim}{v}$ be a vector in R_+^E given by

$$v(e) := dw(e) \quad (e \in S)$$
$$:= 0 \quad (e \in E-S).$$

3° Find a base $\underset{\sim}{y}$ of $\underset{\sim}{v}$ such that $\underset{\sim}{y} - \underset{\sim}{x}^* \in R_+^E$. If $\mathrm{sat}(\underset{\sim}{y}) = S$, then put

$$\underset{\sim}{x}^* := \underset{\sim}{y},$$
$$S^* := S;$$

or else (then $S^* \subsetneq \mathrm{sat}(\underset{\sim}{y}) \subsetneq S$), insert $\mathrm{sat}(\underset{\sim}{y})$ into list L such that $\mathrm{sat}(\underset{\sim}{y})$ is the element next to S^* and that S is next to $\mathrm{sat}(\underset{\sim}{y})$. If $S^* = E$, then the algorithm terminates and $\underset{\sim}{x}^*$ is the lexicographically optimal base; or else, go back to Step 2°.

<u>Remark 3.1</u> : The vector $\underset{\sim}{x}^*$ obtained at the beginning of Step 3° is a base of $\underset{\sim}{P} \cdot S^*$ and, when $\mathrm{sat}(\underset{\sim}{y}) = S$ in Step 3°, $\underset{\sim}{y} - \underset{\sim}{x}^*$ is a base of $(\underset{\sim}{P} \cdot S) \times (S - S^*)$ and

$$\begin{aligned} y(e) - x^*(e) &= dw(e) \qquad (e \in S - S^*) \\ &= 0 \qquad (e \in E - (S - S^*)). \end{aligned}$$

The validity of the above algorithm follows from Theorems 3.1 and 3.2.

<u>Remark 3.2</u> : Since the number of elements in L is at most $|E| + 1$ and since in Step 3°, if the algorithm does not terminates, either the pointer S^* moves to the next in L or the number of elements in L is increased by one, the algorithm terminates after repeating Steps 2° and 3° at most $2|E| - 1$ times and gives the lexicographically optimal base. The present algorithm solves the optimal-flow problem of N. Megiddo [8] with an $O((|S^+| + |S^-|)|V|^3)$ running time by the use of the Dinic-Karzanov maximum-flow algorithm, which is more

efficient than the $O((|S^+| + |S^-|)|A||V|^2)$ algorithm given in [9].

The following theorem shows the relationship between the lexicographically optimal base and the base which minimizes a quadratic objective function among all bases.

Theorem 3.3 : Let $\underset{\sim}{x}^*$ be the lexicographically optimal base of (E,ρ) with respect to a weight vector $\underset{\sim}{w}$ and $\hat{\underset{\sim}{x}}$ be the (unique) optimal solution of the problem $\hat{P}$: Minimize the quadratic function given by

$$f_{\underset{\sim}{w}}(\underset{\sim}{x}) = \frac{1}{2} \sum_{e \in E} x(e)^2/w(e)$$

subject to the constraint that $\underset{\sim}{x}$ is a base of (E,ρ). Then we have $\underset{\sim}{x}^* = \hat{\underset{\sim}{x}}$.

(proof) Define

$$\hat{c}(e) = \hat{x}(e)/w(e) \quad (e \in E)$$

and let the distinct numbers of $\hat{c}(e)$ $(e \in E)$ be given by $c_1 < c_2 < \cdots < c_p$. Then we have

$$\frac{\partial}{\partial x(e)} f_{\underset{\sim}{w}}(\underset{\sim}{x}) \Big|_{\underset{\sim}{x}=\hat{\underset{\sim}{x}}} = \hat{x}(e)/w(e) = \hat{c}(e) \quad (e \in E). \tag{3.16}$$

Therefore, for any e and $e' \in E$, if $\hat{c}(e) < \hat{c}(e')$, we have

$$e' \notin \mathrm{dep}(\hat{x},e), \tag{3.17}$$

since otherwise, because of (3.16), there would exist a base which yields a smaller value of $f_{\underset{\sim}{w}}$ than $\hat{\underset{\sim}{x}}$. Consequently, for each $i = 1, 2, \ldots, p$,

$$\emptyset \neq \mathrm{dep}(\hat{\underset{\sim}{x}},e) \subseteq S_i \equiv \{\hat{e} \mid \hat{e} \in E,\ \hat{c}(\hat{e}) \leqq c_i\} \tag{3.18}$$

for all $e \in S_i$. It follows from (3.18) and Theorem 3.2 that $\hat{\underset{\sim}{x}}$ coincides with the unique lexicographically optimal base $\underset{\sim}{x}^*$ of (E,ρ) with respect to $\underset{\sim}{w}$. This completes the proof of the theorem.

Q.E.D.

Remark 3.3 : For some nonempty subset $S \subsetneq E$, let $\underset{\sim}{x}_1^*$ (resp. $\underset{\sim}{x}_2^*$) be the lexicographically optimal base of the reduction (E,ρ_S) (resp. the contraction (E,ρ^S)) onto S with respect to the weight vector $\underset{\sim}{w}$. Moreover, let $\underset{\sim}{y}_1$ (resp. $\underset{\sim}{y}_2$) be an arbitrary base of the contraction (E,ρ^{E-S}) (resp. the reduction (E,ρ_{E-S})). Then $\underset{\sim}{x}_1^* + \underset{\sim}{y}_1$ and $\underset{\sim}{x}_2^* + \underset{\sim}{y}_2$ are, respectively, optimal solutions of the following problems:

(P_1) to find a base $\underset{\sim}{x}$ of (E,ρ) which lexicographically maximizes $T(\underset{\sim}{c})$ among all bases, where $\underset{\sim}{c}$ is defined by

$$c(e) = x(e)/w(e) \quad (e \in S), \qquad c(e) = 0 \quad (e \in E - S);$$

(P_2) to find a base $\underset{\sim}{x}$ of (E,ρ) which minimizes

$$f(\underset{\sim}{x}) = \frac{1}{2} \sum_{e \in S} x(e)^2/w(e)$$

among all bases.

This shows the fact that Theorem 3.3 does not hold in the limit as the components $w(e)$ $(e \in E-S)$ tend to infinity, which is due to the definition of the lexicographical optimality.

4. Characterization of Weight vectors

In this section, we shall consider the problem of determining the set of weight vectors with respect to which a given base is lexicographically optimal. Such a problem arises in the case where it is required to determine whether or not the obtained lexicographically optimal base remains optimal when the weight vector is altered. A method for solving the problem will be given below.

Let x be a base of a polymatroid (E,ρ). Consider a directed graph $G(E,A^*(\underset{\sim}{x}))$ with a vertex set E and an arc set $A^*(\underset{\sim}{x})$ defined by

$$A^*(\underset{\sim}{x}) = \{(u,v) \mid u,v\varepsilon E, u\neq v, u\varepsilon \mathrm{dep}(\underset{\sim}{x},v)\}. \tag{4.1}$$

Note that $\mathrm{dep}(\underset{\sim}{x},v) \neq \emptyset$ for all $v \in E$ since $\mathrm{sat}(\underset{\sim}{x}) = E$. Moreover, let V_i $(i=1,2,\ldots,q)$ be the strongly connected components of $G(E,A^*(\underset{\sim}{x}))$. A partial ordering $\preceq^*$ is naturally defined on the set $\mathcal{V} = \{V_i \mid i=1,2,\ldots,q\}$ by " $V_i \preceq^* V_j$ if and only if there exists a directed path from a vertex in V_i to a vertex in V_j." From Lemmas 2.1 and 2.3, $G(E,A^*(\underset{\sim}{x}))$ is transitively closed. Therefore, for each strongly connected component V_k and each $u, v \in V_k$ $(u \neq v)$, there is a pair of arcs (u,v) and (v,u) and if $V_i \preceq^* V_j$ there are arcs (u,v) for all $u \in V_i$ and $v \in V_j$.

An ordered pair $(\mathcal{V}_1,\mathcal{V}_2)$ such that $\mathcal{V}_i \subseteq \mathcal{V}$ $(i=1,2)$, $\mathcal{V}_1 \cup \mathcal{V}_2 = \mathcal{V}$ and $\mathcal{V}_1 \cap \mathcal{V}_2 = \emptyset$ is called a <u>monotone dissection</u> of the ordered set $(\mathcal{V},\preceq^*)$ if for any $U_1 \in \mathcal{V}_1$ and $U_2 \in \mathcal{V}_2$ there does not hold

$$U_2 \preceq^* U_1.$$

Now, we have the following.

Lemma 4.1 : Let $\underset{\sim}{x}$ be a base of (E,ρ) and $\mathcal{V}$ be the set of all the strongly connected components of $G(E,A^*(\underset{\sim}{x}))$ as defined above. Then

$$x(U) = \rho(U) \tag{4.2}$$

if and only if

$$U = \cup\{V_i \mid V_i \in \mathcal{V}_1\}, \tag{4.3}$$

where $(\mathcal{V}_1, \mathcal{V}-\mathcal{V}_1)$ is a monotone dissection of $\mathcal{V}$.

(Proof) The lemma easily follows from Lemma 2.4. Q.E.D.

Based on Lemma 4.1, we can show the following.

Theorem 4.1 : Let $\underset{\sim}{x}$ be a base of (E,ρ) and let V_i $(i=1,2,\ldots,q)$ be the strongly connected components of the graph $G(E,A^*(\underset{\sim}{x}))$. A weight vector $\underset{\sim}{w}$ is a vector with respect to which $\underset{\sim}{x}$ is the lexicographically optimal base if and only if $\underset{\sim}{x}$ is expressed as

$$x(e) = \hat{c}_i w(e) \quad (e \in V_i;\ i=1,2,\ldots,q) \tag{4.4}$$

with nonnegative real numbers $\hat{c}_i$ $(i=1,2,\ldots,q)$ satisfying

$$\hat{c}_j \leqq \hat{c}_k \tag{4.5}$$

for any $j, k = 1, 2, \ldots, q$ such that

$$V_j \preceq^* V_k. \tag{4.6}$$

(Proof) The "if" part : Let the distinct numbers of $\hat{c}_i$ $(i=1,2,\ldots,q)$ be given by $c_1 < c_2 < \cdots < c_p$ and define S_i $(i=1,2,\ldots,p)$ by

(3.8). Also, define for each $i = 1, 2, \ldots, p$

$$\mathcal{V}_1^{(i)} = \{V_j \mid V_j \subseteq S_i,\ j=1,2,\ldots,q\}.$$

Since, for each $i = 1, 2, \ldots, p$ and $j = 1, 2, \ldots q$, either $V_j \subseteq S_i$ or $V_j \subseteq E - S_i$, we see from (4.5) and (4.6) that for each $i = 1, 2, \ldots, p$ $(\mathcal{V}_1^{(i)}, \mathcal{V}-\mathcal{V}_1^{(i)})$ is a monotone dissection of $\mathcal{V}$. Since

$$S_i = \cup\{V_j \mid V_j \in \mathcal{V}_1^{(i)}\} \quad (i=1,2,\ldots,p),$$

the validity of the "if" part follows from Lemma 4.1 and Theorem 3.2.

<u>The "only if" part</u> : Let $c(e)$ $(e \in E)$, $c_1 < c_2 < \cdots < c_p$ and $\emptyset \equiv S_0 \subsetneqq S_1 \subsetneqq \cdots \subsetneqq S_p = E$ be those defined as in Theorem 3.2 with respect to the lexicographically optimal base $\underset{\sim}{x}$. It follows from Lemma 4.1 and Theorem 3.2 that, for each $j = 1, 2, \ldots, q$, there exists an integer $i \in \{1,2,\ldots,p\}$ such that $V_j \subseteq S_i - S_{i-1}$ and $c(e) = c_i = \hat{c}_j$ for all $e \in V_j$. Also, for any $j, k = 1, 2, \ldots, q$, if $V_j \preceq^* V_k$, then for some $i \in \{1,2,\ldots,p\}$ either $V_j \cup V_k \subseteq S_i - S_{i-1}$ or $V_j \subseteq S_i$ and $V_k \subseteq E - S_i$ due to Lemma 4.1. In either case, $\hat{c}_j \leqq \hat{c}_k$. Q.E.D.

Given a base $\underset{\sim}{x}$ of (E,ρ), an algorithm for determining the set of weight vectors with respect to which $\underset{\sim}{x}$ is lexicographically optimal is furnished as follows.

If there exist distinct elements $u, v \in E$ such that

$$x(u) = 0, \qquad v \in \mathrm{dep}(\underset{\sim}{x},u),$$

then (from Theorem 4.1) $\underset{\sim}{x}$ cannot be lexicographically optimal with respect to any weight vector (since $x(v) > 0$); otherwise, choose any positive real numbers $w(e)$ $(e \in E)$ such that

$$x(e)/w(e) = \hat{c}_i \quad (e \in V_i;\ i=1,2,\ldots,q),$$

where constants $\hat{c}_i$ $(i=1,2,\ldots,q)$ must satisfy

$$\hat{c}_j \leqq \hat{c}_k$$

for any $j, k = 1, 2, \ldots, q$ such that

$$V_j \preceq^* V_k.$$

Then the weight vector $\underset{\sim}{w} = (w(e))_{e \in E}$ is a desired one. Every weight vector with respect to which a given base is lexicographically optimal can be generated by this procedure.

<u>Remark 4.1</u> : For the optimal-flow problem of N. Megiddo [8], given a flow $\underset{\sim}{f}$ in a network with multiple sources and sinks, the graph $G(S^+, A^*(\underset{\sim}{s}^+_{\underset{\sim}{f}}))$ (resp. $G(S^-, A^*(\underset{\sim}{s}^-_{\underset{\sim}{f}}))$) defined by (4.1) with regard to the supply vector $\underset{\sim}{s}^+_{\underset{\sim}{f}}$ (resp. the demand vector $\underset{\sim}{s}^-_{\underset{\sim}{f}}$) can be constructed by finding the set of strongly connected components (and its partially ordered structure) of an auxiliary graph associated with the flow $\underset{\sim}{f}$ which is usually used for finding flow-augmenting paths in a network. Once the graph is constructed, finding a weight vector $\underset{\sim}{w}$ which satisfies (4.4) $\sim$ (4.6) is an easy task.

<u>Remark 4.2</u> : It should be noted that the notion of lexicographically optimal bases is closely related to that of the principal partition of a graph [5], [6] and its generalization to a matroid [1], [10],

which were originally developed for electrical-network analysis. Let the rank function ρ of a polymatroid be equal to that of a matroid (or a graph) and $\underset{\sim}{x}^*$ be the lexicographically optimal base of the polymatroid with respect to a uniform weight vector $\underset{\sim}{w}$ with $w(e) = 1$ $(e \in E)$. Then the set of strongly connected components of the graph $G(E,A^*(\underset{\sim}{x}^*))$ defined by (4.1) corresponds to the principal partition of the matroid (or the graph) and the graph $G(E,A^*(\underset{\sim}{x}^*))$ expresses the ordered structure of the principal partition (cf. [10]).

Remark 4.3 : The principal partition of a set of random variables is discussed in [4], where a polymatroidal structure is induced on the set of random variables by the entropy function. The notion of lexicographically optimal bases may be useful for the Shannon theory of information (cf. [4]).

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