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OPTIMAL STOPPING PROBLEM WITH A FINITE SEARCH BUDGET

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Abstract The paper deals with a discrete-time optimal stopping problem of a finite planning horizon where the total amount of money that can be invested in the entire search activities throughout the planning horizon is limited and where both the probability of obtaining an offer at each point in time and the probability distribution function of an offer obtained may depend on the search cost invested at that point in time. The objective is to maximize the expected present discounted value of an offer accepted minus that of the search costs paid up to the termination of the search with acceptance of the offer, provided that at most one offer must be accepted within the planning horizon. The optimal decision strategy for the problem incorporates two kinds of decision rules: an *optimal stopping rule*, prescribing how to stop the search with accepting an offer and an *optimal investment rule*, prescribing how much of the currently available search budget to invest in search activities of each point in time. The present paper examines the properties of both rules analytically and numerically. The most interesting as well as the most important properties revealed are: (1) In the recall model, the optimal stopping rule has a *reservation price property*. In addition, if the amount of search budget available can be assumed to be infinite, the reservation price becomes time-independent; i.e., the optimal stopping rule has a *myopic property*, and (2) in both of the recall and no recall model, it is possible that the optimal investment increases or decreases drastically with a slight change in the amount of search budget remaining at that time.

1. Introduction

In almost all optimal stopping problems that have been posed and investigated so far [1]~[7], [9]~[12], it has been implicitly assumed that, in addition to the search cost invested in each point in time being fixed and identical throughout a given planning horizon, the total amount of money available for search activities over the entire planning horizon, i.e., the *search budget*, is either sufficiently large or even infinite to allow for an infinite number of searches. Being realistic for such a relatively small-scale search problem as a job search problem [8],[11], the assumptions might safely be said to be far removed from reality for such a large-scale search problem as an R&D problem [12]. Furthermore, with the exception of [8], the notion of search intensity, i.e., a more profitable offer can be obtained with a higher probability if a higher search cost is invested, has not been taken into consideration. For the above reasons, it can be said that many conventional models of an optimal stopping problem do not reflect real situations. Accordingly, in order to make models of an optimal stopping problem more realistic and applicable even to large-scale stopping problems, we should reformulate the problem, taking the three points into consideration:

(1) *The amount of search budget available throughout the planning horizon is finite.*
(2) *A search cost invested at each point in time is a decision variable that can be controlled by the searcher.*
(3) *A more profitable offer can be obtained with a higher probability if more search cost is invested.*

In the present paper, we set up a model of an optimal stopping problem in which the above three points are adopted and examine the properties of the optimal decision strategy, consisting of the two rules: An *optimal stopping rule*, prescribing how to stop a search by accepting an offer and an *optimal investment rule*, prescribing how much of the search budget that is currently available to allocate to search activities at each point in time.

2. An Example

The following example conveys the flavor of the basic problem examined in the current paper. Suppose a board of directors in a company decides to put a new product on sale in the not too distant future, forms a certain amount of R&D budget for it, and calls on the manager of the R&D department to propose an idea of new product every year with an estimation of the long-run net profit that will be expected. The decisions to be made by the board of directors are as follows:

(1) *Whether or not to take the idea proposed of a certain year as that of the new product to be put on sale.*

(2) *If not, how much to invest in R&D activities to unearth another idea over the next one year.*

For example, suppose that the idea of the new product to be put on sale must be determined up to January 1 of the next year, and assume that, on January 1 of this year, an idea with the expected long run net profit of 4 million dollars is proposed and that 500 million dollars of the search budget remains as of January 1 of this year. Now, postulate that, if the decision to reject the idea is made, then one of five alternatives A, B, C, D, and E of investing, respectively, 100, 200, 300, 400, and 500 million dollars over the year up to January 1 of the next year must be taken. Each alternative generates an idea with probabilities 0.25, 0.45, 0.60, 0.70, and 0.75, respectively, all of which will yield a net profit of 50 million dollars with probability 0.5 and 150 million dollars with probability 0.5, evaluated at January 1 of the next year. Assume, for example, Alternative B has been taken. Then 200 million dollars invested will increase to 245 million dollars $(0.45 \times (0.5 \times 50 + 0.5 \times 150) + 200)$ by January 1 of the next year. Therefore, the expected present discounted net profit evaluated on January 1 of this year becomes $245 \times \beta - 200 = 18.75$ million dollars where the interest rate $r = 0.12$, hence the discount factor $\beta = 1/(1+r) \approx 0.892$.

Table 1

Expected present discounted net profit for each alternative

alternative	investment	total revenue†1	net profit†2		r=0.00	r=0.12	r=0.22†3
present idea	–				4.00	4.00	4.00*
A	100	0.25×100+100=125.00	125×β−100	=	25.00	11.61	2.46
B	200	0.45×100+200=245.00	245×β−200	=	45.00	18.75	0.82
C	300	0.60×100+300=360.00	360×β−300	=	60.00	21.34*	− 4.92
D	400	0.70×100+400=470.00	470×β−400	=	70.00	19.64	−14.75
E	500	0.75×100+500=575.00	575×β−500	=	75.00*	13.38	−28.69

†1: Expected total revenue, evaluated on Jan. 1 of the next year.
†2: Expected present discounted net profit, evaluated on Jan. 1 of this year where β=1/(1+r).
†3: Rate of interest

The Table 1 above shows the expected present discounted net profit for each alternative for the three different interest rates, $r = 0.00$, 0.12, and 0.22. The asterisk * represents the alternatives maximizing the expected present discounted net profit: if $r = 0.00$, investing the whole remaining search budget (Alternative E) is optimal, if $r = 0.12$, investing 300 million dollars (Alternative C) is optimal, and if $r = 0.22$, accepting the present idea is optimal. With, for example, an interest rate $r = 0.12$, the optimal decision of investing 300 million dollars out of 500 million dollars given means the following: invest 200 million dollars in an investment opportunity with $r = 0.12$ (*financial investment*) and invest the residual 300 million dollars in search activities for this year (*search investment*).

Now, we shall consider the five different situations where the amount of search budget that is currently available are, respectively, $i = 100$, 200, 300, 400, and 500 million dollars. It is clear from the table that when $r = 0.12$, it is optimal to invest 100 million dollars if $i = 100$ million dollars, 200 million dollars if $i = 200$ million dollars, and 300 million dollars if $i = 300, 400$, and 500 million dollars. Note here that the optimal investment of this year increases with the amount of the remaining search budget; it will be shown in the later sections that the monotonicity of the optimal investment always holds true. Now, if the deadline up to which an idea to be put on sale must been determined is January 1 of *the year after next* instead of January 1 of *the next year*, is the optimal investment for this year also increases with the amount of the remaining search budget i ? Contrary to our expectation, the answer is negative; we shall demonstrate this by a counter-example in Section 6.2.2.

3. Model

Consider the following discrete-time optimal stopping problem with a finite planning horizon. First, for convenience, let points in time be numbered backward from the final point in time of the horizon as time 0, time $1, \cdots$, and so on, equally spaced; an interval between time t and time $t-1$ is called a period t. Second, assume that a search starts with a finite amount of search budget and if c dollars out of the search budget that is currently available is invested in search activities of each point in time, an offer can be obtained with a known probability $p(c)$ at the next point in time where $p(0) = 0$. Sequentially obtained offers $w, w', w'', \cdots$ are assumed to be stochastically independent random variables having a known distribution function $F(w|c)$, which is dependent on the search cost invested c, with a finite expectation $\mu(c)$ and $F(w|c) = 0$ on $w \leq 0$ for all values of c. Here, let $p = \sup_{c \geq 0} p(c)$, $0 < p \leq 1$, and $\mu = \sup_{c \geq 0} \mu(c) < \infty$. Then postulate that at most one of offers sequentially obtained must be accepted within the time horizon. Finally, let a per-period discount factor be represented by β.

Now, in terms of availability in the future of an offer once inspected and passed up (the offer being available implies that if wanted, the searcher can accept it at his own convenience at any time in the future), we shall consider the following two cases: (1) it becomes instantly and forever unavailable and (2) it becomes forever available. The former is called a *no recall model*, the latter a *recall model* [2],[10].

The objective of the search process is to maximize the *expected present discounted net profit*, i.e., the expected present discounted value of an offer accepted less the total expected present discounted value of search costs that have been invested up to the termination of the search.

4. Optimal Decision strategy

4.1. Preliminaries

Since no offer being obtained can be regarded as an offer 0, the probability $p(c)$ and the distribution function $F(w|c)$ can be combined into the distribution function $G(w|c)$ whose probability density function is

$$g(w|c) = (1 - p(c))\mathrm{I}(w = 0) + p(c)f(w|c)\mathrm{I}(w > 0) \tag{4.1}$$

where $\mathrm{I}(S)$ is an indicator function; that is, $\mathrm{I}(S) = 1$ if a given statement S is true, or else $\mathrm{I}(S) = 0$. Then, for any given function $s(w)$,

$$\int_0^\infty s(w)dG(w|c) = (1 - p(c))s(0) + p(c)\int_{0+}^\infty s(w)dF(w|c) \tag{4.2}$$

where the domain of integration in each of $\int_x^\infty$ and $\int_{x+}^\infty$ is, respectively, $x \leq w < \infty$ and $x < w < \infty$. For any real numbers $i \geq 0$ and x, define

$$K(i, x) = \max_{0\leq c\leq i}\{\beta\int_0^\infty \max\{w, x\}dG(w|c) - c\} - x, \tag{4.3}$$

$$h(i) = \sup\{x|K(i, x) > 0\}, \tag{4.4}$$

$$h^* = \sup\{x|K(\infty, x) > 0\}\ (= h(\infty)). \tag{4.5}$$

Let $\tilde{\beta}$ be the largest value of β for which $K(\infty, 0) = 0$; i.e.,

$$\tilde{\beta} = \max\{\beta\big|K(\infty, 0) = 0\} \tag{4.6}$$

where $\tilde{\beta} \geq 0$ because $K(\infty, 0) = 0$ for $\beta = 0$.

Lemma 1. *We have*

(a) *$K(i, x)$ is nondecreasing in i and nonincreasing in x with $K(i, 0) \geq 0$ for any i,*

(b) *If $\beta < 1$, then $K(i, x)$ is strictly decreasing in x and diverging to $-\infty(\infty)$ as $x \to \infty(-\infty)$ for all i; hence, both $h(i)$ and h^* are given by the unique solutions of, respectively, $K(i, x) = 0$ and $K(\infty, x) = 0$,*

(c) *If $\beta < 1$ and $K(i, 0) > 0$ for a certain i, then $h(j) > 0$ for all $j \geq i$,.*

Proof: (a). The monotonicity in i is obvious, and the monotonicity in x follows from the fact that $\max\{\beta w, (\beta - 1)x\}$ is nonincreasing in x.

(b) is clear from the fact that (4.3) can be transformed into

$$K(i, x) = \begin{cases} \max_{0\leq c\leq i}\{\beta p(c)\int_{0+}^\infty \max\{w - x, 0\}dF(w|c) - c\} + \beta x - x, & x \geq 0, \\ \max_{0\leq c\leq i}\{\beta p(c)\mu(c) - c\} - x & x \leq 0 \end{cases} \tag{4.7}$$

(c) is evident from (b). ***Q.E.D.***

4.2. No Recall Model

Let $v_t(i,y)$ denote the maximal expected present discounted net profit starting from time t with a search budget i and a current offer y. Then, clearly $v_0(i,y) = y$, and

$$v_t(i,y) = \max\{y, V_t(i)\}, \quad t \geq 1, \tag{4.7}$$

in which y is the net profit from stopping and $V_t(i)$ is the maximal expected present discounted net profit from continuing, expressed as

$$V_t(i) = \max_{0\leq c\leq i}\{\beta \int_0^\infty v_{t-1}((i-c)/\beta, w))dG(w|c) - c\} \tag{4.8}$$

Note that the remaining search budget i at time t increases to $(i-c)/\beta$ if less than $100(1-\beta)$ % of the i is invested, or else decreases to $(i-c)/\beta$ and that $V_1(i) = K(i,0) \geq 0$. By substituting (4.7), (4.8) becomes

$$V_t(i) = \max_{0\leq c\leq i}\{\beta \int_0^\infty \max\{w, V_{t-1}((i-c)/\beta)\}dG(w|c) - c\}, \quad t \geq 1. \tag{4.9}$$

Now (4.7) tells us that the *optimal stopping rule* is provided as follows; if $y > V_t(i)$, stop with accepting the current offer y, or else continue. Accordingly, the $V_t(i)$ provides a reservation price of the model. The *optimal investment* $c_t(i)$ is given by $c = c^*$ attaining a maximum in the right hand side of (4.9); if there exist more than one c^*, let $c_t(i)$ be denoted as the smallest of them.

Let $i_n^*(t,i)$, $V_n^*(t,i)$, and $c_n^*(t,i)$ represent the sequences of, respectively, states, reservation prices, and optimal investments over times $n = 1, 2, \cdots, t$, starting from time t with a search budget i. Then it is clear from the definition that $i_t^*(t,i) = i$, $V_t^*(t,i) = V_t(i)$ and $c_t^*(t,i) = c_t(i)$, and

$$\begin{aligned} i_n^*(t,i) &= i_{n+1}^*(t,i) - c_{n+1}^*(t,i) \\ V_n^*(t,i) &= V_n(i_n^*(t,i)) \\ c_n^*(t,i) &= c_n(i_n^*(t,i)) \end{aligned} \tag{4.10}$$

for $n = 1, 2, \cdots, t-1$.

Theorem 1. *We have*

(a) *If* $K(\infty,0) = 0$, *then* $V_t(i) = 0$ *for all* t *and* i, *hence* $V_t(i) = 0$ *on* $0 \leq \beta \leq \tilde{\beta}$ *for all* t *and all* i,

(b) $V_t(i)$ *is nondecreasing in* t *and* i *with* $V_1(i) \geq 0$ *for all* i *and upper-bounded in* i *for all* t,

(c) $V_n^*(t,i)$ *is nondecreasing in* n *for all* i *and* t,

(d) $c_1(i)$ *is nondecreasing in* i.

Proof: (a). Since $K(i,0)$ is nondecreasing in i with $K(0,0) = 0$, it follows from the assumption in the lemma that $K(i,0) = 0$ for all i. Thus $V_1(i) = 0$ for all i. Suppose $V_{t-1}(i) = 0$ for all i. Then, from (4.9) we have $V_t(i) = K(i,0) = 0$ for all i. The latter half is immediate from the definition of $\tilde{\beta}$ in (4.6).

(b). Since $\max\{w, V_1((i-c)/\beta)\} \geq w$, from (4.9) $V_2(i) \geq K(i,0) = V_1(i) \geq 0$ for all i. If $V_t(i) \geq V_{t-1}(i)$ for all i, then from (4.9) $V_{t+1}(i) \geq V_t(i)$ for all i. The monotonicity in i for all t can also be easily proved by induction starting with $V_1(i)$ being nondecreasing in i. The boundedness in i can be proved as follows: First, clearly $V_1(i) \leq K(\infty,0) < \infty$ for all i. Suppose $V_{t-1}(i) \leq (t-1)K(\infty,0) < \infty$. Then from (4.9) we have $V_t(i) \leq K(i,(t-1)K(\infty,0)) + (t-1)K(\infty,0) \leq K(\infty,0) + (t-1)K(\infty,0) = tK(\infty,0) < \infty$.

$$V_t(i,k) = \max_{0\le c\le i}\{\beta\int_0^\infty v_{t-1}\big((i-c)/\beta, \max\{w,k\}\big)dG(w|c) - c\}. \tag{4.14}$$

It is easy to see that $V_1(i,k) = K(i,k) + k$.

Theorem 3. *We have*

(a) $V_t(i,k)$ *is nondecreasing in* $t, i,$ *and* k *with* $V_t(i,k) \ge K(i,k) + k$ *for all* t, i *and* k,

(b) *If* $K(\infty,0) = 0$, *then* $V_t(i,k) \le k$ *for all* $t, i,$ *and* k; *hence,* $V_t(i,k) \le k$ *on* $0 \le \beta \le \beta'$ *for all* $t, i,$ *and* k,

(c) *If* $K(\infty,0) > 0$, *then*

1. $c_1(i,k)$ *is nondecreasing in* i *for all* k.
2. *if* $\beta = 1$, $V_t(i,k) \ge k$ *for all* $t, i,$ *and* k,
3. *if* $\beta < 1$, $V_t(i,k)$ *is upper-bounded in* t *and* i *for all* k.

Proof: (a). First, we have $V_t(i,k) \ge K(i,k) + k = V_1(i,k)\cdots(*)$ from (4.14) because of $v_{t-1}((i-c)/\beta, \max\{w,k\}) \ge \max\{w,k\}$. The monotonicity in t can be easily proved by induction starting with the inequality $(*)$. The other assertions can also be easily verified by induction where $V_1(i,k)$ is nondecreasing in i and k.

(b). $V_1(i,k) = K(i,k) + k \le K(\infty,0) + k = k$ for all i and k; hence, the assertion is true for $t = 1$. Suppose it is true for $t-1$. Then, since $u_{t-1}(i,k) = k$ for all i and k, we have $V_t(i,k) = K(i,k) + k = V_1(i,k) \le k$ for all i and k. The latter half is clear from Lemma 1(d).

(c1). Obvious.

(c2). Clear from $(*)$ in (a) and $K(i,k) \ge 0$ for all i and k.

(c3). Let $s_t = (1 + \beta + \beta^2 + \cdots + \beta^{t-1})K(\infty,0) \le K(\infty,0)/(1-\beta)$ for all t. First, clearly $V_1(i,k) \le K(\infty,0) + k = s_1 + k$ for all i and k. Next, suppose $V_{t-1}(i,k) \le s_{t-1} + k$ for all i and k. Then, since $v_{t-1}((i-c)/\beta, \max\{w,k\}) \le \max\{\max\{w,k\}, s_{t-1} + \max\{w,k\}\} = s_{t-1} + \max\{w,k\}$, arranging (4.14) by substituting the inequality yields $V_t(i,k) \le \beta s_{t-1} + K(i,k) + k \le \beta s_{t-1} + K(\infty,0) + k = s_t + k$. Accordingly, it follows by induction that $V_t(i,k) \le s_t + k \le K(\infty,0)/(1-\beta) + k$ for all t and for all i and k; hence, $V_t(i,k)$ is upper-bounded in t and i. *Q.E.D.*

Lemma 2. *If* $\beta < 1$, *then* $V_t(i,k) - k$ *is strictly decreasing in* k *for all* t *and* i *and diverges to* $-\infty(+\infty)$ *as* $k \to +\infty(-\infty)$.

Proof: Let $Y_t(i,k) = V_t(i,k) - k$ and $y_{t-1}(i,k) = v_{t-1}(i,k) - k$ $(= \max\{0, Y_{t-1}(i,k)\})$. Since $Y_1(i,k) = K(i,k)$, the assertion obviously holds for $t = 1$ from Lemma 1(b). Next, suppose $Y_{t-1}(i,k)$ is strictly decreasing in k for all i with converging to $-\infty$ $(+\infty)$ as $t \to +\infty$ $(-\infty)$. Now, (4.14) becomes

$$\begin{aligned} Y_t(i,k) = \max_{0\le c\le i}\{\beta\int_0^\infty & y_{t-1}\big((i-c)/\beta, \max\{w,k\}\big)dG(w|c) \\ & + \beta\int_0^\infty \max\{w-k, 0\}dG(w|c) - c\} - (1-\beta)k. \end{aligned} \tag{4.15}$$

Accordingly, it immediately follows that the assertion also becomes true for t due to the last term $-(1-\beta)k$ of (4.15). *Q.E.D.*

Define

$$h_t(i) = \sup_{0\le k}\{k | V_t(i,k) - k > 0\} \tag{4.16}$$

where $h_1(i) = h(i)$ from (4.4).

Theorem 4. *If $\beta < 1$, the $h_t(i)$ is a unique, nonnegative solution of $V_t(i,k) - k = 0$ for all t and i, which is nondecreasing and upper-bounded in t and i; hence, $V_t(i,k) < k$ if $k > h(i)$, or else $V_t(i,k) \geq k$.*

Proof: The existence of the unique solution $h_t(i)$ is evident from Lemma 2. That $h_t(i)$ is nonnegative follows from $Y_t(i,0) = V_t(i,0) \geq V_1(i,0) = K(i,0) \geq 0$ for all t and i, and that $h_t(i)$ is nondecreasing and upper-bounded in t and i is from $Y_t(i,k)$ being nondecreasing and upper-bounded in t and i. ***Q.E.D.***

The above theorem indicates that the *optimal stopping rule* can be prescribed as follows; if the best offer so far $k > h_t(i)$, stop with accepting the offer, or else continue. The *optimal investment* is given by the $c = c^*$ attaining the maximum of the right hand side of (4.14). Let it be denoted by $c_t(i,k)$; if there exist more than one c^*, let $c_t(i,k)$ be denoted by the smallest of them.

Let $i_n^*(t,i,k)$, $h_n^*(t,i,k)$, and $c_n^*(t,i,k)$ denote, respectively, states, reservation prices and optimal investments over times $n = 1, 2, \cdots, t$ starting from time t with a search budget i, provided that $k = (k_1, k_2, \cdots, k_t)$, a vector of the best offers k_n over times $n = 1, 2, \cdots, t$ where $k_t \leq k_{t-1} \leq \cdots \leq k_1$. From the definition, $i_t^*(t,i,k) = i$, $h_t^*(t,i,k) = h_t(i)$, $c_t^*(t,i,k) = c_t(i,k_t)$, and

$$\begin{aligned} i_n^*(t,i,k) &= i_{n+1}^*(t,i,k) - c_{n+1}^*(t,i,k) \\ h_n^*(t,i,k) &= h_n(i_n^*(t,i,k)) \\ c_n^*(t,i,k) &= c_n(i_n^*(t,i,k), k_n) \end{aligned} \tag{4.17}$$

for $n = 1, 2, \cdots, t$.

Now, Theorem 3(c3) guarantees that, if $K(\infty, 0) > 0$ and $\beta < 1$, $V_t(i,k)$ converges as $i \to \infty$ and as $i \to \infty$ and then $t \to \infty$. Let the limits be denoted by, respectively, $V_t(k)$ and $V(k)$; furthermore, let $v_t(k) = \max\{k, V_t(k)\}$ and $v(k) = \max\{k, V(k)\}$. Then, from (4.14) we get

$$V_t(k) = \max_{0 \leq c}\{\beta \int_0^\infty v_{t-1}(\max\{w,k\}) dG(w|c) - c\}, \quad t \geq 1, \tag{4.18}$$

where $V_1(k) = K(\infty, k) + k$,

$$V(k) = \max_{0 \leq c}\{\beta \int_0^\infty v(\max\{w,k\}) dG(w|c) - c\}. \tag{4.19}$$

Theorem 5. *Suppose $K(\infty, 0) > 0$ and $\beta < 1$. Then*
(a) *If $h^* \leq k$, then $V_t(k) \leq k$, hence $v_t(k) = k$,*
(b) *If $k \leq h^*$, then $V_t(k) \geq k$, hence $v_t(k) = V_t(k)$,*
(c) *The above two assertions are also true in the limit of t.*

Proof: (a,b). Clear for $t = 1$ because $V_1(k) = K(\infty, k) + k$. Assume that the assertion is true for $t-1$. First suppose $h^* \leq k$. Then, since $\max\{w,k\} \geq k \geq h^*$ for all w, we have $V_{t-1}(\max\{w,k\}) \leq \max\{w,k\}$ for all w from the induction hypothesis, yielding $V_t(k) \leq K(\infty,k) + k \leq k$. Thus (a) holds for t. Now since $V_t(k) \geq K(\infty,k) + k$ from $(*)$ in the proof of Theorem 3(a), it follows that $V_t(k) \geq k$ for $k \leq h^*$. Hence (b) is true for t. Thus the induction is complete. (c) Evident. ***Q.E.D.***

Theorem 6. *If $K(\infty, 0) > 0$ and $\beta < 1$, then $V(k) = h^*$ if $k \leq h^*$.*

Proof: In order to prove the theorem, it would suffice to prove the following: (1) if arranging the right hand side of (4.19) with $k \leq h^*$ by substituting $V(k) = h^*$, $k \leq h^*$, the resultant expression becomes equal to h^*, and (2) Equation (4.19) has a unique solution. First, let us prove (1). Suppose $k \leq h^*$. Then

$$\begin{aligned}\text{r.h.s. of (4.19)} &= \max_{0\leq c}\{\beta\int_0^\infty \big(V(\max\{w,k\})\mathrm{I}(w \leq h^*) + \max\{w,k\}\mathrm{I}(h^* < w)\big)dG(w|c) - c\} \\ &= \max_{0\leq c}\{\beta\int_0^\infty \big(h^*\mathrm{I}(w \leq h^*) + w\mathrm{I}(h^* < w)\big)dG(w|c) - c\} \\ &= \max_{0\leq c}\{\beta\int_0^\infty \max\{w,h^*\}dG(w|c) - c\} \\ &= K(\infty,h^*) + h^* = h^*.\end{aligned}$$

Next, let us prove (2). Suppose (4.19) has two different solutions $U(k)$ and $V(k)$, and let $\Delta = \sup_k|U(k) - V(k)| > 0$. Then we have

$$|U(k) - V(k)| \leq \max_{0\leq c}\beta\int_0^\infty |U(\max\{w,k\}) - V(\max\{w,k\})|dG(w|c) \leq \beta\Delta,$$

from which $\Delta \leq \beta\Delta$, yielding the contradiction of $1 \leq \beta$; hence the solution must be unique. ***Q.E.D.***

5. Numerical Analyses *

5.1. Preliminaries

The section examines, by means of numerical analyses, the properties of the optimal decision strategy of the problem where $F(w|c)$ is assumed to be a c-independent uniform distribution function on $[a,b]$, $0 < a < b < \infty$, and where four cases of offer probabilities $p(c)$ are only considered:

$$\text{Case } 0: \; p(c) = p(1 - e^{-\lambda c}), \quad 1 \geq p > 0, \quad \lambda > 0, \tag{5.1}$$
$$\text{Case } 1: \; p(c) = g(c|p_1,\lambda_1,\rho_1), \tag{5.2}$$
$$\text{Case } 2: \; p(c) = g(c|p_1,\lambda_1,\rho_1) + \mathrm{I}(s_2 < c)g(c - s_2|p_2,\lambda_2,\rho_2), \quad s_2 > 0, \tag{5.3}$$
$$\begin{aligned}\text{Case } 3: \; p(c) = g(c|p_1,\lambda_1,\rho_1) &+ \mathrm{I}(s_2 < c)g(c - s_2|p_2,\lambda_2,\rho_2) \\ &+ \mathrm{I}(s_3 < c)g(c - s_3|p_3,\lambda_3,\rho_3), \quad s_3 > s_2 > 0. \end{aligned} \tag{5.4}$$

where

$$g(c|p,\lambda,\rho) = p(\rho^{\lambda^c} - \rho)/(1-\rho), \quad c \geq 0, \quad 1 \geq p > 0, \quad 1 > \lambda > 0, \quad 1 > \rho > 0. \tag{5.5}$$

---

* The computer used is a min-computer (Data General, MV10000), and the language and compiler used are, respectively, Fortran 77 and F77L (Lahey Computer Systems). In order to make computational error as small as possible, all the variables were defined on double precision. All computation results, enormous if sent to print, were stored into direct files in an external storage, and all the graphs for the results were directly drawn using a 3-dimensional graph drawing software, CORE-PC of (Mitubishi Research Institute), and X-Y plotter (MP3200 of GRAPHTEC).

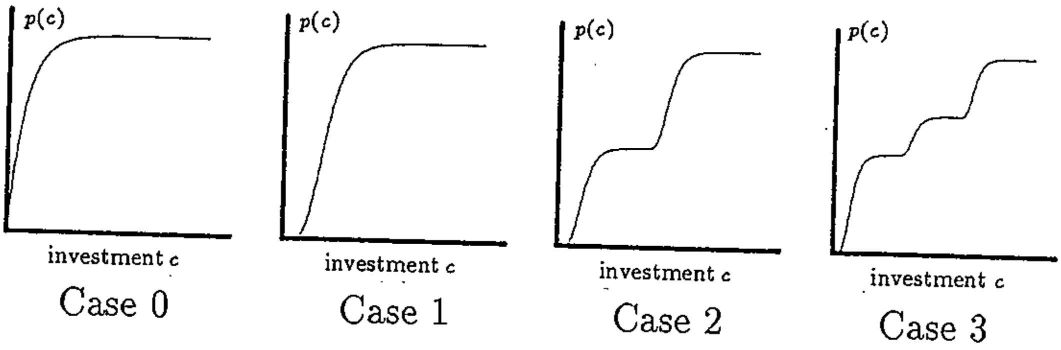


Figure 1

Four cases of offer probabilities $p(c)$ used in the numerical analyses

Case 0: p=1.0, λ=0.5
Case 1: p_1=1.0, λ_1=0.5, $\rho_1=10^{-5}$
Case 2: p_1=0.5, λ_1=0.5, $\rho_1=10^{-5}$, s_1=15, p_2=0.5, λ_2=0.5, $\rho_2=10^{-5}$
Case 3: p_1=0.5, λ_1=0.5, $\rho_1=10^{-5}$, s_1=15, p_2=0.2, λ_2=0.5, $\rho_2=10^{-5}$, s_2=30, p_3=0.3, λ_3=0.5, $\rho_3=10^{-5}$

For all the numerical examples that were computed, we shall assume that the uniform distribution function $F(w)$ is continuous for the no recall model and discrete for the recall model; furthermore, for the discrete distribution, we only consider the case that its mass points w consist of $50, 70, 90, 110, 130$, and 150. For the offer probabilities defined above, it should be noted that Case 0 is concave, and Case 1, Case 2, and Case 3 have, respectively, one, two, and three inflection points as seen in Figure 1. The existence of inflection points in the offer probabilities $p(c)$ reflects a real situation that, in order to make a critical breakthrough in the research of the technological possibility of an idea such as the Josephson device as a logic IC for computers, an investment at least up to a certain level, usually enormous, must be made.

For the convenience of numerical computation, we shall transform the offer probabilities $p(c)$ as follows:

$$p(c) = \begin{cases} p(c) & \text{on } c \le c^*, \\ p(c^*) & \text{on } c \ge c^* \end{cases} \tag{5.6}$$

where $c^* = \max\{c \mid p(c) \le 0.99999 \times p(\infty)\}$. By the transformation, we can replace the constraint $0 \le c \le i$ in the right hand sides of (4.9) and (4.14) with $0 \le c \le \min\{i, c^*\}$. The reason is as follows. First, it is obvious for $i < c^*$. Then let $c^* \le i$.

Now let us express (4.9) and (4.14) as, respectively, $V_t(i) = \max_{0 \le c \le i} D_t(i, c)$ and $V_t(i, k) = \max_{0 \le c \le i} D_t(i, k, c)$ where

$$D_t(i, c) = \beta \int_0^\infty \max\{w, V_{t-1}((i-c)/\beta)\} dG(w|c) - c, \qquad t \ge 1, \tag{5.7}$$

$$D_t(i, k, c) = \beta \int_0^\infty v_{t-1}((i-c)/\beta, \max\{w, k\}) dG(w|c) - c, \quad t \ge 1. \tag{5.8}$$

Since $F(w|c)$ has been assumed to be independent of c and $p(c)$ has been transformed to be independent of $c \ge c^*$, $G(w|c)$ is also independent of $c \ge c^*$. Accordingly, both $D_t(i, c)$ and $D_t(i, k, c)$ are also nonincreasing in $c \ge c^*$ from Theorem 1(b) and Theorem 3(a). This implies that $v_t(i) = \max\{\max_{0\le c\le c^*} D_t(i,c),\ \max_{c^*<c} D_t(i,c)\} = \max_{0\le c\le c^*} D_t(i,c)$ and similarly $v_t(i,k) = \max_{0\le c\le c^*} D_t(i,k,c)$.

In order to calculate the reservation price $V_t(i)$ on $0 \le i \le I$ for certain given I and t using (4.9), $V_{t-1}(i)$ must be computed on $0 \le i \le I/\beta$ in advance of it because of $0 \le (i-c)/\beta \le i/\beta \le I/\beta$, and furthermore, in order to do this, similarly $V_{t-2}(i)$ must be obtained on $0 \le i \le I/\beta^2$ in advance of it, $\cdots$, and, in general, $V_n(i)$ must be calculated on $0 \le i \le l/\beta^{t-n}$, $n = t-1, t-2, \cdots, 1$. The same argument holds for $V_t(i,k)$.

Finally, for convenience of numerical computations, we shall evaluate $V_t(i)$ only for $i \in \mathcal{A} = \{j\Delta | j = 0, 1, \cdots\}$, $\Delta = c^*/N$ for a given N. In the case, the argument of $V_{t-1}(\xi)$, $\xi = (i-c)/\beta$, is not always in $\mathcal{A}$ if $\beta < 1$. If $\xi \notin \mathcal{A}$, we evaluate $V_{t-1}(\xi)$ approximately by interpolation; i.e., if $j\Delta < \xi < (j+1)\Delta$, then

$$V_{t-1}(\xi) \approx \frac{(j+1)\Delta - \xi}{\Delta} V_{t-1}(j\Delta) + \frac{\xi - j\Delta}{\Delta} V_{t-1}((j+1)\Delta). \tag{5.9}$$

Similar for $V_{t-1}(\xi, k)$. It is of course clear that a sufficiently large N must be taken in order to attain a sufficiently reasonable accuracy of the approximation.

5.2. Results

The values of the parameters, p, ρ, λ, a, b, $\cdots$ for the numerical examples that were computed in the present paper are listed on the bottom of each figure.

No Recall Model

Figure 2 shows graphs of optimal reservation prices, $V_t(i)$ and $V_n^*(t,i)$, and optimal investments, $c_t(i)$ and $c_n^*(t,i)$, for Case 0, Case 1, and Case 2 with $\beta = 1.00$, 0.98 where, for $V_n^*(t,i)$ and $c_n^*(t,i)$, the search process is assumed to start from time $t = 10$ with six or seven different starting search budgets i, equally spaced. Figure 3 shows graphs of the optimal investments, $c_1(i)$, $c_2(i)$, and $c_3(i)$, in Graph w of Figure 2, in which shapes are hard to distinguish due to the complexity of the plot.

Figure 4 shows graphs of limiting optimal reservation prices V_t ($= \lim_{i\to\infty} V_t(i)$) and limiting optimal investments c_t ($= \lim_{i\to\infty} c_t(i)$) for $t = 0, 1, \cdots, 10$, representing the relation with t and β. The bold line curves in the graphs are for the first t for which $|(V_t - V_{t-1})/V_t| < 0.000001$ and can be regarded as approximations for the limits of V_t and c_t as $t \to \infty$.

Figure 5 shows graphs exemplifying how the optimal investments $c_n^*(t,i)$ and the optimal reservation prices $V_n^*(t,i)$, starting from time $t = 10$ with a remaining search budget $i = |oa|$, are obtained from $c_t(i)$ and $V_t(i)$. The optimal investment $c_n^*(t,i)$ is obtained as follows. First, the optimal investment of the starting point in time, $c_{10}^*(10,i)$, is given by $|bc|$. Therefore, the remaining search budget of time 9 reduces to $i_9^*(10,i) = |oa| - |bc| = |oa'|$, implying that the optimal investment of time 8 becomes $c_8^*(10,i) = |oa'| - |b'c'| = |oa''|$. On the other hand, the optimal reservation prices, $V_{10}^*(10,i)$, $V_9^*(10,i)$, $\cdots$, are given by $|bd|$, $|b'd'|$, $|b''d''|$ $\cdots$ and so on where the points $a, a', a'', \cdots$ and $b, b', b'', \cdots$ plotted on the (i,t) plane in the graph of $V_n^*(t,i)$ are the same as ones in the graph of $c_n^*(t,i)$.

Recall Model

Figure 6 shows graphs of optimal investments $c_t(i,k)$ for $t = 1, 2, 3, 4$, $k = 50$, 70, 90, 110, 130, 150 and $\beta = 0.98$.

Figure 7 shows graphs of limiting optimal investments $c_t(k)$ of $c_t(i,k)$ as $i \to \infty$ for $k = 50, 90, 130$ on $0 < \beta < 1$ for Case 0 and Case 1. The bold line curves are for the first t for which $\max_\beta \left|(V_t(k) - V_{t-1}(k))/V_t(k)\right| < 0.000001$ for each k and can be regarded as approximations for limits of $c_t(k)$ as $t \to \infty$.

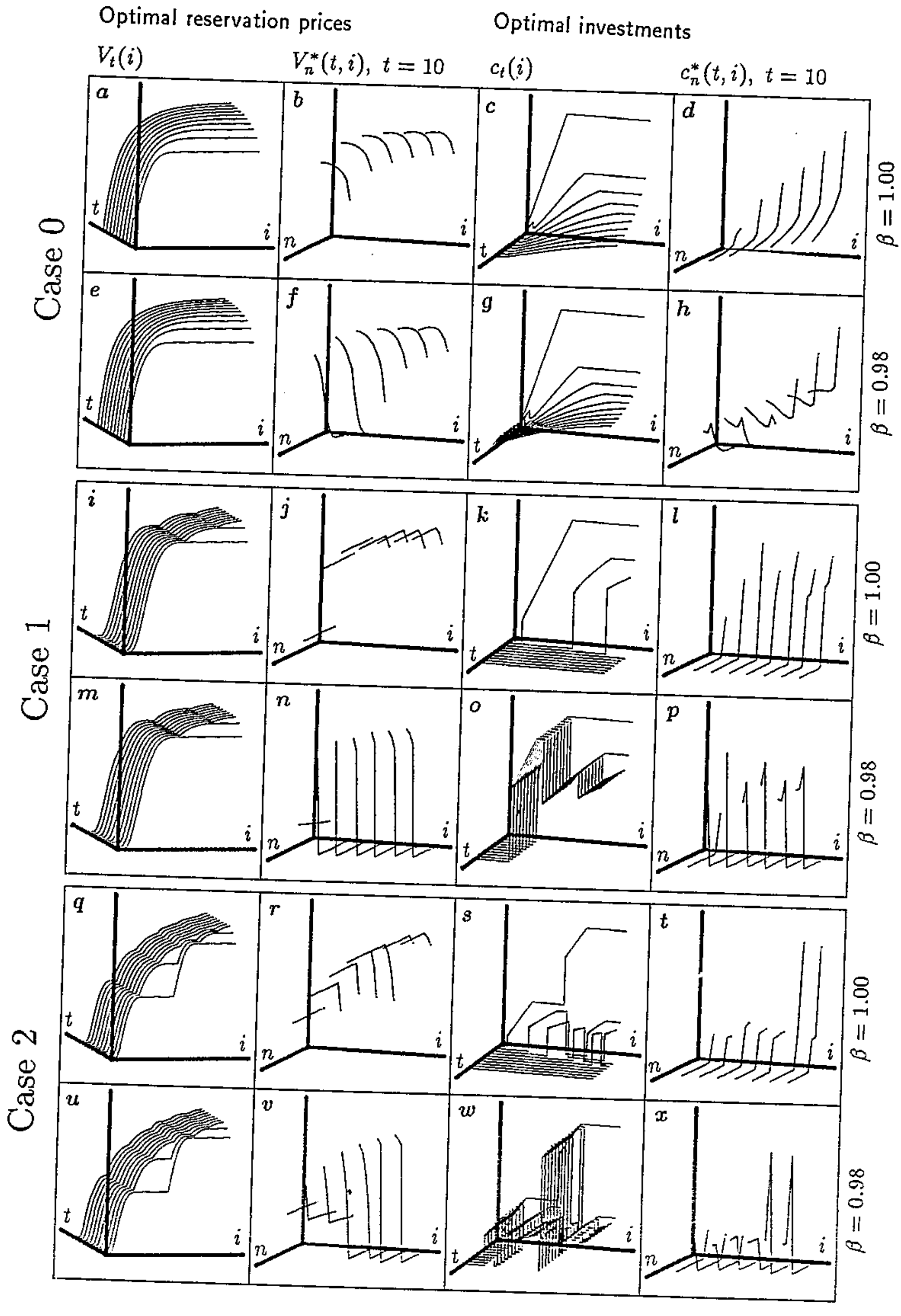


Figure 2

Optimal reservation prices, $V_t(i)$ and $V_n^*(t,i)$, and optimal investments, $c_t(i)$ and $c_n^*(t,i)$

Case 0: p=1.0, λ=0.5, a=50, b=150, N=2000

Case 1: p_1=1.0, λ_1=0.5, $\rho_1=10^{-5}$, a=50, b=150, N=2000

Case 2: p_1=0.5, λ_1=0.5, $\rho_1=10^{-5}$, s_1=15, p_2=0.5, λ_2=0.5, $\rho_2=10^{-5}$, a=50, b=150, N=2000

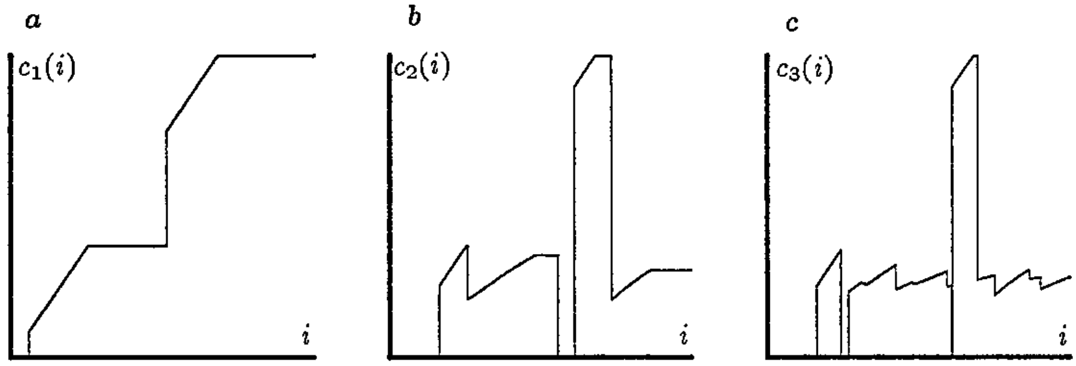


Figure 3
Optimal investments, $c_1(i)$, $c_2(i)$, and $c_3(i)$ in Graph w of Figure 2

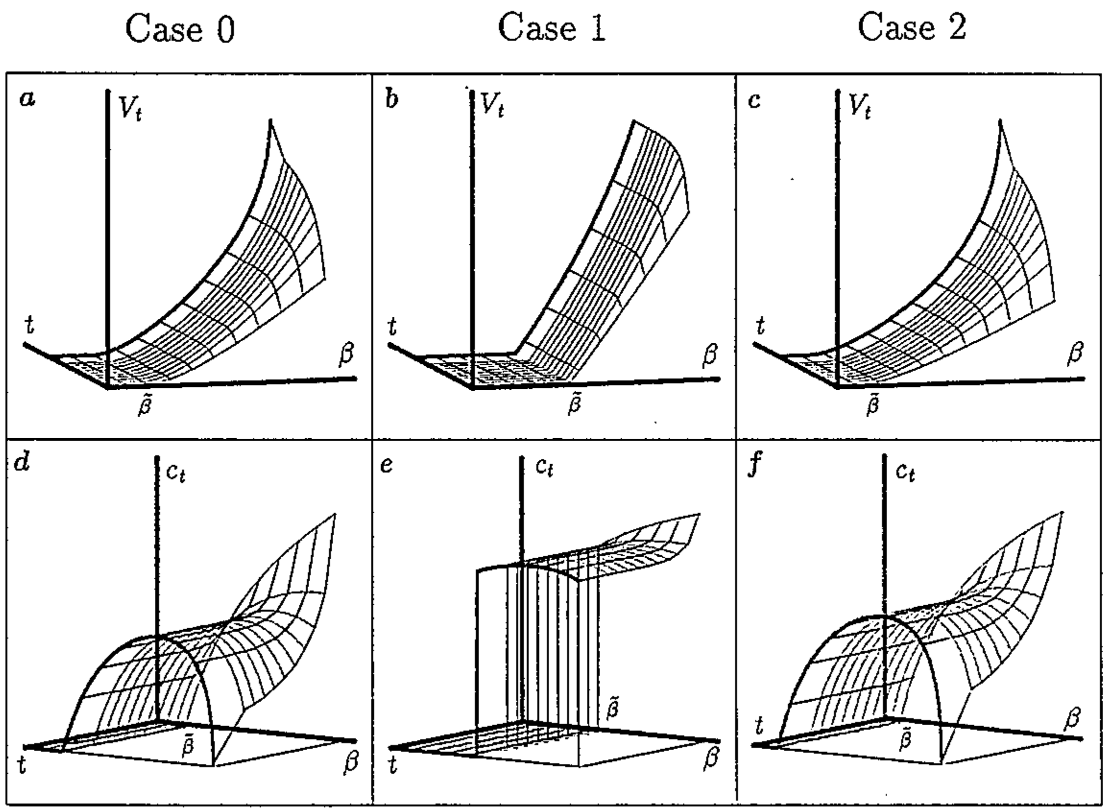
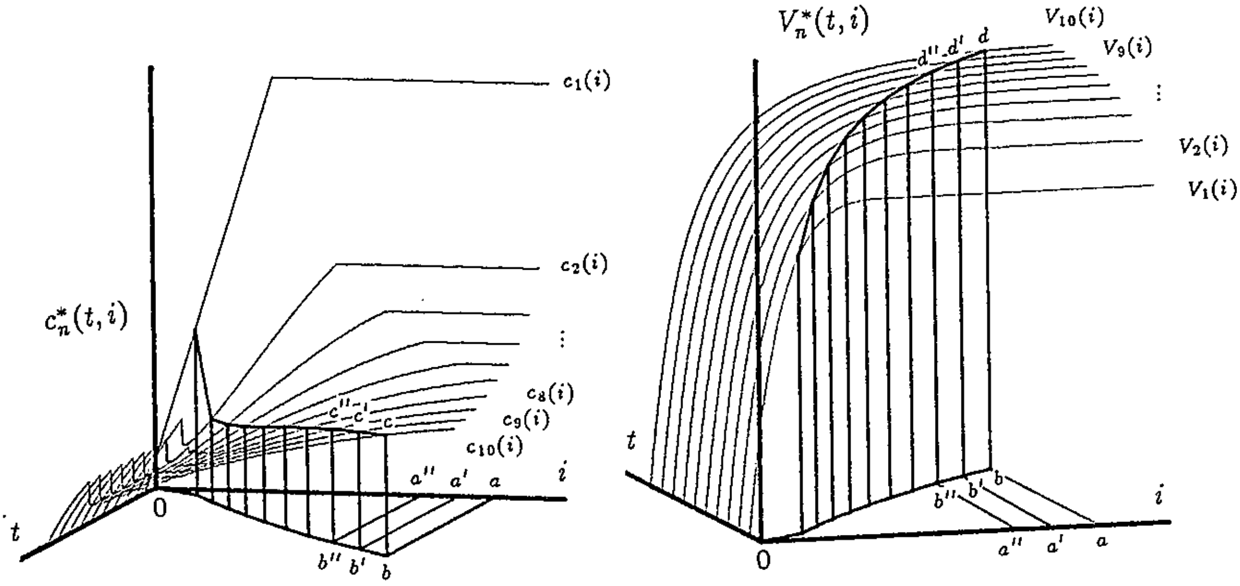


Figure 4
Limiting reservation prices V_t and limiting optimal investments c_t, $t = 0, 1, \cdots, 10$ on $0 < \beta \leq 1$ where the bold curves are graphs of the limits of c_t and V_t as $t \to \infty$.

Case 0: p=1.0, λ=0.5, a=0, b=20
Case 1: p_1=1.0, λ_1=0.8, $\rho_1=10^{-5}$, a=0, b=100
Case 2: p_1=0.7, λ_1=0.8, $\rho_1=10^{-5}$, s_1=30, p_2=0.3, λ_2=0.8, $\rho_2=10^{-5}$, a=0, b=100



Optimal investment $c_n^*(t,i)$ Optimal reservation prices $V_n^*(t,i)$

Figure 5

Relation of the optimal investments $c_n^*(t,i)$ and optimal reservation prices $V_n^*(t,i)$ to $c_t(i)$ and $V_t(i)$ (See Grapf e,g of Figure 2)

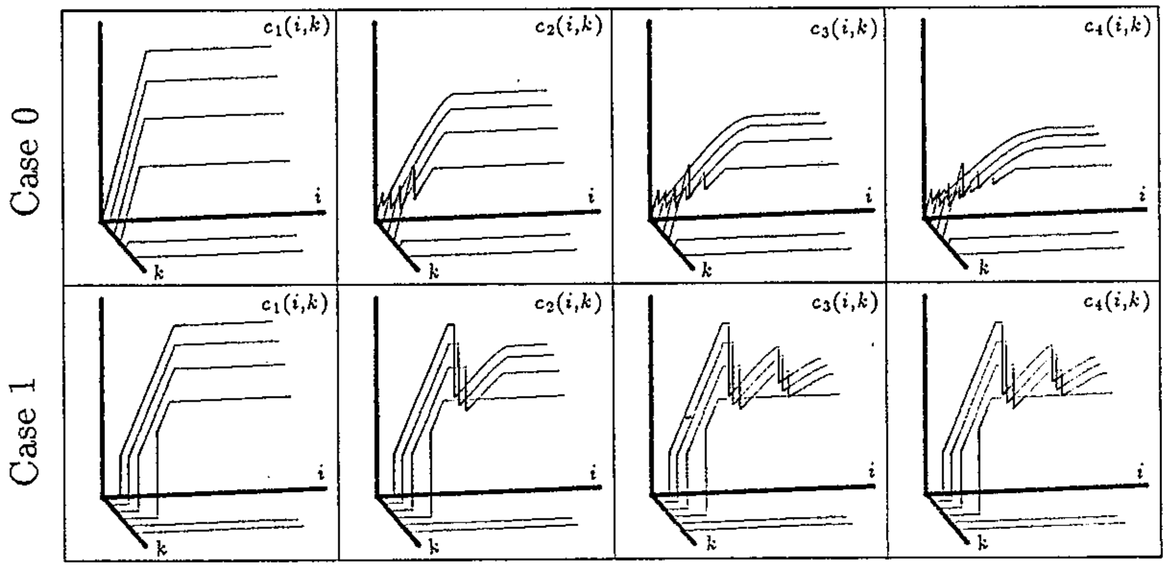


Figure 6

Optimal investments $c_t(i,k)$ for $t = 1,2,3,4$, $k = 50,70,90,110,130,150$, and $\beta = 0.98$ where $F(w)$ is a discrete uniform distribution function with mass points $w = 50,70,90,110,130,150$
Case 0: p=1.0, λ=0.5, Case 1: p_1=1.0, λ_1=0.8, ρ_1=10^{-5}

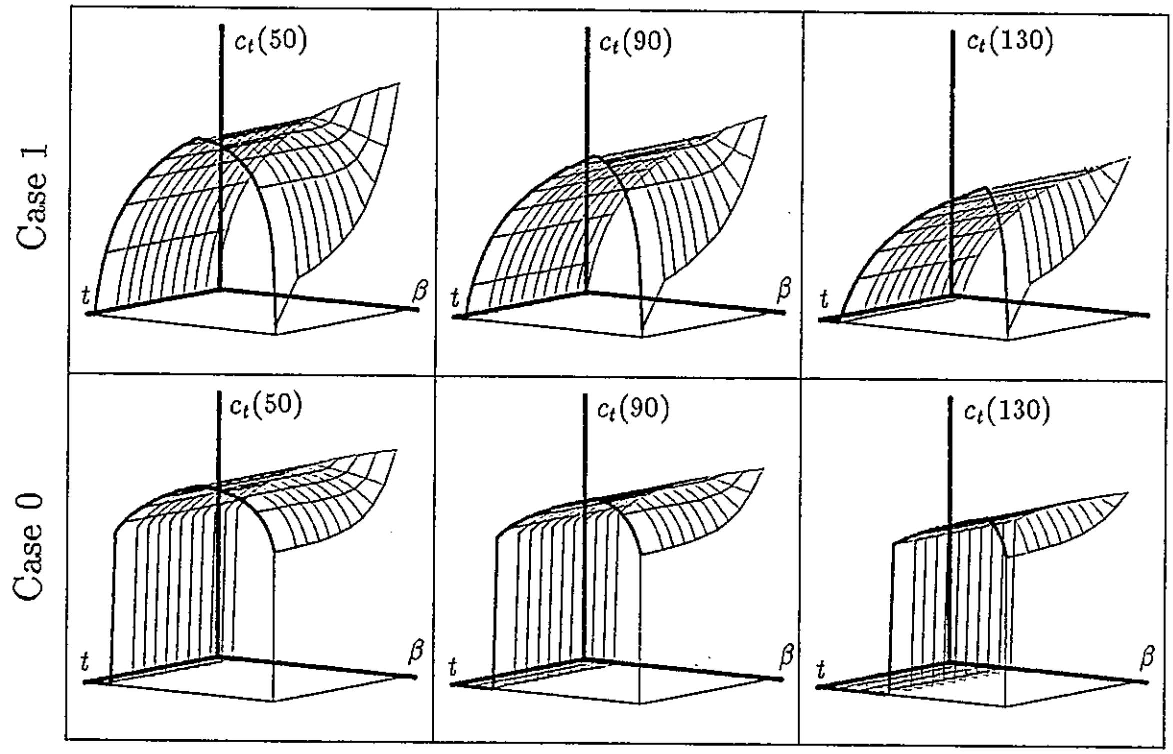


Figure 7

Limiting optimal investment $c_t(k)$, $t = 0, 1, \cdots, 10$ for $k = 50, 90, 130$ on $0 < \beta \leq 1$ where the bold curves are graphs of limits $c(k)$ of $c_t(k)$ as $t \to \infty$.
Case 0: p=1.0, λ=0.5, Case 1: p_1=1.0, λ_1=0.8, ρ_1=10^{-5}

6. Conclusions and Discussion

6.1. Case of $K(\infty, 0) = 0$

1. The equality $K(\infty, 0) = 0$ is equivalent to $\beta p(c)\mu(c) \leq c$ for all $c \geq 0$, implying that the expected present discounted value of an offer obtained at the next point in time by paying *any* search cost c at a certain point in time, $\beta p(c)\mu(c)$, does not make up for the search cost invested c. Therefore, it can be intuitively expected that if the equality holds true, entering the search process will not yield any economical value.

Theorem 1(a) and Theorem 3(b) claims that this is also theoretically true, asserting that if the search process starts with any nonnegative offer x when the equality holds true, since $U_t(i) = 0 \leq x$ in a recall model and $U_t(i, x) \leq x$ in a no recall model, it is optimal to accept the offer without entering the search process. Furthermore, the following can be concluded from the two theorems and the definition of $\tilde{\beta}$. Suppose that there exists a positive $\tilde{\beta}$. Then if an interest rate r is sufficiently large (i.e., a discount factor β is sufficiently small), the above assertion holds true; that is, it is optimal not to enter the process by allocating the whole search budget that is currently available to a financial investment rather than to attempt to get a profit from a search investment.

6.2. Case of $K(\infty, 0) > 0$

6.2.1. Optimal Stopping Rule

No Recall Model

2. The optimal reservation price $V_t(i)$ is nondecreasing in t and i with $V_1(i) \geq 0$ for all i and upper-bounded in i for all t (Theorem 1(b), Graph a,e,i,m,q,u of Figure 2). It should be noted that the nondecreasing patterns are not always concave or convex; for example, it can have such a shape as *a fall of multiply stored cascades* as seen in Graph i,m,q,u of Figure 2.

3. The optimal reservation price $V_n^*(t,i)$, $n = 1, 2, \cdots, t$, starting from time t with a given search budget i, can be proved to be always nondecreasing in n (Theorem 1(c)), but the nondecreasing patterns are not always smooth (Graph b,f,j,n,r,u of Figure 2), possibly altering discontinuously even by a slight change in a starting search budget i and a discount factor β.

4. The optimal reservation price $V_t(i)$ converges to a limit V_t as $i \to \infty$ for all t from Theorem 1(b), and when $\beta < 1$, the limit V_t, nondecreasing in t, converges as $t \to \infty$ to h^*. The h^* is the unique solution of $K(\infty, x) = 0$ (Theorem 2(c)). The $V_t(i)$ converges to a limit $V(i)$ as $t \to \infty$ for all i (Theorem 2(d)). The V_t is nondecreasing in β (Theorem 2(a), Graph a,b,c of Figure 4), implying that, the smaller the interest rate may be, the greater offer should be searched. If $\beta < 1$, since $K(\infty, V) = 0$ from (4.11), it follows that the largest β for which V becomes equal to 0 coincides with the $\bar{\beta}$ defined in section 4.1.

Recall Model

5. If $\beta = 1$, it is always optimal to continue the search up to the end of the planning horizon and accept the best offer obtained so far (Theorem 3(c2)). This is a property that is peculiar in the model; it does not appear at all in any conventional optimal stopping problem with recall.

6. Assume $\beta < 1$. In the case, there exists the nonnegative reservation price $h_t(i)$, which is the unique solution of $V_t(i,k) - k = 0$; it is nondecreasing and upper-bounded in t and i (Theorem 4). Then the optimal stopping rule can be stated as follows. For the best offer k so far, if $k > h_t(i)$, stop with accepting it, or else continue. Accordingly, it follows that the optimal stopping rule of the model has a *reservation price property* as in almost all optimal stopping problems that have been posed so far.

7. As a search budget $i \to \infty$, in whatever point in time on the planning horizon, if the present best offer $k > h^*$, stop with accepting it, or else continue (Theorem 5). Here note that the optimal stopping rule is time-independent, implying that whatever point in time the search process starts from, the optimal stopping rule of the starting point in time is the same as that of time 1 when the search process is terminated at the next point in time. In other words, whatever planning horizon to go there remains, it is optimal to behave, in terms of stopping decision, *as if* there remains only a period of planning horizon to go. This however does not mean that when the best offer $k \leq h^*$, the search process must terminate at the next point in time with accepting an offer at that time; it still proceeds if there remain more than one periods of planning horizon. The property is usually called a *myopic property*, which a conventional optimal stopping problem with recall also has.

8. The limit of $V_t(i)$ as $t \to \infty$ becomes equal to h^* if $k \leq h^*$ (Theorem 6), meaning that, in limiting t and i, if the present best offer $k \leq h^*$, the expected present discounted value from continuing the search is equal to h^*.

6.2.2. Optimal Investment

Due to an intractability of the mathematical treatment of expressions (4.9) and (4.14), it is quite difficult to analytically examine the relation of optimal investments, $c_t(i)$ and $c_t(i,k)$, to t, i, and k, so we investigated it by means of numerical analyses. Although the properties found in terms of the optimal investment are from only a limited number of numerical examples, some of them are quite counterintuitive and beyond our comprehension.

No Recall Model

9. One of the most interesting results obtained in the paper is that the optimal investment $c_t(i)$, $t \geq 2$, does not always become monotone in a search budget i that is currently available; it is proved that $c_1(i)$ is always nondecreasing in i (Theorem 1(d)). Here we shall demonstrate such a phenomenon by the following simple example. Let $\beta = 1$ and the offer probabilities $p(c)$ be

$$p(0.0) = 0.000,\ \ p(0.1) = 0.104,\ \ p(0.2) = 0.198,\ \ p(0.3) = 0.282,\ \ p(0.4) = 0.357,$$
$$p(0.5) = 0.424,\ \ p(0.6) = 0.485,\ \ p(0.7) = 0.539,\ \ p(0.8) = 0.587,\ \ p(0.9) = 0.631.$$

Let an offer have the value of 50 or 150 million dollars with even probabilities, i.e., 0.5; hence its expectation is 100 million dollars. In the case, first we have

$$V_1(i) = \max_{0 \leq c \leq i} \{100p(c) - c\}$$

for $i = 0.0, 0.1, \cdots$ and $c = 0.0, 0.1, \cdots, i$ where the monetary units of i and c are the same, one million dollars. Immediately we obtain

$$V_1(0.0) = 0.00,\ \ V_1(0.1) = 10.3,\ \ V_1(0.2) = 19.6,\ \ V_1(0.3) = 27.9,\ \ V_1(0.4) = 35.3,$$
$$V_1(0.5) = 41.9,\ \ V_1(0.6) = 47.9,\ \ V_1(0.7) = 53.2,\ \ V_1(0.8) = 57.9,\ \ V_1(0.9) = 62.2.$$

Next let us compute $V_2(i)$ and $c_2(i)$ using

$$V_2(i) = \max_{0 \leq c \leq i} \{(1 - p(c))V_1(i - c) + p(c)\{0.5\max\{50, V_1(i - c)\} + 0.5\max\{150, V_1(i - c)\}\} - c\}.$$

Then we have

$$V_1(0.7) = \max\{53.2000,\ 53.2184,\ 53.2038,\ \underline{53.2454},\ 53.2397,\ 53.1896,\ 53.2045,\ 53.2000\},\ \ c_1(0.7) = 0.3$$

$$V_1(0.8) = \max\{57.9000,\ \underline{58.1336},\ 58.0158,\ 57.9842,\ 57.9979,\ 57.9704,\ 57.9940,\ 57.9483,\ 57.9000\},\ \ c_1(0.7) = 0.1.$$

That is, when a remaining search budget $i = 0.7$ million dollars, the optimal investment is 0.3 million dollars, but when the remaining search budget increases to $i = 0.8$ million dollars, the optimal investment decreases to 0.1 million dollars; that is, *an increased investment with a lower residual search budget, and conversely a decreased investment with a larger residual search budget.* The example exemplifies that the optimal investment does not always increases with a remaining search budget even in case of no discounting. I hope that each reader ascertains by himself that the computations above do not include miscalculation or computational error.

In some cases, the non-monotonicity of the optimal investment shows abnormal patterns, resembling *rippling waves lapping a beach* (Graph c,g of Figure 2), *the teeth of a saw* (Graph o of Figure 2), or *ditches in the Grand Canyon* (Graph w of Figure 2). As reflected in these graphs, the non-monotonicity pattern may vary very sensitively to a change in a remaining search budget i; its very slight increment may reduce the optimal investment drastically up to zero (*the bottom of the ravine*) or lift it very high (*the top of the mountain*). We can provide the following interlpetation for the occurrence of the non-monotonicity although it is a little intuitive.

When a remaining search budget is small, it might be more reasonable to attempt to attain a total maximization through accepting a more profitable offer appearing with a higher probability by paying a larger search cost; by doing so, it might be possible to retrieve the disadvantageous situation of a small remaining search budget. Conversely, when a remaining search budget is large, it might be more reasonable to attempt to attain a total maximization through reserving the large remaining search budget by not investing a larger search cost in search activities.

Figure 8 shows another example of the occurrence of the non-monotonicity in the optimal investment $c_2(i)$ in Graph w of Figure 2, or Graph b of Figure 3. The five curves and five dots • on each curve are, respectively, graphs of the expected present discount values $D_2(i,c)$ from investing $c \leq i$ and the optimal investment $c_2(i)$ starting from time $t = 2$ with a search budget $i = 45, 100, 160, 340, 600$. As shown in the figure, the non-monotonicity occurs among $i = 160$, $i = 340$, and $i = 600$, that is, $c_2(160) > c_2(340) < c_2(600)$.

If drawing the graphs on a three dimensional space by taking i-axis as the third axis, the non-monotonicity can be displayed more vividly as seen in Figure 9 where the indented bold lines on the curved surface represents a locus of the coordinates $(i, c_2(i), D_2(i, c_2(i)))$; its vertical projection depicted on (i,c)-plane is the graph of the optimal investment $c_2(i)$ (the same as Graph a,b of Figure 3) and its horizontal projection pictured as an increasing curve $\widetilde{ab}$ is the graph of the optimal reservation price $V_2(i) = D_2(i, c_2(i))$ (the same as Graph u of Figure 2).

10. The limiting optimal investment c_t of $c_t(i)$ as $t \to \infty$ can increase discontinuously (Graph ***d,f*** of Figure 4) or stepwise (Graph e of Figure 4) at the discount factor $\tilde{\beta}$, defined by (4.6). It is very hard to give any appropriate interpretation that can persuade real investment decision makers to believe in such a discontinuity, especially for the stepwise increase.

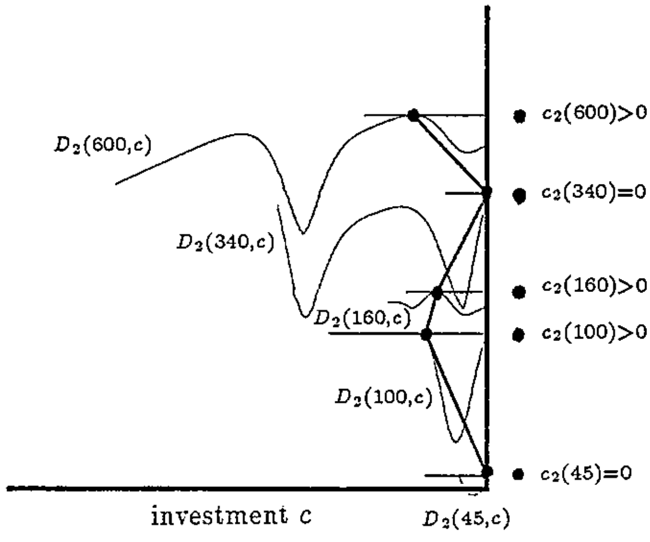


Figure 8

Inversion phenomenon of the optimal investment $c_t(i)$ in the remaining search budget i

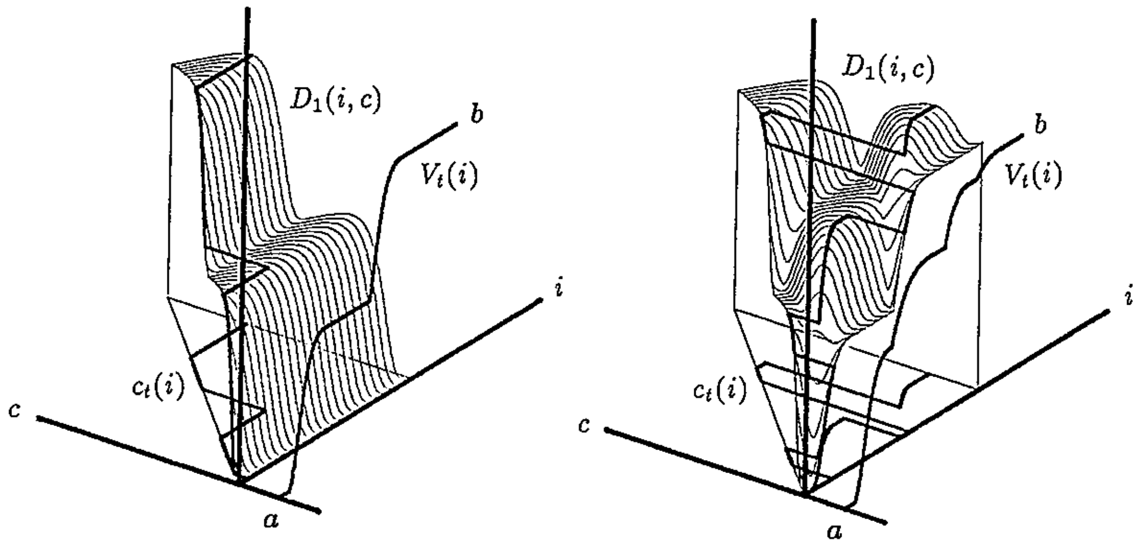


Figure 9
Optimal investment $c_t(i)$ maximizing $D_t(i,c)$ on $0 \le c \le i$ and optimal reservation price $V_t(i)$, $t = 1, 2$
(See Graph a,b of Figure 3 and Graph u of Figure 2)

11. Although the limiting optimal investment c_t increases in β on $(0 \le \beta \le 1)$ for a relatively small planning horizon, if the planning horizon becomes longer than that, there exists a value of β^* for which c_t decreases for $\beta \ge \beta^*$; that is, c_t becomes maximal at the value of β^* (Graph *d,e,f* of Figure 4).

12. When $\beta = 1$, as $t \to \infty$, the c_t converges to 0 in Graph *d,f* of Figure 4, but converges to a positive number in Graph *e* of Figure 4. The former case implies that, if starting with a sufficiently large planning horizon, it becomes optimal to continue to allocate the whole remaining search budget to a financial investment, and if the end of the planning horizon has become visible as time elapses, it becomes optimal to start to gradually invest more of the remaining search budget to a search investment. The latter case implies that it is optimal to continue to allocate a part of the remaining search budget to a search investment even when starting with a sufficiently large planning horizon.

Recall Model

13. In a recall model, the optimal investments, $c_t(i,k)$, and the limiting optimal investment, $c_t(k)$, also depict similar patterns to ones in the recall model as depicted in Figure 6 and Figure 7.

7. Some limitations

In the model, we have abstracted certain other aspects of the optimal stopping problem, and many of the underlying assumptions of the current formulation are unrealistic. For a more realistic modification of the model, the following provisions have to be taken into consideration.

First, both an offer probability $p(c)$ and offer distribution function $F(w|c)$ may depend on the history of past search costs paid and offers obtained so far. In an R&D problem such as in Section 2, the history dependency will be a decisively important factor to be taken into consideration. Second, the assumption that an offer once inspected and passed up is either completely available (*with recall*) or forever lost (*with no recall*) is not realistic. It is more realistic to assume that the availability is uncertain (*with uncertain recall*) [4],[5],[6]. Then, one of the most interesting variations of our model is the introduction of the notion of a free search sequence, which was first introduced by Weitzman [12]. Furthermore, the following provisions should also be introduced: adaptive learning about offer probability and offer distribution function, parallel search, randomly generated new opportunities, and so on.

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References

[1] Chow, Y.S., Robbins, H. and Siegmund,D.: Great Expectations: The Theory of Optimal Stopping, *Houghton Mifflin Company*, Boston (1971)

[2] DeGroot, M.H.: *Optimal Statistical Decisions*, McGraw-Hill, (1970)

[3] Gilbert, J.P. and Mosteller, F.: Recognizing the Maximum of a Sequence, *Journal of the American Statistic Association*, Vol. 61 (1966), 35–73

[4] Ikuta, S.: Optimal Stopping Problem with Uncertain Recall, *Journal of the Operations Research Society of Japan*, Vol. 31 (1988), 145–170

[5] Karlin, S.: Stochastic Models and Optimal Policy for Selling an Asset5 Chap.9, *Studies in Applied Probability and Management Science* (ed. S. Karlin), Stanford University Press (1962), 148–158

[6] Karni, E. and Schwartz, A.: Search Theory: The Case of Search with Uncertain Recall, *Journal of Economic Theory*, Vol. 16 (1977), 38–52

[7] Landsberger, M. and Peled D.: Duration of Offers, Price Structure, and the Gain from Search, *Journal of Economic Theory*, Vol. 16 (1977), 17–37

[8] Lippman, S.A. and McCall, J.J.: Job Search in a Dynamic Economy, *Journal of Economic Theory*, Vol. 12 (1976), 365–390

[9] Rothschild, M.: Searching for the Lowest Price When the Distribution of Prices is Unknown, *Journal of Political Economy*, Vol. 82, 689–711

[10] Sakaguchi, M.: Dynamic Programming of Some Sequential Sampling Design, *Journal of Mathematical Analysis and Applications*, Vol. 2 (1961), 446–466

[11] Sargent, T.J.: *Dynamic Macroeconomic Theory*, Harvard University Press, (1987)

[12] Weitzman, M.L.: Optimal Search for the Best Alternative, *Econometrica*, Vol. 47 (1979), 641–654