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SUBJECTIVE UTILITY WITH UPPER AND LOWER PROBABILITIES ON FINITE STATES

by

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probabilities. Nakamura (1990b) and Fishburn and Nakamura (1991) integrated thresholds into nonlinear utility theory which allows for intransitive preferences.

The second concerns generalizations of subjective expected utility (SEU) to cope with Ellsberg-type paradox, which allow for nonadditivity of subjective probabilities. They adopt the Choquet integration developed by Choquet (1953-54). Those generalizations were initiated by Schmeidler (1984, 1989). He uses the idea of lottery acts (functions from the state space into a convex set of probability distributions on the consequence space) introduced by Anscombe and Aumann (1963).

Since then, several axiomatic characterizations were developed under various set-ups that depend on the nature of state and consequence spaces, including Gilboa (1987), Wakker (1989a, b), and Nakamura (1990c, 1992) and others. Gilboa adopted Savage's basic formulation (see Savage (1954) and Fishburn (1970)). His theory requires that the set of states is continuously divisible and the consequence space includes at least three elements. Wakker (1989a, b) developed topological approaches when the set of states is finite and the consequence space is a connected separable topological space. Nakamura (1990c, 1992) developed algebraic approaches for finite state spaces when the consequence space is *dense* in the following sense: if a consequence x is preferred to a consequence y, then there is a consequence z such that z is between x and y with respect to strict preferences.

In decisions under uncertainty, it is more likely that ambiguity of preference judgments stems from vague probability judgments of uncertain events, so that threshold effects might be characterized by inexact measurements of subjective probabilities. Axiomatic characterizations of imprecise probabilties have been extensively studied (see a survey by Fishburn (1986)). As far as I know, vague probability judgment was first brought into the SEU framework by Suppes (1975). He obtained a utility representation with upper and lower probabilities when preferences are semiordered. However, his model deals with only binary acts that have at most two realizable consequences.

The present paper deals with all acts that are functions from the state space into the consequence space. We assume that the set of states is finite and the consequence space is dense. We say that utility of an act is given by Choquet expected utility with respect to (w.r.t.) a nonadditive probability measure when the Choquet integration is adopted. Our representational form that is different from Suppes' says that an act f is preferred to an act g if and only if Choquet expected utility of f w.r.t. a lower probability measure is greater than Choquet expected utility of g w.r.t. an upper probability measure. A notable feature of the representation is that thresholds between consequences vanish as in monetary-like context. However, we may have thresholds between acts because of vague probability judgements generating upper and lower probabilities.

The paper is organized as follows. In Section 2, we discuss threshold representations under uncertainty. Section 3 introduces axioms to characterize a threshold representation with upper and lower probabilities and presents a representation theorem. Section 4 establishes upper and lower Choquet expected utility representations, then Section 5 provides a proof of the theorem in Section 3. Finally, we conclude in Section 6.

2 Threshold Representations Under Uncertainty

Let S be the set of states. Subsets of S are called events. Denote the complement $S\backslash A$ of $A \in 2^S = \{A : A \subseteq S\}$ by A^c. A nonnegative set function π on 2^S is said to be a *monotone* *measure* when $\pi(\emptyset) = 0$, $\pi(S) = 1$, and for all $A, B \in 2^S$, $\pi(A) \geq \pi(B)$ if $A \supseteq B$. A monotone measure π is not necessarily additive, i.e., we may have $\pi(A \cup B) \neq \pi(A) + \pi(B)$ for some disjoint $A, B \in 2^S$. An act is a function from S into the set X of consequences. $\mathcal{F}$ is the set of all acts. Let $\succ$ on $\mathcal{F}$ be the binary preference relation with $\sim$ and $\succeq$ defined in the usual way: for $f, g \in \mathcal{F}$, $f \sim g$ if neither $f \succ g$ nor $g \succ f$; $f \succeq g$ if $f \succ g$ or $f \sim g$. When $R, R' \in \{\succ, \succeq, \sim\}$, for all $f, g, h \in \mathcal{F}$, we shall write $fRgR'h$ whenever fRg and $gR'h$.

A general threshold representation in decisions under uncertainty may be given as follows: for two real valued functions U and V on $\mathcal{F}$ and for all $f, g \in \mathcal{F}$, $U(f) \geq V(f)$ and

$$f \succ g \text{ iff } V(f) > U(g). \tag{1}$$

Alternatively, $f \succ g$ iff $V(f) > V(g) + \sigma(g)$, where $\sigma(g) = U(g) - V(g) \geq 0$ is a threshold function. It is well known that the preference relation $\succ$ in the model (1) is an interval order, i.e., it is irreflexive and for all $f, g, f', g' \in \mathcal{F}$, if $f \succ g$ and $f' \succ g'$, then $f \succ g'$ or $f' \succ g$.

Since U and V must be specified for all acts in $\mathcal{F}$, the model (1) is too general for practical applications. Therefore, we are concerned with a more tractable representations of U and V as follows: for a real valued function u on X, two monotone measures π^+ and π^- on 2^S, and for all $f \in \mathcal{F}$,

$$U(f) = \int_S u(f(s))d\pi^+(s) \text{ and } V(f) = \int_S u(f(s))d\pi^-(s). \tag{2}$$

The integration in the representation (2) is defined in Choquet's sense to account for monotone measures as follows. Since we are concerned with a finite S, assume that $S = \{s_1, \ldots , s_n\}$, and $u_1 \geq \ldots \geq u_n$, where $u_i = u(f(s_i))$ for $s_i \in S$ and all i. Let π_f denote the decumulative distribution function induced by a monotone measure π through f so that $\pi_f(t) = \pi(\{s : u(f(s)) \geq t\})$ with $\pi_f(t) = 0$ for $t > u_1$ and $\pi_f(t) = 1$ for $t \leq u_n$. Then the integration is given by

$$\int_S u(f(s))d\pi(s) = \sum_{i=1}^{n-1}(u_i - u_{i+1})\pi_f(u_i) + u_n.$$

This means that expected utilities are calculated w.r.t. decumulative distributions. We say that U and V in the model (2) are respectively *upper and lower Choquet expected utility representations*. Since $\pi^+(A) \geq \pi^-(A)$ for all $A \in 2^S$, π^+ and π^- are interpreted respectively as upper and lower probability measures.

The Choquet integration can be defined by the expectations w.r.t. cumulative distributions induced by a monotone measure π through acts. Let π_* be the dual measure of π such that $\pi_*(A) = 1 - \pi(A^c)$ for all $A \in 2^S$. Gilboa (1989) showed that the Choquet integration w.r.t. π using decumulative distributions is equivalent to the Choquet integration w.r.t. π_* using cumulative distributions. Let π_*^+ and π_*^- be the dual measures of π^+ and π^- in the model (2), respectively. Then $\pi_*^- \geq \pi_*^+$, so that π_*^- and π_*^+ are interpreted respectively as upper and lower probability measures. When we apply the Choquet integration using cumulative distributions, the upper (respectively, lower) Choquet expected utility representation is obtained by the lower (respectively, upper) probability measure.

We say that a monotone measure π is *complementarily additive* when $\pi(A) + \pi(A^c) = 1$ for all $A \in 2^S$. It easily follows that if π^+ and π^- are complementarily additive, then π^+, π^-, π_*^+, and π_*^- are identical. Thus $U = V$ in the model (2), so thresholds vanish. In general those four measures are not identical each other, so that we may have two different pair of upper and lower probability measures depending upon the definition of the Choquet integration. To obtain a common pair of upper and lower probability measures in both definitions, it is easy to see that π^+ and π^- must be *complementarily symmetric*, i.e., $\pi^+(A) + \pi^-(A^c) = 1$ for all $A \in 2^S$. Throughout the paper we adopt the Choquet integration using decumulative distributions.

3 Axioms and a Representation Theorem

This section introduces our axioms and presents a representation theorem for the model (2). Throughout the paper, let $S = \{s_1, \ldots, s_n\}$ be the set of finite states. Given $x, y \in X$ and $A \in 2^S$, a binary act is an act f with $f(s) = x$ for all $s \in A$ and $f(s) = y$ for all $s \in A^c$, denoted by xAy. A constant act is an act f with $f(s) = x$ for all $s \in S$ and some $x \in X$, so every $x \in X$ is identified with a constant act. For $a \in X$, define $X_a = \{x : x \succ a\}$, and $X^a = \{x : a \succ x\}$. When $\mathcal{F}_1$ and $\mathcal{F}_2$ are subsets of $\mathcal{F}$ and $R \in \{\succ, \succsim, \sim\}$, $\mathcal{F}_1 R \mathcal{F}_2$ means that $f_1 R f_2$ for all $f_1 \in \mathcal{F}_1$ and all $f_2 \in \mathcal{F}_2$.

We use nine basic axioms. Four of them, that are understood as applying to all $f, g \in \mathcal{F}$, all $x, y, z \in X$, and all $A \in 2^S$, are stated as follows.

Axiom 1 *$\succ$ on $\mathcal{F}$ is a strict partial order, and $\succ$ on X is a weak order.*

Axiom 2 *If $f \succ g$, then $f \succ a \succ g$ for some $a \in X$.*

Axiom 3 *If $xAz \succsim f \succ yAz$, then for some $a \in X$, $f \sim aAz$ and $f \succ wAz$ for all $w \in X^a$; if $xAz \succ f \succsim yAz$, then for some $a \in X$, $f \sim aAz$ and $wAz \succ f$ for all $w \in X_a$.*

Axiom 4 *If $x \succ f(s)$ for all $s \in S$, then $x \succ f$; if $f(s) \succ x$ for all $s \in S$, then $f \succ x$.*

By definitions, a strict partial order means that $\succ$ is irreflexive and transitive, and a weak order means that $\succ$ is asymmetric and negatively transitive. Although the model (2) leads to an interval ordered preference $\succ$ on $\mathcal{F}$, it is sufficient to require that $\succ$ on $\mathcal{F}$ is a strict partial order, since $\succ$ on X is a weak order and X is dense. The density of X follows from Axiom 2. Axiom 3 is a solvability axiom, which is somewhat stronger than the usual one. Axiom 4 is a dominance axiom which says for example that if every consequence of an act f is preferred to a constant act x, then the act is also preferred to x. An implication of Axioms 1–4, that will be used to introduce our Archimedean axiom and weak multi-symmetry axiom later in the section, is given as follows.

Lemma 1 *If Axioms 1–4 hold, then for each $f \in \mathcal{F}$, there are $a, b \in X$ such that $a \succsim b$, and for all $x \in X$, $f \succ x$ if $b \succ x$; $f \sim x$ if $a \succsim x \succsim b$; $x \succ f$ if $x \succ a$.*

Proof. Let $S = \{s_1, \ldots, s_n\}$ and $f(s_i) = x_i$ for $s_i \in S$, $x_i \in X$, and $i = 1, \ldots, n$. With no loss of generality, we assume that $x_1 \succsim \ldots \succsim x_n$. If $f \succ x_1$, then by Axiom 2, $f \succ c \succ x_1$ for some $c \in X$. By Axiom 1, $c \succ f(s)$ for all $s \in S$, so by Axiom 4, $c \succ f$. This is a contradiction. Hence, $x_1 \succsim f$. Similarly, we obtain $f \succsim x_n$.

Suppose that $x_1 \succ f$. Note that $x = xSy$ for all $x, y \in X$. Since $x_1Sy \succ f \succsim x_nSy$, Axiom 3 implies that there is an $a \in X$ such that $f \sim a$ and $x \succ f$ for all $x \in X_a$. Suppose that $f \succ x_n$. Then similarly we obtain that $f \sim b$ for some $b \in X$ and $f \succ x$ for all $x \in X^b$. Since $x_1 \succsim f \succsim x_n$, we note by Axioms 1 and 4 that $x \succ f$ if $x \succ x_1$, and $f \succ x$ if $x_n \succ x$.

It follows from Axiom 1 and the preceding paragraph that there are $a, b \in X$ such that $a \succsim b$, $f \sim \{a, b\}$, and for all $z \in X$, $z \succ f$ if $z \succ a$; $f \sim z$ if $a \succ z \succ b$; $f \succ z$ if $b \succ z$. It remains to show that $f \sim x$ if $x \sim a$, and $f \sim x$ if $x \sim b$. Suppose that $x \sim a$. If $x \succ f$, then by Axiom 2, $x \succ c \succ f$ for some $c \in X$. Thus by Axiom 1, $a \succ c$, so $f \succ a$, a contradiction. If $f \succ x$, then by Axiom 2, $f \succ c \succ x$ for some $c \in X$. Thus by Axiom 1, $c \succ a$, so $f \succ a$, a contradiction. Hence, we must have $f \sim x$. If $x \sim b$, then similarly we obtain that $f \sim x$. □

Lemma 1 implies that there are mappings, m^+ and m^-, from $\mathcal{F}$ into X that assign constant acts, $m^+(f)$ and $m^-(f)$, such that $m^+(f) \succeq m^-(f)$, and for all $x \in X$,

$$\begin{aligned} f \succ x & \quad \text{if} \quad m^-(f) \succ x; \\ f \sim x & \quad \text{if} \quad m^+(f) \succeq x \succeq m^-(f); \\ x \succ f & \quad \text{if} \quad x \succ m^+(f). \end{aligned}$$

Let M^+ and M^- be the sets of all m^+ and all m^-, respectively. For $a, b \in X$, let $[a, b] = \{x \in X : a \succeq x \succeq b\}$ be a *preference interval* in X. Since $\succeq$ on X is a weak order, the left end of a preference interval is at least preferred to the right end of that preference interval. Let $[m^+(f), m^-(f)]$ be a preference interval associated with an act f. When $f = xAy$, we denote $m^+_A(xy) = m^+(f)$ and $m^-_A(xy) = m^-(f)$.

Axioms 1–4 imply that $f \succ g$ if and only if $m^-(f) \succ m^+(g)$. In words, f is preferred to g if and only if the right end of preference interval of f is preferred to the left end of preference interval of g. Suppose that u is an order preserving function on X, i.e., $x \succ y$ iff $u(x) > u(y)$. Then $f \succ g$ iff $u(m^-(f)) > u(m^+(g))$. Thus the model (2) says that for an appropriate order preserving function u on X, there are monotone measures π^+ and π^- on 2^S such that $u(m^+(g)) = \int_S u(g(s))d\pi^+(s)$ and $u(m^-(f)) = \int_S u(f(s))d\pi^-(s)$.

We say that an event $A \in 2^S$ is *weakly null* if for all $x, y \in X$ $xAy \sim y$ whenever $x \succeq y$; *weakly universal* if for all $x, y \in X$, $x \sim xAy$ whenever $x \succeq y$; *strongly null* if for all $x, y \in X$, $z \succ xAy$ for all $z \in X_y$ whenever $x \succeq y$; *strongly universal* if for all $x, y \in X$, $xAy \succ z$ for all $z \in X^x$ whenever $x \succeq y$. It follows from Lemma 1 and Axiom 4 that if A is strongly null (respectively, universal), then A is weakly null (respectively, universal). In terms of preference intervals $[m^+_A(xy), m^-_A(xy)]$, we have

$$\begin{aligned} m^+_A(xy) \succeq y & \quad \text{and} \quad m^-_A(xy) \sim y \text{ when } A \text{ is weakly null;} \\ x \sim m^+_A(xy) & \quad \text{and} \quad x \succeq m^-_A(xy) \text{ when } A \text{ is weakly universal;} \\ m^+_A(xy) \sim y & \quad \text{and} \quad m^-_A(xy) \sim y \text{ when } A \text{ is strongly null;} \\ x \sim m^+_A(xy) & \quad \text{and} \quad x \sim m^-_A(xy) \text{ when } A \text{ is strongly universal.} \end{aligned}$$

The weak and strong notions of null and universal events correspond to the following fact: if $\succ$ is dense and nonempty, and the model (2) is to hold, then

$$\begin{aligned} \pi^-(A) &= 0 \text{ when } A \text{ is weakly null;} \\ \pi^+(A) &= 1 \text{ when } A \text{ is weakly universal;} \\ \pi^+(A) &= 0 \text{ when } A \text{ is strongly null;} \\ \pi^-(A) &= 1 \text{ when } A \text{ is strongly univeral.} \end{aligned}$$

If A is weakly null (respectively, weakly universal), then $\pi^-(A) = 0$ (respectively, $\pi^+(A) = 1$) easily follows from the model. To see the last two claims, we suppose that A is strongly null and $x \succ y$ for some $x, y \in X$. When A is strongly universal, a similar argument gives $\pi^-(A) = 1$.

With no loss of generality, let $u(x) = 0$ and $u(y) = -1$. Let I be the set of all positive integers. Since $\succ$ is dense, we take a decreasing sequence $\{x_i\}$ such that $x \succ x_i \succ x_{i+1} \succ y$ for all $i \in I$. On the contrary, we suppose that $0 < \pi^+(A) \leq 1$. Let $\alpha = \pi^+(A)$ and $u_i = u(x_i)$ for $i \in I$. Since A is strongly null, $x_i \succ xAx_{i+1}$ for all $i \in I$. Then the model requires that $u_i > (1-\alpha)u_{i+1}$ for all $i \in I$. Thus for all $i \in I$, $u_1 > (1-\alpha)^i u_{i+1}$, so that we must have $u_1 = 0$. This contradicts $x \succ x_1$. Hence, $\pi^+(A) = 0$.

The fifth axiom, which applies to all $x, y, z, w \in X$ and all $A, B \in 2^S$, is stated as follows.

Axiom 5 *If A is not weakly null and B is not weakly universal, and if $x \succeq y$ and $z \succeq w$, then $aAy \succ zBw$ for all $a \in X_x$ iff $xAy \succ zBb$ for all $b \in X^w$; if A is not strongly universal and B is not strongly null, and if $x \succ y$ and $z \succ w$, then $xAa \succ zBw$ for all $a \in X_y$ iff $xAy \succ bBw$ for all $b \in X^z$.*

Although Axiom 5 looks complicated, an intuitive meaning is straightfoward from the model(2). If Axiom 5 is to hold, then the left end of preference interval of zBw is not preferred to the right end of preference interval of xAy. Moreover, if $xAy \sim zBw$, then the left end of preference interval of zBw is indifferent to the right end of preference interval of xAy.

The sixth axiom, which applies to all $x, y, z \in X$ and all $A, B \in 2^S$, requires that π^+ and π^- in the model (2) are monotone.

Axiom 6 *If $x \succeq y$ and $A \supseteq B$, then $z \succ xBy$ whenever $z \succ xAy$; $xAy \succ z$ whenever $xBy \succ z$.*

Let N be any set of consecutive integers. Given an event $A \in 2^S$ which is neither weakly null nor strongly universal, we define a *lower standard sequence* as a set $\{a_i : a_i \in X, i \in N\}$ for which there are $a, b \in X$ such that not$(a \sim b)$, either $\{a, b\} \succeq \{a_i\}$ and $m_A^-(aa_i) \sim m_A^-(ba_{i+1})$ for all $i, i+1 \in N$ or $\{a_i\} \succeq \{a, b\}$ and $m_A^-(a_i a) \sim m_A^-(a_{i+1} b)$ for all $i, i+1 \in N$. Given an event $A \in 2^S$, which is neither strongly null nor weakly universal, we define an *upper standard sequence* in a similar manner by replacing m^- by m^+. We say that $\{a_i\}$ is a *standard sequence* if it is either a lower standard sequence or an upper standard sequence. Our Archimedean condition is stated as follows.

Axiom 7 *Every strictly bounded standard sequence is finite.*

A partition of S is a sequence of nonempty events that mutually disjoint and whose union equals S. For an n-partition $P = \{A_1, \ldots, A_n\}$, if $f(s) = x_i$ for all $s \in A_i$ and all i, then we shall respectively write $f_P(x_1 \ldots x_n)$ and $m_P^*(x_1, \ldots, x_n)$ instead of f and $m^*(f)$, where $* \in \{-, +\}$. The following axioms apply to all $x, y, z, x_1, \ldots, x_n \in X$ for $n \geq 1$, all $f \in \mathcal{F}$, all n-partitions, all $m, m' \in M^- \cup M^+$, and all $A, B \in 2^S$.

Axiom 8 *If $x_1 \succeq \ldots \succeq x_n$, $y_1 \succeq \ldots \succeq y_n$, and $x_i \succeq y_i$ for all i, then*

$$m_A(m'_P(x_1 \ldots x_n) m'_P(y_1 \ldots y_n)) \sim m'_P(m_A(x_1 y_1) \ldots m_A(x_n y_n)).$$

Axiom 9 *If $x \succeq y \succeq z$, then*

$$m_B^-(m_A^+(yy) m_A^-(xz)) \sim m_A^-(m_B^+(yx) m_B^-(yz)) \text{ when } y \succeq m_A^-(xz);$$

$$m_B^+(m_A^+(xz) m_A^-(yy)) \sim m_A^+(m_B^+(xy) m_B^-(zy)) \text{ when } m_A^+(xz) \succeq y.$$

Those two axioms are concerned with relations among left and right ends of preference intervals of acts. When either $m, m' \in M^+$ or $m, m' \in M^-$, Axiom 8 is a weak multi-symmetry axiom defined in Nakamura (1992), that extends Pfanzagle's bisymmetry axiom.

The main result of the paper is given as follows.

Theorem 1 *Suppose that S is finite, and that there are $B, C \in 2^S$ such that B is neither weakly null nor strongly universal, and C is neither strongly null nor weakly universal. If Axioms 1–8 hold, then*
(1) there are two monotone measures π^+ and π^- on 2^S and a real valued function u on X such that for all $A \in 2^S$, $\pi^+(A) \geq \pi^-(A)$ and for all $f, g \in \mathcal{F}$,

$$f \succ g \text{ iff } \int_S u(f(s))d\pi^-(s) > \int_S u(g(s))d\pi^+(s).$$

(2) π^+ and π^- are complementarily symmetric if Axiom 9 holds.
(3) π^+ and π^- are unique, and u is unique up to a positive linear transformation.

The proof is deferred to Section 5. We note that Axioms 2, 3, 8, and 9 are not necessary for the model (2), since m^+ and m^- may not exist when X is a set of rational numbers and $\succ = >$ on X. However, it is possible to make those sufficient if we adopt some topological assumption, e.g., X is a connected separable topological space.

4 Upper and Lower Choquet Expected Utility Representations

This section establishes the upper and lower Choquet expected utility representations in the model (2). Throughout the section, we shall assume that Axioms 1–8 hold. We introduce two auxiliary binary relations $\succ^+$ and $\succ^-$ on $\mathcal{F}$, and show that those relations lead to the upper and lower Choquet expected utility representations, respectively. Define $\succ^+$ and $\succ^-$ on $\mathcal{F}$ as follows: for all $f, g \in \mathcal{F}$,

$$\begin{aligned} f \succ^+ g \quad &\text{iff} \quad f \sim x \succ g \text{ for some } x \in X; \\ f \succ^- g \quad &\text{iff} \quad f \succ x \sim g \text{ for some } x \in X. \end{aligned}$$

In terms of preference intervals, we shall see later in Proposition 1 that $\succ^+$ (respectively, $\succ^-$) essentially governs the placements of left (respectively, right) ends of preference intervals in the model (2).

Throughout the section, let $* \in \{+, -\}$. Let $\sim^*$ be the symmetric complement of $\succ^*$. Thus $f \sim^* g$ iff neither $f \succ^* g$ nor $g \succ^* f$. Also, let $\succeq^* = \succ^* \cup \sim^*$. Note that $x \succeq y$ iff $x \succeq^* y$; for each $f \in \mathcal{F}$, $f \sim^* m^*(f)$. If $f \succ g$, then by Axiom 2, $f \succ a \succ g$ for some $a \in X$. Thus $f \sim m^-(f) \succ g$, so $f \succ^+ g$. Similarly, $f \succ^- g$. Therefore, $f \succ g$ implies $f \succ^* g$.

A basic structure for $\succ^*$ is given as follows.

Lemma 2 *$\succ^*$ is an weak order.*

Proof. We show that $\succ^+$ is a weak order. The proof for $\succ^-$ is similar. To show asymmetry of $\succ^+$, suppose that $f \succ^+ g$ and $g \succ^+ f$. Then $f \sim a \succ g$ and $g \sim b \succ f$ for some $a, b \in X$. By Axiom 1, $a \sim b$, so $a \sim b \succ f$. By Axiom 2, $b \succ c \succ f$ for some $c \in X$. Thus $a \sim b \succ c$. Since

$\succeq$ on X is a weak order, $a \succ c$, so by Axiom 1, $a \succ f$. This is a contradiction. Hence, $\succ^+$ is asymmetric.

Next we show negative transitivity of $\succ^+$. Suppose on the contrary that not($f \succ^+ g$), not($g \succ^+ h$), and $f \succ^+ h$. Then $f \sim a \succ h$ for some $a \in X$. If $a \succ g$, then $f \succ^+ g$; if $a \sim g$, then $g \succ^+ h$; if $g \succ a$, then by Axiom 1, $g \succ h$, so $g \succ^+ h$. These are contradictions. Hence, $\succ^+$ is negatively transitive. □

We say that $A \in 2^S$ is *null w.r.t.* $\succeq^*$ if for all $x, y, z \in X$, $xAz \sim^* yAz$ whenever $x \succeq^* y \succeq^* z$, and that A is *universal w.r.t.* $\succeq^*$ if for all $x, y, z \in X$, $xAy \sim^* xAz$ whenever $x \succeq^* y \succeq^* z$. Then we have the following relations of null and universal events.

Lemma 3 *(1) A is weakly null iff A is null w.r.t.* $\succeq^-$.
(2) A is strongly null iff A is null w.r.t. $\succeq^+$.
(3) A is weakly universal iff A is universal w.r.t. $\succeq^+$.
(4) A is strongly universal iff A is universal w.r.t. $\succeq^-$.

Proof. (1) Suppose that A is weakly null. Then for all $x, y \in X$ with $x \succeq y$, $xAy \sim y$. Assume that $x \succeq^+ y \succeq^+ z$. Thus $x \succeq y \succeq z$, so $z \sim \{xAz, yAz\}$. By Axiom 4, $\{xAz, yAz\} \succ w$ for all $w \in X^z$. Therefore, by Lemma 1, $\{xAz, yAz\} \sim^- z$. Since $\sim^-$ is transitive, $xAz \sim^- yAz$. Hence, A is null w.r.t. $\succeq^-$.

Suppose that A is null w.r.t. $\succeq^-$. Then for all $x, y, z \in X$ with $x \succeq^- y \succeq^- z$, $xAz \sim^- yAz$. If we take $y = z$, then $xAz \sim^- z$, so $xAz \sim z$. Hence, A is weakly null.

(2) Suppose that A is strongly null. Then for all $x, y \in X$ with $x \succeq y$, $z \succ xAy$ for all $z \in X_y$. Assume that $x \succeq^+ y \succeq^+ z$. If X_z is empty, then it easily follows from Lemma 1 and Axiom 4 that A is null w.r.t. $\succeq^+$. If X_z is not empty, then $w \succ \{xAz, yAz\}$ for all $w \in X_z$. By Lemma 1, $z \sim \{xAz, yAz\}$, so $z \sim^+ \{xAz, yAz\}$. Since $\sim^+$ is transitive, $xAz \sim^+ yAz$. Hence, A is null w.r.t. $\succeq^+$.

Suppose that A is null w.r.t. $\succeq^+$. Then for all $x, y, z \in X$ with $x \succeq^+ y \succeq^+ z$, $xAz \sim^+ yAz$. If we take $y = z$, then $xAz \sim^+ z$, so by Lemma 1, $w \succ xAz$ for all $w \in X_z$. Hence, A is strongly null.

(3) Similar to (1).

(4) Similar to (2). □

The following proposition establishes the upper and lower Choquet expected utility representations. The proof is deferred to the end of the section.

Proposition 1 *Suppose that S is finite, and that there is an event which is neither null nor universal w.r.t.* $\succeq^*$. *Then there exist a monotone measure* π^* *on* 2^S *and a real valued function* u^* *on X such that for all* $f, g \in \mathcal{F}$,

$$f \succeq^* g \text{ iff } \int_S u^*(f(s))d\pi^*(s) \geq \int_S u^*(g(s))d\pi^*(s).$$

Moreover, π^* *is unique, and* u^* *is unique up to a positive linear transformation.*

To prove Proposition 1, we need the following four lemmas.

Lemma 4 *Suppose that* $z \sim xAy$ *and* $x \succeq y$.
(1) If X^y *is not empty, A is not weakly universal, and* $z \succ xAa$ *for all* $a \in X^y$, *then* $z \sim^- xAy$.
(2) If X_x *is not empty, A is not weakly null, and* $aAy \succ z$ *for all* $a \in X_x$, *then* $xAy \sim^- z$.

Lemma 5 *Suppose that $z \sim xAy$ and $x \succ y$.*
(1) If A is not strongly null, then $z \sim^+ xAy$ iff $z \succ aAy$ for all $a \in X^x$.
(2) If A is not strongly universal, then $xAy \sim^- z$ iff $xAa \succ z$ for all $a \in X_y$.

Lemma 6 *(1) If $b \succeq c$ and $a \succ bAc$, then $a \succ bAx$ for all $x \in X^c$.*
(2) If $b \succ c$ and $a \succ bAc$, then $a \succ xAc$ for all $x \in X^b$.
(3) If $b \succeq c$ and $bAc \succ a$, then $xAc \succ a$ for all $x \in X_b$.
(4) If $b \succ c$ and $bAc \succ a$, then $bAx \succ a$ for all $x \in X_c$.

Lemma 7 *(1) If $xAz \succeq a \succ yAz$, then there is a $b \in X$ such that $x \succeq b \succ y$, $a \sim bAz$, and $a \succ wAz$ for all $w \in X^b$.*
(2) If $xAz \succ a \succeq yAz$, then there is a $b \in X$ such that $x \succ b \succeq y$, $a \sim bAz$, and $wAz \succ a$ for all $w \in X_b$.

Proof of Lemma 4. We show (1). The proof of (2) is similar. Suppose that the hypotheses of the lemma hold. If x_z is empty, then the desired result follows from Lemma 1 and the definition of $\sim^+$. We assume that X_z is not empty. Since S is not weakly null and $b = bSz$, Axiom 5 implies that $b \succ xAy$ for all $b \in X_z$. Hence the desired result follows from Lemma 1 and the definition of $\sim^+$. □

Proof of Lemma 5. We show (1). The proof of (2) is similar. Suppose that the hypotheses of the lemma hold. Assume first that $a \sim^+ xAy$. If X_z is empty, then $z \succeq x \succ y$. By Axioms 1 and 4, $z \succ aAy$ for all $a \in X^x$. If X_z is not empty, then the desired result follows from Lemma 1 and the definition of $\sim^+$. Assume next that $z \succ aAy$ for all $a \in X^x$. Then it similarly follows that $z \sim^+ xAy$. □

Proof of Lemma 6. (1) Suppose that $b \succeq c$ and $a \succ bAc$. Note by Axiom 1 that $x \succ bAc$ for all $x \in X_a$. Since S is not weakly null, if A is not weakly universal, then Axiom 5 gives the desired result. Thus assume that A is weakly universal. Then $bAc \sim b$. By Lemma 1, $a \succ b$. Since $a \succ b \succeq c$, Axiom 4 implies that $a \succ bAx$ for all $x \in X^c$

(2) suppose that $b \succ c$ and $a \succ bAc$. Note by Axiom 1 that $x \succ bAc$ for all $x \in X_a$. If A is not strongly null, then Axiom 5 gives the desired result. Thus assume that A is strongly null. Since $a \succ bAc$, Axiom 2 implies that $a \succ d \succ bAc$ for some $d \in X$. By Axiom 4, $d \succeq c$, so by Axiom 1, $a \succ c$. Since A is strongly null, $a \succ xAc$ for all $x \in X_c$. If $c \succeq x$, then $a \succ c \succeq x$, so by Axiom 4, $a \succ xAc$. Hence $a \succ xAc$ for all $x \in X^b$.

(3) Similar to (1).

(4) Similar to (2). □

Proof of Lemma 7. We show (1). The proof of (2) is similar. Suppose that $xAz \succeq a \succ yAz$. Then Axiom 3 implies that there is a $b \in X$ such that $a \sim bAz$ and $a \succ wAz$ for all $w \in X^b$. Since $xAz \succeq a$, $x \succeq b$. It remains to show that $b \succ y$.

Assume first that $y \succ z$. Then by Lemma 6(2), $a \succ wAz$ for all $w \in X^y$. Since $a \sim bAz$, $b \succeq y$. Suppose that $b \sim y$. Then by Axiom 1, $b \succ z$. Since $a \succ yAz$, Axiom 2 implies that $a \succ c \succ yAz$ for some $c \in X$. By Lemma 6(2), $c \succ wAz$ for all $w \in X^y$. By Axiom 1, $bAz \succeq c$. Thus Axiom 3 implies that $c \sim bAz$. If A is not strongly null, then Axiom 5 implies that $w \succ bAz$ for all $w \in X_c$, so that $a \succ bAz$. If A is strongly null, then $w \succ bAz$ for all $w \in X_z$, so $a \succ bAz$. Since $a \sim bAz$, these are contradictions. Hence $b \succ y$.

Assume next that $z \succeq y$. Then by Lemma 6(1), $a \succ zA^cw$ for all $w \in X^y$. Since $a \sim zA^cb$, $b \succeq y$. Suppose that $b \sim y$. Then by Axiom 1, $z \succeq b$. Since $a \succ yAz$, Axiom 2 implies that $a \succ c \succ yAz$ for some $c \in X$. By Lemma 6(1), $c \succ zA^cw$ for all $w \in X^y$. By Axiom 1, $bAz \succeq c$. Thus by Axiom 3, $c \sim bAz$. If A^c is not weakly universal, then Axiom 5 implies that $w \succ bAz$ for all $w \in X_c$, so that $a \succ bAz$. If A^c is weakly universal, then $zA^cy \sim z$. Thus $a \succ z$, so $a \succ b$. Therefore, by Axiom 4, $a \succ bAz$. These are contradictions. Hence $b \succ y$. □

Proof of Proposition 1. Suppose that S is finite, and that there is an event which is neither null nor universal w.r.t. $\succeq^*$. Before proving the proposition, we define some terminologies. We say that $\succeq^*$ is bounded if for each $f \in \mathcal{F}$, there are $a, b \in X$ such that $a \succeq^* f \succeq^* b$. Let N be any set of consecutive integers. Given an event $A \in 2^S$ that is neither null nor universal w.r.t. $\succeq^*$, a standard sequence w.r.t. $\succeq^*$ is defined as a set $\{a_i : a_i \in X, i \in N\}$ for which there exist $a, b \in X$ such that not($a \sim^* b$), either $\{a,b\} \succeq^* \{a_i\}$ and $aAa_i \sim^* bAa_{i+1}$ for all $i, i+1 \in N$, or $\{a_i\} \succeq^* \{a,b\}$ and $a_iAa \sim^* a_{i+1}Ab$ for all $i, i+1 \in N$.

Since Nakamura (1990c) showed that the proposition holds if $\succeq^*$ on $\mathcal{F}$ satisfies the following six axioms (B1–B6), it suffices to show that B1–B6 are derived from Axioms 1–8. Axioms B1–B6 are understood as applying to all $f \in \mathcal{F}$, all $x, y, z, w, x_1, \ldots, x_n, y_1, \ldots, y_n \in X$, all $n > 1$, all n-partitions P, all $m^* \in M^*$, and all $A, B \in 2^S$.

B1. $\succeq^*$ is bounded, transitive, and complete.
B2. If $xAz \succeq^* f \succeq^* yAz$, then $f \sim^* aAz$ for some $a \in X$.
B3. If A is not null w.r.t. $\succeq^*$ and $\{x,y\} \succeq^* z$, then $x \succeq^* y$ iff $xAz \succeq^* yAz$; if A is not universal w.r.t. $\succeq^*$ and $z \succeq^* \{x,y\}$, then $x \succeq^* y$ iff $zAx \succeq^* zAy$.
B4. If $x \succeq^* y$ and $A \supseteq B$, then $xAy \succeq^* xBy$.
B5. Every strictly bounded standard sequence w.r.t. $\succeq^*$ is finite.
B6. If $x_1 \succeq^* \ldots \succeq^* x_n$, $y_1 \succeq^* \ldots \succeq^* y_n$, and $x_i \succeq^* y_i$ for all i, then

$$f_A(m_P^*(x_1 \ldots x_n) m_P^*(y_1 \ldots y_n)) \sim^* f_P(m_A^*(x_1 y_1) \ldots m_A^*(x_n y_n)).$$

(B1) This follows from Lemmas 1 and 2.

(B2) Suppose that $xAz \succeq^+ f \succeq^+ yAz$. If $f \sim^+ xAz$ or $f \sim^+ yAz$, then the desired result obtains. Thus assume that $xAz \succ^+ f \succ^+ yAz$. Then by B1, $xAz \succ^+ m^+(f) \succ^+ yAz$, so $xAz \succeq m^+(f) \succ yAz$. It follows from Lemma 7(1) that there is an $a \in X$ such that $x \succeq a \succ y$, $m^+(f) \sim aAz$, and $m^+(f) \succ wAz$ for all $w \in X^a$. We are to show that $m^+(f) \sim^+ aAz$, so by B1, $f \sim^+ aAz$. Let $b = m^+(f)$. We have three cases to examine: $x \succ y \succeq z$; $z \succeq x \succ y$; $x \succ z \succ y$.

Case 1 ($x \succ y \succeq z$). By Axiom 1, $a \succ z$. Suppose that A is not strongly null. Then Lemma 5(1) implies that $b \sim^+ aAz$. Suppose that A is strongly null. Then $w \succ aAz$ for all $w \in X_z$. Since $b \sim aAz$, $z \succeq b$. If $z \succ b$, then by Axiom 4, $yAz \succ b$, a contradiction. Thus $b \sim z$. Therefore, $w \succ aAz$ for all $w \in X_b$. Hence $b \sim^+ aAz$.

Case 2 ($z \succeq x \succ y$). Since $x \succeq a \succ y$ and $b \sim aAz$, Axioms 1 and 4 imply $z \succeq b$. If A^c is weakly universal, then $zA^cy \sim z$. Thus by Lemma 1, $zA^cy \succeq b$, so $yAz \succeq b$. This is a contradiction. Therefore, A^c is not weakly universal. Since $b \succ wAz = zA^cw$ for all $w \in X^a$, Lemma 4(1) implies that $b \sim^+ aAz$.

Case 3 ($x \succ z \succ y$). If $a \succ z$, then a similar analysis of Case 1 gives that $b \sim^+ aAz$. If $z \succeq a$, then a similar analysis of Case 2 gives that $b \sim^+ aAz$.

(B3) Suppose that A is not null w.r.t. $\succsim^+$ and $\{x,y\} \succsim^+ z$. The proof for the other case is similar. Then by Lemma 3(2), A is not strongly null and $\{x,y\} \succsim z$. Suppose first that $x \succsim^+ y$. Then $x \succsim y$. We are to show that $xAz \succsim^+ yAz$. Suppose on the contrary that $yAz \succ^+ xAz$. Then $yAz \sim a \succ xAz$ for some $a \in X$. By Lemma 7(1) and Axiom 1, $y \succ x$, a contradiction. Hence, $xAz \succsim^+ yAz$.

Next we suppose that $xAz \succsim^+ yAz$. Assume that $xAz \succ^+ yAz$. Then $xAz \sim a \succ yAz$ for some $a \in X$. Thus by Lemma 7(1) and Axiom 1, $x \succ y$.

Assume that $xAz \sim^+ yAz$. Then it easily follows from Lemma 1 that $a \sim \{xAz, yAz\}$ for some $a \in X$ and $w \succ \{xAz, yAz\}$ for all $w \in X_a$. Suppose $x \succ z$. Then by Lemma 5(1), $a \succ wAz$ for all $w \in X^x$, since A is not strongly null. Thus $y \succsim x$, so by Axiom 1, $y \succ z$. Suppose $y \succ z$. Then similarly $x \succsim y$ and $x \succ z$. Hence, $x \sim y$ if $\{x,y\} \succsim z$.

(B4) Suppose that $x \succsim^* y$ and $A \supseteq B$. Then $x \succsim y$. It follows from Axiom 6 and Lemma 1 that $m^*_A(xy) \succsim m^*_B(xy)$. Hence by B1, $xAy \succsim^* xBy$.

(B5) This easily follows from Axiom 7 and Lemma 2.

(B6) This easily follows from Axiom 8, B1, and Lemma 1. □

5 Proof of the Theorem

Suppose that S is finite, and that there are $B, C \in 2^S$ such that B is neither weakly null nor strongly universal, and C is neither strongly null nor weakly universal. Suppose also that Axioms 1–8 hold. By Lemma 3, B is neither null nor universal w.r.t. $\succsim^-$, and C is neither null nor universal w.r.t. $\succsim^+$.

It follows from Proposition 1 that there are two monotone measures π^+ and π^- on 2^S, and two real valued functions u^+ and u^- on X such that for all $f, g \in \mathcal{F}$,

$$f \succ^+ g \text{ iff} \int_S u^+(f(s))d\pi^+(s) > \int_S u^+(g(s))d\pi^+(s);$$

$$f \succ^- g \text{ iff} \int_S u^-(f(s))d\pi^-(s) > \int_S u^-(g(s))d\pi^-(s).$$

Moreover, π^+ and π^- are unique, and u^+ and u^- are unique up to positive linear transformations, so (3) in the theorem follows. To show (1), it suffices to show that $u^+ = u^-$ and $\pi^+ \geq \pi^-$. By Axiom 8, we have the following weak isometry condition: for all $x, y, z, w \in X$ with $x \succsim y$ and $z \succsim w$,

$$m^+_B(m^-_C(xy)m^-_C(zw)) \sim m^-_C(m^+_B(xz)m^+_B(yw)).$$

Since $\succsim = \succsim^+ = \succsim^-$ on X, B is neither null nor universal w.r.t. $\succsim^-$ and C is neither null nor universal w.r.t. $\succsim^+$, it easily follows from Proposition 1 in Nakamura (1992) that $u^+ = u^-$. Then also $\pi^+ \geq \pi^-$ follows.

To show (2), suppose that Axiom 9 holds. Since $\succ$ is not empty, let $x \succ y$ for some $x, y \in X$. First we assume that $A \in 2^S$ is neither weakly null nor weakly universal. Thus $\pi^-(A) \neq 0$ and $\pi^+(A) \neq 1$. By (1), we obtain that $x \succsim m^-_A(xy) \succ y$ and $x \succ m^+_A(xy) \succsim y$. By Axiom 1, $x \succ m^-_A(xy)$ and $m^+_A(xy) \succ y$. Since $\succ$ is dense, there are $z, w \in X$ such that $x \succ z \succ m^-_A(xy)$ and $m^+_A(xy) \succ w \succ y$.

Case 1 ($x \succ z \succ m_A^-(xy)$). Since $z = m_A^+(zz)$ and $m_A^+(zx) \succeq m_A^-(zy)$, (1) implies that

$$\begin{aligned}
u(m_A^-(m_A^+(zz)m_A^-(xy))) &= \pi^-(A)u(m_A^+(zz)) + (1-\pi^-(A))u(m_A^-(xy)) \\
&= \pi^-(A)u(z) + (1-\pi^-(A))(\pi^-(A)u(x) + (1-\pi^-(A))u(y)) \\
&= \pi^-(A)(1-\pi^-(A))u(x) + \pi^-(A)u(z) + (1-\pi^-(A))^2u(y), \\
u(m_A^-(m_A^+(zx)m_A^-(zy))) &= \pi^-(A)u(m_A^+(zx)) + (1-\pi^-(A))u(m_A^-(zy)) \\
&= \pi^-(A)(\pi^+(A^c)u(x) + (1-\pi^+(A^c))u(z)) \\
&\quad + (1-\pi^-(A))(\pi^-(A)u(z) + (1-\pi^-(A))u(y)) \\
&= \pi^-(A)\pi^+(A^c)u(x) + \pi^-(A)(2-\pi^-(A)-\pi^+(A^c))u(z) \\
&\quad + (1-\pi^-(A))^2u(y).
\end{aligned}$$

By Axiom 9, $u(m_A^-(m_A^+(zz)m_A^-(xy))) = u(m_A^-(m_A^+(zx)m_A^-(zy)))$. Thus it follows from the above two equations that

$$\pi^-(A)((1-\pi^-(A))u(x) + u(z)) = \pi^-(A)(\pi^+(A^c)u(x) + (2-\pi^-(A)-\pi^+(A^c))u(z)),$$

which is rearranged to give $\pi^-(A)(1-\pi^-(A)-\pi^+(A^c))(u(x)-u(z)) = 0$. Since $u(x) > u(z)$, and A is not weakly null, we must have $\pi^-(A) + \pi^+(A^c) = 1$.

Case 2 ($m_A^+(xy) \succ w \succ y$). Since $w = m_A^-(ww)$ and $m_A^+(xw) \succeq m_A^-(yw)$, (1) implies that

$$\begin{aligned}
u(m_A^+(m_A^+(xy)m_A^-(ww))) &= \pi^+(A)u(m_A^+(xy)) + (1-\pi^+(A))u(m_A^-(ww)) \\
&= \pi^+(A)(\pi^+(A)u(x) + (1-\pi^+(A))u(y)) + (1-\pi^+(A))u(w) \\
&= \pi^+(A)^2u(x) + (1-\pi^+(A))u(w) + \pi^+(A)(1-\pi^+(A))u(y), \\
u(m_A^+(m_A^+(xw)m_A^-(yw))) &= \pi^+(A)u(m_A^+(xw)) + (1-\pi^+(A))u(m_A^-(yw)) \\
&= \pi^+(A)(\pi^+(A)u(x) + (1-\pi^+(A))u(w)) \\
&\quad + (1-\pi^+(A))(\pi^-(A^c)u(w) + (1-\pi^-(A^c))u(y)) \\
&= \pi^+(A)^2u(x) + (1-\pi^+(A))(\pi^+(A)+\pi^-(A^c))u(w) \\
&\quad + (1-\pi^-(A))(1-\pi^-(A^c))u(y).
\end{aligned}$$

By Axiom 9, $u(m_A^+(m_A^+(xy)m_A^-(ww))) = u(m_A^+(m_A^+(xw)m_A^-(yw)))$. Thus it follows from the above two equations that

$$(1-\pi^+(A))(u(w) + \pi^+(A)u(y)) = (1-\pi^+(A))((\pi^+(A)+\pi^-(A^c))u(w) + (1-\pi^-(A^c))u(y)),$$

which is rearranged to give $(1-\pi^+(A))(1-\pi^+(A)-\pi^-(A^c))(u(w)-u(y)) = 0$. Since $u(w) > u(y)$, and A is not weakly universal, we must have $\pi^+(A) + \pi^-(A^c) = 1$.

Next we assume that A is weakly null. Then we are to show that A^c is weakly universal, so that $\pi^-(A) + \pi^+(A^c) = 1$. When A is weakly universal, the proof is similar. If A^c is neither weakly null nor weakly universal, then Cases 1 and 2 imply that A is neither weakly null nor weakly universal. This is a contradiction. Therefore, A^c must be either weakly null or weakly universal.

Suppose that A^c is not weakly universal. Since B is neither weakly null nor strongly universal, $0 < \pi^-(B) < 1$. Let $x \succ z$. Then $x \succ m_B^-(xz) \succ z$. Since $\succ$ is dense, $x \succ y \succ m_B^-(xz)$ for some

$y \in X$. It follows from (1) that

$$\begin{aligned} u(m_A^-(m_B^+(yy)m_B^-(xz))) &= u(m_B^-(xz)) \\ &= \pi^-(B)u(x) + (1-\pi^-(B))u(z), \\ u(m_B^-(m_A^+(yx)m_A^-(yz))) &= \pi^-(B)u(m_A^+(yx)) + (1-\pi^-(B))u(m_A^-(yz)) \\ &= \pi^-(B)(\pi^+(A^c)u(x) + (1-\pi^+(A^c))u(y)) + (1-\pi^-(B))u(z). \end{aligned}$$

Since A^c is not weakly universal, $\pi^+(A^c) \neq 1$, so that $m_A^-(m_B^+(yy)m_B^-(xz)) \succ m_B^-(m_A^+(yx)m_A^-(yz))$. This contradicts Axiom 9. Hence A^c must be weakly universal. □

6 Conclusion

The purpose of the paper has been to provide an axiomatic characterization for a threshold representation in decisions under uncertainty. Thresholds are represented by inexact measurement of subjective probabilities, i.e., upper and lower probabilities. In SEU frameworks, various set-ups lead to different axiomatizations. We have assumed that the set of states is finite and the consequence space is dense. Thus we could apply the weak multi-symmetric structure to obtain the threshold representation.

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