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Relations between continuous-time and discrete-time linear systems

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abstract

In this paper the concept of basic linear system(Takahara 1981) is used to explore relations between continuous-time and discrete-time systems from two directions.

Firstly preservation of systems structure through sampling is considered. Two sampling methods are defined and both preserve linearity, causality, stationarity, and finite dimensionality of system core. These preservation makes the resultant discrete-time system a basic linear system.

Secondly we investigate what a basic linear system is the minimal system to embed all of discrete-time basic linear systems made from sampling of a basic linear system. It is said to be minimal in the sense that any element of the set can be embedded into the continuous-time basic linear system and there is no greater basic linear system that can embed each of discrete-time basic linear systems. It will be shown that the original continuous-time basic linear system is the system.

keywords

basic linear system, simple sampling, 0-th hold-sampling, discrete-time system, continuous-time system, category of basic linear systems, minimal embedding of sampled systems

1 INTRODUCTION

In mathematical general systems theory Takahara(1981) has characterized linear systems as basic linear systems. A basic linear system has a unique state space representation, and it makes behavioral consideration of linear system possible. For example, Ikeshoji et al(1981), Takahara et al (1984, 1985), Saito(1987), Nakano et al(1987, 1988), and Mesarovic and Takahara(1989) have shown some of explanatory ability of the basic linear system as a meta-model of linear systems theory.

In this paper the concept of basic linear system is used to explore relations between continuous-time and discrete-time systems from two directions.

Firstly we focus on what properties are preserved by sampling from a continuous-time basic linear system. Two sampling methods are used and each of them preserve linearity, causality, stationarity, and finite dimensionality of system core. These preservation makes the resultant discrete-time system a basic linear system.

Secondly a continuous-time basic linear system is shown to be the minimal system to embed the set of discrete-time basic linear systems made from sampling of the system. It is minimal in the sense that any element of the set can be embedded into the continuous-time basic linear system and there is no greater basic linear system that can embed each of discrete-time basic linear systems.

In the following the proofs of propositions, theorems and corollary are placed in the appendix of this paper.

2 BASIC CONSEPTS

Notation in this paper follows Mesarovic and Takahara(1974, 1989).The time set T is assumed to be R^+(the set of non-negative reals) for continuous-time systems or $\tau Z^+=\{n\tau \mid n$ is in Z^+(the set of non-negative integers)$\}$ for discrete-time ones, where τ is an element of R^+. Let A and B be linear spaces on the same field $\underline{A}$. The zero element of $\underline{A}$ is denoted by 0. The set of all functions from T to A, which is denoted by A^T, makes a linear system over $\underline{A}$ in natural way. The input and output objects of a linear system are represented by $X \subset A^T$ and $Y \subset B^T$, respectively. Both X and Y are always assumed to be linear spaces. A <u>linear time system</u> S over $\underline{A}$ is a linear subspace of $X \times Y$. It is always assumed that $X = \{x \mid (\exists y)((x, y) \in S)\}$ and $Y = \{y \mid (\exists x)((x, y) \in S)\}$.

The following restrictions on a time scale T will also be used

$$T^t = [0, t),\ T_t = [t, \infty),\ T_{tt'} = [t, t'),\ \bar{T}^t = [0, t],\ \bar{T}_{tt'} = [t, t'].$$

The restrictions on $x \in X$ to the above segments are often abbreviated by

$$x^t = x|T^t,\ \bar{x}^t = x|\bar{T}^t,\ x_{tt'} = x|T_{tt'},\ \bar{x}_{tt'} = x|\bar{T}_{tt'}.$$

For $y \in Y$, or in general for sets, a similar notation will be used. According to this notation, we can define an operation called the concatenation as follows. Let $x, x' \in A^T$. For any $t \in T$, another element x' is defined in A^T by

$$x''(t'') = \begin{cases} x(t''), & \text{if } t'' < t \\ x'(t''), & \text{if } t'' \geq t \end{cases}$$

x" is called the <u>concatenation</u> of x^t and x'_t and denoted by $x'' = x^t \cdot x'_t$.

Let $S \subset X \times Y$ be a time-system. Let C be an arbitrary set and $\rho_0 : C \times X \rightarrow Y$ a function. Then if ρ_0 satisfies the condition

$$(x, y)\in S \leftrightarrow (\exists c\in C)(y = \rho_0(c, x))$$

then ρ_0 and C are called an initial response function of S and a state space of S, respectively. If for any $c\in C$, $t\in T$, $x, x'\in X$,

$$x^t = x'^t \rightarrow \rho_0(c, x))|\bar{T}^t = \rho_0(c, x))|\bar{T}^t$$

then is said to be strongly causal. If a time system has a strongly causal initial response function then it is said to be strongly causal.

Let $\underline{X} = \{xt, x\_t, x_{tt'}, x\_{tt'}|x\in X \text{ and } t, t'\in T\}$. Let us define a partial function σ^t: $(\underline{X}) \rightarrow \underline{X}$ for $t\in T$ by

$$\sigma^t(x_{tt'})(\tau) = x_{tt'}(\tau - t) \text{ for } t'' + t \le \tau < t'' + t.$$

Then we call σ^t a time-shift operator. And for each $t\in T$, another operator λ^t: $(\underline{X}) \rightarrow \underline{X}$ defined by $\lambda^t(x) = \sigma^{-t}(x|T_t)$is called a left time-shift operator. It is extensively defined for y and (x, y). If S satisfies $\lambda^t(S)\subset S$ for any $t\in T$, then S is called stationary.

Define $S(x) = \{y \mid (x, y)\in S\}$. S(0) is called the system core of S.

The system we focus on in this paper is a basic linear system, which is defined as follows.

Definition 1 Basic linear system (Takahara 1981)

Let a time set T be R^+ or τZ^+ for some $\tau\in R^+$. A time system $S\subset X\times Y$ over a field $\underline{A}$ is called a basic linear system if the following conditions are satisfied.

(a) Linearity

$(\forall(x, y)\in S)\ (\forall(x', y')\in S)\ (\forall\ \alpha, \beta\in \underline{A})$

$(\ (\alpha x+\beta x', \alpha y+\beta y')\in S\ )$

(b) Stationarity

$(\forall t\in T)\ (\lambda^t(S)\subset S)$

(c) Strong pre-causality

$(\forall\ x, x'\in X)\ (\forall t\in T)\ (x^t = x'^t \rightarrow S(x)|\bar{T}^t = S(x')|\bar{T}^t)$

(d) Finite-dimensionality of system core

$\dim S(0) < \infty$

The concept of a basic linear system is a system-theoretical abstraction of a system described by linear differential or difference equations with constant coefficients.

One of the most profound properties of a basic linear system is that it has the unique state space representation up to isomorphism(Takahara 1981). Also the unique initial response function of a basic linear system $S\subset X\times Y$ exists, which is called the universal response function. Let ρ^*_0: $S(0)\times X \rightarrow Y$ be the universal initial response function of S, whose state space is S(0). The function ρ^*_0 consists of both the state response function $\rho^*_{10}: S(0) \rightarrow Y$ and the input response function $\rho^*_{20}: X \rightarrow Y$ and satisfies the following:

$$(x, y) \in S \leftrightarrow (\exists c \in S(0))(y = \rho^*_0(c, x) = \rho^*_{10}(c) + \rho^*_{20}(x))$$

Both of them are linear functions. Since ρ^*_{20} is strongly causal, that is, for any $t \in T$, x, $x' \in X$,

$$x^t = x'^t \rightarrow \rho^*_{20}(x))|\bar{T}^t = \rho^*_{20}(x))|\bar{T}^t$$

holds, so is ρ^*_0.

Though we do not mention in this paper what the state space representation or the isomorphism is, it is the existence of the unique state space representation for a basic linear system that makes behavioral and structural exploration of, for example, feedback, decoupling, inverse system, and canonical forms for linear systems possible.

For category notion we follow Strecker and Herrich(1979). A category is a quintuple $C = (O, M, dom, cod, \circ)$ where

(1) O is a class whose members are called C-objects,

(2) M is a class whose members are called C-morphisms,

(3) dom and cod are functions from M to O (dom(f) is called the domain of f and cod(f) is called the codomain of f),

(4) $\circ$ is a function from $D = \{(f, g) \mid f, g \in M \text{ and } dom(f) = cod(g)\}$ into M, called the composition law of C ($\circ(f, g)$ is written $f \circ g$ and we say that $f \circ g$ is defined if, and only if, $(f, g) \in D$) such that the following conditions hold:

(i) Matching Condition: If $f \circ g$ is defined, then $dom(f \circ g) = dom(g)$ and $cod(f \circ g) = cod(f)$;

(ii) Associativity Condition: If $f \circ g$ and $h \circ f$ are defined, then $h \circ (f \circ g) = (h \circ f) \circ g$;

(iii) Identity Existence Condition: For each C-object A there exists a C-morphism e such that $dom(e) = A = cod(e)$ and

(a) $f \circ e = f$ whenever $f \circ e$ is defined, and

(b) $e \circ g = g$ whenever $e \circ g$ is defined;

(iv) Smallness of Morphism Class Condition: For any pair (A, B) of C-objects, the class

$$hom_C(A, B) = \{f \mid f \in M, dom(f) = A \text{ and } cod(f) = B\}$$

is a set.

For each C-object A an identity is shown to be unique. So the identity of A is often written as 1_A. The class of C-objects will be denoted by Ob(C), whereas, Mor(C) will stand for the class of C-morphisms. The fact that a morphism f in $hom_C(A, B)$ is denoted as $f: A \rightarrow B$. A subcategory C' of C is a category whose object class Ob(C') and morphism class Mor(C') are subclass of Ob(C) and Mor(C), respectively.

A functor is a "mapping" from a category to another category. Let C and D be categories. A functor F: C→D consists of two functions; a function from Ob(C) to Ob(D) and a function F from Mor(C) to Mor(D) such that the following conditions are satisfied. (1) F preserves identities; i.e., if e is a C-identity, then F(e) is a D-identity. (2) F preserves composition; F(f∘g) = F(f)∘F(g); i.e., whenever dom(f) = cod(g), then dom(F(f)) = cod(F(g)) and the above equality holds.

A sink in category C is a pair $((f_i)_I, X)$, where X is a C-object and $(f_i : X_i \rightarrow X)_I$ is a family of C-morphisms each with codomain X. Let (I, F, C) be a functor. Then a natural sink $((k_i)_{i \in Ob(I)}, K)$ for F is a sink in C such that for each $i \in Ob(I)$, $k_i \in hom_C(F(i), K)$ and for all morphisms $m \in hom_I(i, j)$ the triangle

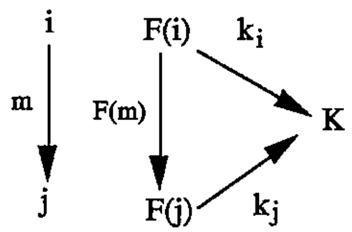

Fig. 1. natural sink of F

commutes. In this case $((k_i)_{i \in Ob(I)}, K)$ is called colimit of F provided that if $((k'_i)_{i \in Ob(I)}, K')$ is another natural sink for F then there exists a unique morphism $h : K \rightarrow K'$ such that for any $j \in Ob(I)$ the triangle

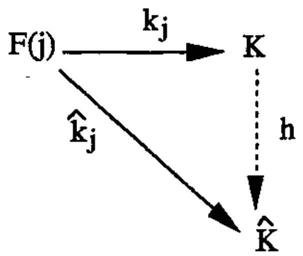

Fig. 2. colimt of F

commutes. That is $k'_j = h \circ k_j$.

3 SAMPLING

Two elementary methods of sampling are defined in this section. And they give two kinds of discrete-time systems.

3.1 sampling without holding

Let $S \subset X \times Y$ be a continuous-time basic linear system, and $x \in X$, $y \in Y$ and $\tau \in R^+$ be arbitrary. The simple sampling function d_τ is defined by the restriction of x or y to τZ^+. That is defined as $d_\tau(x)(t) = x(t)$ or $d_\tau(y)(t) = y(t)$ for any $t \in \tau Z^+$.

Definition 2 Simply sampled system

Let $S \subset X \times Y$ be a continuous-time system. Define the simple sampling function D_τ from $A^{R^+} \times B^{R^+}$ to $A^{\tau Z^+} \times B^{\tau Z^+}$ as $D_\tau(S) = \{(d_\tau(x), d_\tau(y)) \mid (x, y) \in S\}$. $D_\tau(S)$ is called simply sampled system.

The simple sampling is illustrated in Fig. 3.

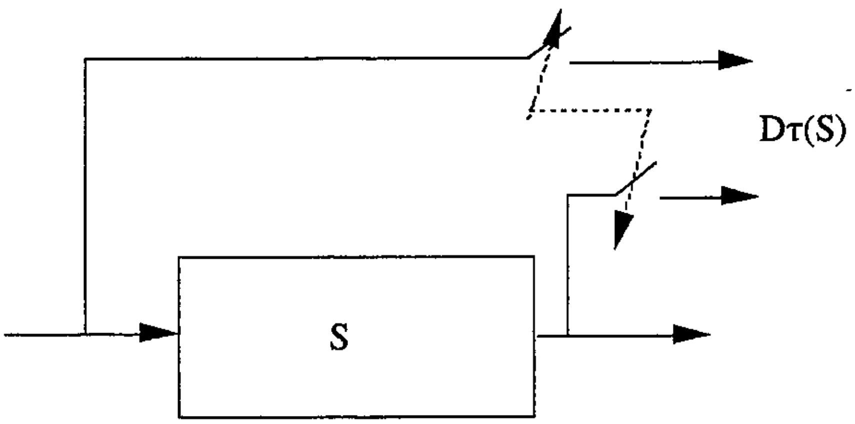

Fig. 3. Simple sampling

3.2 hold sampling

Let $S \subset X \times Y$ be a continuous-time system. Let $x \in X$ and $\tau \in T = R^+$ be arbitrary. The second method of sampling, the 0-th hold $e_\tau : X \rightarrow X$, is defined. We need the following to the definition. For any $n \in Z^+$

$$e_\tau(x)(n\tau+\sigma) = x(n\tau) \text{ for any } 0 \leq \sigma < \tau, \text{ if } \tau \neq 0;$$

$$e_0(x) = x, \text{ if } \tau = 0.$$

$_\tau X$ is defined to be $\cup_{t \in R^+} \lambda^t(e_\tau(X))$. A continuous-time stationary system is always assumed to satisfy $_\tau X \subset X$ for any $\tau \in T$. Notice that $_0X = X$ holds.

Definition 3 System with 0-th hold

The deformed system from S by e_τ is denoted as ${}_\tau S$. That is,

$(x, y) \in {}_\tau S \leftrightarrow x \in {}_\tau X$ and $(x, y) \in S$.

Notice that ${}_0 S = S$ holds and ${}_\tau S$ is a linear subspace of S if S is a linear system. 0-th hold-sampling is illustrated in Fig. 4.

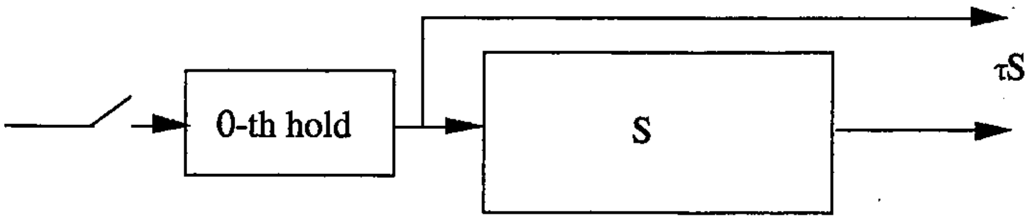

Fig. 4. 0-th hold

Definition 4 0-th hold-sampling system

Let $S \subset X \times Y$ be a continuous-time system and $\tau \in T$ be arbitrary. The 0-th hold-sampling system ${}_\tau \hat{S}$ of S is the restriction of ${}_\tau S$ to τZ^+. That is

$$ {}_\tau \hat{S} = D_\tau({}_\tau S) = \{(d_\tau(x), d_\tau(y)) \mid (x, y) \in {}_\tau S\}. $$

Fig. 5 shows the 0-th hold-sampling system of S. This sampling method is one of the traditional sampling methods in control theory.

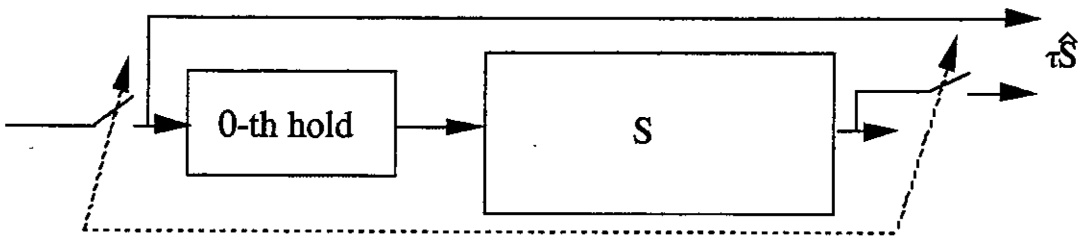

Fig. 5. 0-th hold-sampling

4 PRESERVED PROPERTIES

In this section some of preservations in sampling are considered. And each of ${}_\tau S$, ${}_\tau\hat{S}$ and $D_\tau(S)$ is shown to be a basic linear system with respective time set. The next proposition reveals a relation between two sampling methods, though it is trivially holds.

Proposition 1

Let $S \subset X \times Y$ be a continuous-time system. For any $\tau \in R^+$, $D_\tau(S) \supseteq {}_\tau\hat{S}$ holds.

The followings shows preservation of structural properties of basic linear systems.

Theorem 2

(i) ${}_\tau S$ is a basic linear system with continuous time scale.

(ii) ${}_\tau\hat{S}$ is a basic linear system with discrete time scale ${}_\tau Z^+$.

(iii) $D_\tau(S)$ is a basic linear system with discrete time scale ${}_\tau Z^+$.

The above theorem shows each of structural properties of a basic linear system is preserved by two types of sampling.

Corollary

Let $S \subset X \times Y$ be a continuous-time system. If ${}_\tau\hat{S}$ or $D_\tau(S)$ is not a basic linear system then S is not, either.

That is, if the sampled data does not have the properties of a basic linear system then the original system cannot be a basic linear system.

For a linear system the size of its system core $S(0) = \{y \mid (0, y) \in S\}$ decides its generic dynamic variety. For sampled systems the followings holds.

Proposition 3

Let $S \subset X \times Y$ be a continuous-time linear system. Then

$$\dim S(0) = \dim {}_\tau S(0) \geq \dim {}_\tau\hat{S}(0) = \dim D_\tau(S)(0).$$

5 EMBEDDING OF SAMPLED SYSTEMS INTO A CONTINUOUS-TIME SYSTEM

In this section the minimality of a continuous system that can embed all of sampled data $\{{}_\tau\hat{S} \mid \tau \in R^+\}$ is considered.

We define the category of continuous-time basic linear systems based on Takahara(1981). Firstly we define morphisms of basic linear systems.

Definition 5 <u>Basic linear morphisms</u> (Takahara 1981)

Let $S \subset X \times Y$, $S' \subset X' \times Y'$ be continuous-time basic linear systems. A pair of functions $<h_1, h_2>$, $h_1 : X \to X'$ and $h_2 : Y \to Y'$, is called a <u>basic linear morphism</u> from S to S' if it satisfies the following conditions:

(1) h_1 and h_2 are linear function,

(2) $(\forall (x, y) \in S)\ (\ (h_1(x), h_2(y)) \in S)$

(3) $(\forall t \in T)(\forall (x, y) \in S)\ (\ (h_1 \lambda^t(x), h_2 \lambda^t(y)) = (\lambda^t h_1(x), \lambda^t h_2(y))\ )$

(4) $(\forall t \in T, \forall x \in X, \forall y \in Y)\ (h_1(0^t \cdot x_t)|T^t = 0^t$ and $h_2(0^t \cdot y_t)|T^t = 0^t\ )$

If h_1 and h_2 are injective then called injective morphisms and S' is called an <u>injective model</u> of S (Takahara et al 1983). There is a category BLS_{R^+} whose objects are continuous-time basic linear systems and morphisms basic linear morphisms (Takahara 1981). In BLS_{R^+} the composition of morphisms $h = <h_1, h_2> \in hom_{BLS_{R^+}}(S, S')$ and $h' = <h'_1, h'_2> \in hom_{BLS_{R^+}}(S', S'')$ is defined by $h' \circ h = <h'_1 \cdot h_1, h'_2 \cdot h_2> \in hom_{BLS_{R^+}}(S, S'')$.

For an arbitrary object of BLS_{R^+} a category dS can be made whose objects are ${}_\tau\hat{S}$ for $\tau \in R^+$. We need the following

Proposition 4

Let $S \subset X \times Y$ be a continuous-time stationary system. Let τ and σ be arbitrary elements of R^+. If there exists a positive integer n such that $\tau = n\sigma$ then ${}_\tau X \subset {}_\sigma X$ holds.

Proposition 5

Let S be a continuous-time stationary system and τ and σ arbitrary elements of R^+. If there exists a positive integer n such that $\tau = n\sigma$ then ${}_\tau S \subset {}_\sigma S$ holds.

If the conditions of proposition 5 holds then there is the inclusion $i_{\tau\sigma} : {}_\tau S \to {}_\sigma S$ and $i_{\tau\sigma} \in hom_{BLS_{R^+}}({}_\tau S, {}_\sigma S)$ holds. The inclusion in $hom_{BLS_{R^+}}({}_\tau S, S)$ will be denoted by i_τ. A special category cS can be defined whose objects $Ob(cS) = \{{}_\tau S \mid \tau \in R^+\}$ and morphisms with

$$hom_{cS}({}_\tau S, {}_\sigma S) = \begin{cases} \text{empty, if } \sigma \neq 0 \text{ and } (\forall n\text{:positive integer})(n\sigma \neq \tau) \\ i_{\tau\sigma}, \text{ if } \sigma = 0 \text{ or } (\exists n\text{:positive integer})(n\sigma = \tau) \end{cases}$$

for any τ and σ in R^+. The category cS is a subcategory of BLS_{R^+}.

Let $\tau, \sigma \in R^+$ be arbitrary. If there exists ~~an~~ positive integer n such that $n\sigma = \tau$ or $\sigma = 0$ hold, then we define a mapping $f_{\tau\sigma}$ from ${}_{\tau}\hat{S}$ to ${}_{\sigma}\hat{S}$ by $f_{\tau\sigma}({}_{\tau}\hat{S}) = \{(d_\sigma(x), d_\sigma(y)) \mid (x, y) \in {}_{\tau}S\}$. The following shows that the codomain of $f_{\tau\sigma}$ is ${}_{\sigma}\hat{S}$.

Proposition 6

Let S be a continuous-time basic linear system and ${}_{\tau}\hat{S}$ and ${}_{\sigma}\hat{S}$ be 0-th hold-sampling system. Then $f_{\tau\sigma}({}_{\tau}\hat{S}) \subset {}_{\sigma}\hat{S}$ holds.

Based on this fact a category dS can be defined: $Ob(dS) = \{{}_{\tau}\hat{S} \mid \tau \in R^+\}$, and

$$hom_{dS}({}_{\tau}\hat{S}, {}_{\sigma}\hat{S}) = \begin{cases} \text{empty, if } \sigma \neq 0 \text{ and } (\forall n\text{:positive integer})(n\sigma \neq \tau) \\ f_{\tau\sigma}, \text{ if } \sigma = 0 \text{ or } (\exists n\text{:positive integer})(n\sigma = \tau) \end{cases}$$

Proposition 7

Let S be a continuous-time basic linear system. Then dS is a category.

The structure of dS is depicted in Fig . 6.

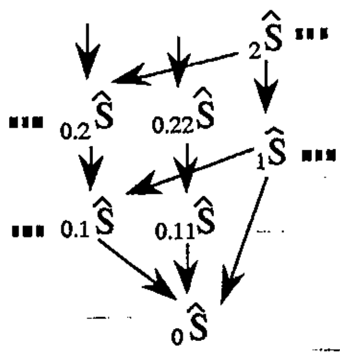

Fig. 6. structure of dS

Now define a functor (dS, C, BLS_{R^+}). C maps ${}_{\tau}\hat{S} \in Ob(dS)$ to ${}_{\tau}S \in Ob(BLS_{R^+})$, and $f_{12} \in hom_{dS}({}_{\tau 1}\hat{S}, {}_{\tau 2}\hat{S})$ to $i_{12} \in hom_{BLS_{R^+}}({}_{\tau 1}S, {}_{\tau 2}S)$. Notice that $C({}_0\hat{S}) = C(S|\{0\}) = {}_0S = S$.

Proposition 8

Let S be a continuous-time basic linear system. Then (dS, C, BLS_{R^+}) is a functor.

The functor C represents an embedding of a discrete-time system into continuous-time system. We call C the hold-embedding functor for S.

In order to compare the variety of systems behavior we introduce a partial order according to Takahara et al (1983). Define an equivalence relation on $Ob(BLS_{R^+})$ as follows: For any S and $S' \in Ob(BLS_{R^+})$, $S \equiv S'$ iff there are $h \in hom_{BLS_{R^+}}(S, S')$ and $k \in hom_{BLS_{R^+}}(S', S)$ such that both are injective. This binary relation $\equiv$ is an equivalence relation. We denote the quotient class of $Ob(BLS_{R^+})$ by $Ob(BLS_{R^+})/\equiv$. And then a partial order $\leq$ on $Ob(BLS_{R^+})/\equiv$ is defined: $[S] \leq [S']$ iff there exists an injective morphism $h \in hom_{BLS_{R^+}}(S, S')$. This order is well-defined and partial(Takahara et al, 1983). That is, it is reflexive, transitive and antisymmetric.

A relation between a continuous-time basic linear system and all of samples systems can be stated as follows.

Theorem 9

Let S be a continuous-time basic linear system. Let (dS, C, BLS_{R^+}) is the hold-embedding functor for S. Then $((i_\tau)_{\tau \in R^+}, S)$ is a colimit of C, where $i_\tau \in hom_{BLS_{R^+}}({}_\tau S, S)$ is the inclusion.

Corollary

Let S be a continuous-time basic linear system. For the set of discrete-time basic linear systems $\{{}_\tau\hat{S} \mid \tau \in R^+\}$ sampled from S, if S' is an injective model for each of $\{C({}_\tau\hat{S}) \mid \tau \in R^+\} = \{{}_\tau S \mid \tau \in R^+\}$ then there is an injective morphism of $hom_{BLS_{R^+}}(S, S')$. That is, S is the smallest model with respect to $\leq$ that can embed all of $\{{}_\tau\hat{S} \mid \tau \in R^+\}$.

6 CONCLUSION

In order to exploit relation between continuous-time and discrete-time linear systems, both preservation of properties through sampling and embedding of discrete-time basic linear systems into a continuous-time basic linear system have been considered.

The two methods of sampling defined preserves each of behavioral properties of basic linear systems. In other words the transformation, sampling, does not make an extraordinary property with respect to the structure of basic linear systems, and then if a sampled system is not a basic linear system then neither the original continuous-time system.

Also if we have the set of sampled data $\{_\tau\hat{S} \mid \tau \in R^+\}$, all of which are made from a continuous-time basic linear system S, then the original continuous-time system S is the smallest system that can produce the data.

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Appendix(proof)

(Proof of Proposition 1)

Let $(x', y') \in \tau\hat{S} = D_\tau(_\tau S)$ be arbitrary. Then there exists $(x, y) \in \tau S \subset S$ such that $(d_\tau(x), d_\tau(y)) = (x', y')$. Therefore we have $(x', y') \in D_\tau(_\tau S) \subset D_\tau(S)$.

(Proof of Theorem 2 (i))

It is clear that τX is strongly stationary. Let $\rho^*_0: S(0) \times X \to Y$ be the universal representation of S. Since $\tau S \subset S$ holds, if $(x, y)\tau S$ if and only if $(\exists c \in S(s))(y = \rho^*_{10}(c) +$

$\rho^*{}_{20}(x)$). The linearity and strong pre-causality of τS are clear from those of $\rho^*{}_0$. The finite-dimensionality dim $\tau S(0) < \infty$ holds because of the fact $\tau S(0) = S(0)$.

We will show that the strongly stationarity of τS. Let $t \in R^+$ be an arbitrary element. Firstly we show that $\lambda^t(\tau S) \subset \tau S$. Let $(x, y) \in \tau S$ be arbitrary. Then there exists $c \in \tau S(0) = S(0)$ such that $y = \rho^*{}_{10}(c) + \rho^*{}_{20}(x)$ holds. Since τX is strongly stationary there exists $x' \in \tau X$ such that $x' = \lambda^t(x)$. Let $c' = \lambda^t c + \lambda^t \rho^*{}_{20}(x) - \lambda^t \rho^*{}_{20}(x')$. Then $c' \in S(0)$ holds. Therefore $\lambda^t \rho^*{}_{10}(c) + \lambda^t \rho^*{}_{20}(x) = \rho^*{}_{10}(c') + \rho^*{}_{20}(x')$. Let $y' = \rho^*{}_{10}(c') + \rho^*{}_{20}(x')$. Then $\lambda^t y = y'$ and $(x', y') \in \tau S$. Thus $\lambda^t(\tau S) \subset \tau S$ holds.

Secondly we show $\tau S \subset \lambda^t(\tau S)$. Let $(x, y) \in \tau S$ be arbitrary. Then there exists $c \in \tau S(0) = S(0)$ such that $y = \rho^*{}_{10}(c) + \rho^*{}_{20}(x)$ holds. Since τX is strongly stationary there exists $x' \in \tau X$ such that $x = \lambda^t(x')$. Let $c' = \rho^*{}_{10}(c) + \rho^*{}_{20}(x) + \lambda^t \rho^*{}_{20}(x')$. Then Then $c' \in S(0)$ holds. Since $\lambda^t(S(0)) = S(0)$ holds, there exists $c'' \in S(0)$ such that $\lambda^t(c'') = c'$. Let $y' = \rho^*{}_{10}(c'') + \rho^*{}_{20}(x')$. Then $\lambda^t(y') = y$ and $(x, y) = \lambda^t(x', y')$ hold. Thus we have $\tau S \subset \lambda^t(\tau S)$. This concludes the proof.

(Proof of Theorem 2 (ii))

1) strong stationarity of $\tau\hat{X}$: Since τX is strongly stationary, so is $\tau\hat{X}$, that is, $\lambda^{n\tau}(\tau\hat{X}) \subset \tau\hat{X}$ holds for any $n \in Z^+$.

2) linearity of $\tau\hat{S}$: It is clear from the linearity of d_τ and ${}_\tau S$.

3) Strong pre-causality: Let $n \in Z^+$ and $\hat{x}, \hat{x}' \in {}_\tau\hat{X}$ be arbitrary. Assume $\hat{x}^{n\tau} = \hat{x}'^{n\tau}$ holds. Then there are x and x' in ${}_\tau X$ such that $d_\tau(x) = \hat{x}$ and $d_\tau(x') = \hat{x}'$. Let $\hat{y}$ be an arbitrary element of $\tau\hat{S}(\hat{x})$ and we will show $\hat{y}|\bar{T}^{n\tau} \in \tau\hat{S}(\hat{x}')|\bar{T}^{n\tau}$. Let $x'' = x^{n\tau} \cdot x'_{n\tau}$. Then we have $x'' \in {}_\tau X$ and $d_\tau(x'') = \hat{x}'$. $(\hat{x}, \hat{y}) \in \tau\hat{S} \rightarrow (\exists y)((x,y) \in \tau S$ and $\hat{y} = d_\tau(y)) \rightarrow (\exists c)(\tau S(0))(y = c + \rho^*{}_{20}(x))$. Let $y'' = c + \rho^*{}_{20}(x'')$. Then

$$
\begin{aligned}
\hat{y}|\bar{T}^{n\tau} &= d_\tau(c + \rho^*{}_{20}(x))|\bar{T}^{n\tau} \\
&= \{(c + \rho^*{}_{20}(x))|\tau Z^+\}|\bar{T}^{n\tau} \\
&= \{(c + \rho^*{}_{20}(x))|\bar{T}^{n\tau}\}|\tau Z^+ \\
&= \{c|\bar{T}^{n\tau} + \rho^*{}_{20}(x)|\bar{T}^{n\tau}\}|\tau Z^+ \\
&= \{c|\bar{T}^{n\tau} + \rho^*{}_{20}(x'')|\bar{T}^{n\tau}\}|\tau Z^+ \quad (\rho^*{}_{20} \text{ is strongly causal}) \\
&= \{(c + \rho^*{}_{20}(x''))|\tau Z^+\}|\bar{T}^{n\tau} \\
&= \{y''|\tau Z^+\}|\bar{T}^{n\tau} \\
&= d_\tau(y)|\bar{T}^{n\tau}
\end{aligned}
$$

Thus we have $(x'', y'') \in \tau S$ and $d_\tau(x'', y'') = (\hat{x}', d_\tau(y''))$. Therefore $d_\tau(y'') \in \tau\hat{S}(\hat{x}')$ holds. Since $\hat{y}|\bar{T}^{n\tau} = (d_\tau(y''))|\bar{T}^{n\tau} = \tau\hat{S}(x'^{\wedge})|\bar{T}^{n\tau}$, we have shown that $\tau\hat{S}(\hat{x}')|\bar{T}^{n\tau} \subset \tau\hat{S}(\hat{x}')|\bar{T}^{n\tau}$. The opposite inclusion can be proved in a similar way.

4) Finite-dimensionality of the system core: Since the restriction $d_\tau|_{\tau S(0)}: \tau S(0) \rightarrow \tau\hat{S}(0)$ is surjective, $\dim \tau\hat{S}(0) \leq \dim \tau S(0) = \dim S(0) < \infty$.

5) Strong stationarity: Let $n \in Z^+$ and $(\hat{x}, \hat{y}) \in \tau\hat{S}$ be arbitrary. Then there exists $(x, y) \in \tau S$ such that $d_\tau(x, y) = (\hat{x}, \hat{y})$. $\lambda^{n\tau}(\hat{x}, \hat{y}) = \lambda^{n\tau}(d_\tau(x, y)) = d_\tau(\lambda^{n\tau}(x, y))$. Since $\lambda^{n\tau}(\tau S) \subset \tau S$ this shows $\lambda^{n\tau}(\hat{x}, \hat{y}) \in \tau\hat{S}$ and then $\lambda^{n\tau}(\tau\hat{S}) \subset \tau\hat{S}$.

(Proof of Theorem 2 (iii))

Since τS is a continuous-time basic linear system, it can be proved in the same way as Theorem 2, (ii).

(Proof of Proposition 3)

Clearly $\tau\hat{S}(0) = D_\tau(S)(0)$ holds. The other part of proof is shown in 4) of the proof for Theorem 2 (ii).

(Proof of Proposition 4)

Let $S \subset X \times Y$ be a continuous-time stationary system. Assume $\tau = n\sigma$ holds. Let $x \in {}_\tau X$ be arbitrary. It suffices to show that $x \in {}_\sigma X$ holds.

By definition of ${}_\tau X$ there exists $t \in R^+$ and $x' \in X$ such that $x = \lambda^{t} \cdot e_\tau(x')$. Since $e_\tau(x') \in {}_\tau X \subset X$ holds, if $\lambda^{t} \cdot e_\sigma \cdot e_\tau(x')(t') = x'(t')$ holds for any $t' \in R^+$, then we have $x \in {}_\sigma X$. Let $t' \in R^+$ be arbitrary. There exist $p \in Z^+$ and σ', $0 \leq \sigma' < \tau$, such that $t' + t = p\tau + \sigma'$ holds. Also there exist $r \in Z^+$ and σ'', $0 \leq \sigma'' < \sigma$, such that $\sigma' = r\sigma + \sigma''$ holds.

$$\begin{aligned} x(t') &= \lambda^{t} \cdot e_\tau(x')(t') \\ &= e_\tau(x')(t'+t) \\ &= e_\tau(x')(p\tau + \sigma') \\ &= x'(p\tau) \end{aligned}$$

$$\begin{aligned} \lambda^{t} \cdot e_\sigma \cdot e_\tau(x')(t') &= e_\sigma \cdot e_\tau(x')(t'+t) \\ &= e_\sigma \cdot e_\tau(x')(t'+t) \\ &= e_\sigma \cdot e_\tau(x')(pn\sigma + r\sigma + \sigma'') \\ &= e_\tau(x')(pn\sigma + r\sigma) \\ &= x'(p\tau) \end{aligned}$$

Thus we have $\lambda^{t} \cdot e_\sigma \cdot e_\tau(x') = x'$, and this concludes the proof.

(Proof of Proposition 5)

Assume $\tau = n\sigma$ holds. Let $(x, y) \in {}_\tau S$ be arbitrary. Then $x \in {}_\tau X$ and $(x, y) \in S$ holds. Since ${}_\tau X \subset {}_\sigma X$ by proposition 4, this implies $x \in {}_\sigma X$ and $(x, y) \in S$. That is, ${}_\tau S \subset {}_\sigma S$.

(Proof of Proposition 6)

Assume that either $\tau = 0$ or $\tau = n\sigma$ holds for some positive integer n. In both cases ${}_\tau S \subset {}_\sigma S$ holds. Then

$f_{\tau\sigma}({}_\tau\hat{S}) = \{(d_\sigma(x), d_\sigma(y)) | (x, y) \in {}_\tau S\} \subset \{(d_\sigma(x), d_\sigma(y)) | (x, y) \in {}_\sigma S\} = {}_\sigma\hat{S}$.

(Proof of Proposition 7)

Let the composition of morphisms be the composition of functions. Then if we assume $cod(f_{\tau\sigma}) = dom(f_{\sigma\upsilon})$ then $f_{\tau\upsilon} = f_{\sigma\upsilon} \cdot f_{\tau\sigma}$ holds. It suffices to show the four conditions on composition. The matching and associativity conditions hold clearly. For any object ${}_\tau S$ the identity function $i_{\tau\tau}$ on it is the identity morphism. Let ${}_\tau S$ and ${}_\sigma S$ be arbitrary. Since $hom_{dS}({}_\tau S, {}_\sigma S)$ has one element at most, the smallness condition holds. This concludes the proof.

(Proof of Proposition 8)

Let ${}_\tau S \in Ob(dS)$ be arbitrary. $C(1_{{}_\tau S}) = C(f_{\tau\tau}) = i_{\tau\tau} = 1_{C({}_\tau S)}$. That is, C preserves identities. Assume $f_{\sigma\upsilon} \cdot f_{\tau\sigma}$ is defined. Then we have that $dom(f_{\tau\upsilon}) = {}_\tau\hat{S}$, $cod(f_{\sigma\upsilon}) = {}_\tau\hat{S}$, $dom(C(f_{\tau\upsilon})) = dom(i_{\tau\upsilon}) = {}_\tau S$, $cod(C(f_{\sigma\upsilon})) = cod(i_{\sigma\upsilon}) = {}_\tau S$. On the other hand we have $C(f_{\sigma\upsilon} \cdot f_{\tau\sigma}) = C(f_{\tau\upsilon}) = i_{\tau\upsilon}$ and $C(f_{\sigma\upsilon}) \circ C(f_{\tau\sigma}) = i_{\sigma\upsilon} \circ i_{\tau\sigma} = i_{\tau\upsilon}$. Then $C(f_{\sigma\upsilon} \cdot f_{\tau\sigma}) = C(f_{\sigma\upsilon}) \circ C(f_{\tau\sigma})$ holds. This concludes the proof.

(Proof of Theorem 9)

1) Since (dS, C, BLS_{R^+}) is a functor, $((i_\tau)_{\tau \in R^+}, S)$ is a natural sink of C.

2) Let $((k_\tau)_{\tau \in R^+}, K)$ is an arbitrary natural sink of C. Define $h = k_0$. Then $k_l = k_0 \circ i_{l0} = k_0 \circ i_l = h \circ i_l$ hold for any τ_l. That is the diagram commutes.

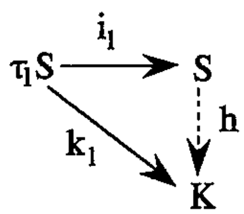

Fig. 7. commutative diagram

It suffices to show the uniqueness of h. Assume that there is a morphism $\bar{h} \in \hom_{BLS_{R^+}}(S,K)$ such that $k_1 = \bar{h} \circ i_1$ hold for any τ_1. Let $\tau_1 = 0$. Then we have $\bar{h} = k_0 = h$.

(Proof of Corollary to Theorem 9)

For any $\tau \in R^+$, $k = h \circ i_\tau$ holds. Let $\tau = 0$. Assume h(s) = h(s') holds. Since i_0= i(identity) on S, then h(i(s)) = h(s) = h(s') = h(i(s')). So k(s) = k(s'). Therefore s = s', and h is injective. Thus S ≤ S'.