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Ratio Equilibrium in an Economy with an Externality

by

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is considered as a specification of Coase's liability rule.

We consider a community which consists of a damaging firm 1, and suffered households, 2, ..., n. Firm 1's revenue function is $f(q)$ with $f(0) = 0$, where activity level q is measured in terms of private cost of production[1]. $f(q)$ is defined on the set of all nonnegative real numbers E_+. When firm 1 has no liability nor needs to compensate households for anything, 1's profit is $f(q) - q$. Household i's utility function $U^i(q, m)$ is defined on the nonnegative orthant of the 2-dimensional Euclidean space E_+^2. q denotes firm 1's activity level and m household i's consumption level. Household i has the initial endowment of money $I_i > 0$ $(i = 2, \ldots, n)$. Note that we call the consumption good "money".

Since a rise of firm 1's activity level increases his revenue and decreases the households' utility levels, a conflict between firm 1 and households 2, ..., n necessarily occurs. For the resolution of such a conflict, it is necessary to specify a liability rule: Who should compensate whom? Liability rule is a device for conflict resolution. We introduce a concept of allowance level $\bar{q} \geqq 0$ as a specification of liability rule as follows.

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1) Ratio equilibrium is determined independently of the measurement of activity level. See Ito and Keneko [5].

The firm can choose freely an activity level if it is not greater than $\bar{q}$. If the firm desires to increase its activity level beyond $\bar{q}$, it must obtain the households' consent. Without their consent, firm 1 can not operate at a higher activity level than the allowance level $\bar{q}$. Hence the firm may compensate them to obtain their consent. Or if households desire to decrease firm 1's activity level smaller than $\bar{q}$, then they compensate firm 1 so that they obtain firm 1's consent, because firm 1 has the right to operate freely at not greater activity level than the allowance level $\bar{q}$. The introduction of allowance level to the community enables us to treat the problem of compensation for external diseconomies in the same way with that of cost share in a public goods economy. Hence we can define ratio equilibrium in our community. In this paper, we consider the behavior of ratio equilibrium in our community when an allowance level is exogeneously given. But we do not consider a decision problem of allowance level.

Let an allowance level $\bar{q} \geqq 0$ be exogeneously given. We denote, by $E(I, \bar{q}) = E(I_2, \ldots, I_n, \bar{q})$, the community with income levels $I = (I_2, \ldots, I_n)$ and allowance level $\bar{q}$. We call $(r, q^*) = (r_1, \ldots, r_n, q^*)$ a <u>ratio equilibrium</u> in $E(I, \bar{q})$ if

$$(1.1) \qquad \sum_{i=1}^{n} r_i = 1 \quad \text{and} \quad q^* \geqq 0,$$

$$(1.2) \quad f(q^*) - \bar{q} - r_1(q^* - \bar{q}) \geqq f(q) - \bar{q} - r_1(q - \bar{q}) \quad \text{for all } q \geqq 0,$$

$$(1.3) \quad \text{for all } i = 2, \ldots, n, \; I_i - r_i(q^* - \bar{q}) \geqq 0 \text{ and}$$

$$U^i(q^*, I_i - r_i(q^* - \bar{q})) \geqq U^i(q, I_i - r_i(q - \bar{q}))$$

$$\text{for all } q \geqq 0 \text{ satisfying } I_i - r_i(q - \bar{q}) \geqq 0.$$

This definition can be interpreted as follows. Under a given allowance level $\bar{q}$ and a ratio $r = (r_1, r_2, \ldots, r_n)$, firm 1 pays (or receives) total compensation $(r_1 - 1)(q - \bar{q})$ and household i receives (pays) compensation $- r_i(q - \bar{q})$ $(i = 2, \ldots, n)$ when firm 1 operates at activity level q. The total compensation that firm 1 pays (receives) should be equal to the sum of compensations received (paid) by households 2, ..., n. That is,

$$(1.4) \quad (r_1 - 1)(q - \bar{q}) = \sum_{i=2}^{n} - r_i(q - \bar{q}).$$

This equation is satisfied because of (1.1). Condition (1.2) means the profit maximization of firm 1 under the assumption that ratio r_1 is fixed, i.e.,

$$f(q) - q - (r_1 - 1)(q - \bar{q}) = f(q) - \bar{q} - r_1(q - \bar{q}) \rightarrow \max.$$

Condition (1.3) means the utility maximization of the households under the assumption that $r_2, \ldots, r_n$ are fixed. The definition requires that the "demands" of all households and firm 1 for firm's activity level coincide.

We can ensure the existence of a ratio equilibrium in the community under natural assumptions:

Proposition 1. Assume that f(q) is a continuous and concave function with $f(\hat{q}) - \hat{q} = \max_{q \geqq 0}\{f(q) - q\}$ for some $\hat{q} \geqq 0$ and that $U^i(q, m)$ $(i = 2, \ldots, n)$ is a continuous, quasi-concave function of (q, m) and is monotonically increasing with respect to m. Then there exists a ratio equilibrium (r, q*) in $E(I, \bar{q})$ for any $\bar{q} \geqq 0$.

Since the ratio equilibria coincide with the Lindahl equilibria in $E(I, \bar{q})$ when the cost function is linear, which is pointed out in Kaneko [8, Lemma 1], it is sufficient to show the existence of a Lindahl equilibrium[2]. But this can be proved by modifying the proof of Bergstrom's existence theorem of a competitive equilibrium [1]. As we can prove it without much difficulty, we omit the proof.

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2) Davis and Winston [3] considered a price mechanism in an economy with externality introducing the concept of allowance level.

The concept of ratio equilibrium is just a mathematical solution, and so we must provide a mechanism through which a ratio equilibrium can be attained as a result. Kaneko [7] designed a decision process called "voting game G(N, W)" for this purpose and proved that the core of the voting game G(N, W) coincides with the set of all ratio equilibria. This means that a ratio equilibrium can be characterized as a result of the majority negotiation according to the rule of the voting game G(N, W). The theorems of [7] are also true in $E(I, \bar{q})$, the formulation and proof of which need to be slightly modified. But we omit the precise formulation and proof of the following proposition.

Proposition 2. Let $\bar{q} \geqq 0$ be an exogenously given allowance level. Assume that $U^i(q, m)$ $(i = 2, \ldots, n)$ is monotonically decreasing and increasing with respect to q and m respectively, and that f(q) is strictly concave and monotonically increasing with $f(\hat{q}) = \max_{q \geqq 0}\{f(q) - q\}$ for some $\hat{q} > \bar{q}$. If Conditions (B)-(L) of [7] hold[3], then Theorem 2

3) Condition (K) must be rewritten in the context of this paper, as follows:

$$(K): U^i(0, 0) < U^i(\bar{q}, I_i) \quad \text{for all } i = 2, \ldots, n.$$

This means that if the households compensate the firm and decrease its' activity level to the 0-level using their total incomes, then the utility levels are lower than those of the state that they give consent to the firm for any activity level without any compensation.

of [7] holds, i.e., the core of the voting game G(N, W) in $E(I, \bar{q})$ coincides with the set of all ratio equilibria in $E(I, \bar{q})$.

In the following, we do not discuss the voting game further but focus our investigation on the relation between ratio equilibria and allowance levels. However we should bear in mind that game G(N, W) exists as a decision mechanism through which a ratio equilibrium can be attained.

Finally we provide a proposition which orients us to our purpose. Let us consider the case where $U^i(q, m)$ can be represented as

$$(1.5) \quad U^i(q, m) = g^i(u^i(q) + m) \quad \text{for all } (q, m) \in E_+^2,$$

where g^i is a monotonic function[4]. This condition means that there exists no income effect in household's demand for firm 1's activity level. Then we have the following proposition:

<u>Proposition 3</u>. Let $E(I^1, \bar{q}^1)$ and $E(I^2, \bar{q}^2)$ be communities with income levels $I^1 = (I_2^1, \ldots, I_n^1)$, $I^2 = (I_2^2, \ldots, I_n^2)$

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4) Kaneko [6] provided a necessary and sufficient condition for a preference relation to be represented as (1.5).

and allowance levels $\bar{q}^1$, $\bar{q}^2$, respectively[5]. We assume that $U^i(q, m)$ is quasi-concave and satisfies (1.5) for all $i = 2, \ldots, n$. If (r, q^*) satisfies

$$I_i^j - r_i(q^* - \bar{q}^j) > 0 \quad \text{for all } i = 2, \ldots, n \text{ and } j = 1, 2, \tag{1.6}$$

then (r, q^*) is a ratio equilibrium in $E(I^1, \bar{q}^1)$ if and only if (r, q^*) is a ratio equilibrium in $E(I^2, \bar{q}^2)$.

Proof. It is noted that when U^i is quasi-concave and satisfies (1.5), u^i given in (1.5) is a concave function, which is proved in [6, Proposition 3].

It is easily verified that (1.7) and (1.8) are equivalent:

$$\begin{aligned} f(q^*) - \bar{q}^1 - r_1(q^* - \bar{q}^1) \\ = \max_{q \geqq 0} \{ f(q) - \bar{q}^1 - r_1(q - \bar{q}^1) \} \end{aligned} \tag{1.7}$$

$$\begin{aligned} f(q^*) - \bar{q}^2 - r_1(q^* - \bar{q}^2) \\ = \max_{q \geqq 0} \{ f(q) - \bar{q}^2 - r_1(q - \bar{q}^2) \} \end{aligned} \tag{1.8}$$

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5) Note that $E(I^1, \bar{q}^1)$ and $E(I^2, \bar{q}^2)$ consist of the same households $2, \ldots, n$ and that household i in $E(I^1, \bar{q}^1)$ has the same utility function with that of i in $E(I^2, \bar{q}^2)$.

We need to prove that the following (1.9) and (1.10) are equivalent:

$$(1.9)\quad u^i(q^*) + I_i^1 - r_i(q^* - \bar{q}^1) = \max\{u^i(q) + I_i^1 - r_i(q - \bar{q}^1) : q \geqq 0 \text{ and } I_i^1 - r_i(q - \bar{q}^1) \geqq 0\},$$

$$(1.10)\quad u^i(q^*) + I_i^2 - r_i(q^* - \bar{q}^2) = \max\{u^i(q) + I_i^2 - r_i(q - \bar{q}^2) : q \geqq 0 \text{ and } I_i^2 - r_i(q - \bar{q}^2) \geqq 0\}.$$

Let $F_i^1 = \{q \geqq 0 : I_i^1 - r_i(q - \bar{q}^1) \geqq 0\}$ and $F_i^2 = \{q \geqq 0 : I_i^2 - r_i(q - \bar{q}^2) \geqq 0\}$. Let $F_i^1 \supset F_i^2$. Then clearly (1.9) implies (1.10). Suppose that (1.10) does not imply (1.9), i.e., there is a q^o such that $u^i(q^o) + I_i^1 - r_i(q^o - \bar{q}^1) > u^i(q^*) + I_i^1 - r_i(q^* - \bar{q}^1)$, $q^o \in F_i^1$ and $q^o \notin F_i^2$. Though $I_i^2 - r_i(q^o - \bar{q}^2) < 0$, it is also true that $u^i(q^o) + I_i^2 - r_i(q^o - \bar{q}^2) > U^i(q^*) + I_i^2 - r_i(q^* - \bar{q}^2)$. Since u^i is concave and (1.6) holds, there is a $\lambda > 0$ such that

$$u^i(\lambda q^* + (1-\lambda)q^o) + I_i^2 - r_i((\lambda q^* + (1-\lambda)q^o) - \bar{q}^2) > u^i(q^*) + I_i^2 - r_i(q^* - \bar{q}^2),$$

and

$$I_i^2 - r_i((\lambda q^* + (1-\lambda)q^o) - \bar{q}^2) > 0.$$

This is a contradiction. When $F_i^1 \subset F_i^2$, the argument is symmetry. (Q.E.D.)

Proposition 3 means that in the absense of income effect a ratio equilibrium (r, q^*) is independent of income levels I and allowance level $\bar{q}$. Of course, we should note that when the income levels or the allowance level in $E(I, \bar{q})$ change, the income distribution derived from a ratio equilibrium also changes. This kind of result is often called Coase's theorem[6]. Our main purpose, however, is to investigate the behavior of ratio equilibrium in $E(I, \bar{q})$ in the presence of income effect.

2. Limit Properties of Ratio Equilibrium

In this section we will consider the "global" behavior of ratio equilibrium in $E(I, \bar{q})$ when the income levels $I_2, \ldots, I_n$ change. That is, we will show two limit properties of ratio equilibrium when $I_2, \ldots, I_n$ become small or large.

First we assume that the firm's revenue function f(q) is a strictly concave, continuous function of $q \geqq 0$ and is continuously differentiable on the open interval $(0, +\infty)$ with

$$(2.1) \qquad \lim_{q \to \infty} \frac{df(q)}{dq} \leqq 0 \quad \text{and} \quad \lim_{q \to +0} \frac{df(q)}{dq} = +\infty .$$

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6) See Inada and Kuga [4].

We would need to explain the meaning of $\lim_{q \to +0} \frac{df(q)}{dq} = +\infty$. When the firm is in a "competitive" market of the good which it produces, its profit function is px - C(x), where x is an amount of the good produced by the firm, p the market price and C(x) the firm's cost function. Since we measure the firm's activity level q by the cost C(x), the profit function is rewritten as pQ(q) - q, where Q(q) is the inverse function of C(x). Hence the revenue function is f(q) = pQ(q). Since $\frac{dC(0)}{dx} = 0$ and $\lim_{q \to +0} \frac{dQ(q)}{dq} = +\infty$ are equivalent under some appropriate conditions, $\lim_{q \to +0} \frac{df(q)}{dq} = +\infty$ means that the marginal cost at the zero level is zero. This assumption is slightly strong, but is rather a technical condition to assure that the ratio equilibria are not in the corners.

Second, we assume that the households' utility functions $U^i(q, m)$, i = 2, ..., n are decreasing functions of q, increasing functions of m, strictly concave, continuous functions of (q, m) in E_+^2 and are continuously differentiable on $\{(q, m) \in E_+^2 : m > 0\}$ with

$$(2.2) \qquad \lim_{m \to +0} \frac{\partial U^i(q, m)}{\partial m} = +\infty \text{ for any } q \geq 0.$$

Note that (2.2) reflects an income effect. As these assumptions are standard and familiar, we would need no explanation.

Let $\hat{q}$ denote the activity level such that

$$(2.3) \quad f(\hat{q}) - \hat{q} = \max_{q \geqq 0} \{f(q) - q\}.$$

This $\hat{q}$ is positive and unique under our assumptions. In the following, we always assume

$$(2.4) \quad 0 \leqq \bar{q} < \hat{q}.$$

If $\bar{q} \geqq \hat{q}$, the firm always chooses $\hat{q}$ as his activity level and no compensation is made, that is, there is no problem.

Lemma 4. Let (r, q^*) be a ratio equilibrium in $E(I, \bar{q})$. Then (i): (r, q^*) is an inner solution[7];

(ii): $r_1 > 1,\ r_2 < 0, \ldots, r_n < 0$.

Proof. (i): Since $I_2, \ldots, I_n > 0$, we have, from (2.2), $0 < I_i - r_i(q^* - \bar{q})$ for all $i = 2, \ldots, n$. Also we have $0 < q^*$ from (2.1).

(ii): Since (r, q^*) is an inner solution, necessary conditions for it to be a ratio equilibrium are

$$(2.5) \quad \frac{\partial U^i}{\partial q} - r_i \frac{\partial U^i}{\partial m} = 0 \quad \text{for all} \quad i = 2, \ldots, n,$$

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7) A ratio equilibrium (r, q^*) is called an inner solution if $q^* > 0$ and $I_i - r_i(q^* - \bar{q}) > 0$ for all $i = 2, \ldots, n$.

$$(2.6) \qquad \frac{df}{dq} - r_1 = 0.$$

If there is at least one i ($2 \leqq i \leqq n$) such that $r_i \geqq 0$, then (2.5) does not hold, because $\frac{\partial U^i}{\partial q} < 0$ and $\frac{\partial U^i}{\partial m} > 0$. Therefore we have $r_i < 0$ for all $i = 2, \ldots, n$ and $r_1 > 1$ because of $\sum_{i=1}^{n} r_i = 1$. (Q.E.D.)

The main result of this section is the following propositions.

<u>Proposition 5</u>. Let $(r(\lambda), q(\lambda))$ be a ratio equilibrium in $E(\lambda I_i\ \bar{q}) = E((\lambda I^2, \lambda I^3, \ldots, \lambda I^n), \bar{q})$ for all $\lambda > 0$. Then we have (i) and (ii):

(i): There is a $\lambda_o > 0$ such that $\bar{q} < q(\lambda) < \hat{q}$ for all $\lambda \leqq \lambda_o$.

(ii): Assume that for any $q \geqq 0$,

$$(2.7) \qquad \lim_{m \to \infty} \left. \frac{\partial U^i/\partial q}{\partial U^i/\partial m} \right|_{(q,m)} = -\infty \text{ for all } i = 2, \ldots, n.$$

Then $\lim_{\lambda \to \infty} q(\lambda) = 0$.

<u>Proof</u>. (<u>i</u>): Suppose that there is a sequence $\{\lambda^\nu\}$ such that $\lambda^\nu > 0$, $q(\lambda^\nu) \leqq \bar{q}$ for all ν and $\lambda^\nu \to 0$ $(\nu \to \infty)$. Since $(r(\lambda^\nu), q(\lambda^\nu))$ is an inner solution by Lemma 4. (i) and $r_i(\lambda^\nu) < 0$, we have $0 < \lambda^\nu I_i - r_i(\lambda^\nu)(q(\lambda^\nu) - \bar{q}) \leqq \lambda^\nu I_i$. Hence $\lambda^\nu I_i - r_i(\lambda^\nu)(q(\lambda^\nu) - \bar{q}) \to 0$ $(\nu \to \infty)$. Therefore we have, by (2.2),

$$\lim_{\nu\to\infty} \frac{\partial U^i}{\partial m}\Bigg|_{(q(\lambda^\nu),\lambda^\nu I_i - r_i(\lambda^\nu)(q(\lambda^\nu)-\bar{q}))} = +\infty .$$

This implies $r_i(\lambda^\nu) \to 0$ $(\nu \to \infty)$. For otherwise, there is a sequence $\{\nu^s\}$ such that $r_i(\lambda^{\nu^s}) \to a < 0$ $(\nu^s \to \infty)$, which implies $\lim_{\nu^s\to\infty} \frac{\partial U^i}{\partial q} = \lim_{\nu^s\to\infty} r_i(\lambda^{\nu^s}) \frac{\partial U^i}{\partial m} = -\infty$ because of necessary condition $\frac{\partial U^i}{\partial q} - r_i(\lambda^\nu) \frac{\partial U^i}{\partial m} = 0$ for all ν. Since $0 < q(\lambda^\nu) \leqq \bar{q}$, this contradicts that U^i is a concave function of q. Since $r_i(\lambda^\nu) \to 0$ $(\nu \to \infty)$ for all $i = 2, \ldots, n$, we have $r_1(\lambda^\nu) \to 1$ $(\nu \to \infty)$. This and necessary condition $\frac{df}{dq} - r_1(\lambda^\nu) = 0$ for all ν imply that $q(\lambda^\nu) \to \hat{q}$ $(\nu \to \infty)$. But this contradicts $q(\lambda^\nu) \leqq \bar{q} < \hat{q}$. Therefore we can not take the above sequence $\{\lambda^\nu\}$.

(ii): From Lemma 4, necessary conditions for a ratio equilibrium are

$$\frac{\partial U^i}{\partial q} - r_i(\lambda) \frac{\partial U^i}{\partial m} = 0 \quad \text{for all } i = 2, \ldots, n,$$

$$\frac{df}{dq} - r_1(\lambda) = 0.$$

Suppose that $q(\lambda) \to 0$ $(\lambda \to \infty)$ does not hold. Then we can take a subsequence $\{\lambda^\nu\}$ such that $q(\lambda^\nu) \to q_o > 0$ and $\lambda^\nu \to \infty$ $(\nu \to \infty)$. Since $r_1(\lambda^\nu) = \frac{df(q(\lambda^\nu))}{dq}$ for any ν, there is a number $M > 0$ such that $r_1(\lambda^\nu) \leqq M$ for any ν. Since $r_i(\lambda^\nu) < 0$ $(i = 2, \ldots, n)$ and $\sum_{i=1}^{n} r_i(\lambda^\nu) = 1$ for all ν, there is a number $M' > 0$ such that $M' \geqq |r_i(\lambda^\nu)|$ for all $i = 2, \ldots, n$ and all ν. Hence $\lambda^\nu I_i - r_i(\lambda^\nu)(q(\lambda^\nu) - \bar{q}) \to$

∞ ($\nu \to \infty$). Using $q(\lambda^\nu) \to q^o > 0$ ($\nu \to \infty$), we have $\lim_{\nu \to \infty} r_i(\lambda^\nu)$

$$= \frac{\partial U^i}{\partial q} \Big/ \frac{\partial U^i}{\partial m} \bigg|_{(q(\lambda^\nu), \lambda^\nu I_i - r_i(\lambda^\nu)(q(\lambda^\nu) - \bar{q}))} = -\infty.$$

This is a contradiction. Hence $\lim_{\lambda \to \infty} q(\lambda) = 0$. (Q.E.D.)

We would need some brief explanation and interpretation of Proposition 5. Proposition 5. (i) says that when the households' income levels are very low, the firm always compensates the households and increases the activity level beyond the allowance level $\bar{q}$ in equilibrium. When the income levels are very low, the households are eager to get income but hardly want to decrease the firm's activity level. Of course, if no compensation is made, they want to do so. This is just an income effect. By this reason the firm can obtain the households' consent by paying small compensations to them, and so the firm can increase the activity level beyond the allowance level. Proposition 5. (ii) says, conversely, that when the income levels are sufficiently high, the households compensate the firm to decrease his activity level to the zero-level. This meaning would be intuitively clear. The case of Proposition 3 (case without income effect) is usually interpreted as a case with sufficient large income levels, i.e., in this case we can approximately regard it as a case without income effect. Hence we can say from Proposition 3 and 5. (ii) that when the income levels are sufficiently high, a ratio equilibrium is approximately independent of the income levels and the allowance level, but the activity level in equilibrium is always small. Finally note that condition (2.7) is necessary for this proposition, but that it would be economically trivial and is satisfied by many plausible examples.

3. Local Property of Ratio Equilibrium

In this section we will investigate relationships between allowance level and ratio equilibrium and effects of some changes of allowance level or the households' income levels upon ratio equilibrium.

We add the following assumption on the households' utility functions U^i, $i = 2, \ldots, n$ to the previous assumptions that U^i's are functions of C^2 and

$$(3.1) \quad \left.\frac{\partial^2 U^i}{\partial q \partial m}\right|_{(q,m)} \leqq 0 \quad \text{for all } (q, m) \text{ with } q \geqq 0 \text{ and } m > 0.$$

Condition (3.1) means that the marginal utility of money is a nonincreasing function of firm's activity level. In other words, when the activity level becomes lower, i.e., the environment for the households becomes better, each household can gain greater (not smaller) additional satisfaction by an additional income.

<u>Lemma 6</u>. (<u>i</u>): $\frac{U_1^i}{U_2^i}(q, m) = \frac{\partial U^i}{\partial q} / \frac{\partial U^i}{\partial m}\Big|_{(q,m)}$ is a nonincreasing function of q and m with $q \geqq 0$ and $m > 0$.[8)]

(<u>ii</u>): For any negative r_i, $\frac{U_1^i}{U_2^i}(q, I_i - r_i(q - \bar{q}))$ is a nonincreasing function of q, where $q \geqq 0$ and $I_i - r_i(q - \bar{q}) > 0$.

(<u>iii</u>) Let $D_i(r_i, I_i, \bar{q})$ denote household i's demand for the firm's activity level in $E(I, \bar{q})$.[9)] For any negative r_i, $D_i(r_i, I_i, \bar{q})$ is a nonincreasing function of $I_i > 0$ and a nondecreasing function of $\bar{q} \geqq 0$.

<u>Proof</u>. We will use the following notations

$$U_1^i = \frac{\partial U^i}{\partial q}, \quad U_2^i = \frac{\partial U^i}{\partial m}, \quad U_{11}^i = \frac{\partial^2 U^i}{\partial q^2}, \quad U_{12}^i = \frac{\partial^2 U^i}{\partial m \partial q},$$

$$U_{22}^i = \frac{\partial^2 U^i}{\partial m^2} .$$

(<u>i</u>): Since $U_1^i < 0$, $U_2^i > 0$ and $U_{11}^i, U_{12}^i, U_{22}^i \leqq 0$, by the assumptions of the previous section and (3.1), we have

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8) We can employ this proposition as an assumption in place of (3.1) for the following discussions. But since Condition (3.1) has a clearer meaning than (i), we employ (3.1) as an assumption.

9) $D_i(r_i, I_i, \bar{q}) = q_i$ is defined by $U^i(q_i, I_i - r_i(q_i - \bar{q}))$ $= \max\{U^i(q, I_i - r_i(q - \bar{q})): q \geqq 0$ and $I_i - r_i(q - \bar{q}) \geqq 0\}$.

$$\frac{\partial}{\partial q}\left(\frac{U_1^i}{U_2^i}\right) = \frac{U_2^i U_{11}^i - U_1^i U_{12}^i}{(U_2^i)^2} \leqq 0,$$

$$\frac{\partial}{\partial m}\left(\frac{U_1^i}{U_2^i}\right) = \frac{U_1^i U_{12}^i - U_1^i U_{22}^i}{(U_2^i)^2} \leqq 0.$$

(ii): It follows from (i) that $\frac{\partial}{\partial q}\left(\frac{U_1^i}{U_2^i}(q, I_i - r_i(q-\bar{q}))\right)$ $= \frac{\partial}{\partial q}\left(\frac{U_1^i}{U_2^i}\right) - r_i \frac{\partial}{\partial m}\left(\frac{U_1^i}{U_2^i}\right) \leqq 0.$

(iii): Let $I_i^1 < I_i^2$, $q^1 = D_i(r_i, I_i^1, \bar{q})$ and $q^2 = D_i(r_i, I_i^2, \bar{q})$. Since $\frac{U_1^i}{U_2^i}$ is a nonincreasing function of m, we have

$$r_i = \frac{U_1^i}{U_2^i}(q^1, I_i^1 - r_i(q^1 - \bar{q}))$$

$$\geqq \frac{U_1^i}{U_2^i}(q^1, I_i^2 - r_i(q^1 - \bar{q})).$$

Hence we have $q^1 \geqq q^2$ because of (ii) and $r_i = \frac{U_1^i}{U_2^i}(q^2, I_i^2 - r_i(q^2 - \bar{q}))$. It is similarly verified that $D_i(r_i, I_i, \bar{q})$ is a nondecreasing function of $\bar{q}$. (Q.E.D.)

Now we analyse the relation of an allowance level $\bar{q}$ and an equilibrium activity level q_R (which is given by a ratio equilibrium) using a concept of marginal social cost[10].

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10) The marginal social cost can be always well-defined, but the concept of social cost itself can not be defined without Assumption (1.5).

If firm 1 increases his activity level from an allowance level from $\bar{q}$ to $\bar{q}$ + dq, then it should compensate household i for extra

$$- \frac{U_1^i}{U_2^i} (\bar{q}, I_i) \, dq,$$

so that household i's utility level holds constant. Hence the marginal social cost is the sum of the total compensation and the private cost (here, 1), i.e.,

$$1 + \sum_{i=2}^{n} \left(- \frac{U_1^i}{U_2^i} (\bar{q}, I_i)\right),$$

while the marginal revenue of firm 1 is

$$\frac{df}{dq} (\bar{q}).$$

If the marginal social cost is smaller (greater) than the marginal revenue, then the allowance level $\bar{q}$ is not a Pareto optimal level and it would be conjectured that an equilibrium activity level is greater (smaller) than $\bar{q}$, because a ratio equilibrium is Pareto optimal. Using these concepts, we have the following criterion for the relation of an allowance level and an equilibrium activity level.

Proposition 7. Let (r, q_R) be a ratio equilibrium in

$E(I, \bar{q})$. Then $q_R \leqq \bar{q}$ if and only if

$$(3.2) \qquad \frac{df}{dq}(\bar{q}) \leqq 1 + \sum_{i=2}^{n} - \frac{U_1^i}{U_2^i}(\bar{q}, I_i).$$

<u>Proof</u>. <u>Necessity</u>: From $q_R \leqq \bar{q}$, we have, by the concavity of f and Lemma 6. (ii),

$$r_1 = \frac{df}{dq}(q_R) \geqq \frac{df}{dq}(\bar{q}) \quad \text{and}$$

$$r_i = \frac{U_1^i}{U_2^i}(q_R, I_i - r_i(q_R - \bar{q})) \geqq \frac{U_1^i}{U_2^i}(\bar{q}, I_i)$$

for all $i = 2, \ldots, n$.

Hence we have

$$1 = \sum_{i=1}^{n} r_i \geqq \frac{df}{dq}(\bar{q}) + \sum_{i=2}^{n} \frac{U_1^i}{U_2^i}(\bar{q}, I_i).$$

<u>Sufficiency</u>: Suppose (3.2) and $q_R > \bar{q}$. Since (r, q_R) is a ratio equilibrium, it must hold that

$$r_1 = \frac{df}{dq}(q_R) \quad \text{and} \quad r_i = \frac{U_1^i}{U_2^i}(q_R, I_i - r_i(q_R - \bar{q}))$$

for all $i = 2, \ldots, n$.

From $q_R > \bar{q}$, we have $r_1 = \frac{df}{dq}(q_R) < \frac{df}{dq}(\bar{q})$. Further we have, by Lemma 6. (ii),

$$r_i = \frac{U_1^i}{U_2^i}(q_R, I_i(q_R - \bar{q})) \leqq \frac{U_1^i}{U_2^i}(\bar{q}, I_i)$$

for all i = 2, ..., n.

Therefore,

$$\sum_{i=1}^{n} r_i < \frac{df}{dq}(q) + \sum_{i=2}^{n} \frac{U_1^i}{U_2^i}(q, I_i) \leqq 1.$$

But this contradicts that (r, q_R) is a ratio equilibrium.

(Q.E.D.)

Note that (3.2) is a condition only at an allocation of an allowance level without compensation and that Samuelson's condition for an allocation to be Pareto optimal is that (3.2) holds in equation at the allocation.

If (3.2) holds in equation, i.e., the marginal social cost is equal to the marginal revenue at an allowance level, then the equilibrium activity level is equal to the allowance level. This allowance level has a special property: In the ratio equilibrium (r, q_R) in the community with the allowance level, not only the equilibrium activity level coincides with the allowance level but also any compensations are not made. That is, if such an allowance level is set, the result without compensations can become a result of the majority bargaining of the voting game G(N, W), which is assured by Proposition 2. We call such an allowance level a neutral allowance level, denoted by

$\bar{q}_N$, i.e., if (r, q_R) is a ratio equilibrium in $E(I, \bar{q}_N)$, then $q_R = \bar{q}_N$.

Proposition 8. Assume that $I = (I_2, \ldots, I_n)$ is fixed. Then there exists one and only one neutral allowance level $\bar{q}_N$, and the following (3.3) is a necessary and sufficient condition for it:

$$\frac{df}{dq}(\bar{q}_N) = 1 + \sum_{i=2}^{n} - \frac{U_1^i}{U_2^i}(\bar{q}_N, I_i). \tag{3.3}$$

Proof.[11] It follows from Proposition 7 that (3.3) is a necessary and sufficient condition for a neutral allowance level. Also if a neutral allowance level exists, it is unique because $\frac{df}{dq}(q)$ is a decreasing function and $\frac{U_1^i}{U_2^i}(q, I_i)$ is a nonincreasing function of q for all $i = 2, \ldots, n$.

So we show the existence of a neutral allowance level. Since U^i is a continuously differentiable on $\{(q, m) \in E_+^2 : m > 0\}$ for all $i = 2, \ldots, n$ and $\lim_{q \to +0} \frac{df}{dq}(q) = +\infty$, we have

$$\frac{df}{dq}(q) + \sum_{i=2}^{n} \frac{U_1^i}{U_2^i}(q, I_i) > 1 \quad \text{for any sufficient small } q > 0.$$

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11) A neutral allowance level exists under the assumptions of Section 2, and the assumptions of this section is not used in the following existence proof.

Also, since $\frac{df}{dq}(\hat{q}) = 1$ and $\frac{U_1^i}{U_2^i}(\hat{q}, I_i) < 0$ for all $i = 2, \ldots, n$, we have

$$\frac{df}{dq}(\hat{q}) + \sum_{i=2}^{n} \frac{U_1^i}{U_2^i}(\hat{q}, I_i) < 1.$$

Further since $\frac{df}{dq}(q) + \sum_{i=2}^{n} \frac{U_1^i}{U_2^i}(q, I_i)$ is a continuous function of q, there exists a $\bar{q}_N$ satisfying (3.3). (Q.E.D.)

<u>Corollary 9</u>. Let $\bar{q}_N$ be the neutral allowance level in the community with I and let (r, q_R) be a ratio equilibrium in $E(I, \bar{q})$. Then $q_R \gtreqless \bar{q}$ if and only if $\bar{q} \lesseqgtr \bar{q}_N$.

<u>Proof</u>. $\frac{df}{dq}(q) + \sum_{i=2}^{n} \frac{U_1^i}{U_2^i}(q, I_i)$ is a decreasing function of q by the strict concavity of f and Lemma 6. (i). Hence $q_R \gtreqless \bar{q}$ if and only if $\bar{q} \lesseqgtr \bar{q}_N$ by Propositions 7 and 8.

(Q.E.D.)

By this corollary we can illustrate the relation of an allowance level, an equilibrium activity level and the neutral allowance level as Figure 1.

We have the following proposition on effects that a change of an allowance level or households' income has on an equilibrium activity level.

<u>Proposition 10</u>. Let $\bar{q}_N(I)$ be the neutral allowance level of the community with I and let $q_R(I, \bar{q})$ be an equilibrium

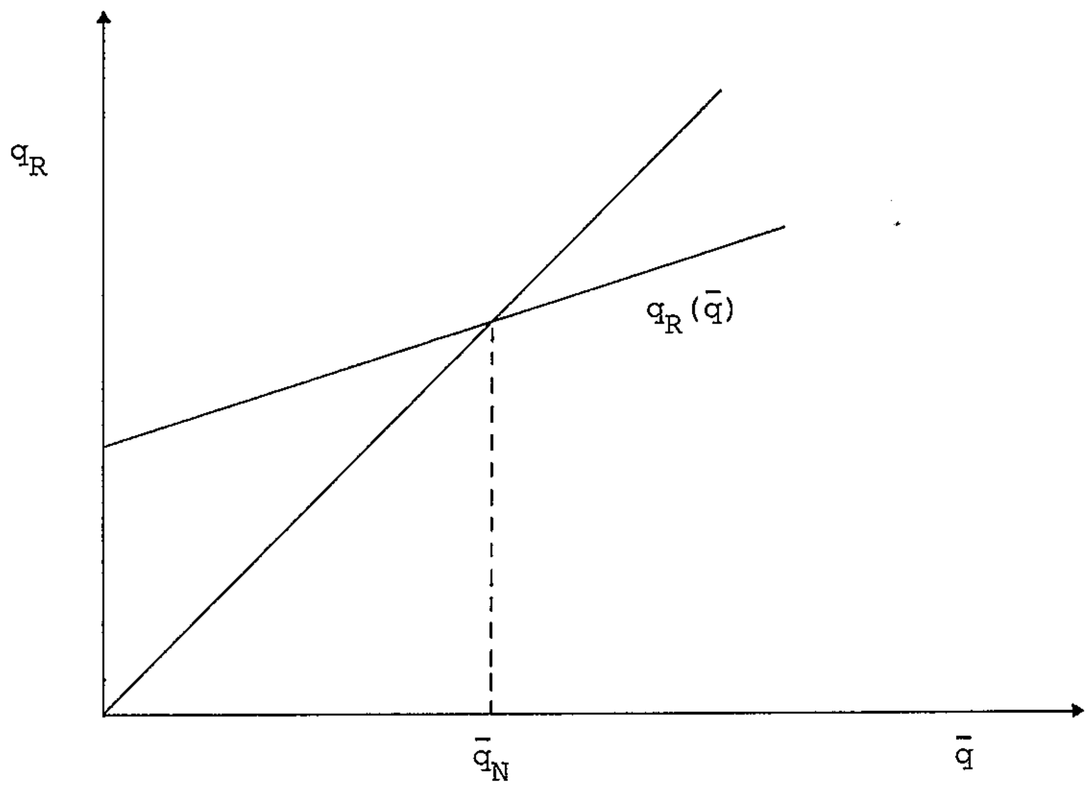

Figure 1.

activity level in $E(I, \bar{q})$. Assume $\bar{q} \geqq \bar{q}_N(I)$. Then

(i): If Δ is a vector such that $\Delta \geqq 0$ and $\Delta \neq 0$, then $q_R(I, \bar{q}) \geqq q_R(I + \Delta, \bar{q})$.

(ii): If $\bar{q} < \bar{q}' < \hat{q}$, then $q_R(I, \bar{q}') \geqq q_R(I, \bar{q})$.

When (3.1) holds in strict inequality, the results of (i) and (ii) holds also in strict inequality.

Proof. (i): Let (r^o, q^o) and (r^1, q^1) be ratio equilibria in $E(I, \bar{q})$ and $E(I + \Delta, \bar{q})$ respectively. Suppose $q^1 > q^o$. Hence we have

$$r_1^o = \frac{df}{dq}(q^o) > \frac{df}{dq}(q^1) = r_1^1.$$

There is an i $(2 \leqq i \leqq n)$ such that $r_i^o < r_i^1$, because $1 = \sum_{i=1}^{n} r_i^o = \sum_{i=1}^{n} r_i^1$. Since $\bar{q} \geqq q^o$ by $\bar{q} \geqq q_N(I)$ and Corollary 9, we have, by Lemma 6. (i) and (ii),

$$\begin{aligned} r_i^1 &= \frac{U_1^i}{U_2^i}(q^1, I_i + \Delta_i - r_i^1(q^1 - \bar{q})) \\ &\leqq \frac{U_1^i}{U_2^i}(q^1, I_i - r_i^1(q^1 - \bar{q})) \\ &\leqq \frac{U_1^i}{U_2^i}(q^o, I_i - r_i^1(q^o - \bar{q})) \\ &\leqq \frac{U_1^i}{U_2^i}(q_o, I_i - r_i^o(q^o - \bar{q})) = r_i^o \end{aligned}$$

This is a contradiction. Therefore $q^o \geqq q^1$.

(ii): As we can prove (ii) analogously to (i), we omit its proof. (Q.E.D.)

We have shown monotonical effects that changes of an allowance level and the households' income level have on an equilibrium activity level. Such monotonical properties, however, may not hold for any allowance level $\bar{q} < \bar{q}_N(I)$. When $\bar{q} < \bar{q}_N(I)$, the equilibrium activity level $q_R(I, \bar{q})$ is higher than $\bar{q}$ by Corollary 9. In this case, there may be the possibility that when the allowance level decreases or the households' income levels increase, the equilibrium activity level increases. It is illustrated as Figure 2. Roughly speaking, when the allowance level decreases, the households agree with a higher activity level so that they can get greater compensations. This possibility is intuitively paradoxical. But we have not succeeded in constructing any counter example for proposition 10 with $\bar{q} < \bar{q}_N(I, \bar{q})$ nor proving it. We can provide only a sufficient condition for it.

Proposition 11. Assume that $D_i(r_i, I_i, \bar{q})$ is a nonincreasing function of $r_i < 0$ for all $i = 2, \ldots, n$. Then (i) and (ii) of Proposition 10 are true.

Proof. (i): Suppose $q^1 \equiv q_R(I + \Delta, \bar{q}) > q_R(I, \bar{q}) \equiv q^o$. By the strict concavity of f, we have

$$r_1^o = \frac{df}{dq}(q^o) > \frac{df}{dq}(q^1) = r_1^1.$$

Hence there is an i ($2 \leqq i \leqq n$) such that $r_i^o < r_i^1$ because $\sum_{i=1}^n r_i^o = \sum_{i=1}^n r_i^1 = 1$. By the assumption and Lemma 6. (iii) We have

$$q^o = D_i(r_i^o, I_i, \bar{q}) \geqq D_i(r_i^1, I_i, \bar{q})$$
$$\geqq D_i(r_i^1, I_i + \Delta_i, \bar{q}) = q^1.$$

This is a contradiction.

(ii): As we can prove (ii) analogously to (i), we omit the proof. (Q.E.D.)

The firm's activity may be thought of as a supply of an environmental good by measuring it in $-(q - \bar{q})$ and $-r_i$ may be considered as the price and so, we can discuss the demand for the environmental good in a standard way. Then the assumption of Proposition 11 is that the demand for the environmental good is a nonincreasing function of $-r_i$. Hence it is just a Giffen's paradox that the assumption of Proposition 11 does not hold. In other words, it is a necessary condition for Proposition 10 for $\bar{q} < \bar{q}_N(I)$ not to hold that the environmental good is a Giffen's good. But it may be reasonable to assume that any environmental good is not a Giffen's good when households' incomes are fairly high.

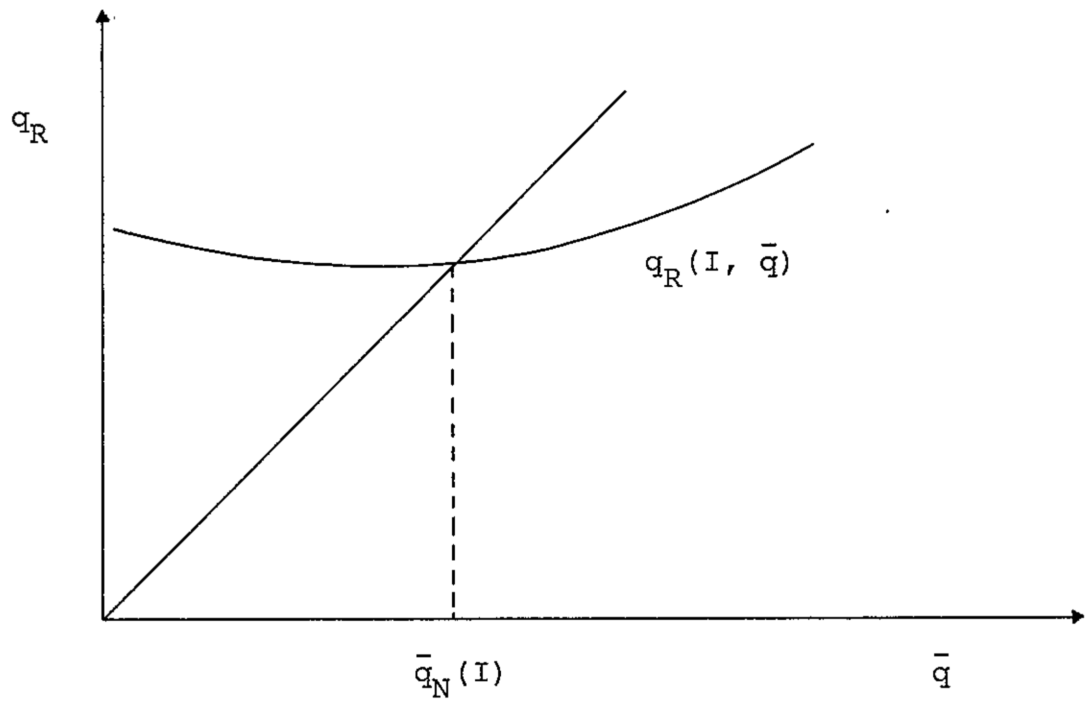

Figure 2.

4. Property of the Neutral Allowance Level

In the previous section, we have characterized local properties of ratio equilibrium when the households' income levels or an allowance level change. As shown in Proposition 7, 10 and Corollary 9, the neutral allowance level plays an important role as a criterion for the local properties. In this section we will investigate the behavior of the neutral allowance level when the households' income levels change.

Proposition 12. Let Δ be a vector in E_{+}^{n-1} such that $\Delta \geqq 0$ and $\Delta \not\leq 0$. Then $\bar{q}_N(I) \geqq \bar{q}_N(I + \Delta)$.

Proof. Suppose $\bar{q}_N < \bar{q}_N(I + \Delta)$. Then $\frac{df}{dq}(\bar{q}_N(I)) >$ $\frac{df}{dq}(\bar{q}_N(I + \Delta))$ by the strict concavity of f, and

$$\sum_{i=2}^{n} \frac{U_1^i}{U_2^i}(\bar{q}_N(I), I_i) \geqq \sum_{i=2}^{n} \frac{U_1^i}{U_2^i}(\bar{q}_N(I + \Delta), I_i + \Delta_i)$$

by Lemma 6. (i). But these can not satisfy (3.3) of Proposition 8. (Q.E.D.)

Hence the neutral allowance level does not have the possibility that it behaves in an irregular form such as the behavior of $q_R(I, \bar{q})$ pointed out in the previous section. Further we can provide limit properties of the neutral allowance level.

Proposition 13. Let $\{I^\nu\} = \{(I_2^\nu, \ldots, I_n^\nu)\}$ be a sequence of income vectors.

(i): If $(I_2^\nu, \ldots, I_n^\nu) \to (+0, \ldots, +0)$, then $\bar{q}_N(I^\nu) \to \hat{q}$ $(\nu \to \infty)$.

(ii): Assume that for any $q \geqq 0$, $\lim_{m \to \infty} \frac{U_1^i}{U_2^i}(q, m) = -\infty$ for all $i = 2, \ldots, n$. If $(I_2^\nu, \ldots, I_n^\nu) \to (\infty, \ldots, \infty)$, then $q_N(I^\nu) \to 0$ $(\nu \to \infty)$.

Proof.[12] (i): For any q $(0 \leqq q \leqq \hat{q})$, we have, by (2.2)

$$\lim_{\nu \to \infty} \frac{\partial U^i}{\partial m}(q, I_i^\nu) = +\infty,$$

and so, by the fact that $0 \leqq \bar{q}_N^\nu \leqq \hat{q}$,

$$\sum_{i=2}^{n} \frac{U_1^i}{U_2^i}(\bar{q}_N^\nu, I_i^\nu) \to 0 \quad (\nu \to \infty).^{13)}$$

Hence we have, by (3.3) of Proposition 8, $\lim_{\nu \to \infty} \frac{df}{dq}(\bar{q}_N^\nu) = 1$. Therefore we have $\bar{q}_N^\nu \to \hat{q}$ $(\nu \to \infty)$.

(ii): For any q $(0 \leqq q \leqq \hat{q})$, we have $\lim_{\nu \to \infty} \frac{U_1^i}{U_2^i}(q, I_i^\nu) = -\infty$. So we have

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12) Note that we do not use the assumption added in Section 3 in the proof of Proposition 13.

13) $\bar{q}_N^\nu$ stands for $\bar{q}_N(I^\nu)$.

$$\lim_{\nu \to \infty} \sum_{i=2}^{n} \frac{U_1^i}{U_2^i} (\bar{q}_N^{\nu}, I_i^{\nu}) = -\infty .$$

Hence $\lim_{\nu \to \infty} \frac{df}{dq} (\bar{q}_N^{\nu}) = \infty$. Therefore we have $\bar{q}_N^{\nu} \to 0$ $(\nu \to \infty)$.

(Q.E.D.)

From Propositions 12 and 13, we can draw Figure 3 which indicates the full behavior of equilibrium activity level when an allowance level and the households' income levels change. All curves do not intersect each other and are nondecreasing in the southeast triangle of the box-diagram of Figure 3, which is shown in Proposition 10, and they may intersect each other and may be decreasing in the north-west triangle, which is pointed out in the previous section. Further, Propositions 5.(ii) and 13 say that curves become flat when they approach the bottom of the box-diagram .

5. Conclusion

We have shown that an economy with external diseconomies can be considered as a variation of a public good economy and can be treated in the same way by introducing the concept of allowance level, and that the theory of ratio equilibrium and the voting game G(N, W) can be directly applied to this economy. It is a device for the resolution of the conflict among the damaging firm and the suffered households

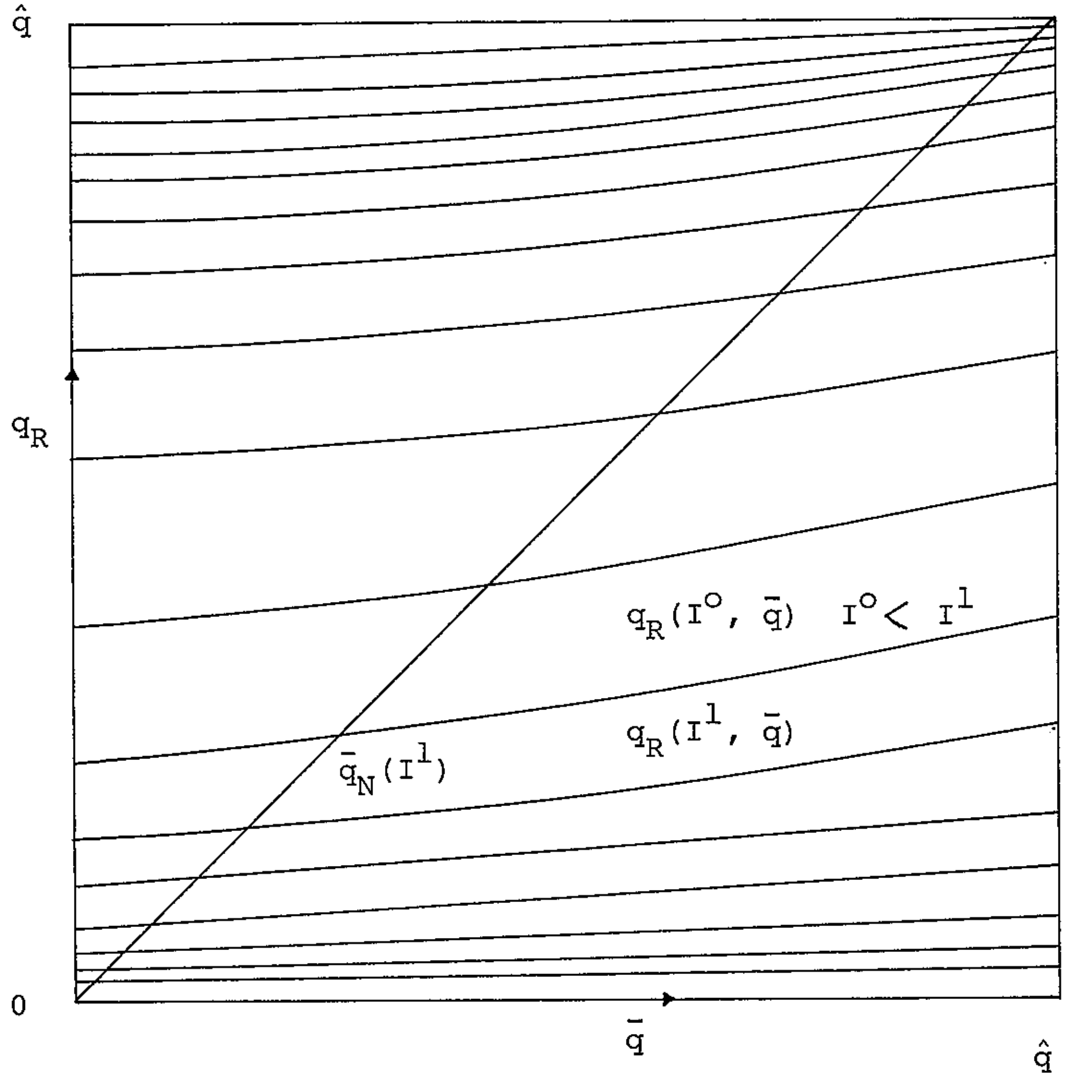

Figure 3.

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