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Finding an ϵ-approximate Solution of Convex Programs with a Multiplicative Constraint

Y. Yamamoto

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Institute of Socio-Economic Planning, University of Tsukuba
Tsukuba, Ibaraki 305, Japan
phone: 0298-53-5001, fax.: 0298-53-5070
e-mail: yamamoto@shako.sk.tsukuba.ac.jp%cunyvm.cuny.edu

3 Approximation of the Feasible Region

For a given $\epsilon > 0$ we shall propose an algorithm for finding an ϵ-approximate solution. For scalars ξ_1 and ξ_2 such that $0 < \xi_1 \leq \xi_2$ we define the following five sets :

$$\begin{aligned}
U_0(\xi_1,\xi_2) &= \{\, u \mid \; u \in R^2; u_1 \leq \xi_2,\ u_2 \leq 1/\xi_1,\ u_1 \cdot u_2 = 1 \,\}, \\
U_1(\xi_1,\xi_2) &= \{\, u \mid \; u \in R^2; (1/\xi_1)u_1 + \xi_1 u_2 \geq 2,\ (1/\xi_2)u_1 + \xi_2 u_2 \geq 2, \\
&\qquad (1/\xi_1)u_1 + \xi_2 u_2 \leq 1 + \xi_2/\xi_1 \,\}, \\
U_2(\xi_1,\xi_2) &= \{\, u \mid \; u \in R^2; u_1 \leq \xi_2,\ u_2 \leq 1/\xi_1, \\
&\qquad (1/\xi_1)u_1 + \xi_2 u_2 \leq 1 + \xi_2/\xi_1 \,\}, \\
U_3(\xi_1,\xi_2) &= \{\, u \mid \; u \in R^2; u_1 \leq \xi_2, u_2 \leq 1/\xi_1 \,\}, \\
U_4(\xi_1,\xi_2) &= \{\, u \mid \; u \in R^2; (1/\xi_1)u_1 + \xi_2 u_2 \leq 1 + \xi_2/\xi_1 \,\}.
\end{aligned}$$

Clearly

$$\begin{aligned}
&U_0(\xi_1,\xi_2) \subseteq U_1(\xi_1,\xi_2) \subseteq U_2(\xi_1,\xi_2) = U_3(\xi_1,\xi_2) \cap U_4(\xi_1,\xi_2), \\
&U_2(\xi,\xi) = U_3(\xi,\xi) = \{\, u \mid u \in R^2;\ u_1 \leq \xi, u_2 \leq 1/\xi \,\}, \\
&U_0(\xi,\xi) = U_1(\xi,\xi) = \{(\xi, 1/\xi)\}.
\end{aligned}$$

For $k = 0, 1, 2, 3$ and 4 let

$$Y_k(\xi_1,\xi_2) = \{\, x \mid x \in R^n;\ (f_1(x), f_2(x)) \in U_k(\xi_1,\xi_2) \,\}.$$

Lemma 3.1 *For $0 < \xi_1 \leq \xi_2$ and $\xi > 0$, the followings hold.*

(a) $Y_0(\xi_1,\xi_2) \subseteq Y_1(\xi_1,\xi_2) \subseteq Y_2(\xi_1,\xi_2) = Y_3(\xi_1,\xi_2) \cap Y_4(\xi_1,\xi_2)$.

(b) $Y_2(\xi,\xi) = Y_3(\xi,\xi) = \{\, x \mid x \in R^n;\ f_1(x) \leq \xi,\ f_2(x) \leq 1/\xi \,\}$.

(c) $Y_0(\xi,\xi) = Y_1(\xi,\xi) = \{\, x \mid x \in R^n;\ f_1(x) = \xi,\ f_2(x) = 1/\xi \,\}$.

(d) $Y_2(\xi_1,\xi_2)$, $Y_3(\xi_1,\xi_2)$ *and* $Y_4(\xi_1,\xi_2)$ *are convex sets.*

(e) $Y_1(\xi_1,\xi_2)$ *is a convex set if both f_1 and f_2 are affine functions.*

(f) If $(\xi_2 - \xi_1)/\xi_1 \leq \epsilon$, then $Y_3(\xi_1,\xi_2) \subseteq Y(\epsilon)$.

(g) If $(1/4)(\xi_2 - \xi_1)^2/\xi_1\xi_2 \leq \epsilon$, then $Y_4(\xi_1,\xi_2) \subseteq Y(\epsilon)$.

Proof: Assertions (a) to (f) are immediate consequences of the definitions. To prove (g), let us consider the problem

$$\begin{aligned}
&\text{maximize} && u_1 \cdot u_2 \\
&\text{subject to} && (1/\xi_1)u_1 + \xi_2 u_2 \leq 1 + \xi_2/\xi_1.
\end{aligned}$$

The optimum solution is given by

$$u_1 = \frac{\xi_1}{2}(1+\frac{\xi_2}{\xi_1}),\ u_2 = \frac{1}{2\xi_2}(1+\frac{\xi_2}{\xi_1})$$

and its objective function value is $1 + (1/4)(\xi_2 - \xi_1)^2/\xi_1\xi_2$. Therefore we see that $Y_4(\xi_1, \xi_2) \subseteq Y(\epsilon)$ if $(1/4)(\xi_2 - \xi_1)^2/\xi_1\xi_2 \leq \epsilon$. □

This lemma shows that when ξ_1 and ξ_2 are sufficiently close to each other, any feasible solution of the problem

$$(P_k(\xi_1,\xi_2)) \qquad \begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & x \in X \cap Y_k(\xi_1, \xi_2) \end{array}$$

is an ϵ-feasible solution of (P). Let

$$\alpha_i = \min\{\, f_i(x) \mid x \in X \,\}$$

for $i = 1, 2$ and let

$$\xi_{min} = \alpha_1 \text{ and } \xi_{max} = \frac{1}{\alpha_2}. \tag{3.1}$$

If $\xi_{min} > \xi_{max}$, then $f_1(x) \cdot f_2(x) > 1$ holds for any $x \in X$, which means that $X \cap Y = \emptyset$. If $\xi_{min} \leq \xi_{max}$, then $X \cap Y \subseteq X \cap Y_2(\xi_{min}, \xi_{max})$ and there is an optimum solution of (P) in $X \cap Y_0(\xi_{min}, \xi_{max}) \subseteq X \cap Y_1(\xi_{min}, \xi_{max})$ if $X \cap Y \neq \emptyset$. When both f_1 and f_2 are affine functions, ξ_{min} and ξ_{max} can be improved as follows. Let $\beta_i = \max\{\, f_i(x) \mid x \in X \,\}$ for $i = 1, 2$ and let $\xi_{min} = \max\{\alpha_1, 1/\beta_2\}$, $\xi_{max} = \min\{1/\alpha_2, \beta_1\}$. If $\xi_{min} \leq \xi_{max}$ then we have $X \cap Y \subseteq X \cap Y_2(\xi_{min}, \xi_{max})$. Then we again find the following four numbers.

$$\begin{aligned}
\alpha_1 &= \min\{\, f_1(x) \mid x \in X;\ 1/\xi_{max} \leq f_2(x) \leq 1/\xi_{min} \,\}, \\
\beta_1 &= \max\{\, f_1(x) \mid x \in X;\ 1/\xi_{max} \leq f_2(x) \leq 1/\xi_{min} \,\}, \\
\alpha_2 &= \min\{\, f_2(x) \mid x \in X;\ \xi_{min} \leq f_1(x) \leq \xi_{max} \,\}, \\
\beta_2 &= \max\{\, f_2(x) \mid x \in X;\ \xi_{min} \leq f_1(x) \leq \xi_{max} \,\}.
\end{aligned}$$

It should be noted that finding α_i and β_i is a convex minimization problem. We then let

$$\xi_{min} = \max\{\xi_{min}, \alpha_1, \frac{1}{\beta_2}\}, \text{ and } \xi_{max} = \min\{\xi_{max}, \frac{1}{\alpha_2}, \beta_1\}.$$

We repeat this procedure until no significant improvement is made.

For ξ_{min} and ξ_{max} thus obtained, we have only to search for an optimum solution either in $X \cap Y \cap Y_3(\xi_{min}, \xi_{max})$ or in $X \cap Y_0 \cap Y_3(\xi_{min}, \xi_{max}) = X \cap Y_0(\xi_{min}, \xi_{max})$.

For $k = 1$ to 4 we can take $\xi_0, \xi_1, \dots \xi_m$ such that $\xi_{min} = \xi_0 < \xi_1 < \cdots < \xi_m = \xi_{max}$, $Y_k(\xi_j, \xi_{j+1}) \subseteq Y(\epsilon)$ and

$$\begin{array}{rll} X \cap Y \cap Y_3(\xi_{min}, \xi_{max}) \subseteq \bigcup_{j=0}^{m-1} Y_k(\xi_j, \xi_{j+1}) & \text{if } k = 2, 3 \text{ or } 4 \\ X \cap Y_0(\xi_{min}, \xi_{max}) \subseteq \bigcup_{j=0}^{m-1} Y_1(\xi_j, \xi_{j+1}) & \text{if } k = 1. \end{array}$$

Therefore take $\bigcup Y_k(\xi_j, \xi_{j+1})$ as $W(\epsilon)$ and apply Corollary 2.3, then we will obtain an ϵ-approximate solution by solving a finite number of convex programs $(P_k(\xi_j, \xi_{j+1}))$, where $(P_1(\xi_j, \xi_{j+1}))$ should be considered only when both f_1 and f_2 are affine functions.

4 Branch-and-Bound Method

As we have seen in the preceding section whichever k we may choose, we can make a finite branch-and-bound method for finding an ϵ-approximate solution of (P). First we define the following procedure $S(k, \epsilon, \omega, w, \xi_1, \xi_2)$ which solves $(P_k(\xi_1, \xi_2))$ and show whether the problem is fathomed or should be branched. Here we denote the incumbent by w and its objective function value by ω. To make the set $Y_k(\xi_1, \xi_2)$ quickly included in $Y(\epsilon)$ we take $\sqrt{\xi_1 \xi_2}$ as the new point separating the interval $[\xi_1, \xi_2]$ into two subintervals.

Procedure $S(k, \epsilon, \omega, w, \xi_1, \xi_2)$

S1: Solve $(P_k(\xi_1, \xi_2))$. Let x be an optimum solution of $(P_k(\xi_1, \xi_2))$ if exists and let z be its objective function value.

S2: If $z \geq \omega$, then return.

S3: If $x \in Y(\epsilon)$, then $w := x$, $\omega := f_0(x)$ and return.

S4: Let $\xi := \sqrt{\xi_1 \xi_2}$ and call Procedure $S(k, \epsilon, \omega, w, \xi_1, \xi)$ and $S(k, \epsilon, \omega, w, \xi, \xi_2)$.

Given ϵ and k the branch-and-bound method is as follows.

Branch-and-Bound Method

1: Solve $(\bar{P})$ and let $\bar{x}$ be an optimum solution of $(\bar{P})$. If $\bar{x} \in Y(\epsilon)$, then $w := \bar{x}$, $\omega := f_0(\bar{x})$ and stop.

2: Find ξ_{min} and ξ_{max} of (3.1).

3: Let $\omega := +\infty$ and call Procedure $S(k, \epsilon, \omega, w, \xi_{min}, \xi_{max})$.

By the choice of ξ in Step S4, we see

$$\frac{\xi-\xi_1}{\xi_1}=\frac{\xi_2-\xi}{\xi}=\sqrt{\frac{\xi_2}{\xi_1}}-1,\ \frac{\xi-\xi_1}{\xi}=\frac{\xi_2-\xi}{\xi_2}=1-\sqrt{\frac{\xi_1}{\xi_2}}.$$

Therefore for the problems of depth d of the branching tree the ratios in Lemma 3.1 (f) and (g) are

$$\frac{\xi_2-\xi_1}{\xi_1}=(\frac{\xi_{max}}{\xi_{min}})^{1/2^d}-1,$$

$$\frac{1}{4}\frac{(\xi_2-\xi_1)^2}{\xi_1\xi_2}=\frac{1}{4}\{(\frac{\xi_{max}}{\xi_{min}})^{1/2^d}-1\}\{1-(\frac{\xi_{min}}{\xi_{max}})^{1/2^d}\},$$

and $(\xi_2-\xi_1)/\xi_1\leq\epsilon$ when $d\geq(\ln\ln(\xi_{max}/\xi_{min})-\ln\ln(1+\epsilon))/\ln 2$. Hence the tree is not branched out below some constant depth and the method terminates within a finite number of iterations.

References

[1] M. Fukushima, *Nonlinear Optimization Theory*, Sangyo Tosho, Tokyo, 1980 (in Japanese).

[2] W.W. Hogan, "Point-to-set maps in mathematical programming", *SIAM Review* 15 (1973) 3, 591-603.

[3] T. Kuno and H. Konno, "A parametric successive underestimation method for convex programming problems with an additional convex multiplicative constraint", Institute of Human and Social Sciences, Tokyo Institute of Technology, Tokyo, Japan (August 1990).

[4] P.T. Thach and R.E. Burkard, "Reverse convex programs dealing with the product of two linear functions", Institute of Mathematics, Graz University of Technology, Graz, Austria (1990).

[5] H. Tuy, "Convex programs with an additional reverse convex constraint", *Journal of Optimization Theory and Applications* 52 (1987) 463-486.