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ABSTRACT

We introduce a continuous deformation algorithm for solving a stationary point problem on a polytope K. The algorithm starts by applying a variable dimension algorithm on $K \times \{0\}$ until an approximate stationary point is found on $K \times \{0\}$. Then by tracing a path of solutions of a system of equations the algorithm virtually follows a path of approximate stationary points in $K \times [0, \infty)$. When the path in $K \times [0, \infty)$ returns to level 0, i.e., $K \times \{0\}$, we again apply the variable dimension algorithm until a new approximate stationary point is found on $K \times \{0\}$. The set $K \times [0, \infty)$ is triangulated so that the approximate stationary point on the path improves the accuracy as we go up the level. Therefore after finitely many steps we obtain a stationary point with a desired accuracy. We also propose a contrivance for a practical implementation of the algorithm.

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1. Introduction

Let K be a polytope in R^n and f be a continuous function from K into R^n. Then the stationary point problem for f on K is to find a point x in K such that

$$(z-x)\cdot f(x)\leq 0 \quad \text{for any} \quad z\in K, \tag{1.1}$$

where $\cdot$ means the inner product of two vectors. We call x a stationary point of f on K. For solving the stationary point problem several variable dimension algorithms have been developed, which originated from the works of Kuhn [9] and Shapley [12] for approximating a fixed point. The algorithms generate a sequence of simplices with variable dimension to yield an approximate stationary point. If the approximation is not good enough, the triangulation is refined to obtain a better approximate stationary point by using the algorithm again. Eaves [4] and Eaves and Saigal [6] developed a homotopy algorithm for computing a zero of a continuous mapping. They use a triangulation of $R^n\times[1,\infty)$ whose mesh size becomes finer as the homotopy parameter $t\in[1,\infty)$ increases. Instead of restarting a variable dimension algorithm on R^n after an approximate solution has been found, one can also continue the algorithm with the simplex yielding the approximate solution by embedding R^n into the set $R^n\times[1,\infty)$. However, the triangulation only allows for a refinement between two subsequent levels of at most two. Van der Laan and Talman [10] and Shamir [11] proposed a new triangulation of $R^n\times[1,\infty)$ built from triangulations between two successive levels so that the refinement can be any size. Recently, Doup and Talman [3] have shown that the variable dimension restart algorithm for solving the stationary point problem on a unit simplex S^1 can be adapted to a continuous deformation algorithm on $S^1\times[1,\infty)$ with an arbitrary refinement.

In this paper we first present a continuous deformation algorithm for solving the stationary point problem on the polytope K in Section 3. Such an algorithm traces a path of zeros of a piecewise linear function g defined on a subdivided $(n+1)$-manifold into R^n. The subdivided $(n+1)$-manifold is made of a generalized primal-dual pair of subdivided manifolds, where the primal sets are determined by the subsets of $K\times[0,\infty)$ .nd the dual are determined by the normal cones of the faces of K. A primal-dual pair of subdivided manifolds is a basic framework in path-following technique for finding fixed points or solving stationary point problems, see for example [1,7,8,13,14]. On condition that the zero of R^n is a regular value of the function g, there exists a path of zeros of g

whose projection on $K \times [0, \infty)$ initiates from $v \times \{0\}$ for an arbitrarily chosen starting point v of K. The projection is unbounded and every point x such that (x, t) lies on the projected path for $t > 0$ is an approximate stationary point of the problem. We also show that after a finitely many iterations an approximate stationary point with the given accuracy will be obtained.

Although the continuous deformation algorithm may have the advantage of using more information about the function values, it is not practical to construct the whole triangulation in advance and keep it in memory when the algorithm is implemented. Therefore in Section 4 we propose another version of the algorithm to make it practical. We build a structure through a combination of a primal-dual pair of subdivide manifolds on K and a pair of triangulations of K. Such a structure can carry out the continuous deformation algorithm with t between some integer l and $l+1$. Therefore by piling up the structure the continuous deformation algorithm could be implemented. However, since it is not yet practical to keep the structure of lower levels in case the path descends the level, we propose an amalgamation of continuous deformation and restart.

2. Preliminaries

In this section we give a brief description of a subdivided manifold, a generalized primal-dual pair of subdivided manifolds and the basic theorem for fixed point algorithms.

We call an m-dimensional convex polyhedron a cell or an m-cell. If a cell X is a face of a cell Y, we write $X \preceq Y$. We denote $X \prec Y$ when X is a proper face of Y. Particularly when an $(m\text{-}1)$-cell X is a face of an m-cell Y, we call X a facet of Y, and denote $X \triangleleft Y$.

Let $\mathcal{L}$ be a finite or countable collection of cells of the same dimension, say m. We denote $\{ X \mid X \text{ is a face of some } m\text{-cell of } \mathcal{L} \}$ by $\bar{\mathcal{L}}$ and $\bigcup \{ X \mid X \text{ is a cell of } \mathcal{L} \}$ by $|\mathcal{L}|$.

DEFINITION 2.1. We call $\mathcal{L}$ a subdivided manifold if it satisfies the following conditions:

(1) for any $X, Y \in \mathcal{L}$ either $X \cap Y = \emptyset$ or $X \cap Y$ is a face of X and Y,

(2) for each $(m\text{-}1)$-cell X of $\bar{\mathcal{L}}$ at most two m-cells of $\mathcal{L}$ containing X as a facet,

(3) each point of $|\mathcal{L}|$ has a neighborhood which intersects finitely many cells of $\mathcal{L}$.

We call the collection of $(m\text{-}1)$-cells of $\bar{\mathcal{L}}$ lying in exactly one m-cell of $\mathcal{L}$ the boundary of $\mathcal{L}$ and denote it by $\partial\mathcal{L}$.

A continuous function g from $|\mathcal{L}|$ into some Euclidean space is said to be a piecewise linear function (abbreviated by *pl*) on $\mathcal{L}$ if the restriction of g to each cell of $\mathcal{L}$ is an affine function.

DEFINITION 2.2. For a subdivided $(n+1)$-manifold $\mathcal{L}$ and a *pl* function $g : |\mathcal{L}| \to R^n$, we say that $c \in R^n$ is a regular value of g if

$$X \in \bar{\mathcal{L}} \quad \text{and} \quad g^{-1}(c) \cap X \neq \emptyset \quad \text{implies} \quad \dim g(X) = n.$$

We now state one of the basic theorems for path following algorithms.

THEOREM 2.1. Let $\mathcal{L}$ be a subdivided $(n+1)$-manifold in R^n and g a *pl* function from $|\mathcal{L}|$ into R^n. Suppose that $c \in R^n$ is a regular value of g and $g^{-1}(c) \neq \emptyset$. Then $g^{-1}(c)$ is a disjoint union of paths and loops, where a path is a subdivided 1-manifold homeomorphic to one of the intervals $(0,1)$, $[0,1)$ and $[0,1]$ and a loop is a subdivided 1-manifold homeomorphic to the 1-dimensional sphere. Furthermore $g^{-1}(c)$ satisfies the following conditions:

(1) $g^{-1}(c) \cap X$ is either empty or a subdivided 1-manifold for each $X \in \mathcal{L}$.

(2) A loop of $g^{-1}(c)$ does not intersect $|\partial\mathcal{L}|$.

(3) $z \in g^{-1}(c)$ is an endpoint of a path if and only if $z \in |\partial\mathcal{L}|$.

(4) If in addition $|\mathcal{L}|$ is closed, every path in $g^{-1}(c)$ which is homeomorphic to either $[0,1)$ or $(0,1)$ is unbounded.

Proof. See Section 9 of Eaves [5]. □

Let $\mathcal{P}$ and $\mathcal{D}$ be two subdivided manifolds. A dual operator, say d, is defined on $\bar{\mathcal{P}}$ and assigns to each cell X of $\bar{\mathcal{P}}$ either the empty set or a cell Y of $\bar{\mathcal{D}}$ such that for some fixed positive integer l, called the degree,

$$\dim X + \dim Y = l$$

holds. We denote the image of $X \in \bar{\mathcal{P}}$ under the operator d by X^d. When a pair of subdivided manifolds $\mathcal{P}$ and $\mathcal{D}$ is linked by such an operator d, we call the triplet $(\mathcal{P}, \mathcal{D}, d)$ a generalized primal-dual pair of subdivided manifolds (abbreviated by GPDM). We allow a dual operator to assign the same cell of $\bar{\mathcal{D}}$ to more than one cell of $\bar{\mathcal{P}}$, that is to be a non-injective dual operator. Let

$$\mathcal{L} = \{ X \times X^d \mid X \in \bar{\mathcal{P}}, X^d \neq \emptyset \}. \tag{2.1}$$

We give the conditions required for $\mathcal{L}$ to be a subdivided manifold in the next lemma.

LEMMA 2.2. Suppose $(\mathcal{P}, \mathcal{D}, d)$ is a GPDM with degree l. Let $\mathcal{L}$ be defined by (2.1). Then $\mathcal{L}$ is a subdivided l-manifold if and only if for any (l-1)-cell $X \times Y$ of $\bar{\mathcal{L}}$:

(1) there are at most two cells Z of $\bar{\mathcal{P}}$ such that

$$X \lhd Z \quad \text{and} \quad Z^d = Y, \tag{2.2}$$

(2) if $Y \lhd X^d$ then there is at most one cell Z of $\bar{\mathcal{P}}$ satisfying (2.2).

Proof. See [1, Lemma 2.2]. □

The following lemma characterizes the cells of the boundary $\partial\mathcal{L}$ of $\mathcal{L}$, which can be proved easily by Lemma 2.2 and the definition of the boundary $\partial\mathcal{L}$.

LEMMA 2.3. An $(l-1)$-cell $X \times Y$ of $\bar{\mathcal{L}}$ belongs to the boundary if and only if the following conditions hold:

(1) if $Y \lhd X^d$, then there is no cell Z of $\bar{\mathcal{P}}$ satisfying (2.2),

(2) if $Y \not\lhd X^d$, then there is exactly one cell Z of $\bar{\mathcal{P}}$ satisfying (2.2).

3. A Continuous Deformation Algorithm

In this section we construct a generalized primal-dual pair of subdivided manifolds underlying the system of the algorithm, and describe the steps of the algorithm to follow the path of solutions of the system.

3.1. Generalized Primal-Dual Pair of Subdivided Manifolds

Before giving the GPDM for the continuous deformation algorithm we rewrite the stationary point problem in (1.1). We consider a convex polytope K in R^n, defined by $K = \{ x \in R^n \mid a^i \cdot x \leq b_i \text{ for } i = 1, \ldots, m \}$. We assume that K is full-dimensional in R^n, that none of the constraints is redundant, and that K is simple so that at each vertex of K, $a^i \cdot x = b_i$ holds for exactly n indices i, $1 \leq i \leq m$. For each face F of K we denote the index set of binding constraints at F by $I(F)$, i.e., $I(F) = \{ i \mid a^i \cdot x = b_i \text{ for all } x \in F \}$. Let F^* be the cone generated by a^i's for $i \in I(F)$, i.e.,

$$F^* = \{ y \mid y = \sum_{i \in I(F)} \mu_i a^i,\ \mu_i \geq 0 \quad \text{for any} \quad i \in I(F) \},$$

where we assume that $F^* = \{0\}$ when $I(F) = \emptyset$. The cone F^* is called the normal cone of the face F. Note that $\dim F = n - \dim F^*$. Then the stationary point problem is a problem of finding a point $x \in K$ and a face F of K such that

$$x \in F \quad \text{and} \quad f(x) \in F^*. \tag{3.1.1}$$

Now let v be an interior point of K. This point serves as a starting point of the algorithm. For each proper face F of K let vF be the convex hull of v and F, i.e., $vF = \{ x \mid x = \alpha v + (1-\alpha) z \text{ for some } z \in F \text{ and some } \alpha \in [0,1] \}$. Note that $\dim vF = \dim F + 1$.

To make a GPDM we define

$$\mathcal{P} = \{ vF \times [0, \infty) \mid F \lhd K \}. \tag{3.1.2}$$

Then $\mathcal{P}$ is a subdivided $(n+1)$-manifold and the collection $\bar{\mathcal{P}}$ is equal to

$$\begin{aligned} \bar{\mathcal{P}} = &\{ vF \times [0,\infty) \mid F \prec K \} \cup \{ F \times [0,\infty) \mid F \prec K \} \\ &\cup \{ \{v\} \times [0,\infty) \} \cup \{ vF \times \{0\} \mid F \prec K \} \\ &\cup \{ F \times \{0\} \mid F \prec K \} \cup \{ \{v\} \times \{0\} \}, \end{aligned} \tag{3.1.3}$$

and

$$|\mathcal{P}| = K \times [0,\infty).$$

Let $\mathcal{D}$ be defined by

$$\mathcal{D} = \{ F^* \mid F \prec K \text{ and } \dim F = 0 \}.$$

Then $\mathcal{D}$ is obviously a subdivided n-manifold,

$$\bar{\mathcal{D}} = \{ F^* \mid F \preceq K \} \tag{3.1.4}$$

and

$$|\mathcal{D}| = R^n.$$

For each proper face F of K let the dual operator d be defined by

$$\begin{aligned}
&(vF \times [0,\infty))^d = \{0\} && \text{if } \dim F = n-1 \\
&(vF \times [0,\infty))^d = \emptyset && \text{if } \dim F \le n-2 \\
&(F \times [0,\infty))^d = F^* && \\
&(\{v\} \times [0,\infty))^d = \emptyset && \\
&(vF \times \{0\})^d = F^* && \\
&(F \times \{0\})^d = \emptyset && \\
&(\{v\} \times \{0\})^d = \emptyset\ . &&
\end{aligned} \tag{3.1.5}$$

Then $\dim X + \dim X^d = n+1$ if $X \in \bar{\mathcal{P}}$ and $X^d \neq \emptyset$, that is the GPDM$(\mathcal{P}, \mathcal{D}, d)$ constructed above has degree $n+1$. Let $\mathcal{L}$ be the collection of subdivided $(n+1)$-manifolds defined by (2.1) for this GPDM$(\mathcal{P}, \mathcal{D}, d)$, i.e.,

$$\begin{aligned}
\mathcal{L} = &\{ vF \times [0,\infty) \times \{0\} \mid F \triangleleft K \} \\
&\cup \{ F \times [0,\infty) \times F^* \mid F \prec K \} \\
&\cup \{ vF \times \{0\} \times F^* \mid F \prec K \}.
\end{aligned} \tag{3.1.6}$$

LEMMA 3.1.1. Any n-cell $X \times Y$ of $\bar{\mathcal{L}}$ derived from (3.1.6) satisfies the conditions (1) and (2) of Lemma 2.2.

Proof. Let $X \times Y$ be an n-cell of $\bar{\mathcal{L}}$, we consider the number of cells Z in such that $X \triangleleft Z$ and $Z^d = Y$. Since the cell $Z \times Z^d$ is an $(n+1)$-cell in $\mathcal{L}$, Z^d is either F^* for some proper face F of K or $\{0\}$. First we consider the case where $Y = Z^d = F^*$ for some proper face F of K. By the dual operator d there are two cells Z in $\bar{\mathcal{P}}$ mapped to F^* as defined in (3.1.5). The two cells must be equal to $vF \times \{0\}$ and $F \times [0, \infty)$. Therefore at most two cells satisfy (2.2). Moreover if X is the common facet of the two cells $vF \times \{0\}$ and $F \times [0, \infty)$, X must be equal to $F \times \{0\}$ and hence $X^d = \emptyset$. Second, we consider the case where $Y = Z^d = \{0\}$. Then X is of n dimension. Since $\mathcal{P}$ is a subdivided manifold and X is an n-cell of $\bar{\mathcal{P}}$, there are at most two $(n+1)$-cells of $\mathcal{P}$ containing X as their facet. By the definition (3.1.5) of the dual operator, every $(n+1)$-cell in $\mathcal{P}$ is mapped to $\{0\}$, therefore there are at most two cells of $\mathcal{P}$ satisfying (2.2). Moreover, suppose $vF_1 \times [0, \infty)$ and $vF_2 \times [0, \infty)$ for some facets F_1 and F_2 of K are two cells satisfying (2.2). Then X must be $vE \times [0, \infty)$, where E is the common facet of F_1 and F_2. Note that $X^d = \emptyset$ in this case. Therefore we have shown that the two conditions of Lemma 2.2 are satisfied. □

Thus we obtain that $\mathcal{L}$ is a subdivided $(n+1)$-manifold as an immediate result of Lemma 2.2. By applying Lemma 2.3 to the GPDM$(\mathcal{P}, \mathcal{D}, d)$ considered here we have the following lemma.

LEMMA 3.1.2.

(i) The boundary of $\mathcal{L}$ is given by

$$\partial \mathcal{L} = \{ \{v\} \times \{0\} \times F^* \mid F \prec K, \dim F = 0 \} \tag{3.1.7}$$

and

$$|\partial \mathcal{L}| = \{v\} \times \{0\} \times R^n. \tag{3.1.8}$$

(ii) $|\mathcal{L}|$ is closed.

Proof. (i) follows immediately from Lemma 2.3.

(ii). By the construction (3.1.6) of $\mathcal{L}$ each cell in $\mathcal{L}$ is closed. Then $|\mathcal{L}|$, the union of finite closed cells of $\mathcal{L}$, is also closed. □

Now let T be a triangulation of $K \times [0, \infty)$ such that the restriction $T|X$ also triangulates X for each cell X of $\bar{\mathcal{P}}$, where $T|X$ is defined by

$$T|X = \{ \sigma \in \bar{T} \mid \sigma \subseteq X, \dim \sigma = \dim X \}.$$

For each t let

$$L(t) = \{ \sigma \mid \sigma \in T, \sigma \subseteq K \times [0, t] \}$$
$$U(t) = \{ \sigma \mid \sigma \in T, \sigma \cap (K \times [t, \infty)) \neq \emptyset \}.$$

ASSUMPTION. The triangulation T satisfies the following conditions:

(i) $L(t)$ has finite many cells for each $t > 0$,

(ii) $\delta = \sup\{\text{diam } \sigma \mid \sigma \in U(t)\}$ tends to zero as t goes to infinity,

where $\text{diam } B = \sup\{\ \|z^1 - z^2\| \mid z^1, z^2 \in B\ \}$ for a set B.

We define

$$\mathcal{M} = \{\ \sigma \times X^d \mid \sigma \in T|X,\ X \in \bar{\mathcal{P}},\ X^d \neq \emptyset\ \}.$$

It is easy to show that

$$|\mathcal{M}| = |\mathcal{L}| \tag{3.1.9}$$

and

$$|\partial\mathcal{M}| = |\partial\mathcal{L}| = \{v\} \times \{0\} \times R^n. \tag{3.1.10}$$

Let $\phi(x,t) = f(x)$ for each $t \in [0,\infty)$. Let Φ be the *pl* approximation of ϕ with respect to the triangulation T. We define a function $g : |\mathcal{M}| \to R^n$ by

$$g(x,t,y) = \Phi(x,t) - y$$

for each $(x,t,y) \in |\mathcal{M}|$. Then the function g is a *pl* function on $\mathcal{M}$. We consider the system of *pl* equations

$$g(x,t,y) = 0, \quad (x,t,y) \in |\mathcal{M}| \tag{3.1.11}$$

as the basic model of our continuous deformation algorithm. By applying Theorem 2.1 to (3.1.11) we have the following theorem.

THEOREM 3.1.3. Suppose the starting point v in K is not a stationary point. Then $(v,0,\phi(v,0))$ lies in $g^{-1}(0) \cap |\partial\mathcal{M}|$. Suppose further that $0 \in R^n$ is a regular value of the function $g : |\mathcal{M}| \to R^n$. Then the connected component S of $g^{-1}(0)$ containing $(v,0,\phi(v,0))$ is an unbounded path. Moreover for any $t > 0$ there exists a point $(x,t,\Phi(x,t))$ lying on S such that x is a stationary point of the *pl* approximation $\Phi(\cdot,t)$ of $\phi(\cdot,t)$.

Proof. Since the starting point v is not a stationary point, $f(v) \neq 0$. It should be noted that $(v,0)$ is a vertex of the triangulation T and hence $\Phi(v,0) = \phi(v,0)$. We obtain from (3.1.11) that $(v,0,\phi(v,0)) \in g^{-1}(0)$. Moreover we see from (3.1.10) that $(v,0,\phi(v,0))$ lies in $|\partial\mathcal{M}|$. By (3) of Theorem 2.1 the connected component S of $g^{-1}(0)$ containing $(v, 0, \phi(v, 0))$ is a path. If the path S is bounded, then according to (2) of Theorem 2.1 the other end point of the path, say (x, t, y), must lie in $|\partial \mathcal{M}| = \{v\} \times \{0\} \times R^n$. This implies $(x, t) = (v, 0)$. Since $(x, t, y) \in g^{-1}(0)$, $y = \Phi(x, t) = \Phi(v, 0) = \phi(v, 0)$. Hence we have $(x, t, y) = (v, 0, \phi(v, 0))$, which contradicts (1) of Theorem 2.1. Therefore the path S is homeomorphic to $[0,1)$ and is unbounded by (3.1.9) and (4) of Theorem 2.1. Since the variable x moves in the compact set K and $\Phi(x, t)$ is a continuous function in x, neither x nor y of points on S can be unbounded. This means that t grows unboundedly along the path S. Since we assume that the number of the cells in $L(t)$ for each $t > 0$ is finite, we see that the path S contains a point (x, t, y) for any $t > 0$. Therefore the point (x, t, y) satisfies either $(x, t) \in vF \times [0, \infty)$, $\Phi(v, t) = y = 0$ for some facet F of K or $(x, t) \in F \times [0, \infty)$, $\Phi(x, t) = y \in F^*$ for some proper face F of K. In both cases the point x is a stationary point of the *pl* approximation $\Phi(\cdot, t)$ of $\phi(\cdot, t)$. □

We know from the above theorem that for any $t > 0$ the point x of (x, t, y) on the path S is a stationary point of the *pl* approximation $\Phi(\cdot, t)$ of $\phi(\cdot, t)$. If $\phi(x, t) = f(x)$ happens to lie in F^* for some face F of K containing the point x, then x is a stationary point of f. Otherwise, it is only an approximate stationary point. If the distance between $f(x)$ and F^* is not satisfactorily small, we continue tracing the path to obtain an approximate stationary point with a higher accuracy. In the following lemma we discuss the accuracy of an approximate stationary point. As norm we use the Euclidean norm in R^n. We define the projection $p(x, t)$ of a point $(x, t) \in K \times [0, \infty)$ on K by $p(x, t) = x$. For a simplex σ of T we denote the set $\{ x \mid (x, t) \in \sigma \text{ for some } t \in [0, \infty) \}$ by $p(\sigma)$.

LEMMA 3.1.4. Let $\gamma(t) = \sup\{ \operatorname{diam} f(p(\sigma)) \mid \sigma \in U(t) \}$. Let x be an approximate stationary point in a face F of K obtained by the algorithm at t. Then $f(x)$ lies in the $\gamma(t)$-neighborhood of F^*, i.e., there is a $y \in F^*$ such that $\|y - f(x)\| \leq \gamma(t)$.

Proof. Let $(x^1, t^1), \ldots, (x^{k+1}, t^{k+1})$ be the vertices of a k-simplex of T containing the point (x, t). Then $\Phi(x, t) = \sum_{j=1}^{k+1} \lambda_j \phi(x^j, t^j)$, where $\lambda_1, \ldots, \lambda_{k+1}$ are convex combination coefficients such that $(x, t) = \sum_{j=1}^{k+1} \lambda_j (x^j, t^j)$. Therefore

$$
\begin{aligned}
\|\Phi(x,t) - f(x)\| &= \|\Phi(x,t) - \phi(x,t)\| \\
&= \|\sum_{j=1}^{k+1} \lambda_j \phi(x^j, t^j) - \phi(x,t)\| \\
&= \|\sum_{j=1}^{k+1} \lambda_j (\phi(x^j, t^j) - \phi(x,t))\| \\
&= \|\sum_{j=1}^{k+1} \lambda_j (f(x^j) - f(x))\| \\
&\leq \gamma(t).
\end{aligned}
$$
□

Let $(x(t), t, y(t))$ be a point on the path S for $t = 1, 2, \ldots$. Since the polytope K is compact, the sequence $x(t)$ has a cluster point $\bar{x}$ in K. For simplicity we assume that this sequence itself converges to $\bar{x}$. Since there is only a finite number of faces of K, there is a face F of K containing the whole sequence. By the construction of the triangulation T, the error $\gamma(t)$ converges to zero as t goes to infinity. Therefore we have that $\bar{x} \in F$ and $f(\bar{x}) \in F^*$ by the closedness of F and F^*.

3.2. Following the Path

We will first describe the path S of solutions of (3.1.11). When $t = 0$, the system (3.1.11) is identical with the system of the simplicial variable dimension algorithm proposed by Talman and Yamamoto [13] by virtue of the construction of $\mathcal{M}$ (see (3.1.6) and the definition of $\mathcal{M}$). Therefore the path S starts with the point $(v, 0, \phi(v, 0)) = (v, 0, f(v))$ for the given point v of K, and leads to a point $(x, 0, \phi(x, 0))$ such that x is an approximate stationary point with respect to the triangulation $T|(K \times \{0\})$. Then S begins to leave the level $t = 0$. The variable t is not necessarily monotonically increasing along the path S and so it may decrease down to zero. Once t vanishes, the path S connects two distinct points $(x', 0, \phi(x', 0))$ and $(x'', 0, \phi(x''\ 0))$ such that x' and x'' are approximate stationary points with respect to $T|(K \times \{0\})$ while t is kept to be zero. Note that neither of these points coincides with x, otherwise S would intersect itself. The path S again begins to leave the level $t = 0$ with the lastly obtained point $(x'', 0, \phi(x'', 0))$.

The system (3.1.11) can be written as

$$g(x,t,y) = \Phi(x,t) - y = 0, \quad (x,t,y) \in \sigma \times X^d, \tag{3.2.1}$$

where σ is a k-simplex of $T|X$ and X is a cell in $\bar{\mathcal{P}}$ such that $X^d \neq \emptyset$. Let $w^1, \ldots, w^{k+1}$ be the vertices of the simplex σ. There exist nonnegative numbers λ_j, $j = 1, \ldots, k+1$, such that $(x,t) = \sum_{j=1}^{k+1} \lambda_j w^j$ and $\sum_{j=1}^{k+1} \lambda_j = 1$. Moreover, if $X^d = F^*$ for some proper face F of K, then there exist nonnegative numbers μ_i, $i \in I(F)$, such that $y = \sum_{i \in I(F)} \mu_i a^i$. Note that $I(F)$ consists of $n+1-k$ indices. We have the following result.

$g^{-1}(0)$ intersects $\sigma \times X^d$ for some k-simplex σ in $T|X$ if and only if the system (3.2.2) has a solution (λ, μ),

$$\begin{aligned} &\sum_{j=1}^{k+1} \lambda_j \begin{pmatrix} \phi(w^j) \\ 1 \end{pmatrix} - \sum_{i \in \bar{I}} \mu_i \begin{pmatrix} a^i \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \\ &\lambda_j \geq 0 \quad \text{for all} \quad j = 1, \ldots, k+1, \\ &\mu_i \geq 0 \quad \text{for all} \quad i \in \bar{I}, \end{aligned} \tag{3.2.2}$$

where $\bar{I}$ is given by

$$\bar{I} = \begin{cases} I(F) & \text{if either} \quad X = vF \times \{0\} \quad \text{or} \quad F \times [0,\infty) \\ \emptyset & \text{if} \quad X = vF \times [0,\infty) \quad \text{for some facet } F. \end{cases}$$

Note that if $X = vF \times \{0\}$ or $X = F \times [0,\infty)$, $k = \dim X = \dim F + 1$ and if $X = vF \times [0,\infty)$ for some facet F of K, $k = \dim X = n+1$. A line segment of solutions (λ, μ) to the system (3.2.2) corresponds to a linear piece of the path S and can be followed by a linear programming (abbreviated by l.p.) pivoting step in the system (3.2.2). At the starting point $(v, 0, \phi(v,0))$ we apply the variable dimension restart algorithm. First we have to find a simplex σ and a cone F^* such that $(v, 0, \phi(v,0)) \in \sigma \times F^*$. It is known that $(v, 0, \phi(v,0))$ lies in $vF \times \{0\} \times F^*$ for some vertex F of K. Assume that v is not a stationary point, we define σ to be the 1-dimensional simplex of $T|(vF \times \{0\})$ containing the starting point $(v,0)$ as a facet, i.e., $\sigma = \{(v,0),(v',0)\}$ with $(v',0)$ a vertex in $T|(vF \times \{0\})$. Then we leave the starting point $(v,0)$ along the path by pivoting into the system

$$\begin{aligned} &\lambda_1 \begin{pmatrix} \phi(v,0) \\ 1 \end{pmatrix} - \sum_{i \in \bar{I}} \mu_i \begin{pmatrix} a^i \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \\ &\lambda_1 \geq 0, \\ &\mu_i \geq 0 \quad \text{for} \quad i \in \bar{I}, \end{aligned}$$

the column $\binom{\phi(v',0)}{1}$ with a nonnegative variable λ_2, where $\bar{I}$ is the index set of binding constraints at the vertex F. In general, the algorithm traces the path of solutions to the system

$$\begin{aligned} &\sum_{j=1}^{k+1} \lambda_j \begin{pmatrix} \phi(v^j,0) \\ 1 \end{pmatrix} - \sum_{i \in \bar{I}} \mu_i \begin{pmatrix} a^i \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \\ &\lambda_j \geq 0 \quad \text{for} \quad j = 1, \ldots, k+1, \\ &\mu_i \geq 0 \quad \text{for} \quad i \in \bar{I}, \end{aligned} \tag{3.2.3}$$

where k is the dimension of some simplex in $T|(vF \times \{0\})$ and $\bar{I} = I(F)$. After a finite number of replacement steps and pivoting steps in the system (3.2.3) we find a point $(x, 0, y)$ in $(vF \times \{0\} \times F^*) \cap g^{-1}(0)$ such that

$$(x, 0, y) \in \tau \times F^* \subseteq F \times \{0\} \times F^* \quad \text{for some face } F \text{ of } K$$

or

$$(x, 0, y) \in \tau \times \{0\} \subseteq vF \times \{0\} \times \{0\} \quad \text{for some facet } F \text{ of } K.$$

It is obvious that the point x in each case is an approximate stationary point and $(x, 0, y)$ lies in the boundary of the cell $vF \times \{0\} \times F^*$.

Suppose the first case occurred. Since $F \times \{0\}$ is also a facet of another cell $F \times [0, \infty)$ in $\bar{P}$, there is a simplex σ in $F \times [0, \infty)$ containing τ as a facet. We continue the algorithm by making an l.p. pivoting step in the system (3.2.3) or (3.2.2) with $\binom{\phi(w)}{1}$, where w is the vertex of σ opposite the facet τ.

Suppose the second case occurred, i.e., $\bar{I} = \emptyset$. The system (3.2.3) becomes

$$\begin{aligned} &\sum_{j=1}^{n+1} \lambda_j \begin{pmatrix} \phi(v^j,0) \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \\ &\lambda_j \geq 0 \quad \text{for} \quad j = 1, \ldots, n+1. \end{aligned}$$

Since $vF \times \{0\}$ is a facet of $vF \times [0, \infty)$ in P, there is a simplex σ in $vF \times [0, \infty)$ containing τ as a facet. The algorithm is continued by pivoting the column $\binom{\phi(w)}{1}$ with a nonnegative variable λ_{n+2} into the above system, where w is the vertex of σ opposite the facet τ.

Until now, we have described how to trace the path S when $t = 0$. Then the path S begins to leave the level $t = 0$, and it corresponds to the solutions of the system (3.2.2) for simplices in T. If we assume nondegeneracy, since the solutions to the system (3.2.2) form a line segment, one of the variables (λ, μ) vanishes at an end point of the line segment. We first consider the case where $\mu_h = 0$ for some $h \in \bar{I}$, then we are at the end point of the line segment in the cell $F \times [0, \infty) \times F^*$ for some proper face F. Suppose $\bar{I} \setminus \{h\} = \emptyset$. Since F is a facet of K, the cell $F \times [0, \infty) \times \{0\}$ is the facet of two $(n+1)$-cells $F \times [0, \infty) \times F^*$ and $vF \times [0, \infty) \times \{0\}$. Therefore there is a simplex in $vF \times [0, \infty)$ such that σ is its facet. The path is followed by making an l.p. pivoting step with $\binom{\phi(\bar{w})}{1}$ in the system (3.2.2) and taking $\bar{I} = \emptyset$, where $\bar{w}$ is the vertex in the new generated simplex opposite σ. Next suppose $\bar{I} \setminus \{h\} \neq \emptyset$. Let E be the face of K such that $I(E) = I(F) \setminus \{h\}$. Therefore σ is a facet of a unique simplex in $E \times [0, \infty)$, and the algorithm is continued by making an l.p. pivoting step with $\binom{\phi(\bar{w})}{1}$ in the system (3.2.2) and taking $\bar{I} = I(E)$, where $\bar{w}$ is the vertex in the new generated simplex opposite σ.

We now consider the case where at an end point of the line segment $\lambda_h = 0$ for some $h \in \{1, \ldots, k+1\}$. Then $(x, t) = \sum_{j=1}^{k+1} \lambda_j w^j$ lies in some facet τ opposite the vertex w^h of σ in some cell X of $\bar{P}$. This facet lies either in the boundary ∂X of X or is a facet of just one other k-simplex of $T|X$ with vertices $\bar{w}^j = w^j$, $j \neq h$ and $\bar{w}^h \neq w^h$. Then in the latter case we make an l.p. pivoting step in the system (3.2.2) with $\binom{\phi(\bar{w}^h)}{1}$ to follow the path S.

Suppose that τ lies in the boundary ∂X of X and $X = vF \times [0, \infty)$ for some facet F of K, then $\bar{I} = \emptyset$. The boundary ∂X of X consists of $F \times [0, \infty)$, $vF \times \{0\}$ and $vE \times [0, \infty)$ for some $E \triangleleft F$. Suppose that τ lies in $F \times [0, \infty)$ or $vF \times \{0\}$. We make a pivot step with $\binom{a^i}{0}$ in the system (3.2.2) and replace $\bar{I}$ with $\bar{I} \cup \{i\}$, where i is the index of the unique binding constraint at the facet F. Furthermore in the latter case, i.e., τ lies in $vF \times \{0\}$, we see that t vanishes and the set $\tau \times F^*$ belongs to $vF \times \{0\} \times F^*$. After the pivoting step we will trace a path of solutions to the system (3.2.3) by the variable dimension algorithm. Suppose that τ lies in $vE \times [0, \infty)$ for some facet E of F. Then there is another $(n+1)$-simplex in the cell $vF' \times [0, \infty)$ with vertices $\bar{w}^j = w^j$, $j \neq h$, and $\bar{w}^h \neq w^h$, where F shares with F the common facet E. The path is followed by making an l.p. pivoting step with $\binom{\phi(\bar{w}^h)}{1}$ in the system (3.2.2).

Suppose τ lies in the boundary ∂X of X and $X = F \times [0, \infty)$ for some face F of K. Then $\partial X = \{\, F \times \{0\} \,\} \cup \{\, E \times [0, \infty) \mid \text{for some } E \triangleleft F \,\}$. Suppose first that $\bar{\tau}$ lies in $F \times \{0\}$, then there is a unique k-simplex $\bar{\sigma}$ in $vF \times \{0\}$ containing $\bar{\tau}$ as a facet. We compute ϕ at the vertex $\bar{w}$ in $\bar{\sigma}$ opposite $\bar{\tau}$ and follow the path by making an l.p. pivoting step with $\binom{\phi(\bar{w})}{1}$ in the system (3.2.2). Then we see that $t = 0$ and the set $\bar{\sigma} \times F^* \subseteq vF \times \{0\} \times F^*$. We will again trace a path of solutions to the system (3.2.3) by the variable dimension algorithm. Suppose second that $\bar{\tau}$ lies in $E \times [0, \infty)$ for some facet E of F. Then the path S is followed by pivoting $\binom{a^i}{0}$ into the system (3.2.2) and replacing $\bar{I}$ with $\bar{I} \cup \{i\}$, where i is the unique index in $I(E)$ not in $I(F)$.

When t decreases to zero, the path we are tracing never returns to the starting point $(v, 0, \phi(v, 0))$ but provides a stationary point, say x, of $\Phi(\cdot, 0)$ by the virtue of the boundary structure of $\mathcal{L}$. Then we start to trace the path of solutions of (3.2.3) by the variable dimension algorithm to obtain another approximate stationary point x^1. It should be noted that x^1 is different from x because otherwise the path of zeros of g would intersect itself. Then we continue to follow the path of solution to the system (3.2.2) with the steps described above.

4. Practical Implementation of the Continuous Deformation Algorithm

We have seen in Section 3 that if the triangulation of $K \times [0, \infty)$ is constructed, the parameter t grows as we trace the path of solutions of (3.2.1) starting from $(v, 0, \phi(v, 0))$. After finitely many iterations an approximate stationary point will be obtained with the accuracy as mentioned in Lemma 3.1.4. It is not practical, however, to construct the whole triangulation in advance and keep it in memory from the view point of implementation. In this section we will show that the combination of a primal-dual pair of subdivided manifolds on K and a pair of triangulations of K provides a structure to carry out the continuous deformation algorithm with t between some integer l and $l+1$. Thus we have only to pile up the structure to implement the continuous deformation algorithm in Section 3. However, it is not yet practical to keep the structure of lower levels in case that the path descends the level. Then we will propose an amalgamation of continuous deformation and restart.

Let H be an n-dimensional polytope of R^n. For each face G of H we choose a point denoted by $c(G)$ from the relative interior of G. For a sequence G_o, $G_1, \ldots, G_k$ of faces of H such that

$$G_o \prec G_1 \prec \cdots \prec G_k,$$

we denote by $\Delta(G_o, \ldots, G_k)$ the convex hull of points $c(G_i)$ for $i = 0, \ldots, k$. Clearly these points $c(G_i)$ are affinely independent and hence $\Delta(G_o, \ldots, G_k)$ is a k-dimensional simplex. Let

$$\mathcal{P}_H = \{ \ \Delta(G_o, \ldots, G_n) \mid \dim G_o = 0, \ G_o \lhd \cdots \lhd G_n = H \ \}. \tag{4.1}$$

Then $\mathcal{P}_H$ is a subdivided n-manifolds and

$$|\mathcal{P}_H| = H. \tag{4.2}$$

In fact, for a given point x of H generate the following sequences of points x^i, real numbers γ_i and faces H_i:

$$\begin{aligned}
&x^o := x\,, \\
&H_o := H\,, \\
&\gamma_i := \max\{ \ \gamma \mid x^i + \gamma(x^i - c(H_i)) \in H_i \ \} && \text{for } i = 0, \ldots, n-1\,, \\
&x^{i+1} := x^i + \gamma_i(x^i - c(H_i)) && \text{for } i = 0, \ldots, n-1\,, \\
&H_{i+1} \text{ is a facet of } H_i \text{ containing } x^{i+1} && \text{for } i = 0, \ldots, n-1\,,
\end{aligned}$$

where when x^i happens to coincide with $c(H_i)$ for some i, $H_{i+1}, \ldots, H_n$ are arbitrarily chosen faces of H_i such that $H_n \lhd \cdots \lhd H_{i+1} \lhd H_i$ and $\gamma_i = \gamma_{i+1} = \cdots = \gamma_{n-1} = +\infty$. Then

$$x = (\prod_{j=o}^{n-1} \frac{1}{1+\gamma_j})c(H_n) + \sum_{i=o}^{n-1}(\prod_{j=o}^{i} \frac{1}{1+\gamma_j})\gamma_i c(H_i),$$

which shows that $x \in \Delta(H_n, \ldots, H_o)$. Let $\mathcal{D}_H$ be the subdivided n-manifold consisting of a single cell H. Introducing the dual operator d from $\bar{\mathcal{P}}_H$ into $\bar{\mathcal{D}}_H \cup \{\emptyset\}$ such that

$$\begin{aligned}
(\Delta(G_o, \ldots, G_k))^d &= G_o \quad \text{if} \quad G_o \lhd G_1 \lhd \cdots \lhd G_k = H, \\
&= \emptyset \quad \text{otherwise.}
\end{aligned}$$

We define

$$\begin{aligned}
\mathcal{L}_H &= \{ \ X^d \times X \mid X \in \bar{\mathcal{P}}_H, \ X^d \neq \emptyset \ \} \\
&= \{ \ G_o \times \Delta(G_o, \ldots, G_k) \mid G_o \lhd \cdots \lhd G_k = H \ \}.
\end{aligned} \tag{4.3}$$

Then we obtain the following lemma.

LEMMA 4.1. $\mathcal{L}_H$ is a subdivided n-manifolds,

$$\bar{\mathcal{L}}_H = \{ \ G \times \Delta(G_o, \ldots, G_k) \mid G \preceq G_o \prec \cdots \prec G_k \preceq H \ \}$$

and

$$\partial \mathcal{L}_H = \{ \ G_o \times \Delta(G_o, \ldots, G_k) \mid G_o \lhd \cdots \lhd G_k \lhd H \ \}.$$

Proof. It is straightforward to show that $\mathcal{L}_H$ satisfies the two conditions of Lemma 2.2. The other statements are also readily seen from the construction of $\mathcal{L}_H$. □

To extend the above structure $\mathcal{L}_H$ to the whole polytope K, we first make a subdivided n-manifold $\mathcal{K}$ which subdivides K into n-dimensional cells. For each cell G of $\bar{\mathcal{K}}$, we choose a point $c(G)$ from its relative interior. If we make the subdivided manifold $\mathcal{L}_H$ for each cell of $\mathcal{K}$ by using these points, the union $\mathcal{L}_{\mathcal{K}}$ of $\mathcal{L}_H$ over all cells H of $\mathcal{K}$ again forms a subdivided manifold. We readily see

$$\bar{\mathcal{L}}_{\mathcal{K}} = \bigcup_{H \in \mathcal{K}} \bar{\mathcal{L}}_H$$

and

$$\partial \mathcal{L}_{\mathcal{K}} = \{ \ G_o \times \Delta(G_o, \ldots, G_k) \mid G_o \lhd \cdots \lhd G_k \lhd H \quad \text{for some} \quad H \in \mathcal{K} \text{ and } G_k \text{ lies in some facet of } K \ \}. \tag{4.4}$$

When we use $\{K\}$ as the subdivision $\mathcal{K}$, i.e., K is not subdivided, $\mathcal{L}_{\mathcal{K}} = \mathcal{L}_K$ and

$$\partial \mathcal{L}_K = \{ \ F_o \times \Delta(F_o, \ldots, F_k) \mid F_o \lhd \cdots \lhd F_k \lhd K \ \},$$

when a triangulation of K is used as $\mathcal{K}$, $\mathcal{L}_{\mathcal{K}}$ is closely related to the triangulation proposed by Doup [2, Chapter 12]. Now we further introduce another dual operator δ from $\bar{\mathcal{L}}_{\mathcal{K}}$ into the subdivided n-manifold $\{ \ F^* \mid F \preceq K \ \}$. Namely, for a cell $G_o \times \Delta(G_o, \ldots, G_k)$ of $\bar{\mathcal{L}}_{\mathcal{K}}$ such that $G_o \lhd \cdots \lhd G_k$, let F be the minimal face of K which contains G_k. Then when $\dim F = \dim G_k$ we assign $G_o \times \Delta(G_o, \ldots, G_k)$ to the normal cone F^* of F by δ. The dual operator δ assigns the other cells of $\bar{\mathcal{L}}_{\mathcal{K}}$ to the empty set. With this dual operator δ we define

$$\begin{aligned}
\mathcal{M}_{\mathcal{K}} =& \{ \ (X_o \times X_1) \times (X_o \times X_1)^{\delta} \mid X_o \times X_1 \in \bar{\mathcal{L}} \quad \text{and} \quad (X_o \times X_1)^{\delta} \neq \emptyset \ \} \\
=& \{ \ G_o \times \Delta(G_o, \ldots, G_k) \times F^* \mid G_o \lhd \cdots \lhd G_k \preceq H \quad \text{for some} \quad H \in \mathcal{K}, \\
& F \quad \text{is the minimal face of } K \text{ containing } G_k \\
& \text{and } \dim F = \dim G_k \ \}.
\end{aligned} \tag{4.5}$$

When $\mathcal{K} = \{K\}$, $\mathcal{M}_{\mathcal{K}}$ is simplified as

$$\mathcal{M}_{\mathcal{K}} = \{ F_o \times \Delta(F_o, \ldots, F_k) \times F_k^* \mid F_o \lhd \cdots \lhd F_k \preceq K \}.$$

LEMMA 4.2. $\mathcal{M}_{\mathcal{K}}$ is a subdivided n-manifold without boundary.

Proof. We give a sketch of the proof. Choose an $(n\text{-}1)$-cell, say $X_o \times X_1 \times Y$, from $\bar{\mathcal{M}}_{\mathcal{K}}$. Then it is a facet of some n-cell $G_o \times \Delta(G_o, \ldots, G_k) \times F^*$ of $\mathcal{M}_{\mathcal{K}}$. Then there are five cases:

(1) $X_o \lhd G_o$, $X_1 = \Delta(G_o, \ldots, G_k)$ and $Y = F^*$,

(2) $X_o = G_o$, $X_1 = \Delta(G_o, \ldots, G_{k-1})$ and $Y = F^*$,

(3) $X_o = G_o$, $X_1 = \Delta(G_o, \ldots, G_{i-1}, G_{i+1}, \ldots, G_k)$ and $Y = F^*$,

(4) $X_o = G_o$, $X_1 = \Delta(G_1, \ldots, G_k)$ and $Y = F^*$,

(5) $X_o = G_o$, $X_1 = \Delta(G_o, \ldots, G_k)$ and $Y \lhd F^*$.

In each case we see that the cell $X_o \times X_1 \times Y$ is also a facet of the following cell.

(1) $X_o \times \Delta(X_o, G_o, \ldots, G_k) \times F^*$,

(2) $G_o \times \Delta(G_o, \ldots, G_{k-1}, G_k') \times F^*$ when G_{k-1} is not contained in a proper face of F, where G_k' is a face of some cell H' of $\mathcal{K}$ such that $G_k' \subseteq F$ and $G_{k-1} \lhd G_k'$,

$G_o \times \Delta(G_o, \ldots, G_{k-1}) \times (F')^*$ when G_{k-1} is contained in a proper face F' of F,

(3) $G_o \times \Delta(G_o, \ldots, G_{i-1}, G_i', G_{i+1}, \ldots, G_k) \times F^*$, where G_i' is another facet of G_{i+1} containing G_{i-1}. Since the boundary of the face G_{i+1} is a subdivided manifold, G_i' is unique.

(4) $G_1 \times \Delta(G_1, \ldots, G_k) \times F^*$,

(5) $G_o \times \Delta(G_o, \ldots, G_k, G_{k+1}) \times (F')^*$, where F' is a face of K whose normal cone $(F')^*$ coincides with Y and G_{k+1} is a face of H such that $G_k \lhd G_{k+1} \subseteq F'$.

It sho ıld be noted that the cells of (1) to (5) are unique an . different from $G_o \times \Delta(G_o, \ldots, G_k) \times F^*$. Therefore we have seen that $\mathcal{M}_{\mathcal{K}}$ ·s a subdivided manifold without boundary. □

Let us define

$$\mathcal{N}_{\mathcal{K}} = \mathcal{M}_{\mathcal{K}} \times [0, 1],$$

then $\partial\mathcal{N}_K = \mathcal{M}_K \times \{0,1\}$ because $\partial\mathcal{M}_K = \emptyset$. Let (x_o, x_1, y, t) be a point of $|\mathcal{N}_K|$, that is

$$(x_o, x_1, y, t) \in G_o \times \Delta(G_o, \ldots, G_k) \times F^* \times [0,1]$$

for some sequence $G_o, \ldots, G_k$ of faces of some cell $H \in K$ and some face F of K such that $G_o \lhd \cdots \lhd G_k \preceq H$ and F is the minimal face of K containing G_k. Let T_o be a triangulation of K which triangulates each cell H of K and let T_1 be a triangulation of K which triangulates each $\Delta(G_o, \ldots, G_k)$ for each cell H of K. Let us further denote by $\Phi_o(\cdot, 0)$ and $\Phi_1(\cdot, 1)$ the piecewise linear approximations of f with respect to T_o and T_1, respectively. Then the basic system for tracing the path of solutions of (3.1.11) for t between 0 and 1 is as follows:

$$(1-t)\Phi_o(x_o, 0) + t\Phi_1(x_1, 1) - y = 0,$$
$$(x_o, x_1, y, t) \in G_o \times \Delta(G_o, \ldots, G_k) \times F^* \times [0,1]. \quad (4.6)$$

Let σ_o be a simplex of T_o which contains x_o and let σ_1 be a simplex of T_1 which contains x_1. Then by the definition of piecewise linear approximation,

$$\Phi_o(x_o, 0) = \sum_{j=1}^{k_o+1} \lambda_{oj} f(u_{oj}), \quad \Phi_1(x_o, 1) = \sum_{j=1}^{k_1+1} \lambda_{1j} f(u_{1j}),$$

where $u_{o1}, \ldots, u_{ok_o+1}$ are the vertices of σ_o, $u_{11}, \ldots, u_{1k_1+1}$ are the vertices of σ_1, and λ_{oj} and λ_{1j} are nonnegative coefficients satisfying

$$x_o = \sum_{j=1}^{k_o+1} \lambda_{oj} u_{oj}, \quad \sum_{j=1}^{k_o+1} \lambda_{oj} = 1,$$
$$x_1 = \sum_{j=1}^{k_1+1} \lambda_{1j} u_{1j}, \quad \sum_{j=1}^{k_1+1} \lambda_{1j} = 1.$$

Thus (4.6) has a solution (x_o, x_1, y, t) if and o ıly if there is a nonnegative solution $(\nu_{o1}, \ldots, \nu_{ok_o+1}, \nu_{11}, \ldots, \nu_{1k_1+1}, \mu_i (i \in I))$ of the linear system

$$\sum_{j=1}^{k_o+1} \nu_{oj} \begin{pmatrix} f(u_{oj}) \\ 1 \end{pmatrix} + \sum_{j=1}^{k_1+1} \nu_{1j} \begin{pmatrix} f(u_{1j}) \\ 1 \end{pmatrix} - \sum_{i \in I} \mu_i \begin{pmatrix} a^i \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad (4.7)$$

where I is the index set of binding constraints at the face F.

In order to make a subdivided manifold on which the continuous deformation algorithm works, we have only to pile up the manifold $\mathcal{N}_K$. More precisely, let K_l be a subdivision of K and T_l be a triangulation of K for $l = 0, 1, \ldots$ such that

(i) T_o subdivides both $\{ vF \mid F \preceq K \}$ and K_o;

(ii) T_l subdivides K_l for $l = 1, 2, \ldots$;

(iii) T_l is finer than T_{l-1} for $l = 1, 2, \ldots$ in the sense that

$$\max_{\sigma \in T_l} \text{diam } \sigma \leq r \cdot \max_{\sigma \in T_{l-1}} \text{diam } \sigma$$

for some positive r less than 1. A refinement of K_{l-1} is typically chosen as K_l. Let us abbreviate $\mathcal{M}_{K_l}$ by $\mathcal{M}_l$. and let $\mathcal{N}_l = \mathcal{M}_l \times [l, l+1]$. After finding an approximate stationary point x and a simplex τ of T_o which contains x by applying the variable dimension algorithm, we start to trace a path of solutions to the system

$$(1-t)\Phi(x_o, 0) + t\Phi(x_1, 1) - y = 0, \quad (x_o, x_1, y, t) \in \mathcal{N}_o. \tag{4.8}$$

In general, we trace a path of solutions to

$$((l+1)-t)\Phi(x_l, l) + (t-l)\Phi(x_{l+1}, l+1) - y = 0, \; (x_l, x_{l+1}, y, t) \in \mathcal{N}_l. \tag{4.9}$$

When t vanishes in tracing the path of (4.8), we will obtain another approximate stationary point, say x. Let τ be the minimal simplex of T_o containing x and let F be the minimal face of K containing τ. Then we have a solution (ν, μ) of

$$\begin{aligned} &\sum_{j=1}^{k+1} \nu_{oj} \begin{pmatrix} f(u_{oj}) \\ 1 \end{pmatrix} - \sum_{i \in I} \mu_i \begin{pmatrix} a^i \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \\ &\nu_{oj} \geq 0 \quad \text{for} \quad j = 1, \ldots, k+1, \\ &\mu_i \geq 0 \quad \text{for} \quad i \in I, \end{aligned} \tag{4.10}$$

where k is the dimension of τ and I is the index set of constraints binding at F. There are two possible cases. The first case is that $F = K$. From the assumption that $\dim K = n$, the normal cone F^* is $\{0\}$, which implies that (4.10) reduces to

$$\sum_{j=1}^{k+1} \nu_{oj} \begin{pmatrix} f(u_{oj}) \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$
$$\nu_{oj} \geq 0 \quad \text{for} \quad j = 1, \ldots, k+1. \tag{4.11}$$

We then find a facet, say F', of K such that the set vF' contains τ. Since K is a simple polytope and the linear inequalities defining K are nonredundant, there is a unique inequality

$$a^i \cdot x \leq b_i$$

binding at the facet F'. We introduce the column $\binom{a^i}{0}$ into the system (4.11) together with the nonnegative variable μ_i. The second case is that F is a proper face of K. In this case we find a simplex σ of $T_o|vF$ which contains τ as a facet. Letting u be the vertex of σ opposite to τ, we introduce the column $\binom{f(u)}{1}$ into the system (4.11) with a nonnegative variable $\lambda_{o,k+2}$. In either case we will start tracing a path of solutions of the basic system (3.2.3) of the variable dimension algorithm based on the triangulation T_o. Since we are now on the path starting from x, which is different from the approximate stationary point we have obtained by the first application of the variable dimension algorithm, we will reach the third approximate stationary point by following the path. Once the third approximate stationary point is obtained, we again start to trace the path of (4.8).

When t reaches 1, we will further trace the path of solution to (4.9) with $l = 1$. To be more precise, we have an approximate stationary point x, a simplex τ of T_1 containing x, a cell H of K_1 and its minimal face G containing τ and also the minimal face F of K containing G. It should be noted that barring the degeneracy $\dim \tau = \dim G = \dim F$ and the cell H is uniquely determined. Denoting the vertices of τ by $u_{11}, \ldots, u_{1k+1}$, and the index set of binding constraints at F by I, (4.9) is rewritten as

$$\sum_{j=1}^{k+1} \nu_{1j} \begin{pmatrix} f(u_{1j}) \\ 1 \end{pmatrix} - \sum_{i \in I} \mu_i \begin{pmatrix} a^i \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$
$$\nu_{1j} \geq 0 \quad \text{for} \quad j = 1, \ldots, k+1,$$
$$\mu_i \geq 0 \quad \text{for} \quad i \in I.$$

We then find the point $c(G)$ of the relative interior of G and introduce the column $\binom{f(c(G))}{1}$ into the above system with a nonnegative variable ν_{21}.

In general, when t reaches $l+1$ in tracing a path of solutions to (4.9), we will do the similar procedure to continue to trace the path in $\mathcal{N}_{l+1}$. When t falls to l, we have first to find the cell H of K_{l-1} containing the current approximate point x, then to find a sequence $G_o \lhd \cdots \lhd G_k$ of faces of H such that $x \in \Delta(G_o, \ldots, G_k)$ and $\dim G_o = 0$. This sequence can be found by applying the procedure we used to show that $|\mathcal{P}_H| = H$. We evaluate the function at the vertex G_o and introduce the column $\binom{f(G_o)}{1}$ into the system

$$\sum_{j=1}^{k+1} \nu_{lj} \begin{pmatrix} f(u_{lj}) \\ 1 \end{pmatrix} - \sum_{i \in I} \mu_i \begin{pmatrix} a^i \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$
$$\nu_{lj} \geq 0 \quad \text{for} \quad j = 1, \ldots, k+1,$$
$$\mu_i \geq 0 \quad \text{for} \quad i \in I,$$

with a nonnegative variable $\nu_{l-1,1}$.

As we have seen in this section, if we fix the subdivision K_l of K, the relative interior points $c(G)$ for each face G of $\bar{K}_l$ and the triangulation T_l which refines K_l for all $l = 0, 1, \ldots$, we will obtain a sequence of simplices τ_{l_k} of T_{l_k} and approximate stationary points x_{l_k} in τ_{l_k}. By choosing an approximate subsequence, we will have a sequence of simplices and approximate stationary points such that $x_l \in \tau_l \in T_l$ for $l = 0, 1, \ldots$. Since the triangulation T_{l+1} is finer than T_l by the factor r, x_l is an approximate stationary point with the accuracy of Lemma 3.1.4 if l is sufficient large.

It is however impractical to keep K_l and $c(G)$'s in memory for all lower levels l in case the path may come down the level. Therefore we here propose an amalgamation of the restart algorithm and the continuous deformation algorithm. Namely, when the parameter t reaches its ceiling $l+1$ while tracing the path in $\mathcal{N}_l$, we continue to trace it in $\mathcal{N}_{l+1}$. This is done by finding a cell of K_{l+1} and its face containing the current simplex and generating a relative interior point of the face at which the function f should be evaluated. The relative interior points could be generated at need, but should be remembered once generated as long as we move in $\mathcal{N}_l$. When t falls down to l, we stop the continuous deformation algorithm and apply the variable dimension algorithm with the approximate stationary point at hand on a finer triangulation than T_l

by the factor r. Clearly this algorithm also generates a sequence of approximate stationary points whose cluster points are stationary points.

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