79-10 (No.46)

The Optimal Progressive Income Tax - the Existence and the Limit Tax Rates

by

Mamoru Kaneko *)

July 1979

Abstract: The purpose of this paper is to consider the problem of optimal income taxation in the domain of progressive income tax functions. We prove the existence of an optimal tax function and that the optimal marginal and average tax rates tend to 100-percents as income level becomes high.

*). Dr. Mamoru Kaneko, Institute of Socio-Economic Planning, University of Tsukuba, Sakura-mura, Niihari-gun, Ibaraki-ken 300-31, Japan

1. Introduction

1). It is commonly recognized that there exists a conflict between equity and efficiency in income taxation. Progressive income taxation is a device to enhance the degree of equity. A highly progressive income taxation, however, obstructs labor incentives of individuals with high productivities or earning abilities, and so it decreases the degree of efficiency. Therefore optimal income taxation should be determined to harmonize these contrary norms. To discuss this problem, Mirrlees provided a model of optimal income taxation in his pioneering work [12]. Since Mirrlees [12], many authors considered this problem in variations of Mirrlees' model.

There are two directions of studies from a technical point of view. One is the approach initiated by Mirrlees himself, where income tax functions of any forms are allowed. In this approach, the method of the calculus of variations or Pontryagin's Maximum Principle is used. Another one is the approach initiated by Sheshinski [19], where only linear tax functions are allowed. The first approach is superior to the second to consider the progressiveness of an optimal income tax function, though there are many problems arising in the optimal taxation which can be discussed in the domain of linear tax functions without losing the essence. The first approach, however, has a weak point, as Mirrlees himself pointed it out. It is the lack of mathematical rigorousness. That is, though several necessary conditions for an optimal tax function are discussed under the assumption of the existence of an optimal income tax function, the existence is doubtful in the domain of arbitrary functions. Further the differentiability of several variables derived from an optimal tax function is also doubtful.[1)] The second approach does not have such a weak point, though we can not fully consider the progressiveness of optimal income tax functions. These two approaches are in completentary relationship but have weak points respectively.

2). In this paper, we shall consider the problem of optimal income taxation in the domain of progressive tax functions. Of course, the progressive tax functions include the linear and proportional tax functions. The model we shall provide is a variation of Mirrlees's model, where we take a public good - government's service - into account explicitly. The merit of our approach is to be able not only to discuss rigorously the existence of an optimal income tax function but also to consider fully it's progressiveness. We shall employ the Nash social welfare function of Kaneko and Nakamura [7] and Kaneko [8] as the welfare criterion, i.e., the government's objective function in the model.[2)]

In Section 2, we shall formulate the model of optimal income

---

1). Mirrlees [12,Section 4] pointed this out. We shall discuss this problem in Section 4.

2). When the Nash social welfare function is replaced by the utilitarian welfare function, the result of this paper can be still gained without essential change.

taxation. The existence theorem of optimal income tax function shall be stated under relatively weak assumptions. The proof of the theorem will be provided in Section 5.

In Section 3,we shall provide a limit theorem on the optimal marginal and average tax rates. The theorem states that the optimal marginal and average tax rates tend to 100-percents as income level becomes high. This theorem shall be proved in Section 6 under stronger assumptions than those used in the proof of the existence theorem.

2. Model and Problem

3). $(X, \mathcal{B}, \mu)$ is a measure space of all individuals, where X is the set of all individuals, $\mathcal{B}$ a σ-algebra of subsets of X and μ a measure on $\mathcal{B}$ with $0 < \mu(X) < +\infty$. We assume that $\{i\} \in \mathcal{B}$ for all $i \in X$ and $\mu(\{i\}) = 0$ for all $i \in X$.

We assume that leisure, a consumption good and a public good enter the individuals' utility functions $U^i(a,b,Q)$, ($i \in X$), where a denotes leisure time, b a level of the consumption good and Q a level of the public good supplied by an economic agent called "government". Every U^i ($i \in X$) is defined on $Y = [0,L] \times E^2_+$ where $L > 0$ is the initial endowment of leisure time and E^2_+ the nonnegative orthant of the 2-dimensional Fuclidean space E^2.
We assume:

(A): For all $i \in X$, $U^i(a,b,Q)$ is a monotonically increasing, continuous and strictly quasi-concave function of (a,b,Q).

(B): For each $(a,b,Q) \in Y$, $U^i(a,b,Q)$ is a measurable function of i.

(C): For each $i \in X$, $U^i(a,b,Q)$ is bounded, i.e., for some M^i $U^i(a,b,Q) \leqq M^i$ for all $(a,b,Q) \in Y$.[3]

Assumption (A) is standard in the equilibrium analysis, so no explanation is necessary. Assumption (B) is just a technical condition, which is also familiar in the theory of market with

---

3). The boundedness from below follows Assumption (A).

a continuum of traders.4) Assumption (C) is the boundedness of utility functions. The boundedness is usually justified by St. Petersburg paradox and is thought of as a natural assumption.5) This assumption is necessary for the sake of use of the Nash social welfare function for a measure space of individuals of Kaneko [8] as the welfare criterion.

Of course, we assume that the utility functions $U^i(a,b,Q)$ ($i \in X$) are measurable utility functions in the sense of von Neumann and Morgenstern [21]. In this paper we do not consider any probability mixtures but only "pure" states. Since pure states are thought of as special cases of mixtures, it makes sense to assume von Neumann-Morgenstern utility functions. If necessary, the government can rotate a roulette and yield probability mixtures. But if the utility functions are concave, only a pure state appears as a result of our theory because of the use of the Nash social welfare function as the government's objective function.

We permit positive linear transformations of U^i ($i \in X$). Exactly speaking, when $\alpha(i)$ and $\beta(i)$ are measurable functions of i with $\alpha(i) > 0$ for all $i \in X$, we can employ

$$(2.1) \quad V^i(a,b,Q) = \alpha(i)U^i(a,b,Q) + \beta(i) \quad \text{for all } i \in X \text{ and all } (a,b,Q) \in Y$$

as the same utility functions. Hence we can gain utility functions V^i ($i \in X$) satisfying the following (2.2) by certain positive

4). See Aumann [1].

5). See Owen [15,Chap.VI], Shapley [17,18] or Aumann [2].

transformations:

(2.2) $V^i(a,b,Q) - V^i(0,0,0)$ is uniformly bounded, i.e., there is an M such that $V^i(a,b,Q) - V^i(0,0,0) \leqq M$ for all $i \in X$ and all $(a,b,Q) \in Y$.

We shall confine us to the class of utility functions satisfying (2.2) for the sake of integrability of the Nash social welfare function.

Further we assume:

(D): For any $\varepsilon > 0$, there is a $\delta > 0$ such that $U^i(\varepsilon,0,0) - U^i(0,0,0) > \delta$ for all $i \in X$.

This assumption is a kind of uniformness of utility functions. This assumption is not necessarily preserved for arbitrary positive linear transformations. In the following we allow only the positive linear transformations satisfying

(2.3) For some $\varepsilon > 0$ and $M > 0$, $\varepsilon \leqq \alpha(i) \leqq M$ for all $i \in X$.

That is, when $\alpha(i)$ satisfies (2.3), the new V^i gained from U^i satisfying (2.2) and (D) also satisfies (2.2) and (D) .[6)]

The following lemma shall be necessary to define the Nash social welfare function.

Lemma 1. Let $(a(i),b(i),Q(i))$ be a measurable function of i such that $(a(i),b(i),Q(i)) \in Y$ for all $i \in X$. Then $U^i(a(i),b(i),Q(i))$ is a measurable function of i .

---

6). Assumptions (A) and (B) are preserved for arbitrary positive linear transformations.

Proof. See Appendix.

4). Each individual $i \in X$ owns a labor production function $f^i(x)$. That is, if he works for time x, he can provide a quantity of labor $f^i(x)$. For simplicity, we assume that $f^i(x)$ coincides with the quantity of the consumption good produced by his labor $f^i(x)$, being independent of the other individuals' labors.[7] We assume:

(E): For all $i \in X$, there is an L^i ($0 < L^i < L$) such that $f^i(x)$ is monotonically increasing on $[0,L^i]$ and is nonincreasing on $[L^i,L]$ with $f^i(0) = 0$.

(F): For all $i \in X$, $f^i(x)$ is a continuous and concave function.

(G): For each $x \in [0,L]$, $f^i(x)$ is a measurable function of i.

(H): $\int_X f^i(L^i)\,d\mu < +\infty$ and for some $c_o > 0$, $c_o \leqq L^i \leqq L-c_o$ for all $i \in X$.[8]

Assumption (E) means that there is an interior point L^i at which i's labor productivity is saturated, i.e., his marginal productivity is zero. Assumption (F) is a standard condition. Assumption (G) is just a technical condition. Assumption (H) is the integrability of $f^i(L^i)$ and a kind of uniformness of L^i.

---

7). For example, labor can be measured in terms of the unit of man-power/hour.

8). It is assured by (E),(F) and (G) that L^i and $f(x(i))$ are measurable functions of i where x(i) is a measurable function with $x(i) \in [0,L]$ for all $i \in X$, which are proved similarly to Lemma 1 and 2.

The government produces the public good using the consumption good as input. We assume that the public good is measured in terms of the consumption good needed to produce it. In other words, the cost function of the public good is $C(Q) = Q$. The government's expenditure coincides with Q when it plans to supply Q-amount of the public good. The government's revenue is determined by a tax function and a level of the public good. 9)

5). We have completedto describe the economic circumstance where we shall work. We are in a position to discuss the problem of optimal taxation.

A <u>tax function</u> T is a real-valued function on the set of nonnegative real numbers E_+ which satisfies

(2.4) $T(y)$ is a monotonically nondecreasing and convex function such that $T(y) \leqq y$ for all $y \in E_+$. 10) 11)

We denote, by $\mathcal{T}$, the set of all tax functions.

---

9). Ito and Kaneko [6] showed that when cost functions of public goods were linearized by measuring the public goods in terms of the costs themselves, ratio equilibrium wasinvariant for such a transformation but not Lindahl equilibrium. The equilibrium concept of this paper has the same property, i.e., is invariant for the linearization of the cost function of the public good. Hence our assumption $C(Q) = Q$ does not lose any generality.

10). This condition implies that $T(y)$ is a continuous function.

11). The monotonicity is not essential, because we proceed the following discussion replacing it by the continuity.

A tax function $T(y)$ means that when an individual i works for time x and earns income $y = f^i(x)$, he must pay income tax $T(y) = Tf^i(x)$ to the government. Hence T must satisfy $y \geqq T(y)$ for all $y \in E_+$. We should note that we admit negative tax, i.e., subsidies. We take income redistribution directly into account beside the supply of the public good. The convexity of tax functions means just the progressiveness of income taxation. In this paper, we admit only progressive income tax functions. But it should be noted that $\mathcal{T}$ includes the proportional and linear tax functions.

Let us suppose that the government employs a tax function T and decides to supply Q-amount of the public good. When each individual $i \in X$ decides to work for time $x(i)$, his gross income is $f^i x(i) = f^i(x(i))$ and his disposable income is $f^i x(i) - Tf^i x(i)$. In this case, the government's revenue is $\int_X Tf^i x(i) \, d\mu$. Since Q is the government's expenditure, it must hold that

$$(2.5) \qquad Q \leqq \int_X Tf^i x(i) \, d\mu .$$

With above explanation in mind, we provide the following definitions. Every individual i maximizes his utility function under the assumption that T and Q are fixed. That is, individual i chooses $x(i) \in [0, L]$ such that

$$(2.6) \qquad U^i(L-x(i), f^i x(i) - Tf^i x(i), Q) \geqq U^i(L-x, f^i(x) - Tf^i(x), Q) \quad \text{for all } x \in [0, L] .$$

Let T_o be the function such that $T_o(y) = 0$ for all $y \in E_+$. Then $(T_o,0)$ is always a feasible tax schedule and an equilibrium tax schedule. $(T_o,0)$ means that the government neither imposes any tax nor supplies the public good. This is a feasible behavior of the government. Hence we have:

Proposition 1. There exist a feasible tax schedule and an equilibrium tax schedule.

The following proposition states that when T is a feasible tax functions, it is always possible to achieve the equation of the government's revenue and expenditure. As we are not directly concerned in this affair, we do not give the proof.

Proposition 2. For every feasible tax function T, there exists a $Q \in E_+$ such that (T,Q) is an equilibrium tax schedule.

Further it is not difficult to verify the following proposition.

Proposition 3. Let T be a feasible tax function. Then:

(i). If $T(0) < 0$, then $T(y) > 0$ for some $y \in E_+$.

(ii). If $\left.\frac{d^-T}{dy}\right|_{y_o} = 1$, then $T(y_o) > 0$. [12]

---

12). Since T is convex and continuous with $y \geqq T(y)$ for all $y \in E_+$, T has the derivatives on the left and the right $\frac{d^-T}{dy}$ and $\frac{d^+T}{dy}$ with $\left.\frac{d^-T}{dy}\right|_y \leqq \left.\frac{d^+T}{dy}\right|_y \leqq 1$ for all $y \in E_+$.

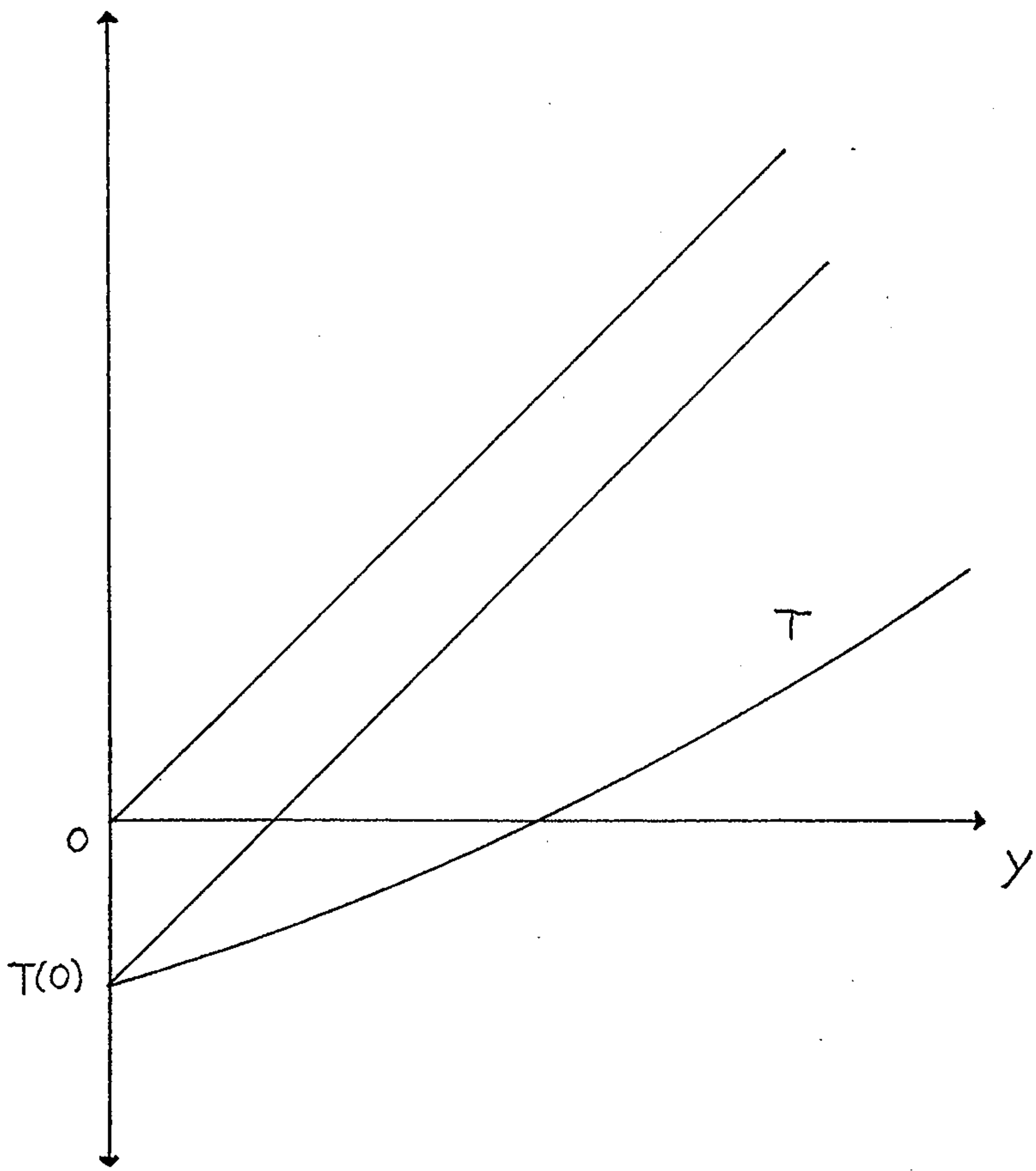

Figure 1.

6). We employ the Nash social welfare function of Kaneko and Nakamura [7] and Kaneko [8] as the government's objective function. To define the Nash social welfare function, it is necessary to set the "origin" in our circumstance. The concept is provided by Kaneko and Nakamura [7] as the socially worst state, which plays the most important role in the theory of the Nash social welfare function. In the circumstance of this paper it is natural to set the origin of Y as the "origin", i.e.,

$$(2.7) \quad O(i) = (0,0,0) \quad \text{for all } i \in X .$$

Let $\tau = (T,Q) \in \mathcal{F}$ and let $(x_\tau(i))_{i \in X}$ be the labor time supplies for (T,Q). Then the Nash social welfare function $W(\tau) = W(T,Q)$ is given as

$$(2.8) \quad W(\tau) = W(T,Q) = \int_X \log(U^i(L-x_\tau(i), f^i x_\tau(i) - Tf^i x_\tau(i), Q) - U^i(0,0,0))\, d\mu \quad {}^{13)}$$

We can assume $U^i(0,0,0) = 0$ for all $i \in X$ without loss of generality. For simplicity, we may write

$$G(L-x_\tau(i), f^i x_\tau(i) - Tf^i x_\tau(i), Q) = \log U^i(L-x_\tau(i), f^i x_\tau(i) - Tf^i x_\tau(i), Q)$$

in the following .

---

13). (2.2), Assumptions (D) and (H) ensure that this function is integrable for all $\tau \in \mathcal{F}$.

The government's objective is to maximize the Nash social welfare function $W(\tau)$. That is, the government chooses a feasible tax schedule $\tau^* = (T^*, Q^*)$ such that

$$(2.9) \quad \max_{\tau \in \mathcal{F}} W(\tau) = W(\tau^*) .$$

We call $\tau^* = (T^*, Q^*) \in \mathcal{F}$ satisfying (2.9) an <u>optimal tax schedule</u>.

The purpose of this paper is to investigate the optimal tax schedules. The first result of this paper is the existence of an optimal income tax schedule. The proof of Theorem I will be provided in Section 5.

<u>Theorem I</u>.(<u>Existence Theorem</u>). Under Assumptions (A)-(H), there exists an optimal tax schedule $\tau^* = (T^*, Q^*)$, which is an equilibrium tax schedule .

<u>7</u>). The concept of optimal tax schedule defined in the above can be also interpreted as noncooperative equilibrium point of a game in extensive form in which the government and the individuals appear as players. The game is formulated as follows. First, the government decides and announces a tax schedule (T,Q) to the individuals. Then every individual independently decides his labor time x(i). The game tree is drawn in Figure 2. The individuals' payoffs are the utilities gained from (T,Q) and x(i)'s . If the government's revenue is smaller than it's expenditure Q , then it suffers a punishment P, which is sufficiently large. In this case, the government suffers P but can supply Q. The government' payoff is

(2.10) Nash social welfare $- \delta P$,

where $\delta = 1$ if the revenue is smaller than Q and $\delta = 0$ otherwise. We complete to define the game in extensive form. Let τ^* be an optimal tax schedule and let $(x^*(i))_{i \in X}$ be the labor time supplies for τ^*. Then $(\tau^*, (x^*(i))_{i \in X})$ is a subgame perfect equilibrium point of the game and vice versa.[14)]

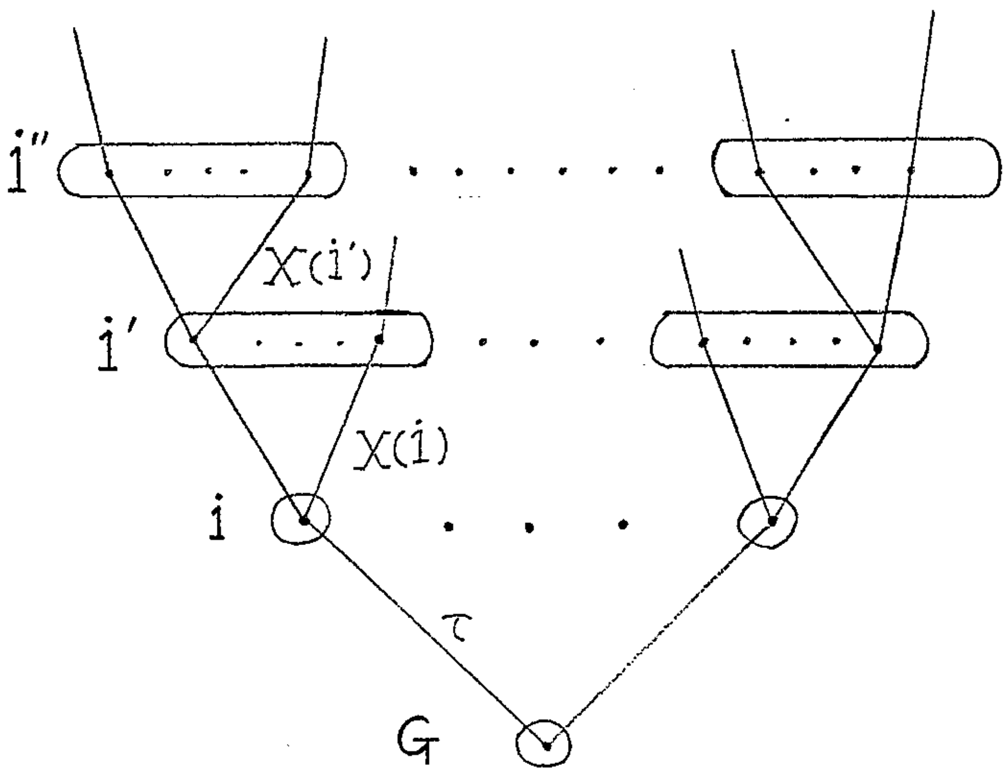

Figure 2.

14). For definition of subgame perfect equilibrium, see Selten [16] .

3. The Limit Marginal and Average Income Tax Rates

8). In the previous section we have shown the existence of an optimal income tax schedule. Our next task is to investigate the shape of the optimal income tax schedule. As we have assumed the progressiveness and the monotonicity on income tax functions, it is not so valuable to investigate local properties of the optimal income tax function. Hence we shall consider the problem of what level the optimal marginal and average tax rates tend to when income level becomes high. The result which we shall prove is the theorem that both the optimal marginal and average tax rates tend to 100 percents. To prove this theorem, we add new assumptions and replace some of the assumptions imposed above by stronger ones.

9). Initially we assume:

(A'): For all $i \in X$, $U^i(a,b,Q)$ is a monotonically increasing, continuously differentiable and strictly concave function of (a,b,Q).

(I): For all $i \in X$, $U^i(a,b,Q)$ is weakly separable with respect to (a,b) and Q, i.e., there are functions g^i and h^i such that $U^i(a,b,Q) = h^i(g^i(a,b),Q)$ for all $(a,b,Q) \in Y$.

(J): For each $Q \geqq 0$, $U^i(L-L^i,b,Q)$ converges uniformly to $M_i \equiv \sup_b U^i(L-L^i,b,Q)$ as $b \to \infty$,i.e., for any $\varepsilon > 0$, there is a b_o such that $U^i(L-L^i,b,Q) \geqq M_i - \varepsilon$ for all $b \geqq b_o$ and all $i \in X$.

(K): For each $Q \geqq 0$, $\left.\frac{1}{b}\cdot\frac{U_1^i}{U_2^i}\right|_{(L-L^i,b,Q)}$ converges uniformly to 0 as $b \to \infty$,i.e., for any $\varepsilon > 0$, there is a b_o such that

$$\left.\frac{1}{b}\cdot\frac{U_1^i}{U_2^i}\right|_{(L-L^i,b,Q)} < \varepsilon \qquad \text{for all } b \geqq b_o \text{ and all } i \in X .$$

Here $U_1^i = \frac{\partial U^i}{\partial a}$, $U_2^i = \frac{\partial U^i}{\partial b}$ and $U_3^i = \frac{\partial U^i}{\partial Q}$.

Though Assumption (A') is stronger than (A), it would be also a standard condition . Assumption (I) means that when $i \in X$ decides his labor time supply, the decision is not influenced by the level of the public good supplied by the government. Since in our model income redistribution is taken into account in tax functions, our public good is just a pure public good like national defense , fire fighting service, etc. . Hence it is not unnatural to assume that the level of the public good does not influence the individual choice of leisure and consumption. Assumption (J) is a kind of uniform boundedness, which is stronger than Assumption (C) and (2.2) . When the utility functions U^i can become identical by positive linear transformations, this assumption, of course, is true. Assumption (K) means that though the marginal rate of substitution of leisure and consumption may tend to infinity as b tends to infinity, the order is smaller than that of b . If this assumption is not true, it happens that the marginal rate of substitution at $(L-L^i,b,Q)$ has magnitude of the same order or greater order than b , which would be implausible. The utility functions which are represented as

$$(3.1)\quad g^i(a,b) = a^\alpha + b^\beta \quad \text{for all } (a,b) \in [0,L] \times E_+ \text{ and } 0 < \alpha, \beta < 1$$

satisfy Assumption (K), but the Cobb-Douglas type functions $g^i(a,b) = a^\alpha b^\beta$ for all $(a,b) \in [0,L] \times E_+$ do not satisfy this assumption . Hence Assumption (K) restricts our consideration to a certain extent.

Next we approximate the labor production functions f^i's by piecewise linear functions. Let n(i) be a measurable function from X to E_+ . We assume :

(E'): For all $i \in X$, f^i satisfies

$$f^i(x) = \begin{cases} n(i)x & \text{if } x \leqq L^i \\ n(i)L^i & \text{if } x \geqq L^i . \end{cases}$$

(L): If $S \subset E_+$ & $\sigma(S) > 0$, then $\mu(\{i \in X : n(i)L^i \in S\}) > 0$, and for any $\varepsilon > 0$, there is a $\beta > 0$ such that

$$\lim_{\alpha \to \infty} \frac{\mu(\{i \in X : n(i)L^i > \alpha + \varepsilon\})}{\mu(\{i \in X : n(i)L^i > \alpha\})} = \beta \,.$$ [16)]

The function $n(i)$ assigns to each individual i his marginal productivity of labor. Assumption (E') means that f^i's ($i \in X$) are approximated by the piecewise linear functions . See Figure 3. This assumption implies (F) and (G) . Assumption (L) means that the distribution of the abilities $n(i)$'s overspreads everywhere of E_+ , and that the density of $n(i)$ converges not rapidly to 0 as $n(i) \to \infty$. When L^i is a constant, i.e., $L^i = L^o$ for all $i \in X$, this limit property is satisfied by many distributions, e.g., the Pareto distribution, the normal distribution, etc..

We are now in a position to state the main result of this section. We shall prove the following theorem in Section 6 .

<u>Theorem II. (Limit Tax Rates Theorem)</u>: Let (T^*, Q^*) be an optimal income tax schedule. Then it holds under Assumptions (A')-(L) that

$$(3.2) \qquad \lim_{y \to \infty} \frac{d^- T^*(y)}{dy} = \lim_{y \to \infty} \frac{d^+ T^*(y)}{dy} = 1 \,.$$

$$(3.3) \qquad \lim_{y \to \infty} \frac{T^*(y)}{y} = 1 \,.$$

---

16). σ denotes the usual Lebesgue measure on E_+ .

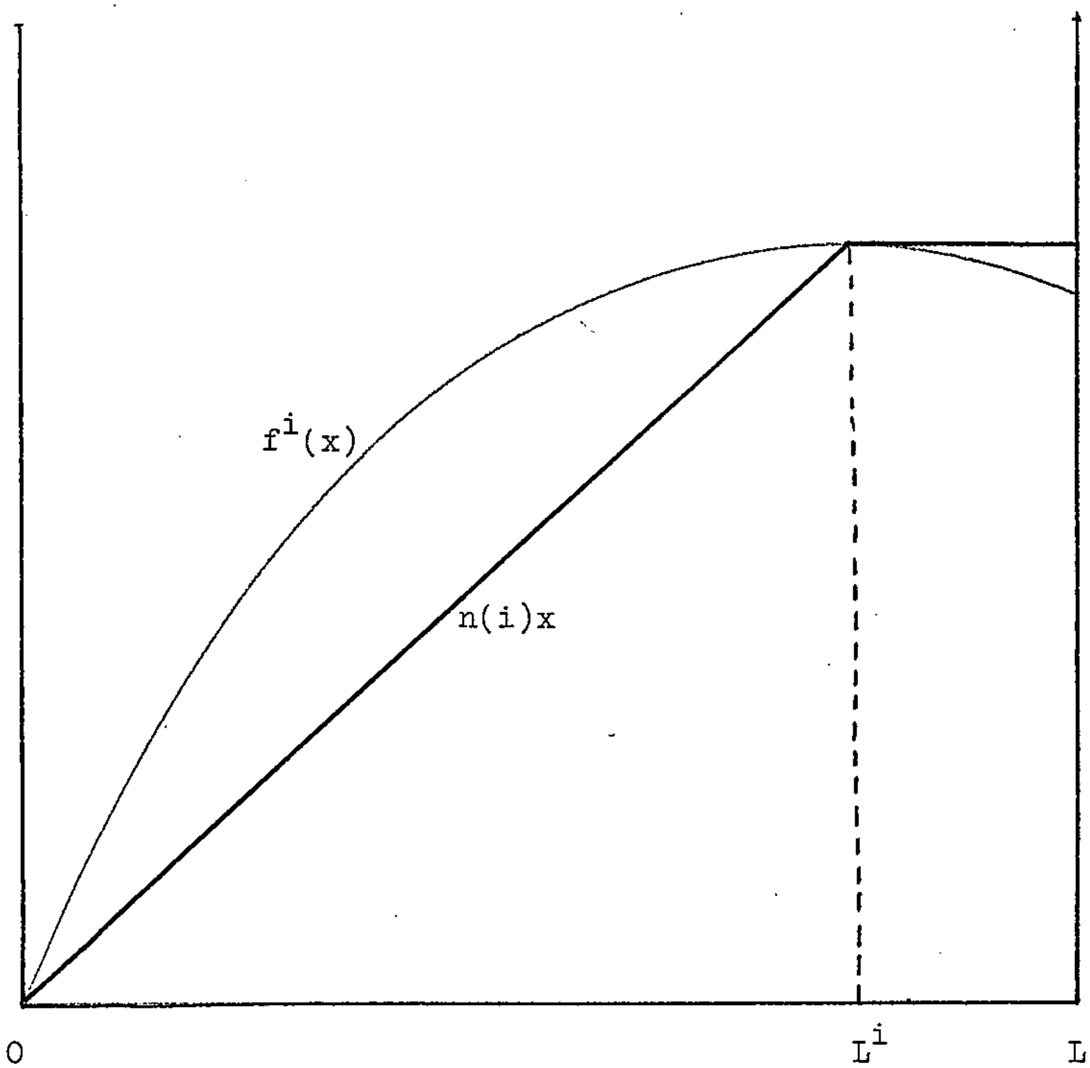

Figure 3 .

Theorem II says that both the optimal marginal and average tax rates tend to 100-percents as income level tends to infinity. It should be, however, noted that disposable income level $y - T^*(y)$ is always a monotonically nondecreasing function of y .

10). (3.3) follows immediately from (3.2) . See Subsection 18. The reason for (3.2) can be intuitively explained as follows. Suppose that the marginal tax rate does not tend to 1 as $y \longrightarrow \infty$. In this case, disposable income $y - T^*(y)$ tends to infinity as gross income tends to infinity with the same order, i.e., $\lim_{y \to \infty} (y-T^*(y))/y > 0$. Hence if an individual i with very high ability works for time L^i , then his disposable income $n(i)L^i - T^*n(i)L^i$ can be also very high. This observation and Assumption (K) imply that individuals with sufficiently high abilities work for L^i's . In this case, if the government increases the marginal tax rate a little in a range of sufficiently high incomes, then individuals with incomes in the range still work for L^i's . Then the government's revenue increases and so the government can increase the level of the public good. This increment of the level of the public good makes the utility levels of individuals with low incomes to increase because the tax function in the range they confront does not change. Even if the utility levels of the individuals with high incomes may decrease, the decrements are smaller than the increments of the utility levels of the individuals with low incomes because of the boundedness of utility functions (Assumptions (C) and (J)). Hence the government can increase the social welfare by increasing a little the marginal tax rate in the range of sufficiently high incomes. This is a contradiction to the optimality of (T^*,Q^*) . Thus we get the result (3.2) of Theorem II . This is also an intuitive sketch of the exact proof of Theorem II which will be provided in Section 6.

11). We compare briefly the result (3.2) of Theorem II with preceding studies. Mirrlees [12] showed many possibilities of the optimal marginal tax rate but concluded that they were much less than 100-percents. Further Sheshinski [19] , Fair [3] , Feldstein [4] , Kesselman [9] and others discussed this problem and provided different upper bounds of the optimal marginal tax rate. But they are also smaller than 100-percents. In our model we take a public good into account explicitly but not in the others. This is, however, not a main reason for the difference. The main reason for it is our assumption of the boundedness of utility functions . This assumption is not used in the other papers. This is a natural assumption which is justified by St.Petersburg paradox. Another reason is the progressiveness of income tax functions. If we admit only linear functions as tax functions, then we can not obtain our result (3.2). Otherwise, it follows from Proposition 3.(ii) that the state that all individuals do not work is the best from the view point of social welfare. This is the reason for the difference between ours and Sheshinski [19] and others in which linear tax functions are only allowed.

12). When T^* satisfies (3.2) and (3.3) of Theorem II, there are two possibilities as follows:

$$(3.4) \qquad \lim_{y \to \infty} (y - T^*(y)) = +\infty ,$$

$$(3.5) \qquad \lim_{y \to \infty} (y - T^*(y)) < +\infty .$$

When (3.4) is true, the disposable income can be as large as desired. When (3.5) is true, the disposable income has the upper bound. In this case, even if an individual with very high ability works for much longer time, his disposable income does not increase or increases a little. Hence he does not work for long time. That is, the tax function obstructs the labor incentives of individuals with high abilities. Thus we would conjecture that (3.4) is true but not (3.5). Regretfully I have not succeeded in proving this conjecture. This is an important open problem.

4. Remarks and Open Problems

13). The first result of this paper is the existence of an optimal income tax function in the domain of progressive income tax function. In order to discuss further properties of the optimal tax functions, we may need to use Pontryagin's maximum Principle or the method of variational calculus. They would be applicable to our model under some stronger assumptions. In this case, our existence theorem may gurantee the existence of an optimal solution and so necessary conditions derived from it would make sense.

In principle, however, the progressiveness of the optimal income tax functions should be derived but not assumed. When we admit tax functions of any forms, we have many difficulties in discussing the optimal income tax functions. Our existence theorem may be generalized to such a case under certain appropriate assumptions, but it would be difficult to discuss further properties of the optimal income tax functions (e.g., Theorem II of Section 3) unless it is proved that they are progressive tax functions as a result . Further, Maximum Principle or the variational method would not be able to be applicable to such a case even if the existence of an optimal income tax function is proved. The reason is as follows. Let T be not a convex function. Then the labor time supply $x(i)$ may not be uniquely determined. That is, even when all the functions given initially are differentiable, solution $x(i)$ satisfying

$$-\frac{\partial U^i}{\partial a} + \frac{\partial U^i}{\partial b} \cdot \frac{d}{dx} (f^i(x) - Tf^i(x)) = 0$$

is not uniquely determined. Hence it would happens that this solution does not have any disirable properties such as continuity or differentiability , which are necessary for the use of Maximum Principle or the variational method.

Thus we have very important open problems as follows:

(1): Whether or not Maximum Principle or the variational method is applicable to our model in the domain of progressive tax functions;

(2): Generalization of the existence theorem to the case of a wider domain including non-progressive tax functions;

(3): If an optimal tax function exists in such a domain, whether or not it is progressive.

5. Proof of Theorem I

14). From (2.2) there exists an $M > 0$ such that $U^i(a,b,Q) \leqq M$ for all $(a,b,Q) \in Y$ and $i \in X$. Hence $W(\tau) = \int_X \log U^i(L-x_\tau(i), f^i x_\tau(i) - Tf^i x_\tau(i), Q)\, d\mu$ is bounded from above. That is,

$$(5.1) \quad \sup_{\tau \in \mathcal{F}} W(\tau) < +\infty .$$

This means that there exists a sequence $\{\tau^s\} = \{(T^s, Q^s)\}$ such that

$$(5.2) \quad (T^s, Q^s) \in \mathcal{F} \text{ for all } s \text{ and } \lim_{s \to \infty} W(\tau^s) = \sup_{\tau \in \mathcal{F}} W(\tau) .$$

Since $Q \leqq \int_X f^i(L^i)\, d\mu < +\infty$ for all $(T,Q) \in \mathcal{F}$ by Assumptions (E) and (H), every Q^s ($s = 1, \ldots,$) belongs to a compact interval. Hence there is a convergent subsequence $\{Q^{s^\nu}\}$ of $\{Q^s\}$. Since $\{(T^{s^\nu}, Q^{s^\nu})\}$ satisfies (5.2), we can assume without loss of generality that $\{Q^s\}$ itself converges to Q^*.

The purpose of Subsections 14 and 15 is to show that we can choose a subsequence of $\{T^s\}$ which converges in a certain sense.

Lemma 3. $\inf_s T^s(0) > -\infty$.

Proof. Suppose $\inf_s T^s(0) = -\infty$. Let $\{(x_s(i))_{i \in X}\}$ be a sequence of labor time supplies for τ^s ($s = 1, \ldots$), i.e., each $x_s(i)$ satisfies (2.6) for τ^s. Then it is clear by Assumption (E) that

$$(5.3) \quad x_s(i) \leqq L^i \quad \text{for all } s \text{ and } i \in X .$$

Let $\tilde{T}^s(y) = y + T^s(0)$ for all $y \in E_+$. Of course, these $\hat{T}^s$ ($s = 1, \ldots$) satisfy (2.4), i.e., $\hat{T}^s \in \mathcal{J}$ for all s. Since T^s is convex and $T^s(y) \leqq y$ for all $y \in E_+$, we have

$$(5.4) \qquad T^s(y) \leqq \tilde{T}^s(y) \quad \text{for all } y \in E_+ .$$

It follows from (5.3) and (5.4) that

$$\int_X \tilde{T}^s f^i(L^i)\, d\mu \geqq \int_X T^s f^i x_s(i)\, d\mu .$$

The left-hand term of this inequality is rewritten as

$$\int_X (f^i(L^i)+T^s(0))\, d\mu = \int_X f^i(L^i)\, d\mu + T^s(0)\mu(X) .$$

By Assumption (H), there is an s for which this value is negative. Since (T^s, Q^s) is a feasible tax schedule, it holds that

$$Q^s \leqq \int_X T^s f^i x_s(i)\, d\mu \leqq \int_X f^i(L^i)\, d\mu + T^s(0)\mu(X) < 0 .$$

This is a contradiction. Q.E.D.

15). Let $K = \inf_s T^s(0)$. Let k be an arbitrary positive integer. Then we define $C[0,k]$ by

$$(5.5) \quad C[0,k] = \left\{ t : \begin{array}{l} t \text{ is a continuous, convex and nondecreasing} \\ \text{function on the interval } [0,k] \text{ which has } \left.\dfrac{d^- t}{dy}\right|_k \leqq 1 \\ \text{and satisfies } K \leqq t(y) \leqq y \text{ for all } y \in [0,k] . \end{array} \right\}$$

It is not difficult to verify the following lemma.

Lemma 4. Let t^s be the restriction of T^s on $[0,k]$ ($s = 1,\dots$). Then every t^s belongs to $C[0,k]$.

We introduce the topology defined by the distance $d(t_1,t_2) = \sup_{y \in [0,k]} |t_1(y)-t_2(y)|$ into $C[0,k]$. Then the following lemma holds:

Lemma 5. $C[0,k]$ is a compact set for any k .
Proof. See Appendix.

Let t^s be the restriction of T^s on $[0,1]$ ($s = 1,\dots$) . Then it follows from Lemma 5 that $\{t^s\}$ has a convergent subsequence $\{\tilde{t}^{1s}\}$. Let T^{1s} be the original tax function on E_+ in $\{T^s\}$ corresponding to $\tilde{t}^{1s}$. Then $\{T^{1s}\}$ converges uniformly on $[0,1]$. Let t^{1s} be the restriction of T^{1s} on $[0,2]$ ($s = 1,\dots$) . In the same way we let $\{\tilde{t}^{2s}\}$ be a subsequence of $\{t^{1s}\}$ such that $\{\tilde{t}^{2s}\}$ converges. Let T^{2s} be the original tax function on E_+ in $\{T^s\}$ corresponding to $\tilde{t}^{2s}$. Then $\{T^{2s}\}$ converges uniformly on $[0,2]$. If we continue this process, we get an array of sequences of the form

$$
\begin{aligned}
\{T^s\} &= \{T^1, T^2, \dots\} ,\\
\{T^{1s}\} &= \{T^{11}, T^{12}, \dots\} ,\\
\{T^{2s}\} &= \{T^{21}, T^{22}, \dots\} ,\\
\{T^{3s}\} &= \{T^{31}, T^{32}, \dots\} ,\\
&\dots\dots\dots\dots\dots\dots
\end{aligned}
$$

in which each sequence is a subsequence of the one directly above it, and for each k the sequence $\{T^{ks}\}$ has the property that $\{T^{ks}\}$ converges uniformly on $[0,k]$. If we define $\tilde{T}^1, \tilde{T}^2, \ldots$ by $\tilde{T}^1 = T^{11}$, $\tilde{T}^2 = T^{22}$, $\tilde{T}^3 = T^{33}$ ..., then the sequence $\{\tilde{T}^s\}$ is the "diagonal" subsequence of $\{T^s\}$. It is clear from this construction that for any k, $\{\tilde{T}^s\}$ converges uniformly on $[0,k]$. Hence $\{\tilde{T}^s\}$ also converges pointwise everywhere on E_+. We can define the function T^* on E_+ by

$$(5.6) \qquad T^*(y) = \lim_{s \to \infty} \tilde{T}^s(y) \quad \text{for all } y \in E_+ .$$

<u>Lemma</u> <u>6</u>. $T^* \in \mathcal{T}$.

<u>Proof</u>. Since each $\tilde{T}^s$ is convex and nondecreasing with $\tilde{T}^s(y) \leqq y$ for all $y \in E_+$, it holds that for any y_1, y_2 $(0 \leqq y_1 \leqq y_2)$ and $\alpha \in [0,1]$,

$$\tilde{T}^s(\alpha y_1 + (1-\alpha) y_2) \leqq \alpha \tilde{T}^s(y_1) + (1-\alpha)\tilde{T}^s(y_2) ,$$

$$\tilde{T}^s(y_1) \leqq \tilde{T}^s(y_2) \quad \text{and } \tilde{T}^s(y_1) \leqq y_1 \quad \text{for all } s .$$

We have, by (5.6),

$$T^*(\alpha y_1 + (1-\alpha) y_2) \leqq \alpha T^*(y_1) + (1-\alpha) T^*(y_2) ,$$

$$T^*(y_1) \leqq T^*(y_2) \quad \text{and } T^*(y_1) \leqq y_1 .$$

Q.E.D.

Let $\{\tilde{Q}^s\}$ be the subsequence of $\{Q^s\}$ corresponding to $\{\tilde{T}^s\}$. This $\{\tilde{Q}^s\}$ also converges to Q^*. Hence we have shown the following lemma.

Lemma 7. $\{\tau^s\} = \{(T^s, Q^s)\}$ has a subsequence $\{(\widehat{T}^s, \widetilde{Q}^s)\}$ which satisfies

(i) $\lim_{s \to \infty} \widetilde{Q}^s = Q^* \geqq 0$, $\lim_{s \to \infty} \widetilde{T}^s(y) = T^*(y)$ for all $y \in E_+$ and $T^* \in \mathcal{J}$.

(ii) $\{\widetilde{T}^s\}$ converges uniformly to T^* on $[0,k]$ for any k .

16). The purpose of this subsection is to prove that (T^*,Q^*) is a feasible tax schedule and

$$(5.7) \qquad W(\tau^*) = W(T^*,Q^*) = \sup_{\tau \in \mathcal{F}} W(\tau) ,$$

which means that $\tau^* = (T^*,Q^*)$ is an optimal tax schedule. That is, if we show the feasiblity and (5.7) , we complete the proof of the existence of an optimal tax schedule.

To show the feasibility, it is sufficient to prove

$$(5.8) \qquad Q^* \leqq \int_X T^* f^i x^*(i)\, d\mu \ ,$$

where $(x^*(i))_{i \in X}$ is the labor time supplies for (T^*,Q^*) .

Lemma 8. Let $\{(\widetilde{T}^s, \widetilde{Q}^s)\}$ be the sequence given in Lemma 7 and let $(\widetilde{x}_s(i))_{i \in X}$ be the labor time supplies for $(\widetilde{T}^s, \widetilde{Q}^s)$ ($s = 1, \dots$). Then $\{\widetilde{x}_s(i)\}$ converges to $x^*(i)$ for each $i \in X$.

Proof. See Appendix.

Let individual $i \in X$ be arbitrarily fixed. By the continuity of f^i, we have $\lim_{s \to \infty} f^i \widetilde{x}_s(i) = f^i x^*(i)$. Since $0 \leqq f^i \widetilde{x}_s(i) \leqq$

$f^i(L^i)$ for all s, we have

$$\lim_{s\to\infty} \tilde{T}^s f^i x_s(i) = T^* f^i x^*(i) ,$$

because $\left|\tilde{T}^s f^i \tilde{x}_s(i) - T^* f^i x^*(i)\right| \leqq \left|\tilde{T}^s f^i \tilde{x}_s(i) - T^* f^i \tilde{x}_s(i)\right| +$
$\left|T^* f^i \tilde{x}_s(i) - T^* f^i x^*(i)\right| \to 0 \quad (s \to \infty)$ by Lemma 7.(ii). 16)

Since $\tilde{T}^s f^i \tilde{x}_s(i) \leqq f^i(L^i)$ for all s,all $i \in X$ and $f^i(L^i)$ is is integrable by Assumption (H), we have, by Lebesgue's dominated convergence theorem,

$$\lim_{s\to\infty} \int_X \tilde{T}^s f^i \tilde{x}_s(i)\, d\mu = \int_X T^* f^i x^*(i)\, d\mu .$$

Since $(\tilde{T}^s, \tilde{Q}^s)$ is feasible, it holds that

$$\tilde{Q}^s \leqq \int_X \tilde{T}^s f^i \tilde{x}_s(i)\, d\mu \quad \text{for all } s .$$

Hence we have

$$Q^* = \lim_{s\to\infty} \tilde{Q}^s \leqq \lim_{s\to\infty} \int_X \tilde{T}^s f^i \tilde{x}_s(i)\, d\mu = \int_X T^* f^i x^*(i)\, d\mu ,$$

which is the feasibility of (T^*, Q^*).

Lemma 9. (5.7) holds.

Proof. Since $\{(\tilde{T}^s, \tilde{Q}^s)\}$ is a subsequence of $\{(T^s, Q^s)\}$ and $\lim_{s\to\infty} W(T^s, Q^s) = \sup_{\tau\in\mathcal{F}} W(\tau)$, it holds that $\lim_{s\to\infty} W(\tilde{T}^s, \tilde{Q}^s) = \sup_{\tau\in\mathcal{F}} W(\tau)$.
For each $i \in X$, $U^i(L-\tilde{x}_s(i), f^i\tilde{x}_s(i)-\tilde{T}^s f^i \tilde{x}_s(i), \tilde{Q}^s) \to U^i(L-x^*(i), f^i x^*(i)-Tf^i x^*(i), Q^*)$ as $s \to \infty$ because of Assumptions (A),(F), Lemma 7

---

16). Note that T^* is a continuous function because $T^* \in \mathcal{J}$.

and Lemma 8. By (2.2), U^i is uniformly bounded. Hence we have, by Lebesgue's dominated convergence theorem,

$$\lim_{s\to\infty}\int_X \log U^i(L-\tilde{x}_s(i), f^i\tilde{x}_s(i)-\tilde{T}f^i\tilde{x}_s(i), \tilde{Q}^s)\ d\mu$$

$$= \int_X \log U^i(L-x^*(i), f^i x^*(i)-T^* f^i x^*(i), Q^*)\ d\mu\ .$$

That is, $\sup_{\tau\in\mathcal{F}} W(\tau) = \lim_{s\to\infty} W(\tilde{T}^s, \tilde{Q}^s) = W(T^*, Q^*)$. Q.E.D.

<u>17</u>). The purpose of the last subsection is to show that (T^*, Q^*) is an equilibrium tax schedule. Suppose $Q^* < \int_X T^* f^i x^*(i)\ d\mu$.

<u>Lemma</u> <u>10</u>. Let $\{q^s\}$ be a sequence which is decreasing and converges to Q^* . Let $(\bar{x}_s(i))_{i\in X}$ be the labor time supplies for (T^*, q^s) ($s = 1, \ldots$). Then $\{\bar{x}_s(i)\}$ converges to $x^*(i)$ for each $i\in X$.

<u>Proof</u>. This lemma can be proved similarly to Lemma 8 .

Hence we have, by Lebesgue's dominated convergence theorem,

$$\lim_{s\to\infty}\int_X T^* f^i \bar{x}_s(i)\ d\mu = \int_X T^* f^i x^*(i)\ d\mu > Q^* = \lim_{s\to} q^s\ .$$

Therefore there is an integer s such that $\int_X T^* f^i \bar{x}_s(i)\ d\mu > q^s > Q^*$. This means that (T^*, q^s) is a feasible tax schedule. For this s, it holds by Assumption (A) that for all $i\in X$,

$$U^i(L-x^*(i), f^i x^*(i)-T^* f^i x^*(i), Q^*) < U^i(L-x^*(i), f^i x^*(i)-T^* f^i x^*(i), q^s)$$

$$\leqq U^i(L-\bar{x}_s(i), f^i\bar{x}_s(i)-T^* f^i\bar{x}_s(i), q^s)\ .$$

Thus we have

$$W(T^*,Q^*) = \int_X \log U^i(L-x^*(i), f^i x^*(i) - T^* f^i x^*(i), Q^*)\, d\mu <$$

$$\int_X \log U^i(L-\overline{x}_s(i), f^i\overline{x}_s(i) - T^* f^i \overline{x}_s(i), q^s)\, d\mu = W(T^*, q^s) \, .$$

This is a contradiction to the optimality of (T^*,Q^*) .

6. Proof of Theorem II

18). Throughout this section we assume that (T^*,Q^*) is an optimal tax schedule. Since T^* is a convex function with $T^*(y) \leqq y$ for all $y \in E_+$, it holds that for all $y > 0$ and $\varepsilon > 0$,

$$(6.1) \quad \left.\frac{d^+T^*}{dy}\right|_{y-\varepsilon} \leqq \left.\frac{d^-T^*}{dy}\right|_{y} \leqq \left.\frac{d^+T^*}{dy}\right|_{y} \leqq \left.\frac{d^-T^*}{dy}\right|_{y+\varepsilon} \leqq 1$$

Hence we have $\displaystyle\lim_{y\to\infty} \left.\frac{d^-T^*}{dy}\right|_{y} = \lim_{y\to\infty} \left.\frac{d^+T^*}{dy}\right|_{y}$.

Suppose $\displaystyle\lim_{y\to\infty} \left.\frac{d^-T^*}{dy}\right|_{y} = 1$. Since T^* is convex, it holds that

$$\left.\frac{d^+T^*}{dy}\right|_{y} \leqq T^*(y+1)-T^*(y) \leqq \left.\frac{d^-T^*}{dy}\right|_{y+1} \qquad \text{for all } y \in E_+ .$$

This implies $\displaystyle\lim_{y\to\infty} (T^*(y+1)-T^*(y)) = 1$. Using the following familiar lemma (Lemma 11), we have

$$(6.2) \quad \lim_{y\to\infty} \frac{T^*(y)}{y} = \lim_{y\to\infty} (T^*(y+1)-T^*(y)) = 1 .$$

Hence it is sufficient to show that $\displaystyle\lim_{y\to\infty} \left.\frac{d^-T^*}{dy}\right|_{y} = 1$.

Lemma 11.(Komatsu [10, Theorem 40,5]). If $f(x)$ and $g(x)$ are defined on E_+ and $g(x)$ is monotonically increasing with $g(x) \to \infty$ ($x \to \infty$), and if $(f(x+1)-f(x))/(g(x+1)-g(x)) \to \alpha$ ($x \to \infty$), then $f(x)/g(x) \to \alpha$ ($x \to \infty$), where α is a real number.

In the following we suppose

$$(6.3) \qquad \lim_{y \to \infty} \left.\frac{d^{-}T^{*}}{dy}\right|_{y} = \lim_{y \to \infty} \left.\frac{d^{+}T^{*}}{dy}\right|_{y} = a < 1 .$$

So we shall derive a contradiction from this supposition.

19). We define a sequence of tax functions $\{T^{s}\}$ by

$$(6.4) \qquad T^{s}(y) = \begin{cases} T^{*}(y) & \text{if } y \leqq sL \\ (a+\delta)(y-sL) + T^{*}(sL) & \text{if } y \geqq sL , \end{cases}$$

where δ is a real number such that $a < a+\delta < 1$. It is clear that every T^{s} belongs to $\mathcal{T}$. See Figure 4.

Lemma 12. There is an integer n_{o} such that for all i with $n(i) \geqq n_{o}$,

$$(6.5) \qquad 1 - a - \delta > \frac{1}{n(i)} \cdot \left.\frac{U_1^i}{U_2^i}\right|_{(L-L^i, n(i)L^i - T^{*}n(i)L^i, Q^{*})}$$

$$(6.6) \qquad 1 - a - \delta > \frac{1}{n(i)} \cdot \left.\frac{U_1^i}{U_2^i}\right|_{(L-L^i, n(i)L^i - T^{s}n(i)L^i, Q^{*})}$$

for all $s \geqq 1$.

Proof. It follows from Assumption (K) that there is a b_{o} such that

$$\frac{1}{b} \cdot \left.\frac{U_1^i}{U_2^i}\right|_{(L-L^i, b, Q^{*})} < \frac{1-(a+\delta)}{L} \quad \text{for all } i \in X \text{ and } b \geqq b_{o}.$$

Hence we have

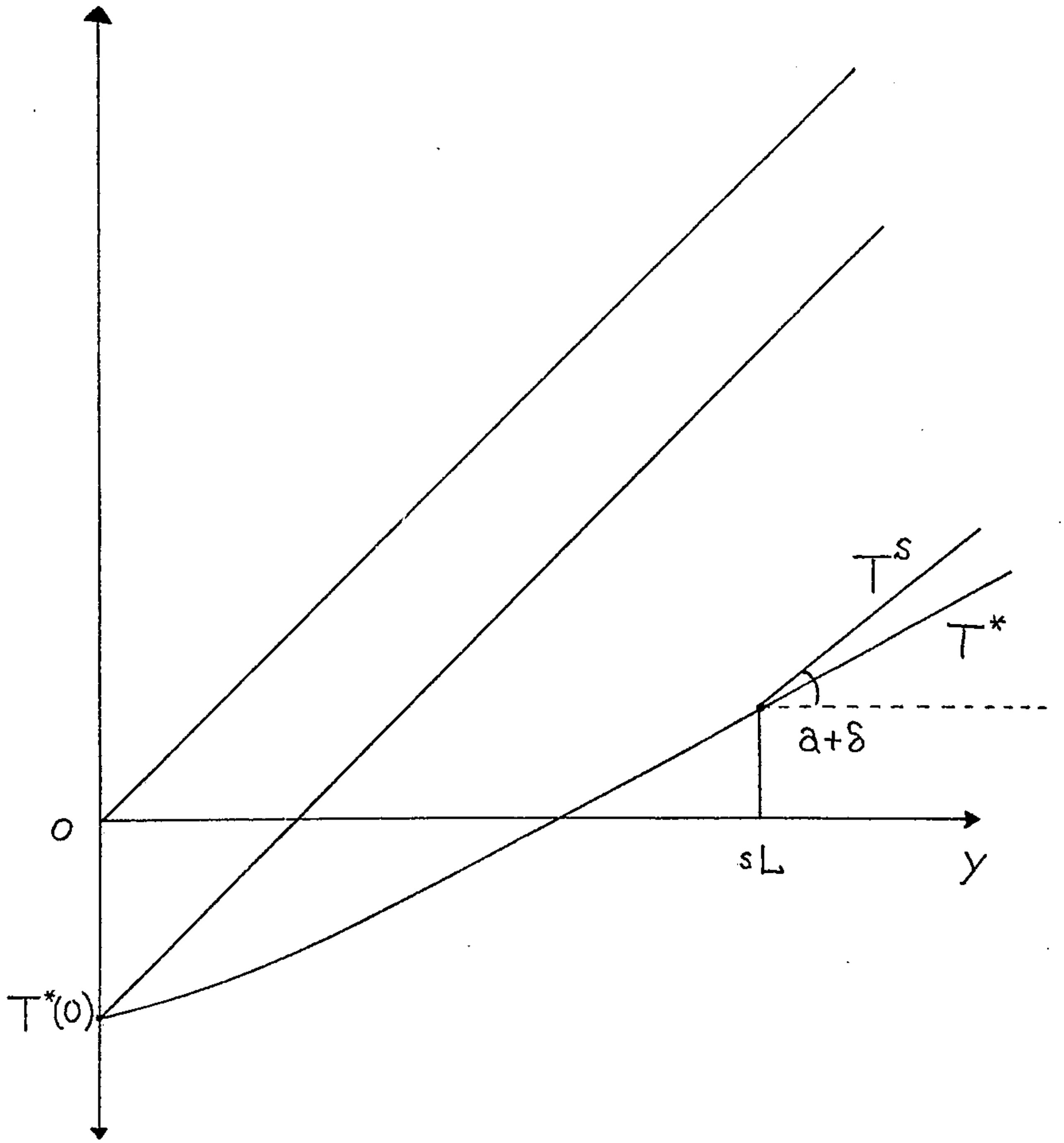

Figure 4.

$$\frac{1}{n(i)L^i - T^1 n(i)L^i} \cdot \frac{U_1^i}{U_2^i}\Bigg|_{(L-L^i,\, n(i)L^i - T^1 n(i)L^i,\, Q^*)} < \frac{1-(a+\delta)}{L}$$

for all i with $n(i)L^i - T^1 n(i)L^i \geqq b_o$.

Since

$$\frac{n(i)L^i - T^1 n(i)L^i}{n(i)} = L^i\left(1 - \frac{T^1 n(i)L^i}{n(i)L^i}\right) \leqq L ,$$

it holds that for all i with $n(i)L^i - T^1 n(i)L^i \geqq b_o$,

$$\frac{1}{n(i)} \cdot \frac{U_1^i}{U_2^i}\Bigg|_{(L-L^i,\, n(i)L^i - T^1 n(i)L^i,\, Q^*)} \leqq$$

$$L \cdot \frac{1}{n(i)L^i - T^1 n(i)L^i} \cdot \frac{U_1^i}{U_2^i}\Bigg|_{(L-L^i,\, n(i)L^i - T^1 n(i)L^i,\, Q^*)} < 1-(a+\delta) .$$

Since $\lim_{n(i) \to \infty} \left(1 - \frac{d^- T^1}{dy}\Big|_{n(i)L^i}\right) = 1-a-\delta > 0$, it holds that $n(i)L^i - T^1 n(i)L^i \to \infty$ as $n(i) \to \infty$. Therefore we can choose n_o such that

$$n(i)L^i - T^1 n(i)L^i \geqq b_o \quad \text{for all i with } n(i) \geqq n_o .$$

It is easily verified that $n(i)L^i - T^{s+1} n(i)L^i \geqq n(i)L^i - T^s n(i)L^i$ for all s and all i . This implies $n(i)L^i - T^s n(i)L^i \geqq b_o$ for all i with $n(i) \geqq n_o$ and all s . Hence (6.6) holds for this n_o . In the above argument, T* can be replaced by T^1 . Q.E.D.

Lemma 13. Let $(x^*(i))_{i \in X}$ and $(x_s(i))_{i \in X}$ be the labor time supplies for (T^*,Q^*) and (T^s,Q^s) ($s = 1,\dots$) respectively, where Q^s is any nonnegative real number. Then it holds that

$$(6.7) \quad x^*(i) = x_s(i) \quad \text{for all } i \in X \text{ and all } s \geqq n_o ,$$

$$(6.8) \quad x^*(i) = L^i \quad \text{for all } i \text{ with } n(i)x^*(i) \geqq n_oL ,$$

where n_o is the integer given in Lemma 12.

Proof. Note that the labor time supplies for a tax schedule (T,Q) does not depend upon Q by Assumption (I), that is, if $(x_1(i))_{i \in X}$ and $(x_2(i))_{i \in X}$ are the labor time supplies for tax schedules (T,Q_1) and (T,Q_2) respectively, then $x_1(i) = x_2(i)$ for all $i \in X$.

Let $s \geqq n_o$. Let i be an individual such that $n(i)x^*(i) < sL$. Since $T^s(y) = T^*(y)$ for all $y \leqq sL$ by (6.4) and T^s is convex, $x^*(i)$ also satisfies

$$U^i(L-x^*(i),n(i)x^*(i)-T^sn(i)x^*(i),Q^*)$$

$$\geqq U^i(L-x,n(i)x-T^sn(i)x,Q^*) \quad \text{for all } x \leqq L^i.$$

In fact, if this inequality is not true, there is an $x^o \leqq L^i$ such that

$$n(i)x^o \geqq sL \text{ and } U^i(L-x^o,n(i)x^o-T^sn(i)x^o,Q^*) >$$

$$U^i(L-x^*(i),n(i)x^*(i)-T^sn(i)x^*(i),Q^*) \quad .$$

This implies $U^i(L-x,n(i)x-T^sn(i)x,Q^*) > U^i(L-x^*(i),n(i)x^*(i)-T^sn(i)x^*(i),Q^*)$ for all x ($x^*(i) < x \leqq x^o$) by the concavity of U^i , which is a contradiction. Hence we have shown that $x^*(i) = x_s(i)$

for all i with $n(i)x^*(i) < sL$.

Let i be an individual such that $n(i)x^*(i) \geqq sL$. Since $n(i) > n(i)x^*(i)/L \geqq s \geqq n_o$, (6.6) of Lemma 12 is true, i.e.,

$$1-a-\delta > \frac{1}{n(i)} \cdot \frac{U_1^i}{U_2^i}\Bigg|_{(L-L^i,n(i)L^i-T^s n(i)L^i,Q^*)} .$$

Using $\left.\frac{d^- T^s}{dy}\right|_{n(i)x^*(i)} = a+\delta$ by (6.4), it follows that

$$\left[-U_1^i + U_2^i \cdot n(i)(1-\frac{d^- T^s}{dy})\right]\Bigg|_{(L-L^i,n(i)L^i-T^s n(i)L^i,Q^*)} > 0 ,$$

which is equivalent to $\left.\frac{d^-}{dx}U^i(L-x,n(i)x-T^s n(i)x,Q^*)\right|_{x=L^i} > 0$.

This and Assumptions (A'),(E') imply $x_s(i) = L^i$. Analogously we can prove that $x^*(i) = L^i$ for all i with $n(i)x^*(i) \geqq n_o L$. This is (6.8) . Hence it holds that $x_s(i) = x^*(i) = L^i$ for all i with $n(i)x^*(i) \geqq sL$. Q.E.D.

<u>Lemma 14</u>. When the government employs T^s ($s \geqq n_o$) in place of T^*, the increment of the government's revenue is not smaller than $\delta \cdot \mu(F(sL+1))$, i.e.,

$$(6.9) \quad \int_X T^s n(i)x_s(i)\, d\mu - \int_X T^* n(i)x^*(i)\, d\mu \geqq \delta \cdot \mu(F(sL+1)),$$

where $F(sL+1) = \left\{ i \in X : n(i)L^i \geqq sL+1 \right\}$.

<u>Proof</u>. Note that $x^*(i) = x_s(i)$ for all $i \in X$. Since $T^s n(i)x^*(i) - T^* n(i)x^*(i) = (a+\delta)(n(i)x^*(i)-sL) + T^*(sL) - T^* n(i)x^*(i) \geqq 0$ for all $i \in X$ by (6.4),

it holds that

$$(6.10)\quad \int_{F(sL,sL+1)} \left[T^s n(i)x^*(i) - T^* n(i)x^*(i)\right] d\mu \geqq 0,$$

where $F(sL,sL+1) = \left\{ i \in X : sL \leqq n(i)x^*(i) < sL+1 \right\}$.

Since $T^* n(i)x^*(i) \leqq a(n(i)x^*(i)-sL) + T^*(sL)$ for all i with $n(i)x^*(i) \geqq sL$ by the definition of a, it holds by (6.4) and (6.8) that

$$(6.11)\quad \int_{F(sL+1)} \left[T^s n(i)x^*(i) - T^* n(i)x^*(i)\right] d\mu$$

$$\geqq \int_{F(sL+1)} \left[(a+\delta)(n(i)x^*(i)-sL)+T^*(sL) - \left[a(n(i)x^*(i)-sL)+T^*(sL)\right]\right] d\mu$$

$$= \int_{F(sL+1)} \delta \left[(n(i)x^*(i)-sL)\right] d\mu = \int_{F(sL+1)} \delta(n(i)L^i-sL)\, d\mu$$

$$\geqq \int_{F(sL+1)} \delta \cdot 1\, d\mu = \delta \cdot \mu(F(sL+1)) .$$

Since $T^s n(i)x^*(i) = T^* n(i)x^*(i)$ for all i with $n(i)x^*(i) \leqq sL$ by (6.4), we have

$$(6.12)\quad \int_{X-F(sL)} \left[T^s n(i)x^*(i) - T^* n(i)x^*(i)\right] d\mu = 0 .$$

Hence it follows from (6.10),(6.11) and (6.12) that

$$\int_X \left[T^s n(i)x^*(i) - T^* n(i)x^*(i)\right] d\mu$$

$$= \int_{F(sL,sL+1)} \left[T^s n(i)x^*(i) - T^* n(i)x^*(i)\right] d\mu$$

$$+ \int_{F(sL+1)} \left[T^s n(i)x^*(i) - T^* n(i)x^*(i)\right] d\mu$$

$$\geqq \delta \cdot \mu(F(sL+1)) .$$

Q.E.D.

20). In the following we assume $s \geqq n_o$ and use Lemma 13 without any remark.

Since $G_3^i(a,b,Q) = \left.\frac{U_3^i}{U^i}\right|_{(a,b,Q)} > 0$ for all $i \in X$ and all $(a,b,Q) \neq (0,0,0)$, there is a $b > 0$ such that

$$(6.13) \quad \mu(H) \geqq \tfrac{1}{2}\mu(F(0,n_oL))$$

$$H = \left\{ i \in F(0,n_oL) : G_3^i(L-x^*(i), n(i)x^*(i)-T^*n(i)x^*(i), Q^*) \geqq b \right\},$$

where $F(0,n_oL) = \left\{ i \in X : 0 \leqq n(i)x^*(i) < n_oL \right\}$.

Since $(x^*(i))_{i \in X}$ is invariant for Q by Assumption (I), we can write

$$(6.14) \quad J^i(Q) = G^i(L-x^*(i), n(i)x^*(i)-T^*n(i)x^*(i), Q)$$

$$\text{for all } i \in X .$$

It holds by Assumption (A') that for all $i \in X$,

$$(6.15) \quad J^i(Q^*+\Delta Q) = J^i(Q^*) + J^{i\prime}(Q^*)\Delta Q + \Delta Q \cdot \varepsilon^i_{\Delta Q}$$

$$\text{and } \varepsilon^i_{\Delta Q} \to 0 \text{ as } \Delta Q \to 0 ,$$

where $J^{i\prime} = \frac{dJ^i}{dQ}$.

Lemma 14. Suppose $\int_H J^{i\prime}(Q^*)\,d\mu < +\infty$.[17] Let $\{q_k\}$ be a decreasing sequence such that $\lim_{k\to\infty} q_k = 0$. Then it holds that

---

17). It is easily verified that dJ^i/dQ is a measurable function of i .

$$(6.16)\quad \int_H \varepsilon^i_{q_k}\, d\mu \longrightarrow 0 \text{ as } k \to \infty .$$

Proof. Let i be fixed. $(J^i(q^*+q_k)-J^i(Q^*))/q_k$ is the inclination of the line connecting two points $(Q^*,J^i(Q^*))$ and $(Q^*+q_k,J^i(Q^*+q_k))$. Since $J^i(Q)$ is a concave function by Assumption (A'), this inclination is a nondecreasing function of k, i.e.,

$$(J^i(Q^*+q_k)-J^i(Q^*))/q_k \leqq (J^i(Q^*+q_{k+1})-J^i(Q^*))/q_{k+1} \quad \text{for all } k .$$

Further it holds that $(J^i(Q^*+q_k)-J^i(Q^*))/q_k \leqq dJ^i(Q^*)/dQ$ for all k. Hence it follows that $\left|\varepsilon^i_{q_k}\right| = \left|(J^i(Q^*+q_k)-J^i(Q^*))/q_k - dJ^i(Q^*)/dQ\right|$ is nonincreasing function of k . Since $(J^i(Q^*+q_k)-J^i(Q^*))/q_k - dJ^i(Q^*)/dQ$ is an integrable function of $i \in H$ and since $\varepsilon^i_{q_k} \to 0$ as $k \to \infty$ for all $i \in H$, we have, by Lebesgue's dominated convergence theorem,

$$\int_H \varepsilon^i_{q_k}\, d\mu \longrightarrow 0 \quad \text{as } k \to \infty .$$

Q.E.D.

Since $n(i)L^i - T^s n(i)L^i \to \infty$ as $n(i)L^i \to \infty$ and $y - T^s(y) \leqq y-T^{s+1}(y)$ for all $y \in E_+$ and all s, there is an $s \geqq n_o$ by Assumption (K) such that

$$(6.16)\quad M_i - \frac{b\cdot\delta\mu(F(0,n_oL))c_1}{4} \leqq G^i(L-L^i,n(i)L^i-T^s n(i)L^i,Q^*) \quad \text{for all } i \text{ with } n(i)L^i \geqq sL ,$$

where c_1 is the positive number defined by

$$(6.17) \qquad c_1 = \lim_{s \to \infty} \mu(F(sL+1))/\mu(F(sL)) .$$

Note that the positiveness of c_1 is ensured by Assumption (L) .

We are now in a position to evaluate the value:

$$(6.18) \qquad W(T^s, Q^* + \delta\mu(F(sL+1))) - W(T^*, Q^*)$$

$$= \int_X G^i(L-x^*(i), n(i)x^*(i)-T^s n(i)x^*(i), Q^*+\delta\mu(F(sL+1))) \; d\mu$$
$$- \int_X G^i(L-x^*(i), n(i)x^*(i)-T^* n(i)x^*(i), Q^*) \; d\mu$$

$$= \int_{F(0,sL)} \Big[G^i(L-x^*(i), n(i)x^*(i)-T^s n(i)x^*(i), Q^*+\delta\mu(F(sL+1)))$$
$$- G^i(L-x^*(i), n(i)x^*(i)-T^* n(i)x^*(i), Q^*)\Big] \; d\mu$$
$$+ \int_{F(sL)} \Big[G^i(L-x^*(i), n(i)x^*(i)-T^s n(i)x^*(i), Q^*+\delta\mu(F(sL+1)))$$
$$- G^i(L-x^*(i), n(i)x^*(i)-T^* n(i)x^*(i), Q^*)\Big] \; d\mu \quad .$$

For a sufficiently large s, the second term of the right hand side can be rewritten as follows:

$$(6.19) \qquad \int_{F(sL)} \Big[G^i(L-x^*(i), n(i)x^*(i)-T^s n(i)x^*(i), Q^*+\delta\mu(F(sL+1)))$$
$$- G^i(L-x^*(i), n(i)x^*(i)-T^* n(i)x^*(i), Q^*)\Big] \; d\mu$$

$$\geqq \int_{F(sL)} \left[M_i - \frac{b\,\delta\mu(F(0,n_o L))c_1}{4} - M_i\right] \; d\mu$$

$$= - \frac{b\,\delta\mu(F(0,n_o L)c_1 \mu(F(sL))}{4} \quad .$$

Let us suppose that $\int_H J^{i'}(Q^*)\,d\mu < +\infty$. Since $T^*(y) = T^s(y)$ for all $y \leqq sL$ by (6.4), the first term of the right hand side of (6.18) can be rewritten as follows:

$$(6.20)\quad \int_{F(0,sL)} \left[J^i(Q^* + \delta\mu(F(sL+1))) - J^i(Q^*)\right] d\mu$$
$$\geqq \int_H \left[J^i(Q^* + \delta\mu(F(sL+1))) - J^i(Q^*)\right] d\mu$$
$$\geqq \int_H b\,\delta\mu(F(sL+1))\,d\mu + \int_H \varepsilon_s^i\,d\mu\cdot\delta\mu(F(sL+1))$$
$$\geqq b\,\delta\mu(F(sL+1))\mu(F(0,n_oL))/2 + \int_H \varepsilon_s^i\,d\mu\cdot\delta\mu(F(sL+1)),$$

where $\varepsilon_s^i = \varepsilon^i_{\delta\mu(F(sL+1))}$ and b is the number defined by (6.13).
Hence it holds for a sufficiently large s that

$$(6.21)\quad W(T^s, Q^* + \delta\mu(F(sL+1))) - W(T^*, Q^*)$$
$$\geqq b\,\delta\mu(F(sL+1))\mu(F(0,n_oL))/2 + \int_H \varepsilon_s^i\,d\mu\cdot\delta\mu(F(sL+1))$$
$$- b\,\delta\mu(F(sL))c_1\mu(F(0,n_oL))/4\ .$$

Since $c_1 = \lim_{s\to\infty} \mu(F(sL+1))/\mu(F(sL)) > 0$ and $\int_H \varepsilon_s^i\,d\mu \to 0$ as $s \to \infty$ by Lemma 14, the right hand side of (6.21) becomes positive for a sufficiently large s. This is a contradiction to the optimality of (T^*,Q^*).

When $\int_H J^{i'}(Q^*)\,d\mu = +\infty$, (6.21) can be rewritten as follows:

$$(6.22)\quad W(T^s, Q^* + \delta\mu(F(sL+1))) - W(T^*, Q^*)$$

$$= \int_H \left[J^i(Q^*+\delta\mu(F(sL+1))) - J^i(Q^*)\right] d\mu$$

$$- b\delta\mu(F(sL))c_1\mu(F(0,n_oL))/4 \quad .$$

Since $\dfrac{\int_H \left[J^i(Q^*+\delta\mu(F(sL+1)))-J^i(Q^*)\right] d\mu}{\delta\mu(F(sL+1))} \longrightarrow \infty$ as $s \longrightarrow \infty$ by the following Lemma 15 and $\int_H J^{i'}(Q^*)\, d\mu = \infty$, and since $\lim_{s\to\infty} \mu(F(sL+1))/\mu(F(sL)) = c_1 > 0$, it holds for a sufficiently large s that

$$\frac{\int_H \left[J^i(Q^*+\delta\mu(F(sL+1)))-J^i(Q^*)\right] d\mu}{\delta\mu(F(sL+1))} - b\cdot\frac{\mu(F(sL))}{\mu(F(sL+1))}\, c_1\cdot(F(0,n_oL))/4$$

$$> 0 \ .$$

This means that the right hand side of (6.22) is positive for such an s . This is a contradiction to the optimality of (T^*,Q^*) .

<u>Lemma</u> <u>15</u>. Let $\{f_s\}$ be a sequence of measurable functions on H such that $0 \leq f_s(i) \leq f_{s+1}(i)$ for all s and all $i \in H$. If there is a measurable function f such that $\lim_{s\to\infty} f_s(i) = f(i)$ for all $i \in H$ and $\int_H f(i)\, d\mu = \infty$, then $\lim_{s\to\infty}\int_H f_s(i)\, d\mu = \infty$.

<u>Proof</u>. See Appendix.

---

18). Since $J^i(Q)$ is convex and monotonically increasing with respect to Q for all $i \in H$, $(J^i(Q^*+\delta\mu(F(sL+1)))-J^i(Q^*))/\delta\mu(F(sL+1))$ is positive and nondecreasing with respect to s for each i .

Proof of Lemma 2: It is not difficult to prove that $x(i)$ is uniquely determined for each fixed $i \in X$. We need to show the measurability of $x(i)$. Let α be an arbitrary real number and let S be the set of all rational numbers. Since U^i is strictly quasi-concave, continuous and T is convex, continuous, it holds that

$x(i) > \alpha$ if and only if there exists an $r \in S$ such that $r > \alpha$ and $U^i(L-r, f^i(r)-Tf^i(r), Q) > U^i(L-\alpha, f^i(\alpha)-Tf^i(\alpha), Q)$.

Therefore we have

$$\{i \in X : x(i) > \alpha\} = \bigcup_{\substack{r \in S \\ r > \alpha}} \{i \in X : U^i(L-r, f^i(r)-Tf^i(r), Q) > U^i(L-\alpha, f^i(\alpha)-Tf^i(\alpha), Q)\} .$$

Since every set of the right-hand side is a measurable set by Lemma 1 and Assumption (G) and S is a countable set, the set of the left-hand side is also a measurable set . Q.E.D.

Proof of Lemma 5: By Ascoli's theorem (Simmons [20, page 126, Theorem C]), it is sufficient to show that $C[0,k]$ is closed, bounded and equicontinuous.

Initially we show that $C[0,k]$ is bounded and equicontinuous. It is clear by definition that

$K \leqq t(y) \leqq k$ for all $y \in [0,k]$ and all $t \in C[0,k]$.

Hence $C[0,k]$ is bounded. Since any $t \in C[0,k]$ is convex and nondecreasing with $\left.\frac{d^-t(y)}{dy}\right|_k \leqq 1$, we have $0 \leqq \frac{d^-t(y)}{dy}, \frac{d^+t(y)}{dy} \leqq 1$ for all $y \in [0,k]$ and $t \in C[0,k]$. This implies

$$t(y+\varepsilon) \leqq t(y) + \varepsilon \quad \text{and} \quad t(y-\varepsilon) \geqq t(y) - \varepsilon \quad \text{for all } y-\varepsilon, y+\varepsilon \in [0,k] .$$

Hence for any $\varepsilon > 0$, let $\delta = \varepsilon$, and so, it holds that for all $y, y' \in [0,k]$ and all $t \in C[0,k]$,

$$|y - y'| \leqq \delta \quad \text{implies} \quad |t(y) - t(y')| \leqq \varepsilon .$$

This means that $C[0,k]$ is equicontinuous.

Next we show that $C[0,k]$ is a closed set. Let $\{t^s\}$ be a convergent sequence of functions in $C[0,k]$. Let $t^o = \lim_{s \to \infty} t^s$. Then it is easily verified that t^o is continuous, convex and nondecreasing. Since t^s is convex with $\left.\frac{d^-t^s(y)}{dy}\right|_k \leqq 1$, we have

$$\frac{t^s(k-\varepsilon) - t^s(k)}{-\varepsilon} \leqq 1 \quad \text{for all } s \text{ and all } \varepsilon > 0 .$$

Hence we have

$$\frac{t^o(k-\varepsilon) - t^o(k)}{-\varepsilon} \leqq 1 \quad \text{for all } \varepsilon > 0 .$$

This implies $\left.\frac{d^-t^o(y)}{dy}\right|_k \leqq 1$. Thus we have shown that $t^o \in C[0,k]$.

Q.E.D.

Proof of Lemma 8: Suppose that $\{\tilde{x}_s(i)\}$ does not converge to $x^*(i)$ for some i. Since $0 \leqq \tilde{x}_s(i) \leqq L^i$ for all s, the sequence has a convergent subsequence $\{\tilde{x}_{s^\nu}(i)\}$ with $\lim_{\nu \to \infty} \tilde{x}_{s^\nu}(i) = a \neq x^*(i)$. By the definition of $\tilde{x}_{s^\nu}(i)$, we have

$$U^i(L-\tilde{x}_{s^\nu}(i), f^i\tilde{x}_{s^\nu}(i)-\tilde{T}^{s^\nu} f^i\tilde{x}_{s^\nu}(i), \tilde{Q}^{s^\nu}) \geqq$$
$$U^i(L-x^*(i), f^i x^*(i)-\tilde{T}^{s^\nu} f^i x^*(i), \tilde{Q}^{s^\nu}) .$$

Since U^i is continuous and $\{(\tilde{T}^{s^\nu}, \tilde{Q}^{s^\nu})\}$ has the properties (i) and (ii) of Lemma 7, it holds that

$$U^i(L-a, f^i(a)-T^*f^i(a), Q^*) \geqq U^i(L-x^*(i), f^i x^*(i)-T^*f^i x^*(i), Q^*) .$$

But $a \neq x^*(i)$. This is a contradiction to the uniqueness of the labor time supply. Q.E.D.

Proof of Lemma 15: Let us construct the sequence of simple functions $\{g_s\}$ such that

$$g_s(i) = \begin{cases} \dfrac{k}{2^s} & \text{if } \dfrac{k}{2^s} \leqq f_s(i) < \dfrac{k+1}{2^s}, \; k = 0,1,\dots, s2^s-1 \\ s & \text{if } f_s(i) \geqq s . \end{cases}$$

Clearly $g_s(i) \leqq f_s(i)$ for all $i \in H$ and all s. Hence $\int_H g_s(i)\, d\mu \leqq \int_H f_s(i)\, d\mu$ for all s. Since $f_s(i) \leqq f_{s+1}(i)$ for all $i \in H$ and all s, it holds that $g_s(i) \leqq g_{s+1}(i)$ for all $i \in H$ and all s. It is also verified that $\lim_{s \to \infty} g_s(i) = f(i)$ for all $i \in H$. By the definition of integral, we have

$\int_H f(i)\, d\mu = \lim_{s \to \infty} \int_H g_s(i)\, d\mu$. Hence we have $\infty = \int_H f(i)\, d\mu$
$= \lim_{s \to \infty} \int_H g_s(i)\, d\mu \leqq \lim_{s \to \infty} \int_H f_s(i)\, d\mu$. Q.E.D.

References

1. Aumann,R.J.,Markets with a continuum of traders, Econometrica 32, 39-50, 1964.

2. ________, The St.Petersburg Paradox: A Discussion of Some Recent Comments, Journal of Economic Theory 14, 443-445.1977.

3. Fair,R.C., The Optimal Distribution of Income, Quarterly Journal of Economics 85, 551-579, 1971.

4. Feldstein,M., On the Optimal Progressivity of the Income Tax, Journal of Public Economics 2, 357-376. 1973.

5. Herstein,I.N.and J.W.Milnor, An Axiomatic Approach to Measurable Utility, Econometrica 21, 291-297. 1953.

6. Ito,Y.and M.Kaneko, Linearization of Cost Functions in a Public Goods Economy, 1978. To appear in J. Econ. Theory.

7. Kaneko,M.and K.Nakamura, The Nash Social Welfare Function, Econometrica 47, 423-435. 1978.

8. Kaneko,M., The Nash Social Welfare Function for a Measure Space of Individuals, 1979.

9. Kessllman,J.R., Egalitarianism of Earnings and Income Taxes, Journal of Public Economics 5, 285-301. 1976.

10. Komatsu,Y., Introduction to Analysis, (Japanease), Hirokawa-shoten,Tokyo, 1962.

11. Marschak,J.,Rational Behavior, Uncertain Prospects, and Measurable Utility, Econometrica 18,111-141, 1950.

12. Mirrlees,J.A., An Exploration in the theory of Optimum Income Taxation, Review of Economic Studies 38, 175-208, 1971.

13. Nash,J.F., The Bargaining Problem, Econometrica 18, 155-162, 1950.

14. ________, Two Person Cooperative Games, Econometrica 21, 128-140, 1953.

15. Owen,G., Game Theory, W.B.Saunder, Philadelphia, 1968.

16. Selten,R., Reexamination of the Perfectness Concept for Equilibrium Points in Extensive Games, International Journal of Game Theory 4, 25-55, 1975.

17. Shapley,L.S., The St.Petersburg Paradox: A Con Game ? , Journal of Economic Theory 14, 439-442, 1977.

18. ___________, Lotteries and Menus: A Comment on Unbounded Utilities, Journal of Economic Theory 14, 446-453, 1977.

19. Sheshinski,E., The Optimal Linear Income-Tax, Review of Economic Studies 39, 297-302, 1972.

20. Simmons,G.F., Introduction to Topology and Modern Analysis, McGraw-Hill, London, 1963.

21. von Neumann,J.and O.Morgenstern, Theory of Games and Economic Behavior, Princeton University Press, Princeton, 3rd ed. 1953.