No.450

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by

Yutaka Nakamura

November 1990

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Yutaka Nakamura
Institute of Socio-Economic Planning
University of Tsukuba
1-1-1 Tennoudai, Tsukuba
305 Ibaraki, Japan

ABSTRACT

Quiggin's anticipated utility, called rank dependent utility in economics, generalizes von Neumann-Morgenstern expected utility to accommodate Allais type violations of preference judgments. His theory and the subsequent axiomatic refinements presume that the underlying consequence spaces are rich, so that certainty equivalents of every gambles exist. This paper develops an axiomatic characterization of rank dependent utility for arbitrary consequence spaces, so that certainty equivalents of gambles do not necessarily exist.

1. INTRODUCTION

In the last decade, growing research activities have been made to generalize expected utility theories of von Neumann and Morgenstern [23] and Savage [18]. Under various frameworks numerous axiom systems and their numerical representations have been developed (see surveys by Fishburn [7] and Machina [12]). In decisions under risk the anticipated utility theory by Quiggin [16] and independently by Yaari [30] initiated much research interests. Their theories are sometimes called a rank dependent utility theory in economics, since expected utility of a gamble is given by the expectation of a utility function with respect to a transformed (de)cumulative probability distribution of that gamble.

There have been made many subsequent refinements and generalizations of Quiggin's approach, i.e., rank dependent approach. They include Chateauneuf [2], Chew [3], Chew, Karni, and Safra [4], Green and Jullien [9], Hilton [10], Luce [11], Nakamura [14], Quiggin [17], Segal [21, 22], Wakker [24] and others. However, those rank dependent approaches do not fully generalize von Neumann-Morgenstern expected utility in the sense that the underlying consequence spaces must be rich, so that certainty equivalents of every gambles exist.

In decisions under uncertainty, a rank dependent approach was initiated by Schmeidler [19, 20]. We say that his approach is a Choquet expectation approach. He generalized subjective expected utility of Anscombe and Aumann [1] to allow for non-additive probabilities over finite states, and adopted Choquet [5] integration. Since then, the Choquet expectation approach motivated various axiomatic characterizations to generalize subjective expected utility theories.

They include Gilboa [8], Nakamura [13, 14] and Wakker [25-29]. Wakker [28] argued that Choquet expectation approaches are more general than rank dependent approaches. Except Gilboa's axiomatization, other Choquet expectation approaches do not apply to a full characterization of rank dependent utility, since similar richness of the consequence spaces is pressumed. Although in Gilboa's approach the consequence space is arbitrary in the sense that it includes at least three consequences that are not mutually indifferent, his axiomatization is complicated to translate it into an axiomatization of rank dependent utility. Therefore we need a different approach to fully characterize rank dependent utility.

The present paper develops such an axiomatic characterization of rank dependent utility in a general set-up such that the consequence space includes at least three consequences that are not mutually indifferent, so that certainty equivalents of gambles do not necessarily exist. Unlike Quiggin and others' rank dependent approaches, the existence of probability equivalents for gambles instead of certainty equivalents plays a key role in our approach. Our axiomatization for probability measures with bounded supports is based on weak multi-symmetric structures developed in Nakamura [14]. We also apply the truncation continuity axiom introduced by Wakker [26, 27] to obtain a rank dependent utility representation for unbounded probability measures. Since a utility function need not be bounded, our axiomatization generalizes the unbounded expected utility representation by Wakker [27].

The paper is organized as follows. Section 2 states rank dependent utility representations for arbitrary consequence spaces. Then Section 3 discusses axioms and presents representation theorems. In Section 4, we prove a rank dependent utility representation when the consequence space is finite. Then Section 5 extends the result in Section 4 to all simple probability measures. Section 6 proves a representation for bounded and unbounded probability neasures. Section 7 concludes the paper.

2. RANK DEPENDENT UTILITY

Let X be a set of consequences. By Γ we denote a Boolean algebra for X, i.e., Γ contains ϕ (empty set) and X and is closed under finite unions and complementation. We shall assume that Γ contains the singleton subset $\{x\}$ for each $x \in X$. A probability measure on Γ is a nonnegative real valued function p on Γ such that $p(X) = 1$ and $p(Y \cup Z) = p(Y) + p(Z)$ whenever $Y, Z \in \Gamma$ are disjoint. One-point measure is a probability measure p on Γ such that $p(\{x\}) = 1$ for $x \in X$. Let P be a convex set of probability measures on Γ that contains every one-point measure. Convexity means that $\lambda p + (1-\lambda)q \in P$ whenever $0 \leq \lambda \leq 1$ and $p, q \in P$. By P^S we denote the convex set of all simple probability measures on Γ, so that each $p \in P^S$ has $\sum_Y p(x) = 1$ for a finite subset $Y \subseteq X$. Each $x \in X$ is identified with a one-point probability measure p that has $p(\{x\}) = 1$. Let $\precsim$ on P be the binary preference relation with $\sim$ and $\prec$ defined in the usual way: for $p, q \in P$, $p \sim q$ if $p \precsim q$ and $q \precsim p$; $p \prec q$ if $p \precsim q$ and not($q \precsim p$). Let N denote the set of all positive integers and $N_n = \{1, \ldots, n\}$ for $n \in N$.

Suppose that V is a mapping from P into $[-\infty, +\infty]$ such that for all $p, q \in P$, $p \precsim q$ if and only if $V(p) \leq V(q)$. Let I be the closed unit interval [0, 1]. A rank dependent utility representation which we shall axiomatize in the subsequent sections has the following form of V: for all $p \in P$,

$$V(p) = \int_{0}^{+\infty} [\psi(1) - \psi(p(\{x \in X: u(x) \leq \tau\}))] d\tau - \int_{-\infty}^{0} [\psi(p(\{x \in X: u(x) \leq \tau\})) - \psi(0)] d\tau, \qquad (1)$$

where u is a real valued function on X and ψ is a strictly increasing and continuous real valued function on I. Also, u and ψ are unique up to positive linear transformations. This is similar to the representation in the Choquet expectation approach by Gilboa [8]. When p is a simple probability measure, (1) is given as follows. Suppose that $x_1 \precsim \ldots \precsim x_n$, $p(\{x_i\}) = \alpha_i$ for $i \in N_n$, and $\Sigma_{i=1}^{n} \alpha_i = 1$. Let $\beta_k = \Sigma_{i=1}^{k} \alpha_i$ for $k \in N_n$ and $\beta_0 = 0$. Then we have

$$V(p) = \sum_{i=1}^{n} (\psi(\beta_i) - \psi(\beta_{i-1})) u(x_i). \qquad (2)$$

When $\psi(\tau) = \tau$ for all $\tau \in I$, (1) and (2) reduce to expected utility representations.

We note that (1) and (2) do not necessarily imply that each p has a certainty equivalent $x_p \in X$, i.e., $V(p) = V(x_p)$. Our approach to axiomatize (1) and (2) is based on the existence of probability equivalent for a bounded $p \in P$ such that $p(\{x \in X: a \lesssim x \lesssim b\}) = 1$ for some $a,b \in X$. For $a,b \in X$ with $a \prec b$, let $P_{ab} = \{p \in P: p(\{x \in X: a \lesssim x \lesssim b\}) = 1\}$. For $p \in P_{ab}$, a probability equivalent of p, denoted $\sigma(p)$, is a probability number $\lambda \in I$ such that $p \sim \lambda a + (1-\lambda)b$. By (1), $\sigma(p)$ on P_{ab} must be unique. With no loss of generality, let $u(a) = 0$ and $u(b) = 1$. Then (1) gives that $V(p) = \psi(1) - \psi(\sigma(p))$, so for all $p,q \in P_{ab}$, $p \lesssim q$ if and only if $\psi(\sigma(q)) \leq \psi(\sigma(p))$. Thus ψ is regarded as a disutility function on P_{ab}. Our approach is first to construct ψ on P_{ab}, and then to obtain u on X such that (1) holds for all bounded $p \in P$. Detainls will be stated in the following sections.

3. AXIOMS AND THEOREMS

This section first shows necessary and sufficient axioms for the rank dependent utility representation (2) and then extends (2) to more general probability measures to obtain the representation (1). To axiomatize the representation (2), a structural assumption for P and X is given as follows.

Assumption 1. $P = P^s$ <u>and there are</u> $a,b,c \in X$ <u>such that</u> $a \prec b \prec c$.

A finite increasing sequence of consequences in X is denoted by $[x_1, \ldots, x_n]$ for distinct $x_i \in X$, $i \in N_n$, and $n \in N$, where $x_1 \lesssim \ldots \lesssim x_n$. Let Λ be the set of all finite increasing sequences of consequences. We shall denote sequences in Λ by capital letters, and write $x \in A$ when $x \in X$ is in A and $A \subseteq B$ when all $x \in A$ are in B. By $|A|$ we denote the number of elements of A. Let Λ^o be the set of all sequences in Λ such that if $A \in \Lambda^o$, then $a \prec b \prec c$ for some $a,b,c \in A$.

By p_A we denote a probability measure with a support $A \in \Lambda$. Given $A = [x_1, \ldots, x_{n+1}] \in \Lambda$, for $\alpha_1, \ldots, \alpha_n \in I$, let $p_A(\alpha_1 \ldots \alpha_n)$ denote a simple probability measure p_A such that $p_A(\{x_1, \ldots, x_i\}) = \alpha_i$ for $i \in N_{n+1}$, where $\alpha_{n+1} = 1$. Thus $\alpha_1 \leq \ldots \leq \alpha_n$. We note that a simple probability measure p has many different forms of $p_A(\alpha_1 \ldots \alpha_n)$ representation as shown in the following examples: $p_A(\alpha \ldots \alpha) = p_B(\alpha)$ when $A = [x_1, \ldots, x_{n+1}]$ and $B = [x_1, x_{n+1}]$; $p_A(\alpha\beta\beta) = p_B(\alpha\beta)$ when $A = [x, y, z, w]$ and $B = [x, y, w]$. However, each $p_A(\alpha_1 \ldots \alpha_n)$ specifies a unique simple probability measure by definition. When we write $p_A(\alpha_1 \ldots \alpha_n)$, we assume throughout the paper that A stands for $[x_1, \ldots, x_{n+1}] \in \Lambda$ for some $x_i \in X$ and $i \in N_{n+1}$, and $0 \leq \alpha_1 \leq \ldots \leq \alpha_n \leq 1$. When P_1 and P_2 are subsets of P, $P_1 \precsim P_2$ means that $p \precsim q$ for all $p \in P_1$ and all $q \in P_2$. We write $p \precsim q \precsim r$ when $p \precsim q$ and $q \precsim r$; $p \prec q \prec r$ when $p \prec q$ and $q \prec r$.

We use seven axioms for the representation (2). Six of them, which are understood as applying to all $p,q,r \in P$, all $A,B \in \Lambda$, all $a,b,x,y,z \in X$, and all $\alpha,\beta,\gamma \in I$, are stated as follows:

Axiom 1. $\precsim$ *on* P *is a weak order*.

Axiom 2. *If* $p \precsim q$ *and* $q \precsim r$, *then* $q \sim \lambda p + (1-\lambda) r$ *for some* $\lambda \in I$.

Axiom 3. *If* $A = [x, y]$ *and* $x \prec y$, *then* $\alpha \leq \beta$ *iff* $p_A(\beta) \precsim p_A(\alpha)$.

Axiom 4. *If* $A = [x, y, z]$ *and* $x \prec y$, *then* $\alpha \leq \beta$ *iff* $p_A(\beta\gamma) \precsim p_A(\alpha\gamma)$; *if* $A = [x, y, z]$ *and* $y \prec z$, *then* $\alpha \leq \beta$ *iff* $p_A(\gamma\beta) \precsim p_A(\gamma\alpha)$.

Axiom 5. *If* $A = [a, x, b]$, $B = [a, y, b]$, *and* $\alpha \prec \beta$, *then* $x \precsim y$ *iff* $p_A(\alpha\beta) \precsim p_B(\alpha\beta)$.

Axiom 6. <u>If</u> $p_A(\{x \in A: x \precsim a\}) \le p_B(\{x \in B: x \precsim a\})$ <u>for all</u> $a \in X$, <u>then</u> $p_B \precsim p_A$.

A weak order means by definition that it is complete and transitive. Axiom 2 is a continuity axiom which says that if p is not preferred to q and q is not preferred to r, then q is indifferent to some convex combination of p and q. Axioms 3 and 4 are monotonicity axioms with respect to probability numbers. We note that Axiom 4 implies Axiom 3 unless there are $a, b \in X$ such that $a \prec x \prec b$ for all $x \in X$ with $x \nsim a$ and $x \nsim b$. A monotonicity axiom with respect to consequences is stated in Axioms 5. Axiom 6 is a dominance axiom.

If $x_1 \prec x_{n+1}$ and $A = [x_1, \ldots, x_{n+1}]$, then by Axiom 6, $x_1 \precsim p_A(\alpha_1 \ldots \alpha_n) \precsim x_{n+1}$. Let $B = [x_1, x_{n+1}]$. Then Axiom 2 implies that $p_A(\alpha_1 \ldots \alpha_n) \sim p_B(\lambda)$ for some $\lambda \in I$. Since $x_1 \prec x_{n+1}$, it follows from Axiom 3 that λ must be unique. In other words, since $p_B(\lambda) = p_A(\lambda \ldots \lambda)$, there exists a unique probability equivalent $\lambda \in I$ such that $p_A(\alpha_1 \ldots \alpha_n) \sim p_A(\lambda \ldots \lambda)$. We shall denote such a λ by $\sigma_A(\alpha_1 \ldots \alpha_n)$. The following lemma shows properties of σ_A.

LEMMA 1. <u>Suppose that</u> $A = [x_1, \ldots, x_{n+1}]$ <u>and</u> $B = [y_1, \ldots, y_{m+1}]$. <u>If Axioms</u> 1-3 <u>and</u> 6 <u>hold</u>, <u>then we have</u>

(1) <u>if</u> $x_1 \prec x_{n+1}$, $x_1 \sim y_1$, <u>and</u> $x_{n+1} \sim y_{m+1}$, <u>then</u>

$p_A(\alpha_1 \ldots \alpha_n) \precsim p_B(\beta_1 \ldots \beta_m)$ <u>iff</u> $\sigma_B(\beta_1 \ldots \beta_m) \le \sigma_A(\alpha_1 \ldots \alpha_n)$.

(2) <u>if</u> $n = m$, $x_1 \prec x_{n+1}$, $x_i \sim y_i$ <u>for</u> $i \in N_{n+1}$, <u>and</u> $\alpha_i \le \beta_i$ <u>for</u> $i \in N_n$, <u>then</u> $\sigma_A(\alpha_1 \ldots \alpha_n) \le \sigma_B(\beta_1 \ldots \beta_n)$.

Proof. (1) This follows from Axioms 1,3 and 6, and the definitions of σ_A and σ_B.

(2) This follows from Axiom 6 and (1). [Q.E.D.]

With Lemma 1 at hand, we introduce the following key axiom, which applies to all $\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_n \in I$, all $n \in N$, and all $A,B \in \Lambda^0$ with $|A| = 3$ and $|B| = n+1$:

Axiom 7. _If_ $\alpha_i \le \beta_i$ _for_ $i \in N_n$, _then_

$$\sigma_A(\sigma_B(\alpha_1\ldots\alpha_n)\sigma_B(\beta_1\ldots\beta_n)) = \sigma_B(\sigma_A(\alpha_1\beta_1)\ldots\sigma_A(\alpha_n\beta_n)).$$

This axiom is a probability equivalent version of the weak multi-symmetry axiom examined in Nakamura [14], since σ_A for all $A \in \Lambda^0$ are $(|A|-1)$-ary operations that map the subsets of the product set $I\times\ldots\times I$ ($|A|-1$ times) into I and give probability equivalents for simple probability measures. In a general set-up, the weak multi-symmetry axiom generalizes the axiom in decisions under risk examined in Fishburn [7, Chap. 3] and the weak commutative axiom proposed by Chew [3]. Chew and Fishburn's axioms are concerned with certainty equivalent versions. Nakamura [13] modified their axioms and applied to decisions under uncertainty.

With Axioms 1-7, we obtain the following representation theorem for all simple probability measures. The proof is deferred to Section 5.

Theorem 1. _Suppose that Assumption_ 1 _holds_. _Then Axioms_ 1-7 _hold if and only if there exist real valued functions_ V _on_ P^S _and_ u _on_ X, _and_ *a strictly increasing and continuous real valued function* Ψ *on* I *such that for all* $p,q\in P^S$, *all* $n\in N$, *all* $A = [x_1,\ldots, x_{n+1}]\in\Lambda$, *and all* $\alpha_1,\ldots,\alpha_n\in I$ *with* $\alpha_1 \leq\ldots\leq \alpha_n$,

$$p \precsim q \text{ iff } V(p) \leq V(q);$$

$$V(p_A(\alpha_1\ldots\alpha_n)) = \sum_{i=1}^{n+1} (\Psi(\alpha_i) - \Psi(\alpha_{i-1}))u(x_i),$$

where $\alpha_0 = 0$ *and* $\alpha_{n+1} = 1$. *Moreover*, u *and* Ψ *are unique up to positive linear transformations*.

When $\Psi(\alpha) = \alpha$ for all $\alpha\in I$, the representation of Theorem 1 reduces to an expected utility representation. To impose such a condition, we need the following axiom, which is understood as applying to all $\alpha,\beta\in I$ and all $A\in\Lambda^o$ with $|A| = 3$.

Axiom 8. *If* $\alpha \leq \beta$ *and* $\gamma = \beta - \alpha$, *then* $\sigma_A(0\beta) = \sigma_A(\sigma_A(0\alpha)\sigma_A(\gamma\beta))$.

Then we obtain the following corollary of Theorem 1. The proof appears at the end of the section.

Corollary 1. *Suppose that Assumption* 1 *holds*. *Axioms* 1-8 *holds if and only if the representation of Theorem* 1 *holds with* $\Psi(\alpha) = \alpha$ *for* $\alpha\in I$.

In the sequel, we extend the representation of Theorem 1 to more general probability measures in P. To do this we need to extend Axiom 6 and add a truncation continuity axiom introduced by Wakker [26, 27]. Following Fishburn [6, Chapter 3], a subset Y of X is said to be a *preference interval* if $z \in Y$ whenever $x, y \in Y$, $x \precsim z$ and $z \precsim y$. When all preference intervals are contained in Γ, we define *upper and lower truncations* of probability measures as introduced by Wakker [26, 27]. Given $a \in X$, an upper truncation of $p \in P$, denoted p^a, is a probability measure on Γ such that

$$p^a(Y) = p(Y) \text{ for all } Y \in \Gamma \text{ with } Y \subseteq \{x \in X: x \prec a\},$$
$$p^a(\{a\}) = p(\{x \in X: a \precsim x\}),$$

and a lower truncation of $p \in P$, denoted p_a, is a probability measure on Γ such that

$$p_a(Y) = p(Y) \text{ for all } Y \in \Gamma \text{ with } Y \subseteq \{x \in X: a \prec x\},$$
$$p_a(\{a\}) = p(\{x \in X: x \precsim a\}).$$

To obtain the representation (1), we need the following structural assumption for Γ, P, and X.

Assumption 2. Γ *contains all preference intervals*, P *contains all upper and lower truncations of probability measures*, *and there are* $a, b, c \in X$ *such that* $a \prec b \prec c$.

A dominance axiom which extends Axiom 6 is stated as follows: for all $p, q \in P$, Axiom 9. *If* $p(\{x \in X: x \precsim a\}) \leq q(\{x \in X: x \precsim a\})$ *for all* $a \in X$, *then* $q \precsim p$.

This axiom is the first order stochastic dominance when X is the real line. The following axiom, which is understood as applying to all $p,q \in P$, is Wakker's truncation continuity axiom.

Axiom 10. *If* $p \prec q$, *then* $p_a \prec q$ *and* $p \prec q^b$ *for some* $a,b \in X$.

A rank dependent utility representation for general probability measures is given as follows. The proof is deferred to Section 6.

Theorem 2. *Suppose that Assumption* 2 *holds*. *Axioms* 1-5, 7, 9, *and* 10 *hold if and only if there exist a function* V *on* P, *a real valued function* u *on* X, *and a strictly increasing and continuous real valued function* Ψ *on* I *such that for all* $p,q \in P$,

$$p \precsim q \text{ iff } V(p) \leq V(q);$$

$$V(p) = \int_0^{+\infty} [\Psi(1) - \Psi(p(\{x \in X: u(x) \leq \tau\}))]d\tau - \int_{-\infty}^{0} [\Psi(p(\{x \in X: u(x) \leq \tau\})) - \Psi(0)]d\tau.$$

Moreover, u *and* Ψ *are unique up to positive linear transformations*.

Proof of Corollary 1. Suppose that Assumption 1 holds. The necessities of Axioms 1-7 follows from Theorem 1. First we show the necessity of Axiom 8. Suppose that the representation of Theorem 1 holds with $\Psi(\alpha)$ =

α for all $\alpha \in I$. Let $A = [a, b, c]$ and $a < b < c$. With no loss of generality, assume that $u(a) = 0$ and $u(c) = 1$. For all $\alpha, \beta \in I$ with $\alpha \leq \beta$, by the definition of σ_A, $p_A(\alpha\beta) \sim p_A(\sigma_A(\alpha\beta)\sigma_A(\alpha\beta))$. Then the representation of Theorem 1 gives $\sigma_A(\alpha\beta) = \alpha u(b) + \beta(1-u(b))$. Therefore, if $\alpha \leq \beta$ and $\gamma = \beta - \alpha$ for $\alpha, \beta \in I$, then

$$\sigma_A(\sigma_A(0\alpha)\sigma_A(\gamma\beta)) = u(b)\sigma_A(0\alpha) + (1-u(b))\sigma_A(\gamma\beta),$$
$$\sigma_A(0\alpha) = \alpha(1 - u(b)),$$
$$\sigma_A(\gamma\beta) = u(b)(\beta - \alpha) + \beta(1 - u(b)).$$

Substituting the last two for the first, we get

$$\sigma_A(\sigma_A(0\alpha)\sigma_A(\gamma\beta)) = \beta(1 - u(b)).$$

Letting $\alpha = 0$, $\sigma_A(\sigma_A(00)\sigma_A(\beta\beta)) = \sigma_A(0\beta) = \beta(1 - u(b))$. Hence Axiom 8 follows.

Next we show the sufficiencies of Axioms 1-8. Suppose that Axioms 1-8 hold. Then the representation of Theorem 1 holds. Let u and ψ be obtained in Theorem 1. Let $A = [a, b, c]$ and $a < b < c$. With no loss of generality, assume that $u(a) = 0$ and $u(c) = 1$. For all $\alpha, \beta \in I$ with $\alpha \leq \beta$, the definition of σ_A implies $p_A(\alpha\beta) \sim p_A(\sigma_A(\alpha\beta)\sigma_A(\alpha\beta))$. By Theorem 1, $\psi(\sigma_A(\alpha\beta)) = \psi(\alpha)u(b) + \psi(\beta)(1 - u(b))$. Therefore, if $\alpha \leq \beta$ and $\gamma = \beta - \alpha$ for $\alpha, \beta \in I$, then we obtain

$$\psi(\sigma_A(\sigma_A(0\alpha)\sigma_A(\gamma\beta))) = \psi(\sigma_A(0\alpha))u(b) + \psi(\sigma_A(\gamma\beta))(1 - u(b)),$$
$$\psi(\sigma_A(0\alpha)) = \psi(0)u(b) + \psi(\alpha)(1 - u(b)),$$

$$\psi(\sigma_A(\gamma\beta)) = \psi(\beta - \alpha)u(b) + \psi(\beta)(1 - u(b)).$$

Substituting the last two for the first, we get

$$\psi(\sigma_A(\sigma_A(0\alpha)\sigma_A(\gamma\beta)))$$
$$= u(b)^2\psi(0) + (1 - u(b))\{u(b)(\psi(\alpha) + \psi(\beta - \alpha)) + (1 - u(b))\psi(\beta)\}.$$

By Axiom 8, $\sigma_A(\sigma_A(0\alpha)\sigma_A(\gamma\beta)) = \sigma_A(0\beta)$, so $\psi(\sigma_A(\sigma_A(0\alpha)\sigma_A(\gamma\beta))) = \psi(\sigma_A(0\beta))$. Noting that $\psi(\sigma_A(0\beta)) = \psi(0)u(b) + \psi(\beta)(1 - u(b))$ and $0 < u(b) < 1$, it follows from the last equation in the preceding paragraph that for all $\alpha,\beta \in I$ with $\alpha \leq \beta$.

$$\psi(\beta) = \psi(\alpha) + \psi(\beta - \alpha) - \psi(0)$$

Since ψ is strictly increasing and continuous, the solution of the above functional equation is given by $\psi(\alpha) = \mu\alpha + \tau$ for all $\alpha \in I$ and real numbers $\mu > 0$ and τ. Since ψ is unique up to a positive linear transformation, we can take $\mu = 1$ and $\tau = 0$, so the desired result follows. [Q.E.D.]

4. A WEAK MULTI-SYMMETRIC STRUCTURE

This section shows that the set of all probability measures with a support $A \in \Lambda^0$ has a weak multi-symmetric structure developed in Nakamura [14], whose numerical structure leads to a rank dependent utility representation when X is finite. Throughout the section we assume that Assumption 1 and Axioms 1-7 hold. Let $I^n = I \times \ldots \times I$ (n times). Denote an element of I^n by $\alpha_1 \ldots \alpha_n$. Let $I^n_+ = \{\alpha_1 \ldots \alpha_n \in I^n: \alpha_1 \leq \ldots \leq \alpha_n\}$. When $A \in \Lambda^o$ and $n = |A| - 1$, σ_A can be regarded as an n-ary operation that maps I^n_+ into I.

Given $A \in \Lambda^o$ with $|A| = n+1$, we shall denote $\sigma^k_A(\alpha\beta) = \sigma_A(\alpha_1 \ldots \alpha_n)$ for $k \in N_n$ when $\alpha_i = \alpha$ for $i = 1,\ldots,k$, and $\alpha_i = \beta$ for $i = k+1,\ldots,n$. Note that $\sigma^n_A(\alpha\beta) = \sigma_A(\alpha \ldots \alpha)$. We say that k is <u>left-inessential</u> if for all $\alpha,\beta,\gamma \in I$, $\sigma^k_A(\alpha\gamma) = \sigma^k_A(\beta\gamma)$ whenever $\alpha \leq \beta \leq \gamma$, and <u>right-inessential</u> if for all $\alpha,\beta,\gamma \in I$, $\sigma^k_A(\alpha\beta) = \sigma^k_A(\alpha\gamma)$ whenever $\alpha \leq \beta \leq \gamma$. Note that n is not left-inessential but right-inessential. When k is not left (right)-inessential, we say that k is <u>left</u> (<u>right</u>)-<u>essential</u>. When k is left and right-essential, we say that k is <u>essential</u>. When there is an essential $k \in N$, we say that σ_A is <u>essential</u>. Left and right essentialities are characterized by the following lemma.

LEMMA 2. <u>Suppose that</u> $A = [x_1,\ldots, x_{n+1}] \in \Lambda^o$. <u>Then for</u> $k \in N_{n-1}$,

(1) k <u>is left-essential iff</u> $x_1 < x_{k+1}$.

(2) k <u>is right-essential iff</u> $x_{k+1} < x_{n+1}$.

Proof. We show (1). The proof of (2) is similar. Given $k \in N_{n-1}$, let $B = [x_1, x_{k+1}, x_{n+1}]$, $C = [x_1, x_{n+1}]$, and $D = [x_{k+1}, x_{n+1}]$. We note that $p_B(\alpha\alpha) = p_C(\alpha)$, $p_B(0\alpha) = p_D(\alpha)$, and $p^k_A(\alpha\beta) = p_B(\alpha\beta)$ for all $\alpha,\beta \in I$.

Suppose first that $k \in N_{n-1}$ is left-essential. By Axiom 6, $p_B(\gamma\gamma) \precsim p_B(\alpha\gamma) \precsim p_B(0\gamma)$, so $p_C(\gamma) \precsim p_B(\alpha\gamma) \precsim p_D(\gamma)$. If $x_1 \sim x_{k+1}$, then by Axiom 6, $p_C(\gamma) \sim p_D(\gamma)$. Thus by Axiom 1, $p_B(\alpha\gamma) \sim p_B(\beta\gamma)$ for all $\alpha,\beta,\gamma \in I$, so $p^k_A(\alpha\gamma) \sim p^k_A(\beta\gamma)$. Therefore, Lemma 1(1) implies that $\sigma^k_A(\alpha\gamma) = \sigma^k_A(\beta\gamma)$, so k is left-inessential. This is a contradiction. Since $x_1 \precsim x_{k+1}$, we must have $x_1 < x_{k+1}$.

Suppose next that $x_1 < x_{k+1}$. If $\alpha < \beta \leq \gamma$, then by Axiom 4, $p_B(\beta\gamma) < p_B(\alpha\gamma)$. Thus by Lemma 1(1), $\sigma_A^k(\alpha\gamma) < \sigma_A^k(\beta\gamma)$, so that k is left-essential. [Q.E.D.]

When I_1 and I_2 are subsets of I, $I_1 \leq I_2$ means that $\alpha \leq \beta$ for all $\alpha \in I_1$ and all $\beta \in I_2$. Let K be any set of consecutive integers. Given $A \in \Lambda^o$ and an essential k, we define a <u>standard sequence</u> as a set $\{\alpha_i: \alpha_i \in I, i \in K\}$ for which there exist $\alpha, \beta \in I$ such that $\alpha \nsim \beta$, either $\{\alpha, \beta\} \leq \{\alpha_i\}$ and $\sigma_A^k(\alpha\alpha_i) = \sigma_A^k(\beta\alpha_{i+1})$ for all $i, i+1 \in K$, or $\{\alpha_i\} \leq \{\alpha, \beta\}$ and $\sigma_A^k(\alpha_i\alpha) = \sigma_A^k(\alpha_{i+1}\beta)$ for all $i, i+1 \in K$. We say that a standard sequence $\{\alpha_i: i \in K\}$ is <u>strictly bounded</u> when $\alpha < \alpha_i < \beta$ for all $i \in K$ and some real numbers α and β.

For $A \in \Lambda^o$, the triple $\langle \leq, \sigma_A, I_n^+ \rangle$ that satisfies B1-B7 in the following proposition is said to be a <u>weak multi-symmetric structure</u>. The proof of the proposition appears at the end of the section.

PROPOSITION 1. <u>Suppose that</u> $A \in \Lambda^o$ <u>and</u> $|A| = n + 1$. <u>Then the following six axioms hold</u>: <u>for all</u> $\alpha, \beta, \gamma, \delta, \alpha_i, \beta_i \in I$, $i \in N_n$, <u>and all</u> $k \in N_n$:

B1. $\leq$ <u>on</u> I <u>is a weak order</u>.

B2. <u>if</u> $\{\alpha, \beta\} \leq \gamma$ <u>and</u> $\sigma_A^k(\alpha\gamma) \leq \delta \leq \sigma_A^k(\beta\gamma)$, <u>then</u> $\delta = \sigma_A^k(\lambda\gamma)$ <u>for some</u> $\lambda \in I$; <u>if</u> $\gamma \leq \{\alpha, \beta\}$ <u>and</u> $\sigma_A^k(\gamma\alpha) \leq \delta \leq \sigma_A^k(\gamma\beta)$, <u>then</u> $\delta = \sigma_A^k(\gamma\lambda)$ <u>for some</u> $\lambda \in I$.

B3. <u>if</u> k <u>is left-essential and</u> $\{\alpha, \beta\} \leq \gamma$, <u>then</u> $\alpha \leq \beta$ <u>iff</u> $\sigma_A^k(\alpha\gamma) \leq \sigma_A^k(\beta\gamma)$; <u>if</u> k <u>is right-essential and</u> $\gamma \leq \{\alpha, \beta\}$, <u>then</u> $\alpha \leq \beta$ <u>iff</u> $\sigma_A^k(\gamma\alpha) \leq \sigma_A^k(\gamma\beta)$.

B4. <u>Every strictly bounded standard sequence is finite</u>.

B5. if $\alpha_1 \leq \ldots \leq \alpha_n$, $\beta_1 \leq \ldots \leq \beta_n$, and $\alpha_i \leq \beta_i$ for $i \in N_n$, then $\sigma_A(\alpha_1 \ldots \alpha_n) \leq \sigma_A(\beta_1 \ldots \beta_n)$.

B6. if $\alpha_1 \leq \ldots \leq \alpha_n$, $\beta_1 \leq \ldots \leq \beta_n$, and $\alpha_i \leq \beta_i$ for $i \in N_n$, then $\sigma_A^k(\sigma_A(\alpha_1 \ldots \alpha_n)\sigma_A(\beta_1 \ldots \beta_n)) = \sigma_A(\sigma_A^k(\alpha_1\beta_1) \ldots \sigma_A^k(\alpha_n\beta_n))$.

The numerical representation of the weak multi-symmetric structure, $\langle \leq, \sigma_A, I_n^+ \rangle$, is given as follows. The proof is deferred to the end of the section.

PROPOSITION 2. For all $A \in \Lambda^o$ with $|A| = n + 1$, there exist real numbers $\lambda_i(A) \in I$ for $i \in N_n$ with $\sum_{i=1}^n \lambda_i(A) = 1$, and a continuous real valued function Ψ on I such that $\lambda_j(A) > 0$ and $\lambda_k(A) > 0$ for some distinct $j, k \in N_n$, and for all $\alpha, \beta, \alpha_1, \ldots, \alpha_n \in I$,

(1) $\alpha \leq \beta$ iff $\Psi(\alpha) \leq \Psi(\beta)$,

(2) if $\alpha_1 \leq \ldots \leq \alpha_n$, then $\Psi(\sigma_A(\alpha_1 \ldots \alpha_n)) = \sum_{i=1}^n \lambda_i(A)\Psi(\alpha_i)$,

(3) $\lambda_i(A)$ for $i \in N_n$ are unique and Ψ is unique up to a positive linear transformation.

To see that Proposition 2 gives the representation (2) for a finite X, we suppose that $X = \{x_1, \ldots, x_{n+1}\}$ and $A = [x_1, \ldots, x_{n+1}] \in \Lambda^o$. Then $P = \{p_A(\alpha_1 \ldots \alpha_n): \alpha_1 \leq \ldots \leq \alpha_n \text{ and } \alpha_i \in I \text{ for } i \in N_n\}$. Assign any numbers $u(x_1)$ and $u(x_{n+1})$ with $u(x_1) < u(x_{n+1})$ to x_1 and x_{n+1}, respectively. Given $\lambda_i(A) \in I$ for $i \in N_n$ obtained in Proposition 2, for $k \in N_{n-1}$, we define

$$u(x_{k+1}) = u(x_1) - (u(x_1) - u(x_{n+1})) \sum_{i=1}^{k} \lambda_i(A).$$

Thus $\lambda_k(A) = (u(x_k) - u(x_{k+1}))/(u(x_1) - u(x_{n+1}))$ for $k \in N_n$. Since $\lambda_k(A)$ for $k \in N_n$ are unique, u on X is unique up to a positive linear transformation. The representation (2) easily follows from Lemma 1(1) and Proposition 2. Hence we obtain Theorem 1 for a finite X.

Proof of Proposition 1. Suppose that $A = [x_1, \ldots, x_{n+1}] \in \Lambda^0$. For $k \in N_n$, we shall write $\sigma_k(\alpha\beta) = \sigma_A(\alpha_1 \ldots \alpha_n)$ and $p_k(\alpha\beta) = p_A(\alpha_1 \ldots \alpha_n)$ when $\alpha_i = \alpha$ for $i = 1, \ldots, k$ and $\alpha_i = \beta$ for $i = k+1, \ldots, n$. B1 follows since $\leq$ on I is a linear order.

We show the first part of B2. The second part of B2 similarily follows. Suppose that $\{\alpha, \beta\} \leq \gamma$, and $\sigma_k(\alpha\gamma) \leq \delta \leq \sigma_k(\beta\gamma)$. We are to prove that $\delta = \sigma_k(\lambda\gamma)$ for some $\lambda \in I$. By Lemma 1(1) and the definition of σ_A, $p_k(\beta\gamma) \precsim p_k(\delta\delta) \precsim p_k(\alpha\gamma)$. It follows from Axiom 2 that $p_k(\delta\delta) \sim \theta p_k(\beta\gamma) + (1-\theta)p_k(\alpha\gamma)$ for some $\theta \in I$. Let $\lambda = \theta\beta + (1-\theta)\alpha$, so $\lambda \leq \gamma$. Since $p_k(\lambda\gamma) = \theta p_k(\beta\gamma) + (1-\theta)p_k(\alpha\gamma)$, $p_k(\delta\delta) \sim p_k(\lambda\gamma)$. Thus by definition, $\delta = \sigma_k(\lambda\gamma)$.

To show B3, suppose that k is left-essential. If $k = n$, B3 easily follows. Thus assume $k \in N_{n-1}$. Let $B = [x_1, x_{k+1}, x_{n+1}]$. By Lemma 2(1), $x_1 < x_{k+1} \precsim x_{n+1}$. If $\{\alpha, \beta\} \leq \gamma$, then by Axiom 4, $\alpha \leq \beta$ iff $p_B(\beta\gamma) \precsim p_B(\alpha\gamma)$. Hence by Lemma 1(1), $\alpha \leq \beta$ iff $\sigma_k(\alpha\gamma) \leq \sigma_k(\beta\gamma)$, so the first part of B3 follows. A similar proof applies to get the second part of B3.

To show B4, let $\{\alpha_i: i \in K\}$ be a standard sequence. Since $0 \leq \alpha_i \leq 1$ for all $i \in K$, every standard sequence is strictly bounded. Suppose that for $\alpha, \beta \in I$, $\beta < \alpha$, $\{\alpha, \beta\} \leq \alpha_i$, and $\sigma_k(\alpha\alpha_i) = \sigma_k(\beta\alpha_{i+1})$ for all $i, i+1 \in K$ and an essential k. The proofs for other cases are similar. Assume that $\{\alpha_i\}$ is infinite. Since k is essential, $\beta < \alpha$, and $\sigma_k(\alpha\alpha_i) = \sigma_k(\beta\alpha_{i+1})$, B3 implies that $\alpha_i < \alpha_{i+1}$ for all $i \in K$, so that $\{\alpha_i\}$ is a bounded and strictly increasing sequence in I. Let α' be the least upper bound of $\{\alpha_i\}$. Noting that $\beta < \alpha$, $\alpha_i < \alpha'$, and $\sigma_k(\alpha\alpha_i) = \sigma_k(\beta\alpha_{i+1})$ for all $i \in K$, it follows from B3 that $\sigma_k(\alpha\alpha_i) < \sigma_k(\beta\alpha') < \sigma_k(\alpha\alpha')$ for all $i \in K$. Thus by B2 and B3, $\sigma_k(\beta\alpha') = \sigma_k(\alpha\beta')$ for some $\beta' \in I$ with $\alpha_i < \beta' < \alpha'$. On the other hand, since α' is the least upper bound of $\{\alpha_i\}$, there exists an $\alpha'' \in \{\alpha_i\}$ such that $\beta' < \alpha'' < \alpha'$. Therefore, by B3, $\sigma_k(\alpha\beta') < \sigma_k(\alpha\alpha'')$. Since $\alpha'' \in \{\alpha_i\}$, we obtain $\sigma_k(\alpha\alpha'') < \sigma_k(\beta\alpha')$. Thus $\sigma_k(\alpha\beta') < \sigma_k(\beta\alpha')$. This contradicts $\sigma_k(\beta\alpha') = \sigma_k(\alpha\beta')$. Hence $\{\alpha_i\}$ must be finite.

B5 follows from Lemma 1(2). If k = n, B6 follows from the definition of σ_A. Therefore, to show B6, we assume $k \in N_{n-1}$. Let $B = [x_1, x_{k+1}, x_{n+1}]$. We note that $p_k(\alpha\beta) = p_B(\alpha\beta)$ for all $\alpha, \beta \in I$. Thus $\sigma_k(\alpha\beta) = \sigma_B(\alpha\beta)$. Hence B6 follows from Axiom 7. [Q.E.D.]

Proof of Proposition 2. Suppose that $A = [x_1, \ldots, x_{n+1}] \in \Lambda^0$. By the definition of Λ^0, $x_1 < x_{j+1} < x_{n+1}$ for some $j \in N_{n-1}$. Thus Lemma 2 implies that j is essential, so σ_A is essential. We note by definition that σ_A is idempotent, i.e., $\sigma_A(\alpha\ldots\alpha) = \alpha$ for all $\alpha \in I$. Since by Proposition 1, $\langle \leq, \sigma_A, I_+^n \rangle$ is a weak multi-symmetric structure with σ_A essential and idempotent, it follows from Theorem 1 in Nakamura [14] that there are real numbers $\lambda_i(A) \in I$ for $i \in N_n$ with $\sum_{i=1}^n \lambda_i(A) = 1$, and a real valued function Ψ_A on I such that $\lambda_j(A) > 0$ and $\lambda_k(A) > 0$ for some distinct $j, k \in N_n$, and for all $\alpha, \beta, \alpha_1, \ldots, \alpha_n \in I$,

$$\alpha \leq \beta \text{ iff } \Psi_A(\alpha) \leq \Psi_A(\beta),$$

$$\Psi_A(\sigma_A(\alpha_1 \dots \alpha_n)) = \sum_{i=1}^{n} \lambda_i(A)\Psi_A(\alpha_i).$$

Moreover, $\lambda_i(A)$ for $i \in N_n$ are unique and Ψ_A is unique up to a positive linear transformation.

For $A,B \in \Lambda^0$ with $|A| = n+1$ and $|B| = m+1$, Axiom 7 implies that for essential $k \in N_n$ and essential $j \in N_m$,

$$\sigma_A^k(\sigma_B^j(\alpha\beta)\sigma_B^j(\gamma\delta)) = \sigma_B^j(\sigma_A^k(\alpha\gamma)\sigma_A^k(\beta\delta)),$$

which is a weak isometry condition defined in Nakamura [14]. Thus Proposition 1 in Nakamura [14] implies that $\Psi_A = \Psi_B$. Let $\Psi = \Psi_A$ for all $A \in \Lambda^0$, so (1), (2) and (3) follow.

It remains to show that Ψ is continuous. Suppose that Ψ is not left continuous. Let $\{\alpha_i\}$ be any strictly increasing sequence in I such that $\lim_{i\to\infty} \alpha_i = \alpha^*$. Then $\lim_{i\to\infty} \Psi(\alpha_i) < \Psi(\alpha^*)$, since Ψ is strictly increasing. Let $A = [x, y, z] \in \Lambda^0$. Since $\alpha_i < \alpha_{i+1} < \alpha^*$, it follows from the definition of σ_A and B3 in Proposition 1 that $\sigma_A(\alpha_i\alpha^*) < \sigma_A(\alpha_{i+1}\alpha^*)$ and $\alpha_i < \sigma_A(\alpha_i\alpha^*) < \alpha^*$. Thus $\lim_{i\to\infty} \sigma_A(\alpha_i\alpha^*) = \alpha^*$, so $\lim_{i\to\infty} \Psi(\sigma_A(\alpha_i\alpha^*)) = \lim_{i\to\infty} \Psi(\alpha_i)$. By (2),

$$\Psi(\sigma_A(\alpha_i\alpha^*)) = \lambda_1(A)\Psi(\alpha_i) + (1-\lambda_1(A))\Psi(\alpha^*).$$

Taking the limit, we obtain

$$\lim_{i\to\infty} \psi(\alpha_i) = \lambda_1(A) \lim_{i\to\infty} \psi(\alpha_i) + (1-\lambda_1(A))\psi(\alpha^*),$$

so that $\lambda_1(A) = 1$. Since $A \in \Lambda^0$, $0 < \lambda_1(A) < 1$. This is a contradiciton. Therefore, ψ is left continuous. Right continuity of ψ similarly follows. Hence ψ is continuous. [Q.E.D.]

4. PROOF OF THEOREM 1

Throughout the section we assume that Assumption 1 holds. The necessities of Axioms 1-6 easily follow, so we show the necessity of Axiom 7. Suppose that the representation of the theorem holds. For $A = [x_1, \ldots, x_{n+1}] \in \Lambda^0$, let $\lambda = \sigma_A(\alpha_1 \ldots \alpha_n)$. Then by the definition of σ_A, $V(p_A(\alpha_1 \ldots \alpha_n)) = V(p_A(\lambda \ldots \lambda))$, so the representation gives

$$\psi(\lambda)(u(x_1) - u(x_{n+1})) = \sum_{i=1}^{n} \psi(\alpha_i)(u(x_i) - u(x_{i+1})).$$

Since $u(x_1) \neq u(x_{n+1})$, we obtain

$$\psi(\sigma_A(\alpha_1 \ldots \alpha_n)) = \sum_{i=1}^{n} \psi(\alpha_i)\frac{u(x_i) - u(x_{i+1})}{u(x_1) - u(x_{n+1})}.$$

Noting the above expression, A7 easily follows.

Next we show the sufficiencies of Axioms 1-7. Suppose that Axioms 1-7 hold. For $a,b \in X$ with $a < b$, define

$$X_{ab} = \{x \in X: a \precsim x \precsim b\}.$$

For all $A = [x_1, \ldots, x_{n+1}] \in \Lambda^0$, let ψ and $\lambda_i(A)$ for $i \in N_n$ be obtained in Proposition 2. Define a real valued function λ_{ab} on X_{ab} as follows: for $A = [a, x, b] \in \Lambda$,

$$\lambda_{ab}(x) = \lambda_1(A) \text{ if } a < x < b,$$
$$= 0 \quad \text{if } x \sim a,$$
$$= 1 \quad \text{if } x \sim b.$$

Note by Proposition 2 that if $A = [a, x, b] \in \Lambda^0$, then $0 < \lambda_1(A) < 1$.

Before providing the sufficiency proof of the theorem, we need the following properties of λ_{ab}.

LEMMA 3. (1) <u>For all</u> $x, y \in X_{ab}$, $x \precsim y$ <u>iff</u> $\lambda_{ab}(x) \leq \lambda_{ab}(y)$.
(2) <u>For all</u> $A = [x_1, \ldots, x_{n+1}] \in \Lambda^0$, <u>if</u> $x_1 \sim a$ <u>and</u> $x_{n+1} \sim b$, <u>then for all</u> $\alpha_1, \ldots, \alpha_n \in I$ <u>with</u> $\alpha_1 \leq \ldots \leq \alpha_n$,

$$\psi(\sigma_A(\alpha_1 \ldots \alpha_n)) = \sum_{i=1}^{n} (\lambda_{ab}(x_{i+1}) - \lambda_{ab}(x_i))\psi(\alpha_i).$$

(3) <u>For all</u> $x, y, z \in X_{ab}$, <u>if</u> $y \in X_{xz}$, <u>then</u>

$$\lambda_{xz}(y) = \frac{\lambda_{ab}(y) - \lambda_{ab}(x)}{\lambda_{ab}(z) - \lambda_{ab}(x)}.$$

Proof. (1) Suppose that $x, y \in X_{ab}$ and $a < \{x, y\} < b$. Let $A = [a, x, b]$ and $B = [a, y, b]$. Note that if $\alpha < \beta$, then $\psi(\alpha) < \psi(\beta)$. Then it follows from Axiom 5, Lemma 1(1), and Proposition 2 that for $\alpha, \beta \in I$ with $\alpha < \beta$,

$x \lesssim y$ iff $p_A(\alpha\beta) \lesssim p_B(\alpha\beta)$

iff $\sigma_B(\alpha\beta) \leq \sigma_A(\alpha\beta)$

iff $\lambda_{ab}(y)\Psi(\alpha) + (1-\lambda_{ab}(y))\Psi(\beta) \leq \lambda_{ab}(x)\Psi(\alpha) + (1-\lambda_{ab}(x))\Psi(\beta)$

iff $\lambda_{ab}(x) \leq \lambda_{ab}(y)$.

Since $0 < \lambda_{ab}(x) < 1$ for $a < x < b$, the definition of λ_{ab} gives that for all $x,y \in X_{ab}$, $x \lesssim y$ iff $\lambda_{ab}(x) \leq \lambda_{ab}(y)$.

(2) Suppose that $a \sim x_1$, $b \sim x_{n+1}$, and $A = [x_1, \ldots, x_{n+1}] \in \Lambda^o$. It follows from Proposition 2 that

$$\Psi(\sigma_A(\alpha_1 \ldots \alpha_n)) = \sum_{i=1}^{n} \lambda_i(A)\Psi(\alpha_i).$$

We are to show that $\lambda_i(A) = \lambda_{ab}(x_{i+1}) - \lambda_{ab}(x_i)$ for $i \in N_n$. For $k \in N_n$, let $\sigma_k(\alpha\beta) = \sigma_A(\alpha_1 \ldots \alpha_n)$ and $p_k(\alpha\beta) = p_A(\alpha_1 \ldots \alpha_n)$ when $\alpha_i = \alpha$ for $i = 1, \ldots, k$ and $\alpha_i = \beta$ for $i = k+1, \ldots, n$. Then

$$\Psi(\sigma_k(\alpha\beta)) = [\sum_{i=1}^{k} \lambda_i(A)]\Psi(\alpha) + [\sum_{i=k+1}^{n} \lambda_i(A)]\Psi(\beta).$$

For $k \in N_{n-1}$, let $B_k = [a, x_{k+1}, b]$ and $C_k = [x_1, x_{k+1}, x_{n+1}]$. Then by Lemma 1(2), $\sigma_{B_k}(\alpha\beta) = \sigma_{C_k}(\alpha\beta)$. Since $p_k(\alpha\beta) = p_{C_k}(\alpha\beta)$, $\sigma_k(\alpha\beta) = \sigma_{C_k}(\alpha\beta)$. Thus $\sigma_k(\alpha\beta) = \sigma_{B_k}(\alpha\beta)$ for $k \in N_{n-1}$. It follows from Proposition 2 that if $a < x_{k+1} < b$ for $k \in N_{n-1}$, then

$$\Psi(\sigma_k(\alpha\beta)) = \Psi(\sigma_{B_k}(\alpha\beta))$$
$$= \lambda_{ab}(x_{k+1})\Psi(\alpha) + (1-\lambda_{ab}(x_{k+1}))\Psi(\beta).$$

Note by Axiom 6 and Lemma 1(1) that $\sigma_{B_k}(\alpha\beta) = \beta$ if $a \sim x_{k+1}$, and $\sigma_{B_k}(\alpha\beta) = \alpha$ if $x_{k+1} \sim b$. Thus by the definition of λ_{ab}, the above equation holds for all $k \in N_{n-1}$. Hence, $\lambda_{ab}(x_{k+1}) = \sum_{i=1}^{k}\lambda_i(A)$ for $k \in N_{n-1}$, which are solved with respect to $\lambda_i(A)$ for $i \in N_{n-1}$ to give $\lambda_i(A) = \lambda_{ab}(x_{i+1}) - \lambda_{ab}(x_i)$ for $i \in N_{n-1}$. Since $\sum_{i=1}^{n}\lambda_i(A) = 1$, $\lambda_n(A) = 1 - \lambda_{ab}(x_n)$.

(3) Suppose that $x,y,z \in X_{ab}$ and $y \in X_{xz}$. If either $y \sim x$ or $y \sim z$, then the desired result follows from (1) and the definition of $\lambda_{xz}(y)$. Thus assume that $x < y < z$. Suppose that $a \nsim x$ and $b \nsim z$. When $a = x$ or $b = z$, the proof is similar. Let $A = [a, x, y, z, b]$ and $B = [x, y, z]$. Then $A,B \in \Lambda^o$. It follows from Proposition 2 that for all $\alpha,\beta,\gamma,\delta \in I$,

$$\sigma_B(\alpha\beta) \leq \sigma_B(\gamma\delta)$$
$$\text{iff } \lambda_{xz}(y)\Psi(\alpha) + (1-\lambda_{xz}(y))\Psi(\beta) \leq \lambda_{xz}(y)\Psi(\gamma) + (1-\lambda_{xz}(y))\Psi(\delta).$$

Since $p_B(\alpha\beta) = p_A(0\alpha\beta 1)$, (2) and Lemma 1(1) imply

$$\sigma_B(\alpha\beta) \leq \sigma_B(\gamma\delta)$$
$$\text{iff } p_B(\gamma\delta) \underset{\sim}{<} p_B(\alpha\beta)$$
$$\text{iff } p_A(0\gamma\delta 1) \underset{\sim}{<} p_A(0\alpha\beta 1)$$
$$\text{iff } \sigma_A(0\alpha\beta 1) \leq \sigma_A(0\gamma\delta 1)$$
$$\text{iff } \frac{\lambda_{ab}(y) - \lambda_{ab}(x)}{\lambda_{ab}(z) - \lambda_{ab}(x)}\Psi(\alpha) + [1 - \frac{\lambda_{ab}(y) - \lambda_{ab}(x)}{\lambda_{ab}(z) - \lambda_{ab}(x)}]\Psi(\beta)$$

$$\leq \frac{\lambda_{ab}(y) - \lambda_{ab}(x)}{\lambda_{ab}(z) - \lambda_{ab}(x)} \psi(\gamma) + [1 - \frac{\lambda_{ab}(y) - \lambda_{ab}(x)}{\lambda_{ab}(z) - \lambda_{ab}(x)}]\psi(\delta).$$

Hence by Proposition 2(3), we obtain the desired result. [Q.E.D.]

Sufficiency Proof. Given $a,b \in X$ with $a < b$, assign any numbers $u_{ab}(a)$ and $u_{ab}(b)$ with $u_{ab}(a) < u_{ab}(b)$ to a and b, respectively. Then define a real valued function u_{ab} on X as follows:

$$\begin{aligned} u_{ab}(x) &= u_{ab}(a) - \lambda_{ab}(x)(u_{ab}(a) - u_{ab}(b)) && \text{if } a \lesssim x \lesssim b, \\ &= \frac{u_{ab}(a) - \lambda_{xb}(a)u_{ab}(b)}{1 - \lambda_{xb}(a)} && \text{if } x < a, \\ &= \frac{u_{ab}(b) - u_{ab}(a)(1 - \lambda_{ax}(b))}{\lambda_{ax}(b)} && \text{if } a < x. \end{aligned}$$

Given u_{ab} and u_{cd} on X as defined above, we show that u_{cd} is a positive linear transformation of u_{ab}. To do this, it suffices to show that when $c \lesssim a < b \lesssim d$, $u_{cd} = \alpha u_{ab} + \beta$ on X_{cd} for some numbers $\alpha > 0$ and β.

Assume first that $a < x < b$. By Lemma 3(3), we obtain

$$\lambda_{ab}(x) = \frac{\lambda_{cd}(x) - \lambda_{cd}(a)}{\lambda_{cd}(b) - \lambda_{cd}(a)}.$$

By definition, $\lambda_{cd}(x) = (u_{cd}(c) - u_{cd}(x))/(u_{cd}(c) - u_{cd}(d))$ and $\lambda_{ab}(x) = (u_{ab}(a) - u_{ab}(x))/(u_{ab}(a) - u_{ab}(b))$. Then substituting those for the above, we get

$$\frac{u_{ab}(a) - u_{ab}(x)}{u_{ab}(a) - u_{ab}(b)} = \frac{u_{cd}(a) - u_{cd}(x)}{u_{cd}(a) - u_{cd}(b)}.$$

Assume next that $c \precsim x < a$. By Lemma 3(3), we obtain

$$\lambda_{xb}(a) = \frac{\lambda_{cd}(a) - \lambda_{cd}(x)}{\lambda_{cd}(b) - \lambda_{cd}(x)}.$$

By definition, $\lambda_{cd}(x) = (u_{cd}(c) - u_{cd}(x))/(u_{cd}(c) - u_{cd}(d))$ and $\lambda_{xb}(a) = (u_{ab}(a) - u_{ab}(x))/(u_{ab}(b) - u_{ab}(x))$. Then substituting those for the above, we get

$$\frac{u_{ab}(a) - u_{ab}(x)}{u_{ab}(b) - u_{ab}(x)} = \frac{u_{cd}(x) - u_{cd}(a)}{u_{cd}(x) - u_{cd}(b)}.$$

Rearrangements of the last equations in the preceding two paragraphs give

$$u_{cd}(x) = \frac{u_{cd}(b) - u_{cd}(a)}{u_{ab}(b) - u_{ab}(a)} u_{ab}(x) + \frac{u_{ab}(b)u_{cd}(a) - u_{ab}(a)u_{cd}(b)}{u_{ab}(b) - u_{ab}(a)}.$$

This also follows when $b < x \precsim d$. Hence the desired result obtains.

Under appropriate positive linear transformations we can take $u = u_{ab}$ on X for all $a,b \in X$ with $a < b$. We note that u is unique up to a positive linear transformation and for all $x,y,z \in X$ with $x < y < z$,

$$\lambda_{xz}(y) = \frac{u(x) - u(y)}{u(x) - u(z)}.$$

For $A \in \Lambda$, there is a $B \in \Lambda^0$ such that $A \subseteq B$. Let $B = [x_1, \ldots, x_n]$ and $A = [x_{i_1}, \ldots, x_{i_{m+1}}]$. Then $p_A(\alpha_1 \ldots \alpha_m) = p_B(\alpha'_1 \ldots \alpha'_n)$ when

$$\begin{aligned} \alpha'_j &= 0 \quad \text{for } 1 \le j < i_1 \\ &= \alpha_k \quad \text{for } i_k \le j < i_{k+1} \\ &= 1 \quad \text{for } i_{m+1} \le j \le n. \end{aligned}$$

Then define V on P^0 as follows:

$$\begin{aligned} &V(p_A(\alpha_1 \ldots \alpha_m)) \\ &\quad = (u(x_1) - u(x_{n+1}))\Psi(\sigma_B(\alpha'_1 \ldots \alpha'_n)) - u(x_1)\Psi(0) + u(x_{n+1})\Psi(1). \end{aligned}$$

Let $a = x_1$, $b = x_{n+1}$, $\alpha_0 = 0$, and $\alpha_{m+1} = 1$. It follows from Lemma 3(2) and the preceding paragraph that

$$\begin{aligned} \Psi(\sigma_B(\alpha'_1 \ldots \alpha'_n)) &= \sum_{i=1}^{n} (\lambda_{ab}(x_{i+1}) - \lambda_{ab}(x_i))\Psi(\alpha'_i) \\ &= \sum_{i=1}^{n} \frac{u(x_i) - u(x_{i+1})}{u(a) - u(b)} \Psi(\alpha'_i) \\ &= \frac{1}{u(a)-u(b)}\{\sum_{k=1}^{m+1} (\Psi(\alpha_k) - \Psi(\alpha_{k-1}))u(x_{i_k}) \\ &\qquad + u(a)\Psi(0) - u(b)\Psi(1)\}. \end{aligned}$$

Therefore,

$$V(p_A(\alpha_1 \ldots \alpha_m)) = \sum_{i=1}^{m+1} (\Psi(\alpha_k) - \Psi(\alpha_{k-1}))u(x_{i_k}),$$

which does not depend on the choice of B as long as $A \subseteq B$ and $B \in \Lambda^0$.

Given $A, B \in \Lambda$, there is a $C \in \Lambda^0$ such that $A \subseteq C$ and $B \subseteq C$. Let $p_A(\alpha_1 \ldots \alpha_k) = p_C(\alpha_1' \ldots \alpha_n')$ and $p_B(\beta_1 \ldots \beta_m) = p_C(\beta_1' \ldots \beta_n')$. Then it follows from Lemma 1(1), Proposition 2, and the definition of V that

$$
\begin{aligned}
p_A(\alpha_1 \ldots \alpha_k) \precsim p_B(\beta_1 \ldots \beta_m) \text{ iff } & p_C(\alpha_1' \ldots \alpha_n') \precsim p_C(\beta_1' \ldots \beta_n') \\
\text{iff } & \sigma_C(\beta_1' \ldots \beta_n') \le \sigma_C(\alpha_1' \ldots \alpha_n') \\
\text{iff } & \psi(\sigma_C(\beta_1' \ldots \beta_n')) \le \psi(\sigma_C(\alpha_1' \ldots \alpha_n')) \\
\text{iff } & V(p_A(\alpha_1 \ldots \alpha_k)) \le V(p_B(\beta_1 \ldots \beta_m)).
\end{aligned}
$$

Hence the representation of the theorem obtains [Q.E.D.]

6. PROOF OF THEOREM 2

Suppose that Assumption 2 holds. The necessity proof easily follows, so we give the sufficiency proof. We shall assume that Axioms 1-5, 7, 9, and 10 hold. Since $P^S \subseteq P$, the representation of Theorem 1 holds. Let u and ψ be obtained in Theorem 1. The uniqueness part of the theorem follows from Theorem 1. Since ψ is unique up to a positive linear transformation, with no loss of generality we assume that $\psi(0) = 0$ and $\psi(1) = 1$. Let R be the real line. Note that $\{x \in X: u(x) \le \tau\}$ for all $\tau \in R$ are preference intervals. Let $f_p(\tau) = p(\{x \in X: u(x) \le \tau\})$ for $p \in P$, so $0 \le f_p(\tau) \le 1$. Then define

$$
\begin{aligned}
A_p(\alpha) &= \{\tau \in R: f_p(\tau) > \alpha\}, \\
B_p(\alpha) &= \{\tau \in R: f_p(\tau) < \alpha\}.
\end{aligned}
$$

To prove the sufficiencies of the axioms, we need the following lemma.

Lemma 4. _Let_ $\tau^+ = \inf A_p(\alpha)$ _and_ $\tau^- = \sup B_p(\alpha)$. _Then for each_ $\varepsilon > 0$, _there are_ $a,b \in X$ _such that_ $\tau^+ \leq u(a) < \tau^+ + \varepsilon$ _and_ $\tau^- - \varepsilon < u(b) \leq \tau^-$.

Proof. First we show that $\tau^+ \leq u(a) < \tau^+ + \varepsilon$ for some $a \in X$. Suppose on the contrary that there is no $x \in X$ such that $\tau^+ \leq u(x) < \tau^+ + \varepsilon$. Then by the definition of f_p, $f_p(\tau) = f_p(\tau^+)$ for $\tau^+ < \tau < \tau^+ + \varepsilon$. Since $\tau \in A_p(\alpha)$ for $\tau^+ < \tau < \tau^+ + \varepsilon$, $f_p(\tau) > \alpha$. Thus $f_p(\tau^+) > \alpha$. However, $f_p(\tau^+) = \sup_{\tau < \tau^+} f_p(\tau) \leq \alpha$, since $\tau^+ = u(x)$ for no $x \in X$. This is a contradiction. Hence for each $\varepsilon > 0$, $\tau^+ \leq u(a) < \tau^+ + \varepsilon$ for some $a \in X$.

Next we show that $\tau^- - \varepsilon < u(b) \leq \tau^-$ for some $b \in X$. Suppose on the contrary that there is no $x \in X$ such that $\tau^- - \varepsilon < u(x) \leq \tau^-$. Then by the definition of f_p, $f_p(\tau) = f_p(\tau^-)$ for $\tau^- - \varepsilon < \tau < \tau^-$. Since $\tau \in B_p(\alpha)$ for $\tau^- - \varepsilon < \tau < \tau^-$, $f_p(\tau) < \alpha$. Thus $f_p(\tau^-) < \alpha$.

If there is no $x \in X$ such that $\tau^- < u(x) < \tau^- + \varepsilon$, then $f_p(\tau) = f_p(\tau^-)$ for $\tau^- < \tau < \tau^- + \varepsilon$, so $f_p(\tau) < \alpha$. However, for $\tau^- < \tau$, $f_p(\tau) \geq \alpha$, a contradiction. Therefore, for each $\varepsilon > 0$, there is a $c \in X$ such that $\tau^- < u(c) < \tau^- + \varepsilon$. Thus we can take a decreasing sequence $\{\tau_i\}$ such that $\tau^- < \tau_{i+1} < \tau_i$ for $i \in N$, $\lim_{i\to\infty} \tau_i = \tau^-$, and $\tau_i = u(x_i)$ for some $x_i \in X$. Let $g_p(\tau) = p(\{x \in X: u(x) > \tau\})$, so $f_p(\tau) + g_p(\tau) = 1$. Noting that $g_p(\tau) \leq 1 - \alpha$ if $\tau^- < \tau$, we obtain $\sup_{i \in N} g_p(\tau_i) \leq 1 - \alpha$. Thus $g_p(\tau^-) \leq 1 - \alpha$, since $\tau^- = u(x)$ for no $x \in X$. However, it follows from the preceding paragraph that $g_p(\tau) > 1 - \alpha$ if $\tau \leq \tau^-$. This is a contradiction. Hence $\tau^- - \varepsilon < u(b) \leq \tau^-$ for some $b \in X$. [Q.E.D.]

Sufficiency Proof. We prove the sufficiencies of the axioms in three steps. We define two subsets of P as follows.

$$P^o = \{p \in P: p(\{x \in X: a \precsim x \precsim b\}) = 1 \text{ for some } a,b \in X\},$$
$$P^* = \{p \in P: a \precsim p \precsim b \text{ for some } a,b \in X\}.$$

By definitions, $P^S \subseteq P^o \subseteq P^* \subseteq P$. The first step is concerned with the convex set P^o of all probability measures with bounded supports. The second step extends the first step to the convex set P^* of all probability measures that are bounded in preferences. Finally the third step covers all $p \in P$. Since $\psi(0) = 0$ and $\psi(1) = 1$, define

$$E_\psi(u, p) = \int_0^{+\infty} [1 - \psi(p(\{x \in X: u(x) \leq \tau\}))]d\tau$$
$$- \int_{-\infty}^{0} \psi(p(\{x \in X: u(x) \leq \tau\}))d\tau.$$

In the following three steps, we are to show that for all $p \in P$, $E_\psi(u, p)$ are well defined and for all $p,q \in P$, $p \precsim q$ iff $E_\psi(u, p) \leq E_\psi(u, q)$.

Step 1 (all $p \in P^o$). Suppose $p \in P^o$. Then there are $a,b \in X$ such that $p(\{x \in X: a \precsim x \precsim b\}) = 1$. By Axiom 9, $a \precsim p \precsim b$, so by Axiom 2, $p \sim \lambda a + (1-\lambda)b$ for some $\lambda \in I$. Then define $V(p) = \psi(\lambda)u(a) + (1-\psi(\lambda))u(b)$. Thus by Axiom 1, for all $p,q \in P^o$, $p \precsim q$ iff $V(p) \leq V(q)$. Since p has a bounded support, $E_\psi(u, p)$ is well defined.

We are to show that $V(p) = E_\psi(u, p)$ for all $p \in P^o$. Suppose that $p(\{x \in X: a \precsim x \precsim b\}) = 1$. Since u is unique up to a positive linear transformation, we assume with no loss of generality that $u(a) = 0$ and $u(b) = 1$. Then it suffices to show that

$$V(p) = \int_0^1 [1 - \psi(f_p(\tau))]d\tau.$$

To do this we construct two sequences of simple probability measures that converge uniformly to p from above and below, respectively.

For $k \in N_{n-1}$ and $n \geq 2$, let $\tau_{kn} = \inf A_p(\psi^{-1}(\frac{k}{n}))$ and $\tau_{nn} = 1$, so $0 \leq \tau_{1n} \leq \ldots \leq \tau_{nn}$. Let $M = \{k \in N_{n-1}: \tau_{k+1,n} - \tau_{kn} > 0\}$ and $\tau_n^+ = \min_{k \in M}\{\tau_{k+1,n} - \tau_{kn}\}$, so $0 < \tau_n^+ < 1$. Suppose that $0 < \varepsilon_n < \tau_n^+$ and $0 < \delta_n < \varepsilon_n \tau_n^+$. It follows from Lemma 4 that there are $a_{kn} \in X$ for all $k \in N_{n-1}$ such that $\tau_{kn} \leq u(a_{kn}) < \tau_{kn} + \delta_n$. When $\tau_{kn} = u(c)$ for some $c \in X$, we take $a_{kn} = c$. Define $x_{kn} \in X$ for $k \in N_n$ recursively from n to 1 as follows:

$$\begin{aligned} x_{nn} &= b \\ x_{kn} &= a_{kn} \quad \text{if } k \in M \\ &= x_{k+1,n} \quad \text{if } k \notin M. \end{aligned}$$

Denote $M = \{k_1, \ldots, k_m\} \subseteq N_{n-1}$. Then let $A = [a_{k_1 n}, \ldots, a_{k_{m+1} n}]$ and $\alpha_i = \psi^{-1}(\frac{k_i}{n})$ for $i \in N_m$, where $a_{k_{m+1} n} = b$. By Axiom 9, $p \precsim p_A(\alpha_1 \ldots \alpha_m)$. Thus $V(p) \leq V(p_A(\alpha_1 \ldots \alpha_m))$, so by Theorem 1, we have

$$V(p) \leq 1 - \sum_{i=1}^{m} \psi(\alpha_i)(u(a_{k_{i+1} n}) - u(a_{k_i n}))$$

$$= 1 - \sum_{i=1}^{m} \frac{k_i}{n}(u(a_{k_{i+1}n}) - u(a_{k_in}))$$
$$= 1 - \sum_{i=1}^{n-1} \frac{i}{n}(u(x_{i+1,n}) - u(x_{in})),$$

since $u(x_{i+1,n}) - u(x_{in}) = 0$ when $i \notin M$. For $k \in M$, we obtain

$$\begin{aligned} \varepsilon_n \tau_n^+ > \delta_n &\Rightarrow \varepsilon_n(\tau_{k+1,n} - \tau_{kn}) > \delta_n \\ &\Rightarrow \varepsilon_n(\tau_{k+1,n} - \tau_{kn}) > \frac{k}{n}\delta_n \\ &\Rightarrow (\frac{k}{n} - \varepsilon_n)(\tau_{k+1,n} - \tau_{kn}) < \frac{k}{n}(\tau_{k+1,n} - \tau_{kn} - \delta_n) \\ &\Rightarrow (\frac{k}{n} - \varepsilon_n)(\tau_{k+1,n} - \tau_{kn}) < \frac{k}{n}(u(x_{k+1,n}) - u(x_{kn})). \end{aligned}$$

Thus $V(p) < 1 - \sum_{k=1}^{n-1} (\frac{k}{n} - \varepsilon_n)(\tau_{k+1,n} - \tau_{kn})$.

For $k = 2, \ldots, n$, let $\tau'_{kn} = \sup B_p(\psi^{-1}(\frac{k-1}{n}))$, $\tau'_{1n} = 0$, and $\tau'_{n+1,n} = 1$, so $\tau'_{1n} \leq \ldots \leq \tau'_{n+1,n}$. Let $M = \{k \in N_n : \tau'_{k+1,n} - \tau'_{kn} > 0\}$ and $\tau_n^- = \min_{k\in M}\{\tau'_{k+1,n} - \tau'_{kn}\}$, so $0 < \tau_n^- < 1$. Suppose that $0 < \varepsilon'_n < \tau_n^-$ and $0 < \delta'_n < \varepsilon'_n \tau_n^-$. It follows from Lemma 4 that there are $b_{kn} \in X$ for $k \in N_n$ such that $\tau'_{kn} - \delta'_n < u(b_{kn}) \leq \tau'_{kn}$. When $\tau'_{kn} = u(c)$ for some $c \in X$, we take $b_{kn} = c$. Define $y_{kn} \in X$ for $k \in N_{n+1}$ recursively from n+1 to 1 as follows:

$$\begin{aligned} y_{n+1,n} &= b \\ y_{kn} &= b_{kn} \quad \text{if } k \in M \\ &= y_{k+1,n} \quad \text{if } k \notin M. \end{aligned}$$

Denote $M = \{k_1, \ldots, k_m\} \subseteq N_n$. Then let $B = [b_{k_1 n}, \ldots, b_{k_{m+1} n}]$ and $\beta_i = \psi^{-1}(\frac{k_i}{n})$ for $i \in N_m$, where $b_{k_{m+1} n} = b$. By Axiom 9, $p_B(\beta_1 \ldots \beta_m) \precsim p$. Thus $V(p_B(\beta_1 \ldots \beta_m)) \le V(p)$, so by Theorem 1, we have

$$\begin{aligned} V(p) &\ge 1 - \sum_{i=1}^{m} \psi(\beta_i)(u(b_{k_{i+1} n}) - u(b_{k_i n})) \\ &= 1 - \sum_{i=1}^{m} \frac{k_i}{n}(u(b_{k_{i+1} n}) - u(b_{k_i n})) \\ &= 1 - \sum_{i=1}^{n} \frac{i}{n}(u(y_{i+1,n}) - u(y_{in})), \end{aligned}$$

since $u(y_{i+1,n}) - u(y_{in}) = 0$ when $i \notin M$. For $k \in M$, we obtain

$$\begin{aligned} \varepsilon_n' \bar{\tau}_n > \delta_n' &\Rightarrow \varepsilon_n'(\tau_{k+1,n}' - \tau_{kn}') > \delta_n' \\ &\Rightarrow \varepsilon_n'(\tau_{k+1,n}' - \tau_{kn}') > \frac{k}{n}\,\delta_n' \\ &\Rightarrow (\frac{k}{n} + \varepsilon_n')(\tau_{k+1,n}' - \tau_{kn}') > \frac{k}{n}\,(\tau_{k+1,n}' - \tau_{kn}' + \delta_n') \\ &\Rightarrow (\frac{k}{n} + \varepsilon_n')(\tau_{k+1,n}' - \tau_{kn}') > \frac{k}{n}\,(u(y_{k+1,n}) - u(y_{kn})). \end{aligned}$$

Thus $V(p) > 1 - \sum_{k=1}^{n} (\frac{k}{n} + \varepsilon_n')\,(\tau_{k+1,n}' - \tau_{kn}')$.

It follows from the preceding paragraphs that

$$\sum_{k=1}^{n-1} (\frac{k}{n} - \varepsilon_n)\,(\tau_{k+1,n} - \tau_{kn}) < 1 - V(p) < \sum_{k=1}^{n} (\frac{k}{n} + \varepsilon_n')\,(\tau_{k+1,n}' - \tau_{kn}').$$

Since $\varepsilon_n > 0$ and $\varepsilon_n' > 0$ are arbitrarily small, when n gets large, we must have

$$1 - V(p) = \int_0^1 \psi(f_p(\tau))d\tau,$$

so that the desired representation obtains.

Step 2 (all $p \in P^*$). Suppose that $a \precsim p \precsim b$ for some $a,b \in X$. Let $A = [a, b]$. By Axiom 2, $p \sim p_A(\lambda)$ for some $\lambda \in I$. Then define $V(p) = \psi(\lambda)u(a) + (1-\psi(\lambda))u(b)$. Thus by Axiom 1, for all $p,q \in P^*$, $p \precsim q$ iff $V(p) \leq V(q)$. We are to show that $V(p) = E_\psi(u, p)$. We have two cases to examine: p is either unbounded above or unbounded below; p is unbounded from both sides.

Case 1 (either unbounded above or unbounded below). Suppose that p is unbounded above but bounded below. Then $p(\{x \in X\colon a \precsim x\}) = 1$ for some $a \in X$. When p is unbounded below but bounded above, the proof is similar. By Axiom 9, $p^x \prec p$ for all $x \in X$. Since p^x for all $x \in X$ are in P^0, $\sup_{x \in X} V(p^x) \leq V(p)$. We note that $E_\psi(u, p)$ is well defined. By Step 1 and the definition of $E_\psi(u, p)$, we have

$$\sup_{x \in X} V(p^x) = \sup_{x \in X} E_\psi(u, p^x) = E_\psi(u, p).$$

Therefore, $E_\psi(u, p) \leq V(p)$. We show that the equality holds. Assume that $a \precsim p \precsim b$ and $E_\psi(u, p) < V(p)$. With no loss of generality, let $u(a) = 0$ and $u(b) = 1$. Let $A = [a, b]$ and $p \sim p_A(\lambda)$. Then $E_\psi(u, p) < 1 - \psi(\lambda)$. Since ψ is strictly increasing and continuous, there is a $\lambda' \in I$ such that $E_\psi(u, p) < 1 - \psi(\lambda') < 1 - \psi(\lambda)$. Thus $p_A(\lambda') \prec p$. By Axiom

10, $p_A(\lambda') \prec p^c$ for some $c \in X$. Since $E_\psi(u, p^c) \le E_\psi(u, p)$, $E_\psi(u, p^c) < 1 - \psi(\lambda')$. Noting that p^c and $p_A(\lambda')$ are in P^o, it follows from Step 1 that $p^c \prec p_A(\lambda')$. This is a contradiction. Hence $V(p) = E_\psi(u, p)$.

Case 2 (unbounded from both sides). Suppose that $a \precsim p \precsim b$. If $E_\psi(u, p)$ is well defined, then a similar proof of Case 1 applies to obtain that $V(p) = E_\psi(u, p)$. Thus it suffices to verify that $E_\psi(u, p)$ is well defined. Suppose that $E_\psi(u, p)$ is undefined, so that

$$\int_0^{+\infty} [1 - \psi(p(\{x \in X: u(x) \le \tau\}))]d\tau = \int_{-\infty}^{0} \psi(p(\{x \in X: u(x) \le \tau\}))d\tau = +\infty.$$

Since p is unbounded below, there is a $c \in X$ such that $c \prec a$. Thus $c \prec p$. By Axiom 10, $c \prec p^d$ for some $d \in X$. Since p^d is bounded above, Case 1 implies that $V(c) < V(p^d) = E_\psi(u, p^d) = -\infty$. This is a contradiction. Hence $E_\psi(u, p)$ is well defined.

Step 3 (all $p \in P$). Suppose that p is not contained in P^*. Then there are no $a, b \in X$ such that $a \precsim p \precsim b$. Thus for all $x \in X$, either $x \prec p$ or $p \prec x$. A similar proof of Case 2 in Step 2 gives that $E_\psi(u, p)$ is well defined. It suffices to show that if we define $V(p) = E_\psi(u, p)$, then for all $p, q \in P$, $p \precsim q$ iff $V(p) \le V(q)$. To do this, it suffices to verify the following claim.

Claim 1. (1) _If for all_ $x \in X$, _either_ $x \prec \{p, q\}$ _or_ $\{p, q\} \prec x$, _then_ $p \sim q$ _and_ $E_\psi(u, p) = E_\psi(u, q)$.

(2) *If* $x \prec p$ *for all* $x \in X$, *then* $E_\psi(u, p) \geq \sup_{x \in X} u(x)$.

(3) *If* $p \prec x$ *for all* $x \in X$, *then* $E_\psi(u, p) \leq \inf_{x \in X} u(x)$.

Proof. (1) Suppose that $x \prec \{p, q\}$ for all $x \in X$. When $\{p, q\} \prec x$ for all $x \in X$, the proof is similar. First we show that $p \sim q$. Assume $p \prec q$. Then by Axiom 10, $p \prec q^a$ for some $a \in X$. By Axiom 9, $q^a \precsim a$, so by Axiom 1, $p \prec a$. This is a contradiction. Hence $q \precsim p$. If $q \prec p$, then similarly we obtain a contradiction, so $p \precsim q$. Hence $p \sim q$.

Next we show that $E_\psi(u, p) = E_\psi(u, q)$. Assume that $E_\psi(u, p) < E_\psi(u, q)$. Since $E_\psi(u, p) = \sup_{x \in X} E(u, p^x)$ and $E_\psi(u, q) = \sup_{x \in X} E(u, q^x)$, there is an $a \in X$ such that $E_\psi(u, p^x) < E_\psi(u, q^a)$ for all $x \in X$. Since p^x and q^a are bounded above, it follows from Axiom 9 and the hypothesis of the claim that $y \precsim p^x \precsim x$ and $z \precsim q^a \precsim a$ for some $y, z \in X$. Thus Step 2 implies that $p^x \prec q^a$ for all $x \in X$. Since q is unbounded above, $a \prec b$ for some $b \in X$. Then we have $q^a \prec b$. By the hypothesis of the claim, $b \prec q$, so $q^a \prec q$. Since $p \sim q$, $q^a \prec p$. By Axiom 10, $q^a \prec p^c$ for some $c \in X$. This is a contradiction. When $E_\psi(u, q) < E_\psi(u, p)$, we obtain a similar contradiction. Hence $E_\psi(p, u) = E_\psi(q, u)$.

(2) Suppose that $x \prec p$ for all $x \in X$. Assume $\sup_{x \in X} u(x) > E_\psi(u, p)$. Then there is an $a \in X$ such that $u(a) > E_\psi(u, p)$. Since $a \prec p$, Axiom 10 implies that $a \prec p^b$ for some $b \in X$. By Axiom 9, $a \prec p^b \precsim b$, so Step 2 gives $u(a) < E_\psi(u, p^b)$. Since $E_\psi(u, p)$ is well defined, $E_\psi(u, p^b) \leq E_\psi(u, p)$. Thus $u(a) < E_\psi(u, p)$, a contradiction. Hence $E_\psi(u, p) \geq \sup_{x \in X} u(x)$.

(3) Similar to (2). [Q.E.D.]

7. CONCLUSIONS

The purpose of this paper has been to provide an axiomatic characterization of rank dependent utility for arbitrary consequence spaces. First we established a rank dependent utility representation for all simple probability measures, where the consequence space includes at least three elements that are not mutually indifferent. A key axiom is a probability equivalent version of the weak multi-symmetry axiom. Then we applied Wakker's truncation continuity axiom to obtain the representation for general probability measures, where a utility function need not be bounded.

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