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Comparisons of AHP with other methods in binary paried comparisons

by

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Title : Comparisons of AHP with other methods in binary paried comparisons

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Abstract :

The essence of AHP (Analitic Hierarchy Process) is to estimate true degrees of importance of objects from paired comparison ratios. It's feature of estimation is to use eigen vector method (EVM). There are other estimation methods based on parired comparison. The typical of them is logarithmic least square (LLS).

Further we often encounter cases where some paired comparisons are missing , that is incomplete information cases (IIC). For IIC we have Harker-Takeda method (HTM) in AHP. We propose another method called two stage method (TSM) for IIC.

We compare these methods through some typical types of data groups in binary cases (→§1). The comparison method is based on rank correlation (RC)with a standard method (in binary cases) called counting method (CM). A method giving higher RCs can be considered better one. We have various informations shown in §4 and §5 based on simulations of these data groups.

1. Methods to be compared in complete and incomplete information cases

The essence of AHP is to estimate the true value w_i (>0) of object i by the i-th component of principal (corresponding to the maximal eigen value) eigen vector $w=(w_1,w_2,\ldots,w_n)$ of the paired comparison matrix

$$A = \begin{bmatrix} 1 & a_{12} & \cdots\cdots & a_{1n} \\ a_{21} & 1 & \cdots\cdots & a_{2n} \\ & \cdots\cdots\cdots & & \\ a_{n1} & a_{n2} & \cdots\cdots & 1 \end{bmatrix} \qquad (1)$$

where $a_{ij}(>0)$ is the ratio of object i to object j and $a_{ji} = 1/a_{ij}$. So this can be called as eigen vector method (EVM).

There will be other methods to estimate w_i based on the elements of comparison matrix A. The most popular of them is the logalismic least square (LLS). This goes as follows. We assume

$$a_{ij}= w_i/w_j \cdot e_{ij} \qquad i,j=1,\ldots,n,\ i<j \qquad (2)$$

where e_{ij} (>0) is an error factor. Taking logarithm we have

$$\log a_{ij} = \log w_i - \log w_j + \log e_{ij} \qquad (3)$$

Determining $w_i (i=1,\ldots,n)$ to minimize $\sum (\log e_{ij})^2$ (under the condition of $\sum \log w_i = 0$)we have

$$w_i = (\prod_{j=1}^{n} a_{ij})^{1/n} \quad , i=1,\ldots,n \qquad (4)$$

So in this complete information case (CIC) (later we show iocomplete information case), LLS gives exactly same results as geometric mean.

If a_{ij} can take continuous values in $(0,\infty)$, model (2) is reasonable and (4) must be the best estimation method. But in AHP a_{ij} is assumed to take only discrete values which are to be 1,2,...,9 and their reciprocals by T. Saaty's proposal. So we cannot theoretically conclude which method is the best.

Instead of T.Saaty's proposal, the author would like to assume that a_{ij} takes values, $\theta^0,\theta^1,\theta^2,\ldots$ and their reciprocals, where θ (>1) is a parameter whose actual value depends on problems. In the simplest case a_{ij} $(i\neq j)$ takes only θ and $1/\theta$, which we call binary case. When we can say object i is more impotant (or better) than object j, we have $a_{ij}=\theta$ or $a_{ji}=1/\theta(\theta>1)$. (Besides if a_{ij} $(i\neq j)$ can take $\theta^0=1$, it is called ternary case. Of course $a_{ij}=1$ means object i and object j are equally important or equivalent.)

We compare some estimation methods through various examples in the field of the simplest binary case, which will clearly reveal distinctions among these methods, we think. Further the binary case is directly applicable to various sport games where an object corresponds to a team and $a_{ij}=\theta$ means team i defeats team j.

We often encounter <u>incomplete information cases</u> (IIC) where data for some pairs are missing so we are not given the value of a_{ij} for all (i,j) (i,j=1,...,n) ; for example

$$A = \begin{bmatrix} 1 & a_{12} & a_{13} & a_{14} \\ a_{21} & 1 & a_{23} & (\ ) \\ a_{31} & a_{32} & 1 & (\ ) \\ a_{41} & (\ ) & (\ ) & 1 \end{bmatrix} \qquad a_{ij}=1/a_{ji} \tag{5}$$

For IIC we have a nice estimation method in AHP, <u>Harker-Takeda method</u> (HTM) which is also based on EVM [3]. But it has some problems we later show, so we propose another method called <u>two stage method</u> (TSM)below.

We illustlate it through the example (5). At the first stage we take the first approximation $\hat{w}_i$ as geometric mean in i-th row of comparison matrix A;

$$\hat{w}_1=(1.a_{12}.a_{13}.a_{14})^{1/4} \quad , \quad \hat{w}_2=(a_{21}.1.a_{23})^{1/3}$$
$$\hat{w}_3=(a_{31}.a_{32}.1)^{1/3} \quad , \quad \hat{w}_4=(a_{41}.1)^{1/2} \tag{6}$$

Then we estimate missing a_{ij} by $\hat{w}_i/\hat{w}_j$;

$$a_{24}=\hat{w}_2/\hat{w}_4 \ (a_{42}=\hat{w}_4/\hat{w}_2), \quad a_{34}=\hat{w}_3/\hat{w}_4 \ (a_{43}=\hat{w}_4/\hat{w}_3) \tag{7}$$

So we have complete comparison matrix A;

$$\hat{A} = \begin{bmatrix} 1 & a_{12} & a_{13} & a_{14} \\ a_{21} & 1 & a_{23} & \hat{w}_2/\hat{w}_4 \\ a_{31} & a_{32} & 1 & \hat{w}_3/\hat{w}_4 \\ a_{41} & \hat{w}_4/\hat{w}_2 & \hat{w}_4/\hat{w}_3 & 1 \end{bmatrix} \tag{8}$$

At the second stage we estimate w_i (i=1,...,n) as the i-th component of the principal eigen vector w of $\hat{A}$ like an ordinary EVM.

Of course LLS is also applicable to IIC. Given example (5) we have

$$\begin{aligned} \bar{a}_{12} &= \bar{w}_1 - \bar{w}_2 &&+ \bar{e}_{12} \\ \bar{a}_{13} &= \bar{w}_1 - \bar{w}_3 &&+ \bar{e}_{13} \\ \bar{a}_{14} &= \bar{w}_1 - \bar{w}_4 &&+ \bar{e}_{14} \\ \bar{a}_{23} &= \bar{w}_2 - \bar{w}_3 &&+ \bar{e}_{23} \\ 0 &= \bar{w}_1 + \bar{w}_2 + \bar{w}_3 + \bar{w}_4 && \end{aligned} \tag{9}$$

corresponding to (3), where $\bar{a}_{ij}=\log a_{ij}$, $\bar{w}_i=\log w_i$, etc.

The normal equation of (9) is

$\bar{w}_1$	$\bar{w}_2$	$\bar{w}_3$	$\bar{w}_4$	
4	0	0	0	$\bar{a}_{12}+\bar{a}_{13}+\bar{a}_{14}$
0	3	0	1	$-\bar{a}_{12}+\bar{a}_{23}$
0	0	3	1	$-\bar{a}_{13}-\bar{a}_{23}$
0	1	1	2	$-\bar{a}_{14}$

(10)

From the solution $\bar{w}_i$ (i=1,...,4) of (1o) we have LLS estimates

$w_i = e^{\bar{w}_i}$ (i=1,...,4).

To summarize we have methods;

EVM and LLS in CIC, HTM, TSM and LLS in IIC.

2. Simple examples to show discrepancies among methods

Hereafter we often represent data in binary case by a graph whose points correspond to objects (or teams) and if $a_{ij}=\theta(>1)$ we have an arrow from point i to point j.

Ex. 1 CIC

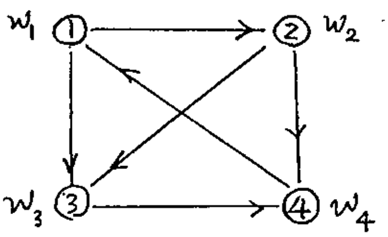


Table 1

	EVM	LLS
w_1	1	1
w_2	0.940	1
w_3	0.772	0.707
w_4	0.690	0.707

(θ=2)

Ex. 2 IIC

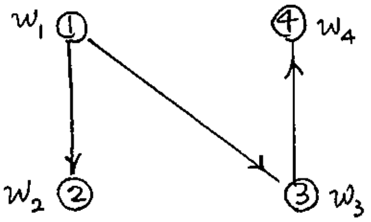


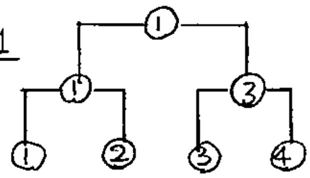
Table 2

	HTM	TSM	LLS
w_1	1	1	1
w_2	0.5	0.442	0.5
w_3	0.5	0.631	0.5
w_4	0.25	0.396	0.25

(θ=2)

This example is the result of the tournament game shown in fig. 1., in which only survival teams can play matches.

Fig1



We shall have such a common agreement that the first winer is of course teaml 1 and the second winner is team 3. So TSM seems more reasonable than HTM and LLS.

Ex. 3 IIC

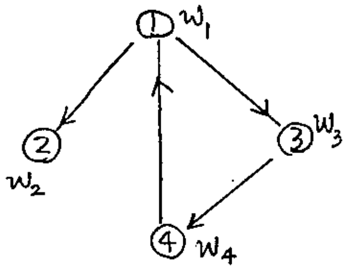


Table 3

	HTM	TSM	LLS
w_1	1	1	1
w_2	0.355	0.537	0.5
w_3	1.08	0.928	1
w_4	1.06	0.946	1

(θ=2)

This example again shows that TSM is better than HTM and LLS. Although team 1 wins 2 matches and loses 1 match whereas team 3 wins only 1 and loses 1, team 3 is highly evaluated by HTM.

(Here we note that through all theses examples we take the value of θ as 2, and other values of θ give different results but the ranking order of teams does not change.)

Then can we generally conclude TSM is better than HTM and LLS ? Of course we cannot state general conclution based on such a few examples.

So we make some typical groups of examples and in each example we compare these methods with a standard method in binary case which we call <u>countig method</u> (CM).

3. Counting method (CM)

We describe the algolithm of CM through the example shown in fig.2.

First we count the number of wins minus the number of losings of each team, which is called <u>first score</u>. For example ① wins 3 matches and loses 1 match, so the first score of ① is 3-1 = 2.First score of ⓘ is equal to the number of outgoing arrows from ⓘ minus the number of coming arrows to ⓘ .

Of course teams having higher first score are to have higher rankings. But we have to rank teams having the same first score. We think, values of wins (or losings) depends on opponents' strength. For example as for ⑤,defeating ① seems to have higher value than that of defeating ⑦.

Table 4

	①	②	③	④	⑤	⑥	⑦	⑧
first score	2	−1	−2	0	2	0	−2	1
second 〃	1	/	0	0	2	0	0	/
third 〃	/	/	2	2	/	3	3	/
ranking	2	6	8	5	1	4	7	3

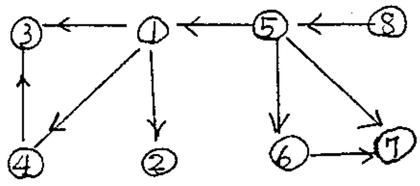


Fig. 2

Accounting these affairs we introduce second scor. Consider paths of length 2 starting at ⓘ ; paths of length 2 starting at ① are

①→③←④ (−), ①→④→③ (+), ①←⑤→⑥ (+), ①←⑤→⑦ (+), ①←⑤←⑧ (−) (11)

And consider the direction of the second arrow on each of these paths. Let the second scor of ⓘ be the number of forward second arrows minus the number of backward second arrows; second score of ① is 3 − 2 = 1 (as shown in (11)).

Thus for example ① and ⑤ have the same first score 2, but ⑤ has higher second score so ⑤ has higher ranking than ① (fig.2)

By the same way we introduce third score, in which we consider paths of length 3 (without a cycle); For example paths starting at ③ are

③←④←①$\xrightarrow{+}$② , ③←①←⑤$\xrightarrow{+}$⑥, ③←①←⑤$\xrightarrow{+}$⑦,

③←①←⑤$\xleftarrow{-}$⑧

and so ③ has third score 3 - 1 = 2.

The rankings calculated by CM of the example in fig.2 are shown in Table 4.

We adot CM as a stndard method to determine the ranks of teams (or objects) in a binary case. Of course we cannot state CM is absolutely reliable, but it will be commonly recognized that CM is reasonable. So we will have such a view that methods near CM are good. We take Spearmans' rank correlation (RC) as a measure of the nearness.

Given any method of EVM,LLS,HTM or STM and a comparison matrix of an example, we calculate $w_1,...,w_n$ by the method and rank teams according to $w_1,...,w_n$. Then we can calculate rank correlation of this rankings and that obtaind by CM.

Ex. 4 8-teams survival tournament match. Fig.3 shows a result of 8-teams survival tournament. The rankings obtained by CM are shown in Table 5. And Table 6 shows rankings calculated by LLS, HTM and TSM and the rank correlations with that of CM. Again TSM is nearest to CM.

Table 5

	①	②	③	④	⑤	⑥	⑦	⑧
first score	3	-1	0	-1	1	-1	0	-1
second 〃	/	2	2	-1	/	0	0	-1
third 〃	/	/	/	2	/	/	/	0
ranking	1	5	3	7	2	6	4	8

Table 6 ranking and RC

teams	①	②	③	④	⑤	⑥	⑦	⑧	RC
LLS	1	2	2	5	2	5	5	8	.821
HTM	1	2	2	5	2	5	5	8	.821
TSM	1	5	3	8	2	6	4	7	.976
CM	1	5	3	7	2	6	4	8	1

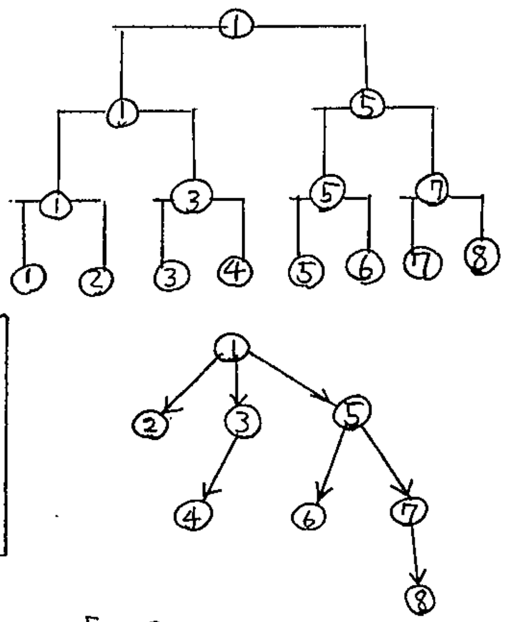


Fig. 3

4. Simulations on typical groups of examples

In CIC we have only to investigate EVM and LLS in logically inconsistet cases. In the case of logical consistency (in which $a_{ij}=\theta$ and $a_{jk} = \theta$ casse $a_{jk} = \theta$ for any i,j,k) the results of EVM and LLS are completely same.[2] So we select the following typical groups of examples and calculate RC with CM in each example.

<u>Type 1</u> CIC (EVM , LLS)

We construct 10 examples with randomly selected arrow directions for each n = 4,...,10. (n is the number of teams).

<u>Type 2</u> IIC (HTM , LLS , TSM)

We select the examples in Type 1 for n = 9 and 10. For each example we delete arrows randomly leaving 27, 18 and 9 arrows for n = 9 and 35, 25 and 15 for n = 10.

<u>Type 3</u> Survival tournament games (HTM, LLS, TSM) for n = 4,...,9

For each n we construct all patterns of matchings and results.

<u>The simulation results of Type 1</u> For any n = 4,...,10 both of EVM and LLS have high RC around 0.96 1.00, but LLS has a slightly higher ones than EVM.

<u>The simulation results of Type 2</u> For dense comparison matrices HTM, LLS and TSM have almost same RCs on average. But for sparse ones TSM has higher RCs than HTM and LLS.

<u>The simulation results of Type 3</u> For smaller n the three methods are almost same. But for the larger n , TSM has higher RCs than HTM and the latter has higher ones than LLS.

5. Conclution

We compared varaous methods to estimate values of objects based on paired comparisons, especially in the field of binary cases in order to get clear results. We rvaluated each method by its RC with CM considered as a standard criterion. It was found that; EVM and LLS have almost equal effect in CIC; TSM is rather better than HTM and LLS in IIC especially in the case of sparse information; In examples of tournament type the order of goodness is TSM HTM LLS, and the differences become greater for the larger n.

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